2014年度
「リーマン面・不連続群論」研究集会

大阪大学中之島センター

2015年2月14日 – 16日

プログラム＋アブストラクト＋講演スライド

Program

14 February (Saturday)

13:30 – 14:20 Yuuki Tadokoro (Kisarazu National College of Technology)
The period matrix of the hyperelliptic curve $w^2 = z^{2g+1} - 1$

14:30 –15:20 Yuichi Kabaya (Kyoto University)
Exotic components in linear slices of quasi-Fuchsian groups

15:50 – 16:40 Masakazu Shiba (Hiroshima University)
Some new problems in the theory of conformal mappings of an open Riemann surface of finite genus

15 February (Sunday)

9:00 – 9:50 Hirokazu Shimauchi (Tohoku University)
Numerical quasiconformal mappings by certain linear systems

10:00 –10:50 Lijie Sun (Tohoku University)
Notes on complex hyperbolic triangle groups of type (m, n, ∞)

11:10 – 12:00 Daisuke Yamaki (Tokyo Institute of Technology)
Holomorphic 1-cochains and combinatorial periods

Lunch

13:30 – 14:20 Yohei Komori (Waseda University)
Projective embeddings of the Teichmüller spaces

14:30 –15:20 Dariusz Partyka (The John Paul II Catholic University of Lublin)
The Schwarz type inequalities for harmonic mappings in the unit disc with boundary normalization

15:50 – 16:40 Ken-ichi Sakan (Osaka City University)
On quasiconformality and some properties of harmonic mappings in the unit disk

Banquet

16 February (Monday)

9:00 – 9:50 Masahiro Yanagishita (Waseda University)
Complex analytic structure on the p-integrable Teichmüller space

10:00 –10:50 Katsuhiko Matsuzaki (Waseda University)
The barycentric extension of circle diffeomorphisms

11:10 – 12:00 Yi Huang (The University of Melbourne)
Flipping numbers and curves

Lunch

13:30 – 14:20 Ryuji Abe (Tokyo Polytechnic University)
Diophantine approximation via Gaussian integers

14:30 –15:20 Sachiko Hamano (Fukushima University)
On the reproducing kernel for the space of semi-exact analytic differentials

15:50 – 16:40 Yukitaka Abe (Toyama University)
Analytic study of singular curves

Abstract

Yuuki Tadokoro (Kisarazu National College of Technology)
The period matrix of the hyperelliptic curve $w^2 = z^{2g+1} - 1$

Our talk consists of two parts. First, we explicitly obtain the period matrix of the hyperelliptic curve defined by the affine equation $w^2 = z^{2g+1} - 1$, its entries being elements of the $(2g+1)$-st cyclotomic field. Second, we introduce an algorithm for obtaining the period matrix for a compact Riemann surface, which is a p-cyclic covering of $\mathbb{C}P^1$ branched over 3 points.

Yuichi Kabaya (Kyoto University)
Exotic components in linear slices of quasi-Fuchsian groups

The linear slice of quasi-Fuchsian punctured torus groups is defined by fixing the length of some simple closed curve to be a fixed positive real number. It is known that the linear slice is a union of disks, and it has one standard component containing Fuchsian groups. Komori-Yamashita proved that there exist non-standard components if the length is sufficiently large. In this talk, I give another proof based on the theory of complex projective structures.

Masakazu Shiba (Hiroshima University)
Some new problems in the theory of conformal mappings of an open Riemann surface of finite genus

Let R be an open (=noncompact) Riemann surface of finite genus g. If a closed (=compact) Riemann surface R' of genus g contains R as a subregion, R' is historically called a "compact continuation of the same genus" of R, but we prefer to use a shorter term a "closing." We give a precise definition in modern terminology and construct a closing of R with a remarkable hydrodynamic property. These closings are used to comprehend the totality $\mathcal{C}$ of the closings of R; if $g = 1$ in particular, we use the modulus of a torus to describe $\mathcal{C}$ as a closed disk M in $\mathbb{H}$. We generalize this result to $g > 1$. The hyperbolic diameter $\sigma_H(R)$ of M is called the hyperbolic span of R. If $R = R_t$ moves holomorphically so that the set $\{(R_t, t) \,|\, t \in \mathbb{D}\}$ is pseudoconvex, $\sigma_H(R_t)$ is a subharmonic function.

Hirokazu Shimauchi (Tohoku University)
Numerical quasiconformal mappings by certain linear systems

In this talk, we propose a numerical method for quasiconformal self mappings of the unit disk. The unit disk is triangulated in a simple way and the quasiconformal mappings are approximated by piecewise linear mappings. The images of the vertices of the triangles are defined by an overdetermined system of linear equations. Further the sequence of the approximation converges to the true solution, at least in the case where the Beltrami coefficients are in C^1. We will also present several numerical experiments. This talk is based on a joint work with R. Michael Porter (CINVESTAV).

