TOHOKU UNIVERSITY

Graduate School of Information Sciences

MASTER THESIS

On Univalent Functions with Half-integer Coefficients

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Preface

The theory of univalent functions is one of the most beautiful subjects in geometric function theory. Its origin can be traced to 1851, when the well-known mapping theorem was formulated by Riemann in his Ph.D. thesis. The *Riemann mapping theorem* states that if D is a non-empty *domain* (simply connected open subset in the complex plane $\mathbb{C}$), then there exists an *injective* and *holomorphic mapping* f which maps D onto the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. This function is known as the *Riemann mapping*. However, his proof was incomplete. The first complete proof was given by Carathéodory in 1912 and used Riemann surfaces. It was simplified by Koebe two years later in a way which did not require these. (See e.g. [1, 8, 12, 20, 25].)

A single-valued function f is said to be *univalent* (or *schlicht*) in a domain $D \subset \mathbb{C}$ if it is injective in D. Without loss of generality, we may assume that f is normalized by $f(0) = f'(0) - 1 = 0$ and defined on $\mathbb{D}$, that is, functions *analytic* and univalent have a Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

on $\mathbb{D}$. Most of this thesis is concerned with the class $\mathcal{S}$ of such functions. An important example of a function in this class is the *Koebe function*

$$\frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n,$$

because it plays the extremal role in many problems. Closely related to $\mathcal{S}$ is the class Σ of functions

$$g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}$$

analytic and univalent in the domain $\mathbb{D}^* = \{z \in \mathbb{C} : |z| > 1\}$ exterior to $\bar{\mathbb{D}}$, with the exception of a single pole at ∞ with residue 1.

In 1909 Koebe showed that the class $\mathcal{S}$ is *compact* with respect to the topology of locally uniform convergence. Since a_n is a continuous functional, the maximum defined as

$$A_n := \max_{f \in \mathcal{S}} |a_n(f)|, \quad n = 2, 3, \ldots$$

exists. Gronwall [13] obtained the first result with respect to the coefficient problem in 1914. The *area theorem* is an inequality that expresses the relation between the range of a function $g \in \Sigma$ and the area of its complement. This is fundamental to the theory of univalent functions. Without knowing Gronwall's work, Bieberbach [5] proved the same relation and derived the coefficient result within the class $\mathcal{S}$ in 1916. It states that the sharp bound of the second coefficient a_2 of a function in the class $\mathcal{S}$ is $|a_2| \leq 2$. This result is deduced from the relation $|b_1| \leq 1$, which is a consequence of the area theorem considering the class Σ. In a footnote, he wrote "Vielleicht ist überhaupt $A_n = n$. (Perhaps it is generally $A_n = n$.)". Since the Koebe function plays the extremal role in so many problems for the class $\mathcal{S}$ as we mentioned above, it is natural to suspect that it maximizes $|a_n|$ for all n. This is the famous conjecture of Bieberbach, first proposed in 1916, which remained one of the major problems of this field.

For many years this problem stood as a challenge and has inspired the development of ingenious methods which now form the backbone of the entire subject. There are usually two ways to approach Bieberbach's conjecture. The first one is to investigate the coefficients for a certain value n. For example, in 1923 Loewner [18] proved $|a_3| \leq 3$, in 1955 Garabedian and Schiffer [11] proved $|a_4| \leq 4$, in 1968 Pederson [23] and Ozawa [21] proved $|a_6| \leq 6$ and in 1972 Pederson and Schiffer [24] proved $|a_5| \leq 5$. The second way is to analyze it for some special univalent functions which include starlike, convex, spirallike, close-to-convex functions, and so on.

This conjecture remained unsolved until 1985, when de Branges [7] gave a remarkable proof. Many partial results were obtained in the intervening years, including results for special subclasses of $\mathcal{S}$ and for particular coefficients, as well as asymptotic estimates and estimates for general n. For example, in 1925 Littlewood [17] proved $|a_n| < e \cdot n$, in 1965 Milin [19] proved $|a_n| < 1.243 \cdot n$ and in 1972 FitzGerald [9] proved $|a_n| < \sqrt{7/6} \cdot n = 1.0801 \ldots \cdot n$.

The purpose of Chapter 1 in this thesis is to review the general principles underlying this thesis. After giving the notation, definitions and some coefficient estimates, we introduce a univalence criterion for polynomials.

Chapter 2 is devoted to investigate the generalizations of the area theorem mentioned above. In particular, the proof of the area theorem can be generalized to produce a system of inequalities called the *Grunsky inequalities*, which are necessary and sufficient conditions for the univalence of the associated function. These inequalities contain a wealth of useful information about the coefficients of univalent functions, leading to an elementary proof of Bieberbach's conjecture for $n = 4$.

In the last chapter, we consider Friedman's theorem, which is a part of Salem's theorem on univalent functions. In 1945 Salem [28] proved a theorem on univalent functions with integer coefficients, which states that if $f \in \mathcal{S}$ and there exists an index p such that for $n \geq p$ all coefficients a_n are rational integers or integers of an imaginary quadratic field, then $f(z)$ is rational. Spencer mentioned this result in a seminar on univalent functions held at New York University and wondered whether it was possible to prove this theorem by elementary means. Friedman [10] proved a part of Salem's theorem, which states that if all coefficients of $f \in \mathcal{S}$ are rational integers then f has only nine forms. Linis [16] gave a short proof of Friedman's theorem and extended to the Gaussian integer ring. One year later Royster [27] extended the method of the proof given by Linis to quadratic fields with negative discriminant. Since these previous results are very interesting, we investigate what happens if all coefficients of $f \in \mathcal{S}$ are half-integers, that is, $2a_n \in \mathbb{Z}$ and we show that such a function has only 19 forms. For this aim, we assemble theories given in advance with complicated calculations using the computational software program Mathematica.

Contents

Chapter 1

Univalent Function Theory

This chapter introduces the class $\mathcal{S}$ and Σ of *univalent functions*. Most of the elementary results concerning the first class are direct consequences of the *area theorem*, which may be regarded as the foundation of the entire subject.

1.1 Introduction

A *domain* is an open connected set in the complex plane $\mathbb{C}$. The *unit disk* $\mathbb{D}$ consists of all points $z \in \mathbb{C}$ with $|z| < 1$. A single-valued function f is said to be *univalent* in a domain $D \subset \mathbb{C}$ if it is injective; that is, if $f(z_1) \neq f(z_2)$ for all points z_1 and z_2 in D with $z_1 \neq z_2$. Furthermore the function f is said to be *locally univalent* at a point $z_0 \in D$, if it is univalent in some neighborhood of z_0. For analytic functions f, the condition $f'(z_0) \neq 0$ is equivalent to local univalence at z_0.

Without loss of generality, we may assume that f is normalized by $f(0) = f'(0) - 1 = 0$ and defined on $\mathbb{D}$, that is, functions *analytic* and univalent have a Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

on $\mathbb{D}$. Most of this thesis is concerned with the class $\mathcal{S}$ of such functions. An important example of a function in the class $\mathcal{S}$ is the *Koebe function*

$$\frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n,$$

because it plays the extremal role in many problems for the class $\mathcal{S}$.