Lijie Sun (Tohoku University)
Notes on complex hyperbolic triangle groups of type (m, n, ∞)

The triangle groups are not necessarily discrete in complex hyperbolic space which is different from the real hyperbolic case. Many authors investigated the discreteness of ideal triangle groups and the triangle groups of type (n, n, ∞). The difficult point for giving discrete cases is that there are no totally geodesic real hypersurfaces in $\mathbb{H}^2_{\mathbb{C}}$. In this talk we mainly consider the complex hyperbolic triangle groups of type (m, n, ∞) and give some discrete cases using the complex hyperbolic version of Klein's combination theorem. From the results more explicit conclusions about non-discrete triangle groups of type (m, ∞, ∞) will also be given.

Daisuke Yamaki (Tokyo Institute of Technology)
Holomorphic 1-cochains and combinatorial periods

We discuss holomorphic 1-cochains and periods of holomorphic 1-cochains. Holomorphic 1-cochains are defined on Riemann surfaces with triangulations and satisfy Riemann's bi-linear relation. Using holomorphic 1-cochains, Wilson defined combinatorial period matrices and showed that for a triangulated Riemann surface, the combinatorial period matrix converges to the (conformal) period matrix as the mesh of the triangulation tends to zero. In this talk, we give another relation between combinatorial period matrices and (conformal) period matrices and study its applications.

Yohei Komori (Waseda University)
Projective embeddings of the Teichmüller spaces

Let X be an orientable hyperbolic surface of genus g with n punctures and r holes. Then the Teichmüller space $\mathcal{T}(X)$ of X is homeomorphic to the real affine space V of $\dim V = 6g - 6 + 2n + 3r$. I have been considering the following question:

> Can we find $\dim V + 1$-number of simple closed geodesics whose hyperbolic lengths embed $\mathcal{T}(X)$ into the finite dimensional real projective space $P(V)$?

Because of the PL-Structure of the Thurston boundary, we might expect that the image of $\mathcal{T}(X)$ should be the interior of some convex polyhedron in $P(V)$.

In this talk I will answer this question for surfaces having at least one hole, with few exceptional cases.

Dariusz Partyka (The John Paul II Catholic University of Lublin)
The Schwarz type inequalities for harmonic mappings in the unit disc with boundary normalization

This talk is intended to give an exposition of the Schwarz type inequalities for harmonic self-mappings of the unit disc with certain additional properties. However this time the classical normalization condition, with the origin as a fixed point, is replaced by certain boundary conditions. In particular, the case is considered, where a harmonic mapping is injective and has a continuous extension to the closed unit disk which keeps the cube roots of unity fixed. Some other cases of this type are also discussed, especially in the context of quasiconformal mappings.

Ken-ichi Sakan (Osaka City University)
On quasiconformality and some properties of harmonic mappings in the unit disk

In this talk we give a summary of our results on quasiconformality and some properties of harmonic mappings in the unit disk which have been obtained jointly with D. Partyka.To begin with we first give brief explanations of Lewy's theorem and Radó-Kneser-Choquet theorem. Next we state (A) (primitive) Schwarz's lemma for harmonic mappings. Moreover, for sense-preserving injective harmonic mappings of the unit disk onto itself, we state (B) (primitive) Heinz's inequality and (C) a theorem by Pavlović on quasiconformality of such mappings.We then explain that under appropriate assumptions we could obtain many improved or modified forms of the results (A),(B) and (C), respectively.

Masahiro Yanagishita (Waseda University)
Complex analytic structure on the p-integrable Teichmüller space

The p-integrable Teichmüller space is a metric subspace of the Teichmüller space of Teichmüller equivalence classes containing Beltrami coefficients with finite hyperbolic L^p-norm. If a Riemann surface R is analytically finite, then the p-integrable Teichmüller space of R coincides with the Teichmüller space of R. Hence, this study has a significance for Riemann surfaces of analytically infinite type. Cui, Takhtajan-Teo and Tang considered the complex analytic structure on the p-integrable Teichmüller space of the unit disk for $p \geq 2$. In this talk, we extend their results to the case of hyperbolic Riemann surfaces.

Katsuhiko Matsuzaki (Waseda University)
The barycentric extension of circle diffeomorphisms

The barycentric extension due to Douady and Earle gives a conformally natural extension of a quasisymmetric automorphism of the circle to a quasiconformal automorphism of the unit disk. In this talk, we consider such extensions for circle diffeomorphisms of Hoelder continuous derivatives and show that this operation is continuous with respect to an appropriate topology for the space of corresponding Beltrami coefficients.

Yi Huang (The University of Melbourne)
Flipping numbers and curves

Solutions to the equation $x^2 + y^2 + z^2 = xyz$ satisfy the following property: given one solution (x, y, z), we can easy write down a new "flipped" solution given by $(x, y, xy - z)$. In particular, this means that an integer solution is flipped to another integer solution. These integer solutions are well-known as Markoff triples, and arise in beautiful results in geometry and number theory.

In recent work with Paul Norbury, we discover similar phenomena for solutions to the equation $(a+b+c+d)^2 = abcd$ — called Markoff quads. We begin with a gentle motivating survey of several famous results related to Markoff triples, before introducing their Markoff quad analogues.