Closely related to $\mathcal{S}$ is the class Σ of functions

$$g(\zeta) = \zeta + b_0 + \sum_{n=1}^{\infty} b_n \zeta^{-n}$$

analytic and univalent in the domain $\mathbb{D}^* = \{\zeta \in \mathbb{C} : |\zeta| > 1\}$ exterior to $\bar{\mathbb{D}}$, except for a single pole at ∞ with residue 1. For each $f \in \mathcal{S}$, the function

$$g(\zeta) = \{f(1/\zeta)\}^{-1} = \zeta - a_2 + (a_2^2 - a_3)\zeta^{-1} + \cdots$$

belongs to the class Σ. This transformation is called an *inversion.*

In the end of this section, we mention the *square-root transformation* $f(z) \longmapsto \sqrt{f(z^2)}$ for $f \in \mathcal{S}$. Since $f(z) = 0$ only at the origin, a single-valued branch of the square root may be chosen as

$$\begin{aligned}\phi(z) &= \sqrt{f(z^2)} = z\sqrt{1 + a_2 z^2 + a_3 z^4 + \cdots} \\ &= z + \frac{a_2}{2} z^3 + \left(\frac{a_3}{2} - \frac{a_2^2}{8}\right) z^5 + \cdots .\end{aligned}$$

The function ϕ is an odd function, i.e. $\phi(-z) = -\phi(z)$. Since f is univalent on $\mathbb{D}$, if $\phi(z_1) = \phi(z_2)$, that is, if $f(z_1^2) = f(z_2^2)$, then $z_1^2 = z_2^2$, which implies $z_1 = \pm z_2$. But if $z_1 = -z_2$, then

$$\phi(z_1) = \phi(-z_2) = -\phi(z_2) = -\phi(z_1).$$

Thus $\phi(z_1) = 0$ and $z_1^2 = 0$. This shows that $z_1 = z_2 = 0$, so that ϕ is univalent. Therefore $\phi \in \mathcal{S}$.

More generally, let $\mathcal{S}^{(m)}$ be the subclass of $\mathcal{S}$ consisting of all functions

$$f(z) = z + \sum_{\nu=1}^{\infty} a_{m\nu+1} z^{m\nu+1},$$

with m-fold symmetry, where $m = 2, 3, \ldots$. Then the mth-root transform $g(z) = \{f(z^m)\}^{-m}$ is univalent and so belongs to the subclass $\mathcal{S}^{(m)}$ of all functions in the class $\mathcal{S}$ with m-fold symmetry. Conversely, every $f \in \mathcal{S}^{(m)}$ is the mth-root transform of some $f \in \mathcal{S}$.

1.2 The Area Theorem

Gronwall [13] obtained the first result with respect to the coefficient problem in 1914. The univalence of the function

$$g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}$$

in the class Σ restricts the value of the Laurent coefficients b_n, which can be observed below.

Theorem 1.1 (Area Theorem)**.** *If $g \in \Sigma$ then*

$$\sum_{n=1}^{\infty} n|b_n| \leq 1.$$

Proof. Let E be the complement in $\mathbb{C}$ of the image domain of g. Set $r > 1$ and let C_r be the image of the circle $|z| = r$ under g. Since g is univalent, C_r is a simple closed curve which encloses a domain $E_r \supset E$. An application of Green's theorem shows that the area A_r of E_r is given by

$$A_r = \frac{i}{2} \iint_{E_r} d(w d\bar{w}) = \frac{i}{2} \int_{C_r} w d\bar{w} = \frac{i}{2} \int_{|z|=r} g(z)\overline{g'(z)dz}.$$

Since $g(z)\overline{g'(z)dz} = g(z)\overline{izg'(z)d\theta}$ and using the Laurent series expansion of g, we have

$$\begin{aligned} A_r &= \frac{i}{2} \int_0^{2\pi} -i \left(z + \sum_{m=0}^{\infty} b_m z^m \right) \overline{\left(z + \sum_{n=0}^{\infty} n b_n z^n \right)} d\theta \\ &= \pi \left(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n} \right). \end{aligned}$$

Letting r tend to 1, we obtain

$$m(E) = \pi \left(1 - \sum_{n=1}^{\infty} n|b_n|^2 \right)$$

where $m(E)$ is the area (i.e. the Lebesgue measure) of $g(E)$ and therefore $m(E) \geq 0$. This proves the theorem. □

1.3 Bieberbach's Conjecture

Bieberbach [5] proved a very important coefficient relation within the class $\mathcal{S}$ in 1916, which was considered separately by Gronwall [13].

Theorem 1.2 (Bieberbach's Theorem). *If $f \in \mathcal{S}$ then $|a_2| \leq 2$. Equality occurs if and only if f is the Koebe function or one of its rotations.*

This result can be deduced from the relation $|b_1| \leq 1$, which is a consequence of the area theorem as follows:

Proof. A square-root transformation and an inversion applied to $f \in \mathcal{S}$ produces a function

$$g(z) = \sqrt{f(1/z^2)} = z - \frac{a_2}{2} z^{-1} + \cdots,$$

which belongs to the class Σ. Using the area theorem (Theorem 1.1), we have

$$|b_1| = \left| \frac{a_2}{2} \right| \leq 1.$$

Therefore this proves the theorem. □

Bieberbach formulated the following famous conjecture. It is based on the fact that $a_n = n$ for the Koebe function.

Conjecture 1.3. *If $f \in \mathcal{S}$ then $|a_n| \leq n$ for $n \geq 2$.*

For many years this famous problem has stood as a challenge and inspired the development of ingenious methods which now form the backbone of the entire subject. This conjecture remained unsolved until 1985, when de Branges [7] gave a remarkable proof.

Theorem 1.4. *If $f \in \mathcal{S}$ then*

$$|a_n| \leq n \tag{1.1}$$

for $n \geq 2$. Equality occurs if and only if f is the Koebe function or one of its rotations.

Many partial results were obtained in the intervening years, including results for special subclasses of $\mathcal{S}$ and for particular coefficients, as well as asymptotic estimates and estimates for general n.

In the end of this section, we give another coefficient estimate, which is a univalence criterion for normalized polynomials in the class $\mathcal{S}$.

Lemma 1.5. *Let $f(z) = z + \sum_{n=2}^{N} a_n z^n$. If $f \in \mathcal{S}$ then*

$$|a_N| \leq \frac{1}{N}. \tag{1.2}$$

Proof. Let $p(z) = z + \sum_{n=2}^{N} a_n z^n \in \mathcal{S}$. Then, from the local univalence of p, $p'(z) \neq 0$ in $\mathbb{D}$. In other words, the roots of the equation

$$p'(z) = 1 + 2a_2 z + \cdots + N a_N z^{N-1} = 0$$

must have modulus greater than unity. From the fundamental theorem of algebra, this equation has exactly $N-1$ complex roots with multiplicity. Let $\zeta_1, \ldots, \zeta_{N-1}$ be the roots of the equation. Applying Viète's formulas, we have

$$\zeta_1 \cdots \zeta_{N-1} = (-1)^{N-1} \frac{1}{N a_N}.$$

Since $|\zeta_n| \geq 1$ for $n = 1, 2, \ldots, N-1$, we obtain

$$|\zeta_1 \cdots \zeta_{N-1}| = \left| \frac{1}{N a_N} \right| \geq 1$$

Therefore, we have

$$|a_N| \leq \frac{1}{N}.$$

□

From this lemma, we can conclude whether a polynomial is univalent or not by only looking at the coefficient of maximum degree.

Chapter 2

Generalizations of the Area Principle

This chapter is devoted to investigate the generalizations of the area theorem mentioned previously.

2.1 Prawitz' Inequality

The proof of the main theorem of this thesis is based on an inequality discovered by Prawitz [26]. Before we mention Prawitz' inequality, we introduce two lemmas. These are also given by Prawitz in [26].