Ryuji Abe (Tokyo Polytechnic University)
Diophantine approximation via Gaussian integers

The Markoff spectrum for the rational number field $\mathbb{Q}$ is defined by means of the minimum of binary indefinite quadratic forms with real coefficients and the Lagrange spectrum is defined with respect to approximation of real numbers by rational ones. It is well-known that the discrete parts of them coincide.

In this talk, we show that there exists an analogy between the Markoff spectrum for the imaginary quadratic number field $\mathbb{Q}(i)$ and the Lagrange spectrum by rational numbers of Gaussian integers, using a geometric characterization of the Markoff spectrum for $\mathbb{Q}(i)$ by simple closed geodesics in an immersed totally geodesic twice punctured torus in the Borromean rings complement.

Sachiko Hamano (Fukushima University)
On the reproducing kernel for the space of semi-exact analytic differentials

We shall discuss the reproducing kernel for the Hilbert space $S(R)$ of all semi-exact L^2-analytic differentials on a finite bordered Riemann surface R. We show that the Bergman kernel restricted to $S(R)$ has a close relation to the L_1-constant with two logarithmic poles, and then discuss a problem related to a conjecture of Suita type for $S(R)$.

Yukitaka Abe (Toyama University)
Analytic study of singular curves

We study singular curves from analytic point of view. The classical theory of compact Riemann surfaces and their Jacobi varieties was generalized to singular curves and generalized Jacobi varieties by algebraic way. It seems to us that there is no analytic study of them. We treat singular curves and generalized Jacobi varieties completely analytically. We give analytic proofs of the Serre duality and generalized Abel's theorem without any help from algebra. Generalized Jacobi varieties are considered as complex Lie groups. We investigate their properties.

The period matrix of the hyperelliptic curve $w^2 = z^{2g+1} - 1$

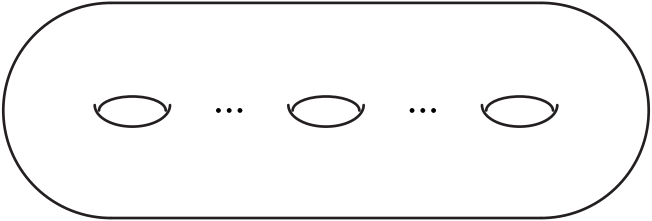
Yuuki TADOKORO

Kisarazu National College of Technology

14 Feb. 2015 @Osaka

Overview

Riemann surface is an important object from analytic, algebraic, geometric, and topological viewpoints.

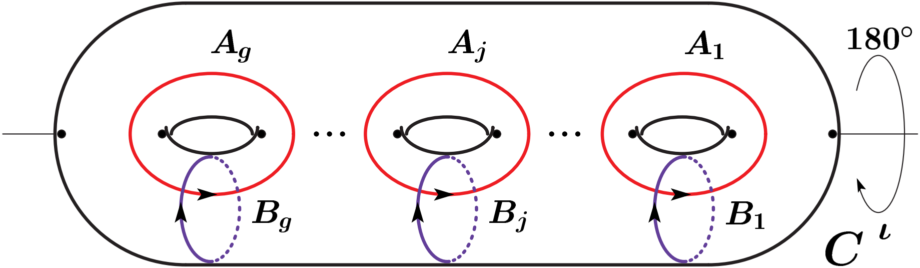


We put emphasis on a complex analytic invariant,
Period matrix

Overview

First part

- C_g: hyperelliptic curve $w^2 = z^{2g+1} - 1$ of genus $g \geq 2$.
- $\{A_i, B_i\}_{i=1,\ldots,g} \subset H_1(C_g; \mathbb{Z})$: a fixed symplctic basis(natural type)



Overview

First part

- C_g: hyperelliptic curve $w^2 = z^{2g+1} - 1$ of genus $g \geq 2$.
- $\{A_i, B_i\}_{i=1,\ldots,g} \subset H_1(C_g;\mathbb{Z})$: a fixed symplctic basis(natural type)
- τ_g: period matrix of C_g with respect to $\{A_i, B_i\}$
  A complex analytic invariant of Riemann surfaces

$\Longrightarrow$ **We explicitly determine τ_g**

Overview

Second part

- $X_{p,l,m}$: compact Riemann surface $w^p = z^l(1-z)^m$ of genus $g = (p-1)/2$.
- $F = F_N$: Fermat curve $w^N = 1 - z^N$ of genus $g = (N-1)(N-2)/2$.