Lemma 2.1. *Let Γ be an analytic Jordan curve, bounding a finite domain D. R and Φ shall be polar coordinates with respect to their origin $O \in D$ for a non-negative and monotonic function $g(R)$. Then*

$$\int_\Gamma g(R)d\Phi \geq 0. \tag{2.1}$$

Lemma 2.2. *Let the origin O for the polar coordinates R and Φ lie outside of the domain D bounded by the Jordan curve Γ. Then*

$$\int_\Gamma g(R)d\Phi \geq 0 \quad \text{if } g(R) \text{ is non-decreasing},$$
$$\int_\Gamma g(R)d\Phi \leq 0 \quad \text{if } g(R) \text{ is non-increasing},$$

for a non-negative function $g(R)$.

Let

$$f(z) = Re^{i\Phi} = z + a_2 z^2 + \cdots$$

be an analytic and univalent function of $z = re^{i\phi}$ and set $r < 1$ and C_r be the image of the circle $|z| = r$ under f. Since C_r is an analytic Jordan curve, the inequality (2.1) holds for every non-negative monotonic function $g(R)$. From the Cauchy-Riemann differential equations, we have

$$d\Phi = \frac{\partial \Phi}{\partial \phi} d\phi = \frac{r}{R} \frac{\partial R}{\partial r} d\phi.$$

Setting $g(R) = R$, the function $G'(R)$ or equivalently

$$g(R) = \frac{d}{d \log R} G(R)$$

becomes

$$\int_{C_r} g(R) d\Phi = r \int_0^{2\pi} G'(R) \frac{\partial R}{\partial r} d\phi = r \frac{\partial}{\partial r} \int_0^{2\pi} G(R) d\phi \geq 0. \tag{2.2}$$

Now, it is necessary to choose a non-increasing function for $g(R)$. Especially choosing $g(R) = R^{-\alpha}$, results in $G(R) = -R^{-\alpha}/\alpha$ and 2.2 turns into

$$\frac{\partial}{\partial r} \int_0^{2\pi} R^{-\alpha} d\phi \leq 0 \tag{2.3}$$

for $r < 1$.

Now we set

$$\left(\frac{f(z)}{z} \right)^{-\frac{\alpha}{2}} = \sum_{n=0}^{\infty} \sigma_n z^n.$$

Since $(R/r)^{-\alpha} = |f(z)/z|^{-\alpha}$, we have

$$R^{-\alpha} = r^{-\alpha} \left(\sum_{n=0}^{\infty} \sigma_n z^n \right) \left(\sum_{n=0}^{\infty} \bar{\sigma}_n \bar{z}^n \right).$$

Thus (2.3) follows

$$-\alpha r^{-\alpha-1} + (2-\alpha)|\sigma_1|^2 r^{1-\alpha} + (4-\alpha)|\sigma_2|^2 r^{3-\alpha} + \cdots \leq 0$$

and therefore

$$\sum_{n=0}^{\infty} \frac{(2n-\alpha)}{\alpha} |\sigma_n|^2 r^{2n} \leq 1.$$

Letting r tend to 1, we finally obtain the theorem below.

Theorem 2.3 (Prawitz' Inequality)**.** *Let* $f \in \mathcal{S}$ *and* $[z/f(z)]^{\alpha/2} = \sum_{n=0}^{\infty} \sigma_n z^n$. *Then*

$$\sum_{n=0}^{\infty} \frac{(2n-\alpha)}{\alpha} |\sigma_n|^2 \leq 1$$

for all real α.

In particular, for $\alpha = 2$ we have the following.

Corollary 2.4. *Let* $f \in \mathcal{S}$ *and* $z/f(z) = \sum_{n=0}^{\infty} c_n z^n$. *Then*

$$\sum_{n=1}^{\infty} (n-1)|c_n|^2 \leq 1. \tag{2.4}$$

This is essentially equivalent to the area theorem, because $c_n = b_{n-1}$. Since

$$\begin{aligned}
\frac{z}{f(z)} &= \frac{z}{z + a_2 z^2 + a_3 z^3 + \cdots} \\
&= \frac{1}{1 - (-a_2 z - a_3 z^2 - \cdots)} \\
&= 1 + (-a_2 z - a_3 z^2 - \cdots) + (-a_2 z - a_3 z^2 - \cdots)^2 + \cdots \\
&= 1 - a_2 z + (a_2^2 - a_3) z^2 + \cdots ,
\end{aligned}$$

we particularly have

$$\begin{aligned}
&c_0 = 1, \quad c_1 = -a_2, \quad c_2 = a_2^2 - a_3, \quad c_3 = -a_2^3 + 2a_2 a_3 - a_4, \\
&c_4 = a_2^4 - 3a_2^2 a_3 + a_3^2 + 2a_2 a_4 - a_5, \\
&c_5 = -a_2^5 + 4a_2^3 a_3 - 3a_2 a_3^2 - 3a_2^2 a_4 + 2a_3 a_4 + 2a_2 a_5 - a_6, \\
&c_6 = a_2^6 - 5a_2^4 a_3 + 6a_2^2 a_3^2 - a_3^3 + 4a_2^3 a_4 \\
&\qquad\qquad - 6a_2 a_3 a_4 + a_4^2 - 3a_2^2 a_5 + 2a_3 a_5 + 2a_2 a_6 - a_7, \\
&c_7 = -a_2^7 + 6a_2^5 a_3 - 10a_2^3 - 10a_2^3 a_3^2 + 4a_2 a_3^3 - 5a_2^4 a_4 + 12a_2^2 a_3 a_4 - 3a_3^2 a_4 \\
&\qquad - 3a_2 a_4^2 + 4a_2^3 a_5 - 6a_2 a_3 a_5 + 2a_4 a_5 - 3a_2^2 a_6 + 2a_3 a_6 + 2a_2 a_7 - a_8.
\end{aligned}$$

Alternatively, we also can represent the coefficients c_n as follows:

Lemma 2.5. *Let $f \in \mathcal{S}$ and $z/f(z) = \sum_{n=0}^{\infty} c_n z^n$. Then the coefficient c_n for $n \geq 1$ can be computed by the determinant*

$$c_n = (-1)^n \begin{vmatrix} a_2 & 1 & & \cdots & 0 \\ a_3 & a_2 & 1 & \cdots & 0 \\ a_4 & a_3 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n+1} & a_n & & \cdots & a_2 \end{vmatrix}.$$

Proof. Let

$$\frac{1}{1 + a_2 z + a_3 z^2 + \cdots} = 1 + c_1 z + c_2 z^2 + \cdots .$$

It follows

$$1 = \left(1 + \sum_{n=1}^{\infty} a_{n+1} z^n\right)\left(1 + \sum_{m=1}^{\infty} c_m z^m\right).$$

This implies that

$$a_{k+1} + c_k + \sum_{l=1}^{k-1} a_{k-l+1} c_l = 0 \tag{2.5}$$

for every $k \geq 1$. This equation is equivalent to

$$\begin{pmatrix} 1 & & & \cdots & 0 \\ a_2 & 1 & & \cdots & 0 \\ a_3 & a_2 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_k & a_{k-1} & \cdots & a_2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} -a_2 \\ -a_3 \\ -a_4 \\ \vdots \\ -a_{k+1} \end{pmatrix}.$$

Applying Cramer's rule, we obtain

$$c_k = \begin{vmatrix} 1 & & \cdots & 0 & -a_2 \\ a_2 & 1 & \cdots & 0 & -a_3 \\ a_3 & a_2 & \cdots & 0 & -a_4 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_k & a_{k-1} & \cdots & a_2 & -a_{k+1} \end{vmatrix} = (-1)^k \begin{vmatrix} a_2 & 1 & & \cdots & 0 \\ a_3 & a_2 & 1 & \cdots & 0 \\ a_4 & a_3 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k+1} & a_k & & \cdots & a_2 \end{vmatrix}.$$

□

2.2 Grunsky's Inequality

This section introduces *Grunsky's inequality*, which has become one of the most powerful tools in the theory of univalent functions. The proof is simple. However, a great difficulty is constructing Grunsky's coefficients. They are very complicated expressions in the terms of coefficients b_n of the function $g \in \Sigma$. Therefore, it is convenient first to introduce the *Faber polynomials.*

Let $g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n} \in \Sigma$. For a fixed $w \in \mathbb{C}$, the function $(g(\zeta) - w)/\zeta$ is analytic for large ζ and vanishes at ∞. Hence, we can write

$$\log \frac{g(\zeta) - w}{\zeta} = -\sum_{n=1}^{\infty} \frac{1}{n} F_n(w) \zeta^{-n}. \tag{2.6}$$

If we differentiate both sides with respect to ζ and put $F_0(w) \equiv 1$, we obtain

$$\frac{g'(\zeta)}{g(\zeta) - w} = \sum_{n=0}^{\infty} F_n(w) \zeta^{-n+1}.$$

Thus, we have

$$\zeta - \sum_{n=1}^{\infty} n b_n \zeta^{-n} = \left(\zeta + b_0 - w + \sum_{n=1}^{\infty} b_n \zeta^{-n} \right) \left(1 + \sum_{n=1}^{\infty} F_n(w) \zeta^{-n} \right)$$

$$= \zeta + b_0 - w + \sum_{n=1}^{\infty} b_n \zeta^{-n} + (\zeta + b_0 - w) \sum_{n=1}^{\infty} F_n(w) \zeta^{-n} + \sum_{n=2}^{\infty} \sum_{k+l=n} b_k F_l(w) \zeta^{-n}.$$

Comparing the coefficients, we obtain $F_1(w) = w - b_0$ and the recursion formula

$$F_{n+1}(w) = (w - b_0) F_n(w) - \sum_{m=1}^{n-1} b_{n-m} F_m(w) - (n+1) b_n.$$

It follows by induction that $F_n(w)$ is a polynomial of degree n of the form

$$F_n(w) = (w - b_0)^n - n b_1 (w - b_0)^{n-2} + \cdots .$$

We call $F_n(w)$ the n-th *Faber polynomial* of the function $g(z)$. In particular, we obtain

$$\begin{aligned} &F_0(w) = 1, \quad F_1(w) = w - b_0, \quad F_2(w) = (w - b_0)^2 - 2b_1, \\ &F_3(w) = (w - b_0)^3 - 3b_1(w - b_0) - 3b_2, \\ &F_4(w) = (w - b_0)^4 - 4b_1(w - b_0)^2 - 4b_2(w - b_0) + 2b_1^2 - 4b_3. \end{aligned} \tag{2.7}$$

The Faber polynomials play an important role in complex approximation theory.

Now we introduce *Grunsky's coefficients.* We can write

$$\log \frac{g(\zeta) - g(z)}{\zeta - z} = -\sum_{k,l=1}^{\infty} \beta_{k,l} z^{-k} \zeta^{-l}.$$

where $|z| > R \geq 1$ and $|\zeta| > R$, because $g(z) \neq g(\zeta)$ with $z \neq \zeta$ and $g'(z) \neq 0$ so that the function on the left-hand side is analytic in this domain. We call $\beta_{k,l}$ *Grunsky's coefficients* of the function $g(z)$. It is clear that $\beta_{k,l} = \beta_{l,k}$. Inserting $w = g(z)$ into equation (2.6), we obtain

$$\begin{aligned}\log \frac{g(\zeta) - g(z)}{\zeta - z} &= \log \frac{g(\zeta) - g(z)}{\zeta} - \log \frac{\zeta - z}{\zeta} \\ &= -\sum_{k=1}^{\infty} F_k(g(z))\zeta^{-k} + \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{z}{\zeta}\right)^k \\ &= -\sum_{k=1}^{\infty} \frac{1}{k}(F_k(g(z)) - z^k)\zeta^{-k}.\end{aligned}$$

Therefore we have

$$F_k(g(z)) = z^k + k \sum_{l=1}^{\infty} \beta_{k,l} z^{-l}. \tag{2.8}$$

From equations (2.7) and (2.8) we can compute

$$\begin{aligned}&\beta_{1,k} = \beta_{k,1} = b_k, \\ &\beta_{2,2} = b_3 + \frac{1}{2}b_1^2, \quad \beta_{2,3} = b_4 + b_1 b_2, \quad \beta_{2,4} = b_5 + b_1 b_3 + \frac{1}{2}b_2^2, \\ &\beta_{3,3} = b_5 + b_1 b_3 + b_2^2 + \frac{1}{3}b_1^3, \quad \beta_{3,4} = b_6 + b_1 b_4 + 2b_2 b_3 + b_1^2 b_2, \\ &\beta_{4,4} = b_7 + b_1 b_5 + 2b_2 b_4 + 2b_1 b_2^2 + b_1^2 b_3 + \frac{3}{2}b_3^2 + \frac{1}{4}b_1^4.\end{aligned}$$

The following theorem is also a generalization of the area theorem.

Theorem 2.6 (Grunsky's Inequality). *Let $g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n} \in \Sigma$, $\beta_{k,l}$ be Grunsky's coefficients of g and $\lambda_k \in \mathbb{C}$. Then*

$$\sum_{k=1}^{\infty} k \left| \sum_{l=1}^{N} \beta_{k,l} \lambda_l \right|^2 \leq \sum_{k=1}^{N} \frac{|\lambda_k|^2}{k}$$

for every $N \in \mathbb{N}$.

Proof. We consider the polynomial

$$h(w) = \sum_{k=1}^{m} \frac{\lambda_k}{k} F_k(w).$$

We conclude from (2.8) that

$$\phi(z) := h(g(z)) = \sum_{k=1}^{m} \frac{\lambda_k}{k} z^k + \sum_{l=1}^{\infty} \beta_{k,l} z^{-l} = \sum_{k=1}^{m} \frac{\lambda_k}{k} z^k + \sum_{k=1}^{\infty} d_k z^{-k}$$

where $d_k = \sum_{l=1}^{m} \beta_{k,l} \lambda_l$. Let E be the complement in $\mathbb{C}$ of the image domain of g. Set $r > 1$ and let $C_r = \{z \in \mathbb{C} : |z| = r\}$. Since g is univalent, $g(C_r)$ is a simple closed curve which encloses a domain $E_r \supset E$. An application of Green's theorem and the substitution $w = g(z), z = re^{i\theta}$ shows that

$$\begin{aligned}
0 &\leq \frac{1}{\pi} \iint_{E_r} |h'(w)|^2 du dv \\
&= \frac{1}{2\pi i} \iint_{E_r} d(\bar{h} dh) = \frac{1}{2\pi i} \int_{g(C_r)} \bar{h} dh \\
&= \frac{1}{2\pi i} \int_{C_r} \bar{\phi} d\phi = \frac{1}{2\pi} \int_0^{2\pi} \overline{\phi(z)} \phi'(z) z d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k=1}^{m} \frac{\bar{\lambda}_k}{k} \bar{z}^k + \sum_{k=1}^{\infty} \bar{d}_k \bar{z}^{-k} \right) \left(\sum_{k=1}^{m} \lambda_k z^k - \sum_{k=1}^{\infty} k d_k z^{-k} \right) d\theta \\
&= \sum_{k=1}^{m} \frac{|\lambda_k|^2}{k} r^{2k} - \sum_{k=1}^{\infty} k |d_k|^2 r^{-2k}
\end{aligned}$$

Letting r tend to 1, we obtain

$$0 \leq \frac{1}{\pi} \int_E |h'(w)|^2 d\Omega = \sum_{k=1}^{m} \frac{|\lambda_k|^2}{k} - \sum_{k=1}^{\infty} k |d_k|^2$$

where $d\Omega$ is the two-dimensional Lebesgue measure. Therefore this proves the theorem. $\square$

Setting $\lambda_m = 1$ and $\lambda_l = 0$ for $l \neq m$, we have

$$\sum_{k=1}^{\infty} k|\beta_{k,m}| \leq \frac{1}{m}.$$

Since $\beta_{k,1} = b_k$, it is easy to see that this inequality for $m = 1$ is equivalent to the area theorem.