$\Longrightarrow$ **We made a program which computes**

$(p, l, m) \longrightarrow$ period matrix of $X_{p,l,m}$

$N \longrightarrow$ period matrix of F

Definition of Period matrix

- X : a compact Riemann surface of genus $g \geq 1$
- $\{\omega_1, \ldots, \omega_g\}$: a basis of $H^{1,0}(X) \cong \mathbb{C}^g$
- $\{a_i, b_i\}_{i=1,\ldots,g}$:a symplectic basis of $H_1(X;\mathbb{Z})$
- $\Omega_A = \left(\int_{a_j} \omega_i\right), \ \Omega_B = \left(\int_{b_j} \omega_i\right)$: Periods

$$\tau_X := \Omega_A^{-1}\Omega_B \in M_g(\mathbb{C})$$

Properties of τ_X

- A complex analytic invariant of X.
- It depends only on the choice of a symplectic basis of $H_1(X;\mathbb{Z})$.
- It is symmetric and its imaginary part is positive definite.
  $\tau_X \in \mathcal{H}_g$: Siegel upper halfspace

period map $\varphi : \mathbb{M}_g \to \mathrm{Sp}_{2\mathrm{g}}(\mathbb{Z}) \backslash \mathcal{H}_g$

Motivation

1. Torelli's theorem
   - X, Y : compact Riemann surfaces of genus g
   - $J(X) = \mathbb{C}^g/(\mathbb{Z}^g + \tau_X\mathbb{C}^g)$: its Jacobian varieties

   $\boldsymbol{X \cong Y \Leftrightarrow J(X) \cong J(Y)}$ as p.p.a.v.

2. For generic genus, few examples of period matrices are known.
   The difficulty is in finding a symp. basis
   - only three types of hyperelliptic curves C
   - no examples of **nonhyperelliptic curves** (for generic genus)

   We are trying to compute these examples using our program.

Motivation

1. Torelli's theorem
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Schindler's results(1993)

Method: Action of $\mathrm{Aut}\, C$, $(z, w) \mapsto (\zeta z, w)$

① $\boldsymbol{C : \omega^2 = z^{2g+2} - 1}$

$(\zeta = \zeta_{2g+2} = \exp(2\pi\sqrt{-1}/(2g+2)))$

$$\tau_X = \left(\frac{1}{g+1} \sum_{k=1}^{g} \frac{\zeta^k(\zeta^{-2ik} - 1)(\zeta^{2kj} - 1)}{1 - \zeta^{2k}} \right)_{i,j}$$

Schindler's results(1993)

② $C \cong C_g : \omega^2 = z(z^{2g+1} - 1)$
$(\zeta = \zeta_{2g+1})$

Schindler's results(1993)

② $\boldsymbol{C \cong C_g : \omega^2 = z(z^{2g+1} - 1)}$
$(\zeta = \zeta_{2g+1})$

$$\begin{cases} t_1 &= (-1)^g \zeta^{g^2}, \quad t_2 = t_1\zeta/(1+\zeta), \\ t_{i+1} &= t_1 \left(1 - \sum_{k=2}^{i} \zeta^{g-i+k-1} t_k t_{i-k+2}\right) \Big/ (1+\zeta^{-i}) \end{cases}$$

Schindler's results(1993)

② $C \cong C_g : \omega^2 = z(z^{2g+1} - 1)$
$(\zeta = \zeta_{2g+1})$

Theorem ((i,j)-th entry of τ_g^S)

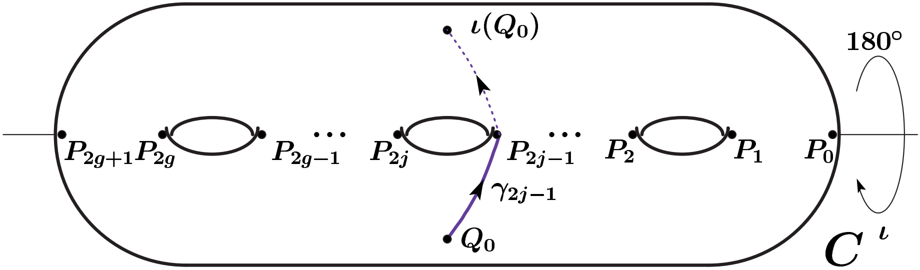
$$s_{i,j} = 1 - \sum_{k=1}^{i} t_k t_{j-i+k}/t_1$$

for $1 \leq i \leq j \leq g$ *and* $s_{j,i}$ *for* $g \geq i > j \geq 1$.

recurrence expression
③ $C : \omega^2 = z(z^{2g} - 1)$ more complex expression

A symplectic basis of hyperelliptic curves

$C \overset{2:1}{\to} \mathbb{C}P^1$: a hyperelliptic curve, ι: its involution,
$\gamma_j : [0,1] \to C$: path from Q_0 to $\iota(Q_0)$

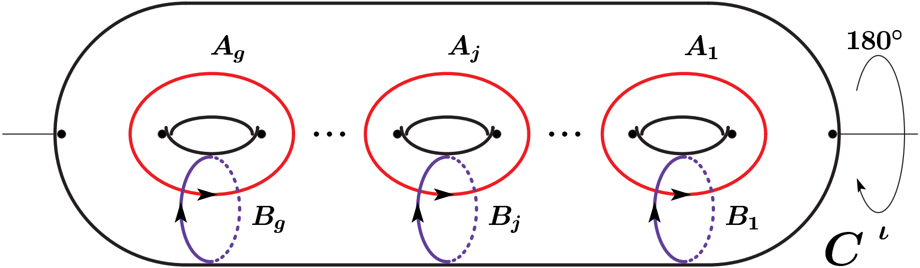


$$\begin{cases} A_i & := & \gamma_{2i-1} \cdot \gamma_{2i}^{-1} \\ B_i & := & \gamma_{2i-1} \cdot \gamma_{2i-2}^{-1} \cdot \cdots \cdot \gamma_1 \cdot \gamma_0^{-1} \end{cases}$$