We now observe that Theorem 2.6 implies

$$\sum_{k=1}^{N} k \left| \sum_{l=1}^{N} \beta_{k,l}\lambda_l \right|^2 \leq \sum_{k=1}^{N} \frac{|\lambda_k|^2}{k}.$$

Setting $\lambda_k / k^{1/2} = z_k$, we have

$$\begin{aligned}\sum_{k=1}^{N} k \left| \sum_{l=1}^{N} \beta_{k,l}\lambda_l \right|^2 &= \sum_{k=1}^{N} k \sum_{l=1}^{N} \beta_{k,l}\lambda_l \sum_{m=1}^{N} \bar{\beta}_{k,m}\bar{\lambda}_m \\ &= \sum_{l=1}^{N} \sum_{k=1}^{N} k\sqrt{lm}\beta_{k,l}\bar{\beta}_{k,m} z_l \bar{z}_m \\ &\leq \sum_{k=1}^{N} z_k \bar{z}_k = \sum_{l,m=1}^{N} \delta_{l,m} z_l \bar{z}_m\end{aligned}$$

where $\delta_{l,m}$ is the Kronecker delta. Thus, we obtain

$$\sum_{l,m=1}^{N} \left(\delta_{l,m} - \sum_{k=1}^{N} k\sqrt{lm}\beta_{k,l}\bar{\beta}_{k,m} \right) z_l \bar{z}_m \geq 0.$$

Seting $A^{(N)}_{l,m} = \delta_{l,m} - \sum_{k=1}^{N} k\sqrt{lm}\beta_{k,l}\bar{\beta}_{k,m}$ and $A^{(N)} = \{A^{(N)}_{l,m}\}_{l,m=1}^{N}$, we can construct a matrix, which we call *Grunsky's matrix* of the function g. Grunsky's matrix $A^{(N)}$ is *Hermitian*, so that it is positive semi-definite. Therefore we have the following lemma.

Lemma 2.7. *Let N be a positive integer and $A^{(N)}$ be Grunsky's matrix of the function $g \in \Sigma$. Then every principal minor of $A^{(N)}$ is non-negative.*

Let $f \in \mathcal{S}$, $a_n \in \mathbb{R}$, $g \in \Sigma$ and

$$\frac{z}{f(z)} = \sum_{n=0}^{\infty} c_n z^n.$$

Since $b_n = c_{n+1} \in \mathbb{R}$, we particularly obtain the following, which are detailed values of the determinant of Grunsky's matrix for $N \leq 3$.

$$\begin{aligned}
\det A^{(1)} =& 1 - c_2^2, \\
\det A^{(2)} =& c_2^6 + 4c_4c_2^4 - c_2^4 - 4c_3^2c_2^3 + 4c_4^2c_2^2 - 4c_4c_2^2 \\
& - c_2^2 - 8c_3^2c_4c_2 + 4c_3^4 - 4c_3^2 - 4c_4^2 + 1, \\
\det A^{(3)} =& - (c_2^6 - c_2^5 + 5c_4c_2^4 - c_2^4 - 5c_3^2c_2^3 - 5c_4c_2^3 + 3c_6c_2^3 + 3c_3^2c_2^2 + 3c_4^2c_2^2 \\
& - 3c_4c_2^2 - 12c_3c_5c_2^2 - 3c_6c_2^2 + c_2^2 - 3c_3^2c_2 - 6c_4^2c_2 - 6c_5^2c_2 + 12c_3^2c_4c_2 \\
& + c_4c_2 + 12c_3c_5c_2 + 6c_4c_6c_2 - 3c_6c_2 + c_2 - 6c_3^4 - 6c_4^3 + 5c_3^2 \\
& + 3c_4^2 + 6c_5^2 - 6c_3^2c_4 + 2c_4 + 12c_3c_4c_5 - 6c_3^2c_6 - 6c_4c_6 + 3c_6 - 1) \\
& (c_2^6 + c_2^5 + 5c_4c_2^4 + c_2^4 - 5c_3^2c_2^3 + 5c_4c_2^3 + 3c_6c_2^3 + 2c_2^3 - 3c_3^2c_2^2 + 3c_4^2c_2^2 \\
& + 3c_4c_2^2 - 12c_3c_5c_2^2 + 3c_6c_2^2 + c_2^2 + 3c_3^2c_2 + 6c_4^2c_2 - 6c_5^2c_2 + 12c_3^2c_4c_2 \\
& + 5c_4c_2 - 12c_3c_5c_2 + 6c_4c_6c_2 + 3c_6c_2 + c_2 - 6c_3^4 - 6c_4^3 + c_3^2 - 3c_4^2 \\
& - 6c_5^2 + 6c_3^2c_4 + 2c_4 + 12c_3c_4c_5 - 6c_3^2c_6 + 6c_4c_6 + 3c_6 + 1).
\end{aligned}$$

These values are used in the proof of the main theorem of this thesis.

Chapter 3

Main Result

The subclasses of $\mathcal{S}$ whose coefficients a_n belong to a quadratic field have been studied by Friedman [10] and Bernardi [4]. Linis [16] gave a short proof of Friedman's theorem. In this chapter, we investigate Friedman's theorem and introduce its extensions including a new result.

3.1 Friedman's Theorem and its Extensions

Salem [28] proved the following theorem for power series with integer coefficients.

Theorem 3.1. *Let the function $f(z)$ be meromorphic in $\mathbb{D}$ and let its expansion be*

$$\sum_{n=-k}^{\infty} a_n z^n.$$

Suppose that there exists an index p such that for $n \geq p$ all coefficients a_n are rational integers or integers of an imaginary quadratic field. Let α be any complex or real number. If there exists two positive numbers δ, η ($\eta < 1$) such that $|f(z) - \alpha| > \delta$ at every point z in the ring $1 - \eta < |z| < 1$, then f is rational.

From this theorem, we immediately obtain the theorem below for univalent functions with integer coefficients.

Theorem 3.2 (Salem's Theorem)**.** *If $f \in \mathcal{S}$ and there exists an index p such that for $n \geq p$ all coefficients a_n are rational integers or integers of an imaginary quadratic field, then $f(z)$ is rational.*

Friedman [10] proved the following theorem which is a part of Salem's theorem.

Theorem 3.3 (Friedman's Theorem)**.** *Let $f \in \mathcal{S}$. If all coefficients a_n are rational integers, then $f(z)$ is one of the following nine functions:*

$$z, \quad \frac{z}{1 \pm z}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{(1 \pm z)^2}, \quad \frac{z}{1 \pm z + z^2}.$$

Proof. Set $F(z) = z/f(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, then the coefficients c_n are rational integers. Since $c_1 = -a_2$ and $|a_2| \leq 2$, it follows that $|c_1| \leq 2$. Applying inequality (2.4), we have $|c_2| \leq 1$ and $c_n = 0$ for $n \geq 3$. Therefore, the possible values for c_n are:

$$c_1 = 0, \pm 1, \pm 2; \;\; c_2 = 0, \pm 1; \;\; c_n = 0 \text{ for } n \geq 3.$$

From the combination of these values we obtain 15 functions. However, the following six functions must be rejected as having zeros in $\mathbb{D}$:

$$1 \pm 2z, \quad 1 \pm 2z - z^2, \quad 1 \pm z - z^2.$$

The remaining nine functions prove the theorem. □

This method of the proof was given by Linis [16]. He also proved the theorem below.

Theorem 3.4. *Let $f \in \mathcal{S}$. If all coefficients a_n are Gaussian integers, then f has 15 forms.*

Note that Gaussian integers are complex numbers whose real and imaginary part are both rational integers, i.e. elements of the set $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ where $i = \sqrt{-1}$.

Royster [27] extended the method of the proof given by Linis to quadratic fields with negative discriminant as follows:

Theorem 3.5. *Let $f \in \mathcal{S}$. If all coefficients a_n are algebraic integers in the quadratic field $\mathbb{Q}(\sqrt{d})$ for some square-free rational negative integer d, then f has 36 forms.*

Algebraic integers in $\mathbb{Q}(\sqrt{d})$ are given by

$$\begin{cases} a + b\sqrt{d} & \text{where } a, b \in \mathbb{Z} & \text{if } d \equiv 2 \text{ or } 3 \pmod 4 \\ \dfrac{a + b\sqrt{d}}{2} & \text{where } a, b \in 2\mathbb{Z} \text{ or } a, b \in 2\mathbb{Z} + 1 & \text{if } d \equiv 1 \pmod 4 \end{cases}$$

As we can see, Gaussian integers are a case of algebraic integers in $\mathbb{Q}(\sqrt{d})$ with $d = -1$.

Linis [16] and Royster [27] have obtained new results by replacing the condition of Theorem 3.3 "rational integers" with other conditions. We shall now consider what happens if all coefficients a_n of the function $f \in \mathcal{S}$ are half-integers. Here, a_n is said to be a *half-integer* if $2a_n$ is a rational integer.

From Lemma 1.5, it is easy to see that all of univalent polynomials having half-integer coefficients which belong to the class $\mathcal{S}$ are:

$$z, \quad z \pm \frac{1}{2}z^2.$$

The following is the main theorem of this thesis.

Theorem 3.6. *Let $f \in \mathcal{S}$. If all coefficients a_n are half-integers then $f(z)$ is one of the following 19 functions:*

$$z, \quad z \pm \frac{1}{2}z^2, \quad \frac{z}{1 \pm z}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{(1 \pm z)^2}, \quad \frac{z}{1 \pm z + z^2},$$
$$\frac{z(2 \pm z)}{2(1 \pm z)}, \quad \frac{z(2 \pm z^2)}{2(1 \pm z^2)}, \quad \frac{z(2 \pm z)}{2(1 \pm z)^2}, \quad \frac{z(2 \pm z + z^2)}{2(1 \pm z + z^2)}.$$

The proof we are going to show in the next section is based on Prawitz inequality (Theorem 2.3) especially on Corollary 2.4 and requires complicated calculations. We set

$$F(z) = \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

In the proof of Theorem 3.3, it was possible to determine all possible values for c_n. Since we cannot obtain them immediately to prove Theorem 3.6, we use the computational software program Mathematica. We only consider the case in which coefficients a_n are half-integers, so that c_n are rational numbers (moreover, $2^n c_n$ are rational integer), because of Lemma 2.5. Thus, the computation by Mathematica gives exact values.

3.2 Proof of Theorem 3.6

Let

$$F(z) = \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

We observe that (2.4) implies

$$\sum_{n=1}^{N} (n-1)|c_n|^2 \leq 1 \tag{3.1}$$

for some positive integer N. If a_n are specified for all $n \leq N$, i.e. c_n are given for $n \leq N-1$, (3.1) is equivalent to a quadratic inequality of a_{N+1} because of (2.5). Furthermore, (3.1) follows

$$(N-1)|c_N|^2 + \sum_{n=1}^{N-1} (n-1)|c_n|^2 \leq 1. \tag{3.2}$$

From (2.5), we have

$$c_N = -a_{N+1} - \sum_{n=1}^{N-1} a_{N-n+1} c_n,$$

and inserting this into (3.2), we obtain

$$(N-1)|a_{N+1} + P_{N-1}|^2 \leq 1 - \sum_{n=1}^{N-1} (n-1)|c_n|^2,$$

where $P_{N-1} = \sum_{n=1}^{N-1} a_{N-n+1} c_n$, because we are considering $a_n, c_n \in \mathbb{R}$. Setting $Q_{N-1} = 1 - \sum_{n=1}^{N-1} (n-1)|c_n|^2$, we have

$$|a_{N+1} + P_{N-1}|^2 \leq \frac{Q_{N-1}}{N-1}.$$

Since $Q_{N-1} \leq 1$ and a_{N+1} must be a half-integer, the coefficient a_{N+1} is uniquely determined, if

$$\frac{2}{\sqrt{N-1}} < \frac{1}{2} \tag{3.3}$$

namely, if $N > 17$. Thus we can specify all univalent functions with half-integer coefficients by calculating up to a_{18}. From Theorem 1.4, it is easy to see that there exist only a finite number of such functions.

From Theorem 1.2, we need to examine the following nine cases:

$$a_2 = 0,\ \pm\frac{1}{2},\ \pm 1,\ \pm\frac{3}{2},\ \pm 2.$$

Moreover, it is sufficient to consider the cases in which $a_2 \geq 0$. Otherwise we may consider $-f(-z)$ which is again univalent with a_2 non-negative. Again from Theorem 1.2, the case when $a_2 = 2$ must be the Koebe function

$$\frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n,$$

so that we have to examine the following only four cases:

$$a_2 = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2}.$$

We now show how to determine the coefficients. For example, we consider the case $a_2 = 3/2$. (3.1) follows

$$\sum_{n=1}^{2} (n-1)|c_n|^2 = \left|\frac{9}{4} - a_3\right|^2 \leq 1.$$

Therefore we have

$$\frac{5}{4} \leq a_3 \leq \frac{13}{4}.$$

However it should be $|a_3| < 3$, so that the possibilities of a_3 when $a_2 = 3/2$ are:

$$a_3 = \frac{3}{2},\ 2,\ \frac{5}{2}$$

because of Theorem 1.4. Choosing one of them, we can determine the coefficient a_4 when $a_2 = 3/2$ and a_3 is specified. Using this method, it is theoretically possible to determine all coefficients one by one. For this aim, we use the following two criteria.

Criterion 3.7. *Suppose that there exists a positive integer N such that for $n \leq N$ all coefficients a_n are specified, i.e. c_n for $n \leq N-1$ are given. If equality is attained in (3.1), then $c_n = 0$ for $n \geq N$.*

Criterion 3.8. *Suppose that there exists a positive integer N such that for $n \leq N$ all coefficients a_n are specified, i.e. c_n for $n \leq N-1$ are given. If there is no possibility of the coefficient a_{N+1} which satisfies* (3.1)*, then this case should be eliminated.*

Using these criteria and the computational software program Mathematica, we obtain Table 3.1.