A symplectic basis of hyperelliptic curves

$$\begin{cases} A_i & := & \gamma_{2i-1} \cdot \gamma_{2i}^{-1} \\ B_i & := & \gamma_{2i-1} \cdot \gamma_{2i-2}^{-1} \cdot \cdots \cdot \gamma_1 \cdot \gamma_0^{-1} \end{cases}$$

$\Rightarrow \{A_i, B_i\}_{i=1,2,\ldots,g}$: a symp. basis of $H_1(C;\mathbb{Z})$



Periods

- C_g: hyperelliptic curve $w^2 = z^{2g+1} - 1$ $(g \geq 2)$.
- $\{A_i, B_i\}_{i=1,\dots,g} \subset H_1(C_g; \mathbb{Z})$: the fixed symp. basis
- $\{\omega_i = z^{i-1}dz/w\}_{i=1,\dots,g} \subset H^{1,0}(C_g)$: a basis
- $\zeta := \zeta_{2g+1} = \exp(2\pi\sqrt{-1}/(2g+1))$
- τ_g: period matrix of C_g with respect to $\{A_i, B_i\}$

$\Omega_A = \left(\int_{A_j} \omega_i\right)$, $\Omega_B = \left(\int_{B_j} \omega_i\right)$ were obtained by Tashiro, Yamazaki, Ito, and Higuchi(1996). Moreover $\det \Omega_A$ and $\det \Omega_B$ too.

Periods

Case: $g = 3$

$$\Omega_A =$$

$$\begin{pmatrix} 1-\zeta & 1-\zeta+\zeta^2-\zeta^3 & 1-\zeta+\zeta^2-\zeta^3+\zeta^4-\zeta^5 \\ 1-\zeta^2 & 1-\zeta^2+\zeta^4-\zeta^6 & 1-\zeta^2+\zeta^4-\zeta^6+\zeta^8-\zeta^{10} \\ 1-\zeta^3 & 1-\zeta^3+\zeta^6-\zeta^9 & 1-\zeta^3+\zeta^6-\zeta^9+\zeta^{12}-\zeta^{15} \end{pmatrix}$$

$$\Omega_B =$$

$$\begin{pmatrix} 1-\zeta^2 & 1-\zeta+\zeta^2-\zeta^4 & 1-\zeta+\zeta^2-\zeta^3+\zeta^4-\zeta^6 \\ 1-\zeta^4 & 1-\zeta^2+\zeta^4-\zeta^8 & 1-\zeta^2+\zeta^4-\zeta^6+\zeta^8-\zeta^{12} \\ 1-\zeta^6 & 1-\zeta^3+\zeta^6-\zeta^{12} & 1-\zeta^3+\zeta^6-\zeta^9+\zeta^{12}-\zeta^{18} \end{pmatrix}$$

Periods

$$\Omega_A =$$

$$\begin{pmatrix} 1-\zeta & 1-\zeta+\zeta^2-\zeta^3 & 1-\zeta+\zeta^2-\zeta^3+\zeta^4-\zeta^5 \\ 1-\zeta^2 & 1-\zeta^2+\zeta^4-\zeta^6 & 1-\zeta^2+\zeta^4-\zeta^6+\zeta^8-\zeta^{10} \\ 1-\zeta^3 & 1-\zeta^3+\zeta^6-\zeta^9 & 1-\zeta^3+\zeta^6-\zeta^9+\zeta^{12}-\zeta^{15} \end{pmatrix}$$

$$= \begin{pmatrix} -1+\zeta & & \\ & -1+\zeta^2 & \\ & & -1+\zeta^3 \end{pmatrix}$$

$$\begin{pmatrix} \zeta & & \\ & \zeta^2 & \\ & & \zeta^3 \end{pmatrix} \begin{pmatrix} 1 & \zeta^2 & \zeta^4 \\ 1 & \zeta^4 & \zeta^8 \\ 1 & \zeta^6 & \zeta^{12} \end{pmatrix}$$

Key lemma(Knuth's book)

- $a_1, \ldots, a_n$: distinct complex constants.
- $V_n = \left(a_i^{j-1}\right)_{i,j}$: A Vandermonde matrix.
- $\sigma_i(a_1, a_2, \ldots, a_n)$: i-th symmetric polynomial.

$$\Rightarrow V_n^{-1} = \left((-1)^{i-1} \frac{\sigma_{n-i}(a_1, \ldots, \widehat{a_j}, \ldots, a_n)}{\prod_{m=1, m\neq j}^{n} (a_m - a_j)} \right)_{i,j}$$

Key lemma(Knuth's book)