	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
Case 1.1	3/2	2	5/2	3	7/2	4	9/2	5	11/2
Case 1.2	3/2	3/2	3/2	3/2	3/2	3/2	3/2	3/2	3/2
Case 1.3	1	2	$\cdots$						
Case 1.4	1	3/2	2	5/2	3	7/2	4	9/2	5
Case 1.5	1	1	1	1	3/2	$\cdots$			
Case 1.6	1	1	1	1	1	1	1	1	1
Case 1.7	1	1	1	1	1/2	$\cdots$			
Case 1.8	1	1/2	0	0	1/2	1	1	1/2	0
Case 1.9	1	1/2	0	0	0	0	0	0	0
Case 1.10	1	0	$\cdots$						
Case 1.11	1/2	1	1/2	1	1/2	1	1/2	1	1/2
Case 1.12	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2
Case 1.13	1/2	1/2	0	0	0	0	0	0	0
Case 1.14	1/2	0	0	0	0	0	0	0	0
Case 1.15	1/2	0	-1/2	-1/2	0	1/2	1/2	0	-1/2
Case 1.16	1/2	-1/2	-1	-1/2	1/2	1	1/2	-1/2	-1
Case 1.17	0	1	$\cdots$						
Case 1.18	0	1/2	0	1/2	0	1/2	0	1/2	0
Case 1.19	0	1/2	0	0	0	0	0	0	0
Case 1.20	0	0	1/2	0	0	1/2	0	0	1/2
Case 1.21	0	0	1/2	0	0	0	0	0	0
Case 1.22	0	0	0	0	1/2	$\cdots$			
Case 1.23	0	0	0	0	0	0	0	0	0
Case 1.24	0	0	0	0	-1/2	$\cdots$			
Case 1.25	0	0	-1/2	0	0	1/2	0	0	-1/2
Case 1.26	0	0	-1/2	0	0	0	0	0	0
Case 1.27	0	-1/2	0	1/2	0	-1/2	0	1/2	0
Case 1.28	0	-1	$\cdots$						

Table 3.1: Possibilities of coefficients obtained by using (1.1) and (2.4)

Table 3.1 consists of the possibilities of coefficients which we can obtain only using inequalities (1.1) and (2.4). Although we have to examine the possibilities of a_n for $n \leq 18$ as we mentioned above, it is consequently enough to calculate up to $n = 7$, because left-hand side of (3.3) depends on the value Q_{N-1} and we use (1.1) as a criterion. For some convenience, we list them for $n \leq 10$.

The symbol "$\cdots$" in Table 3.1 means "it continues". Only calculating the coefficients a_n up to just before the symbol "$\cdots$", equality is attained in (3.1). Thus we do not need to calculate any more coefficients because of Criterion 3.7. For example, we examine Case 1.3. From Lemma (2.5), we can see that $c_1 = c_2 = -1$. Therefore we have

$$\sum_{n=1}^{2}(n-1)|c_n|^2 = 1.$$

Since equality is attained in (2.4), $c_n = 0$ for $n \geq 3$ because of Criterion 3.7, so that

$$F(z) = \frac{z}{f(z)} = 1 - z - z^2$$

and therefore

$$f(z) = \frac{z}{1 - z - z^2}.$$

Since $1 - z - z^2$ has a zero in $\mathbb{D}$, so that $f(z)$ does not belong to the class$\mathcal{S}$ and therefore it should be eliminated.

Using the similar way, we can prove that Cases 1.5, 1.7, 1.10, 1.17, 1.22, 1.24 and 1.28 are:

$$\frac{2z}{2-2z-z^5}, \quad \frac{2z}{2-2z+z^5}, \quad \frac{z}{1-z+z^2}, \quad \frac{z}{1-z^2}, \quad \frac{2z}{2-z^5}, \quad \frac{2z}{2+z^5}, \quad \frac{z}{1+z^2}$$

respectively. However, the following four functions

$$\frac{2z}{2-2z-z^5}, \quad \frac{2z}{2-2z+z^5}, \quad \frac{2z}{2-z^5}, \quad \frac{2z}{2+z^5}$$

have to be rejected as not being univalent in $\mathbb{D}$.

We now eliminate the possibilities which do not belong to the class $\mathcal{S}$ from Table 3.1 by applying Lemma 2.7 and using Mathematica. Cases 1.2 and 1.16, and Cases 1.4, 1.8, 1.9, 1.13, 1.19, 1.21 and 1.26 should be rejected because

$$\det A^{(2)} < 0, \quad \text{and} \quad \det A^{(3)} < 0$$

respectively. However, Case 1.11, and Cases 1.20 and 1.25 should also be eliminated since

$$\det A^{(11)} < 0, \quad \text{and} \quad \det A^{(12)} < 0.$$

The values $\det A^{(11)}$ and $\det A^{(12)}$ each require to examine coefficients up to $n = 22$ and $n = 24$. We can observe that these three cases are

$$a_{2i} = \frac{1}{2}, \quad a_{2i+1} = 1;$$
$$a_{3j-1} = a_{3j} = 0, \quad a_{3j+1} = \frac{1}{2};$$
$$a_{3k-1} = a_{3k} = 0, \quad a_{3k+1} = -\frac{1}{2};$$

for $i \leq 11$ and $j, k \leq 8$ respectively.

Removing Cases which we have already examined from Table 3.1, we obtain Table 3.2.

	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
Case 2.1	3/2	2	5/2	3	7/2	4	9/2	5	11/2
Case 2.2	1	1	1	1	1	1	1	1	1
Case 2.3	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2
Case 2.4	1/2	0	0	0	0	0	0	0	0
Case 2.5	1/2	0	-1/2	-1/2	0	1/2	1/2	0	-1/2
Case 2.6	0	1/2	0	1/2	0	1/2	0	1/2	0
Case 2.7	0	0	0	0	0	0	0	0	0
Case 2.8	0	-1/2	0	1/2	0	-1/2	0	1/2	0

Table 3.2: Possibilities of coefficients obtained after using Lemma 2.7

There are 8 cases remaining. We again investigate them one by one.

Case 2.1

In this case, we could not obtain all possibilities by only calculating the coefficients a_n for $n \leq 10$ (actually, for $n \leq 18$). However, there seems to be a pattern on the coefficients. It is natural to suspect that $a_n = (n+1)/2$ for every positive integer n, and we can prove it by induction in the following way.

We may suppose that

$$f(z) = z + \frac{3}{2}z^2 + 2z^3 + \cdots + a_k z^k + \cdots ,$$

where k is an index with $k \geq 7$ such that for $n \leq k-1$ all coefficients a_n are specified and $a_n = (n+1)/2$. Summation of the first $k-1$ terms gives

$$f(z) = \frac{z(2-z) - z^k(k+1-kz)}{2(1-z)^2} + a_k z^k + O(z^{k+1}),$$

where O is the Landau symbol. Then

$$\begin{aligned} F(z) = \left(\frac{f(z)}{z}\right)^{-1} &= \left[\frac{2-z-z^{k-1}(k+1-kz)}{2(1-z)^2} + a_k z^{k-1} + O(z^k)\right]^{-1} \\ &= \frac{2(1-z)^2}{2-z-z^{k-1}(k+1-kz)}\left[1 + \frac{2(1-z)^2(a_k z^{k-1} + O(z^k))}{2-z-z^{k-1}(k+1-kz)}\right]^{-1} \\ &= \frac{2(1-z)^2}{2-z-z^{k-1}(k+1-kz)}\left[1 - a_k z^{k-1} + O(z^k)\right] \\ &= \frac{2(1-z)^2}{2-z} + \left(\frac{k+1}{2} - a_k\right) z^{k-1} + O(z^k) \\ &= 1 - \frac{3}{2}z + \frac{1}{2^2}z^2 + \frac{1}{2^3}z^3 + \cdots + \left(\frac{1}{2^{k-1}} + \frac{k+1}{2} - a_k\right) z^{k-1} + O(z^k). \end{aligned}$$