Case: $n = 3$

$$\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{bc}{(b-a)(c-a)} & -\frac{ac}{(a-b)(c-b)} & \frac{ab}{(a-c)(b-c)} \\ \frac{b+c}{(b-a)(c-a)} & -\frac{a+c}{(a-b)(c-b)} & \frac{a+b}{(a-c)(b-c)} \\ \frac{1}{(b-a)(c-a)} & -\frac{1}{(a-b)(c-b)} & \frac{1}{(a-c)(b-c)} \end{pmatrix}$$

Result(2014)

Theorem ((i, j)-th entry of τ_g)

$$\sum_{k=1}^{g} \frac{(-1)^{i+g}}{2g+1}(1-\zeta^{2kj})\sigma_{g-i}(\zeta^2, \ldots, \widehat{\zeta^{2j}}, \ldots, \zeta^{2g}) \prod_{m=g-k+1}^{2g-k}(1-\zeta^{2m})$$

Result(2014)

$$\tau_3 = \begin{pmatrix} -\zeta^5 & -2-\zeta^2-\zeta^4-\zeta^5 & \zeta+\zeta^3+\zeta^5 \\ -2-\zeta^2-\zeta^4-\zeta^5 & \zeta+2\zeta^3-\zeta^4+\zeta^5 & 1+\zeta^2+\zeta^3+\zeta^5 \\ \zeta+\zeta^3+\zeta^5 & 1+\zeta^2+\zeta^3+\zeta^5 & \zeta^2 \end{pmatrix}$$

Result(2014)

A relation between Schindler's result and τ_g

$$L_g = \begin{pmatrix} & & & -1 \\ & & -1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ -1 & & & \end{pmatrix} \in M_g(\mathbb{Z})$$

$$\Rightarrow \tau_g^S = L_g \tau_g L_g$$

∵) See the symplectic basis for Schindler's period matrix

Algorithms and programs

Algorithm

- Tretkoff and Tretkoff
  Hurwitz system and Frobenius method
- Kamata $\subset$ T.T. **for Fermat type curves**
- Ours $\subset$ T.T. **Chord slide method for** $X_{p,l,m}$

Programs

Ours	Maple `algcurves`
$X_{p,l,m}$	$f(x,y)=0$
$\mathbb{Q}(\zeta)$	Approximate value
elementary	complex

A program

- p : prime, $0 < l, m < p-1$: coprime
- $X_{p,l,m} := \{w^p = z^l(1-z)^m\}$:
  a compact Riemann surface of $g = (p-1)/2$.
- $\pi : X_{p,l,m} \ni (z, w) \mapsto z \in \mathbb{C}P^1$:
  p-cyclic covering branched over $0, 1, \infty \subset \mathbb{C}P^1$

Using "Chord Slide Method(CSM)", we obtain a geometric algorithm for finding symp. basis of $X_{p,l,m}$'s.

A program

- p : prime, $0 < l, m < p-1$: coprime
- $X_{p,l,m} := \{w^p = z^l(1-z)^m\}$:
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  p-cyclic covering branched over $0, 1, \infty \subset \mathbb{C}P^1$

Using "Chord Slide Method(CSM)", we obtain a mathematica program for calculating period matrices of $X_{p,l,m}$'s.

Demonstration

- $X_{p,l,m}$: compact Riemann surface $w^p = z^l(1-z)^m$ of genus $g = (p-1)/2$.

$\Longrightarrow$ **We made a program which computes**

$$(p, l, m) \longrightarrow \text{period matrix of } X_{p,l,m}$$

Intersection matrix(Outline)

- $\sigma(z,w)=(z,\zeta w)$: automorphism with order p.
- Define $c_i : [0,1] \to X_{p,l,m}$ $(i=1,2,\dots,2g)$ paths

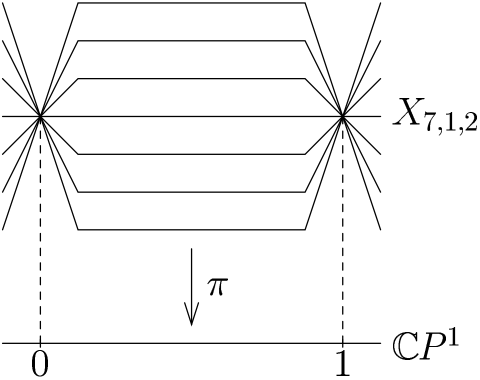
$\Rightarrow A=(c_i \cdot c_j)$ intersection matrix

1. p-cyclic covering of $\mathbb{C}P^1$
2. Dessin d'enfants
3. Chord diagram on S^1

Sample: $X_{7,1,2}=\{w^7=z(1-z)^2\}$: Klein quartic
$K_4 := \{X^3Y+Y^3Z+Z^3X=0\} \subset \mathbb{C}P^2$
$(z=X^3Y^{-2}Z^{-1}+1,\ w=-XY^{-1})$

Intersection matrix(Details)

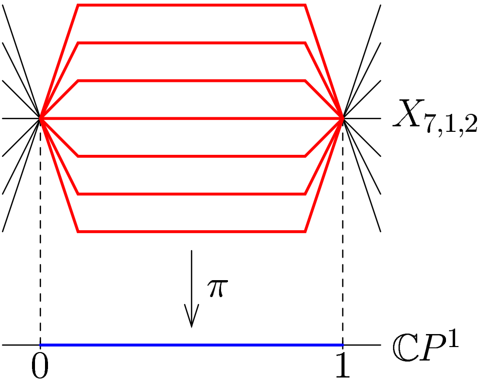
p-cyclic covering of $\mathbb{C}P^1$→DD→CD



$\pi : X_{p,l,m} \ni (z, w) \mapsto z \in \mathbb{C}P^1$, p-cyclic covering

Intersection matrix(Details)

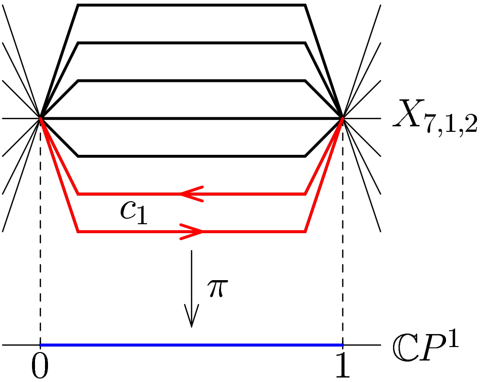
p-cyclic covering of $\mathbb{C}P^1$→DD→CD



lifts of $[0,1] \subset \mathbb{C}P^1$

Intersection matrix(Details)

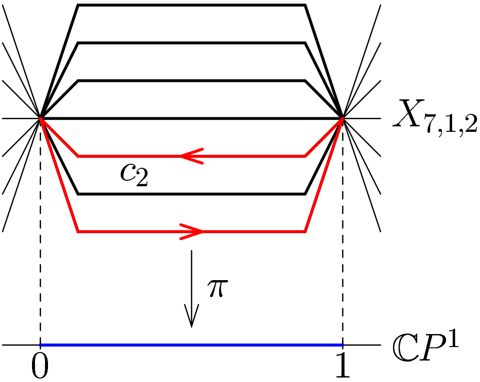
p-cyclic covering of $\mathbb{C}P^1$ → DD → CD



$c_1 = I_0 \cdot \sigma(I_0)^{-1}$

Intersection matrix(Details)

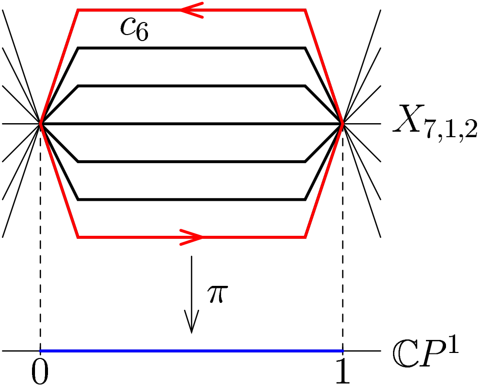
p-cyclic covering of $\mathbb{C}P^1$→DD→CD



$$c_2 = I_0 \cdot \sigma^2(I_0)^{-1}$$

Intersection matrix(Details)

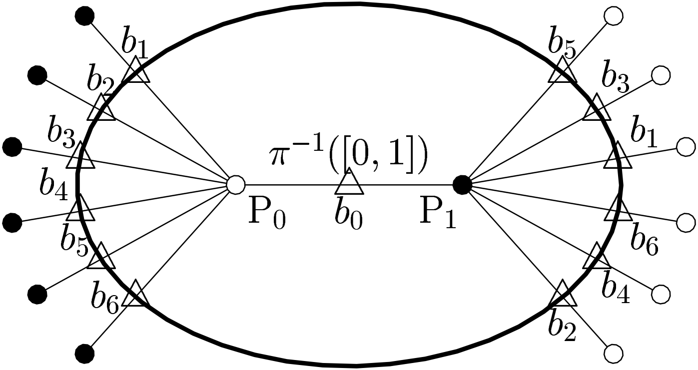
p-cyclic covering of $\mathbb{C}P^1$→DD→CD



$c_6 = I_0 \cdot \sigma^6(I_0)^{-1}$

Intersection matrix(Details)

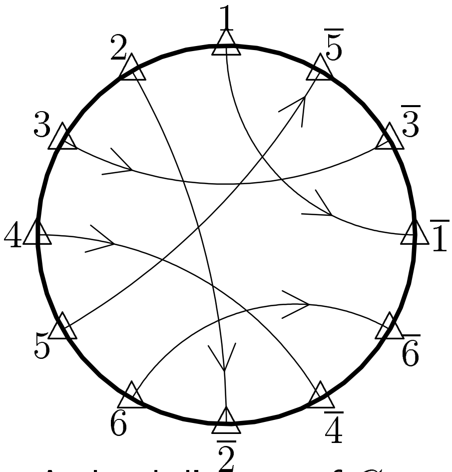
CC→Dessin d'enfants→CD



A dessin d'enfants of $C_{7,2}$

Intersection matrix(Details)