Applying inequality (2.4), we have

$$(k-2)\left|\frac{1}{2^{k-1}} + \frac{k+1}{2} - a_k\right|^2 \leq 1.$$

Since $k \geq 7$, we obtain

$$\left|\frac{1}{2^{k-1}} + \frac{k+1}{2} - a_k\right| \leq \frac{1}{\sqrt{5}} < \frac{1}{2},$$

so that the possibilities of a_k are:

$$\frac{k+1}{2}, \quad \frac{k+2}{2}$$

because a_k should be a half-integer. However for the case $a_k = (k+2)/2$ does not satisfy (3.1) because $k \geq 7$. For example $k = 7$, the left-hand side of (3.1) is

$$\sum_{n=1}^{6}(n-1)|b_n|^2 = \sum_{n=1}^{5}\frac{n-1}{2^{2n}} + 5\left(\frac{1}{2} - \frac{1}{2^5}\right) = \frac{1237}{1024} > 1.$$

Thus $a_k = (k+1)/2$. Again the proof does not depend on the particular value of k and therefore we may show successively that

$$a_{k+1} = \frac{k+2}{2}, \ \ a_{k+2} = \frac{k+3}{2}, \ \ \ldots \ .$$

Thus, we obtain

$$f(z) = z + \sum_{n=2}^{\infty} \frac{n+1}{2} z^n = \frac{z(2-z)}{2(1-z)^2}.$$

We now observe that

$$f(z) - f(w) = 0$$

implies $z = w$, because

$$\frac{z(2-z)}{2(1-z)^2} - \frac{w(2-w)}{2(1-w)^2} = \frac{(z-w)(2-w-z)}{2(1-z)^2(1-w)^2}$$

and $z, w \in \mathbb{D}$. Therefore this function is univalent in $\mathbb{D}$.

For Cases 2.2, 2.3, 2.4 and 2.7, we can prove that

$$f(z) = \frac{z}{1-z}, \quad \frac{z(2-z)}{2(1-z)}, \quad z + \frac{z^2}{2}, \quad z$$

respectively by using the same method. These functions are also univalent in $\mathbb{D}$.

Case 2.5.

Again we could not finish determining all possibilities for this case. However, there also seems to be a pattern with period 6 on the coefficients. The examination uses six inductions and assembles them.

We may suppose that

$$f(z) = z + \frac{1}{2}z^2 - \frac{1}{2}z^4 - \frac{1}{2}z^5 + \frac{1}{2}z^7 + \cdots + a_k z^k + \cdots ,$$

where k is an index with $k \geq 6$ such that for $n \leq k-1$ all coefficients a_n are specified and expressed by

$$a_n = \begin{cases} \frac{1}{2} & \text{if } n = 6m-4, 6m+1 \\ 0 & \text{if } n = 6m-3, 6m \\ -\frac{1}{2} & \text{if } n = 6m-2, 6m-1 \end{cases}$$

for some positive integer m. We consider six cases $k = 6j-4, 6j-3, 6j-2, 6j-1, 6j, 6j+1$ for $j = 1, 2, \dots$. For example, here we prove the case when $k = 6j-4$ by using the same method as in Case 2.1.

Let

$$f(z) = z + \frac{1}{2}z^2 - \frac{1}{2}z^4 - \frac{1}{2}z^5 + \frac{1}{2}z^7 + \cdots + a_{6j-4}z^{6j-4} + \cdots .$$

Summation of the first $6j-5$ terms gives

$$f(z) = \frac{z(2-z+z^2) + z^{6j-4}(z-1)}{2(1-z+z^2)} + a_{6j-4}z^{6j-4} + O(z^{6j-3}).$$

Then,

$$\begin{aligned} F(z) &= \frac{2(1-z+z^2)}{2-z+z^2+z^{6j-5}(z-1)} \left[1 - a_{6j-5}z^{6j-5} + O(z^{6j-4})\right] \\ &= \frac{2(1-z+z^2)}{2-z+z^2} + \left(\frac{1}{2} - a_{6j-4}\right) z^{6j-5} + O(z^{6j-4}) \end{aligned}$$

Let λ_n be the n-th coefficient of the function

$$\frac{2(1-z+z^2)}{2-z+z^2}.$$

Then note that λ_n can be expressed by the recurrence relation

$$\lambda_1 = -\frac{1}{2}, \quad \lambda_2 = \frac{1}{4}, \quad \lambda_n = \frac{1}{2}(\lambda_{n-1} - \lambda_{n-2}).$$

From (3.1) and $j \geq 2$, we have

$$\left|\lambda_{6j-5} - a_{6j-4} + \frac{1}{2}\right| \leq \frac{1}{\sqrt{6}} < \frac{1}{2}$$

for every positive integer j. Now we can see that the sequences $\{|\lambda_{3n}|\}_{n=1}^{\infty}$, $\{|\lambda_{3n+1}|\}_{n=1}^{\infty}$ and $\{|\lambda_{3n+2}|\}_{n=1}^{\infty}$ each are decreasing and $0 < |\lambda_n| < 5/32$ for $n \geq 8$. Since the coefficients a_n should be half-integers and the sequence $\{\lambda_{6j-5}\}_{j=1}^{\infty}$ is alternating, we observe that $a_{6j-4} = 1/2$ for every positive integer j.

Similarly we can prove that

$$a_{6j+1} = \frac{1}{2}, \quad a_{6j-3} = a_{6j} = 0, \quad a_{6j-2} = a_{6j-1} = -\frac{1}{2}$$

for $j = 1, 2, \ldots$, so that we have

$$f(z) = \frac{z(2 - z + z^2)}{2(1 - z + z^2)}$$

and therefore this is univalent in $\mathbb{D}$.

For Cases 2.6 and 2.8, we can use the same method and prove that

$$f(z) = \frac{z(2 + z^2)}{2(1 + z^2)}, \ \frac{z(2 - z^2)}{2(1 - z^2)},$$

respectively. Furthermore these functions are univalent in $\mathbb{D}$.

We have obtained the following 12 functions:

$$z, \quad z + \frac{1}{2}z^2, \quad \frac{z}{1 - z}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{(1 + z)^2}, \quad \frac{z}{1 - z + z^2},$$
$$\frac{z(2 - z)}{2(1 - z)}, \quad \frac{z(2 \pm z^2)}{2(1 \pm z^2)}, \quad \frac{z(2 - z)}{2(1 - z)^2}, \quad \frac{z(2 - z + z^2)}{2(1 - z + z^2)}.$$

Considering $-f(-z)$, we finally get the other seven functions specified in the statement. Therefore the proof is now completed. □

3.3 Further Problems

There are three problems remaining in this field. The first one is to simplify the proof. The method of the proof for Theorem 3.6 requires complicated calculations. Although we used Mathematica, it is desirable to avoid it. For this aim, we need to apply or obtain more powerful univalence criteria. The author has tried to give an easier proof for a long time, but consequently it could not be accomplished. It seems that applying the coefficient estimates within the class $\mathcal{S}$ does not work in this field, because we handle coefficients with small values.

The second problem is to investigate the case in which ka_n are rational integers with $k \geq 3$. If we used the same method as in the proof of Theorem 3.6, it would require a huge effort. Furthermore, to classify all univalent functions with rational number coefficients is also an interesting problem related to this.

The last one is about Theorem 3.2. Since half-integers are not rational integers or integers of an imaginary quadratic field, we do not know whether all univalent functions having half-integer coefficients are rational or not. As a result, all of them are rational. Although it should be possible to prove it, the author has not achieved the proof, and therefore this problem is also remaining.

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