CC→DD→Chord diagram



A chord diagram of $C_{7,2}$

Intersection matrix(Details)

We obtain the intersection matrix $A = (c_i \cdot c_j)$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}$$

Chord diagram methods

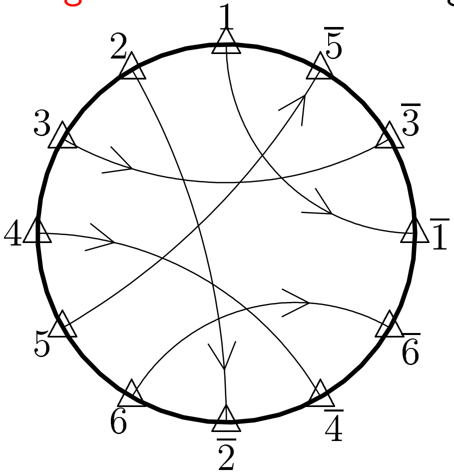
Find $T \in M_{2g}(\mathbb{Z})$ s.t. $TA^tT = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$.

Then, we have a symplectic basis

$$(a_1, \ldots, a_g, b_1, \ldots, b_g) = (c_1, c_2, \ldots, c_{2g})^tT$$

Chord diagram methods

Chord diagram→Linear Chord Diagrams



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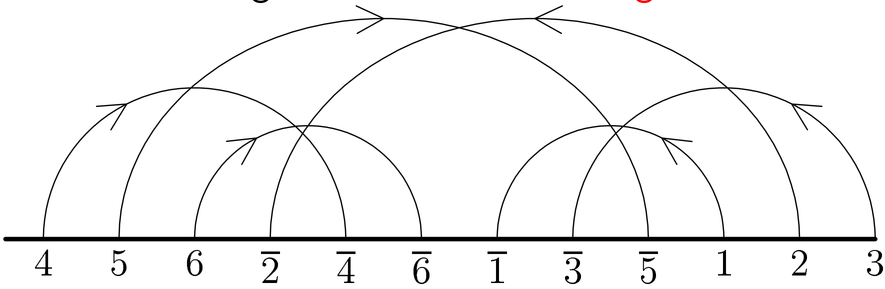
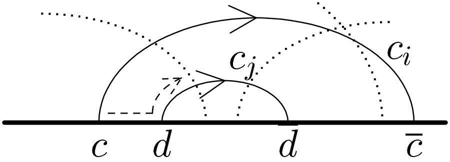


A linear chord diagram of $C_{7,2}$

Chord diagram methods



CSM(Base Change)

$$c'_k = \begin{cases} c_i - c_j & (k = i), \\ c_k & (k \neq i). \end{cases}$$

The advantage of CSM is its applicability to other curves for generic genus. In fact, we obtain another symplectic basis of C_g different to $\{A_i, B_i\}$.

Chord diagram methods

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 1 \end{pmatrix} \qquad TA^tT = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

A basis of holomorphic 1-forms

- $\alpha_l = \lfloor nl/p \rfloor$, $\alpha_m = \lfloor nm/p \rfloor$
- $d_n = \lfloor n(l+m)/p \rfloor - \alpha_l - \alpha_m - 1$
- $\omega_{n,d} = z^{\alpha_l}(1-z)^{\alpha_m} z^d dz/w^n$
- $S := \{(n,d):\ 0 \leq d \leq d_n \text{ and } 1 \leq n \leq p-1\}$

Theorem (Bennama(1998))

$\{\omega_{n,d}\}_{(n,d)\in S}$: *a basis of* $H^{1,0}(X)$

(n,d)	$(3,0)$	$(5,0)$	$(6,0)$
$\omega_{n,d}$	dz/w^3	$(1-z)dz/w^5$	$(1-z)dz/w^6$

Periods

- $\Omega_A = \left(\int_{a_j} \omega_i\right)$, $\Omega_B = \left(\int_{b_j} \omega_i\right)$: Periods

$$\Omega_A = \begin{pmatrix} 1-\zeta & \zeta-\zeta^2 & 1-\zeta^2+\zeta^3-\zeta^5 \\ 1-\zeta^2 & \zeta^2-\zeta^4 & 1-\zeta^4+\zeta^6-\zeta^{10} \\ 1-\zeta^4 & \zeta^4-\zeta^8 & 1-\zeta^8+\zeta^{12}-\zeta^{20} \end{pmatrix}$$

$$\Omega_B = \begin{pmatrix} 1-\zeta^3 & 1-\zeta^4 & \zeta-\zeta^2+\zeta^4-\zeta^6 \\ 1-\zeta^6 & 1-\zeta^8 & \zeta^2-\zeta^4+\zeta^8-\zeta^{12} \\ 1-\zeta^{12} & 1-\zeta^{16} & \zeta^4-\zeta^8+\zeta^{16}-\zeta^{24} \end{pmatrix}$$

Period matrix

- $\tau_g = \Omega_A^{-1}\Omega_B$
- $A^{-1} = \underbrace{\dfrac{1}{\det A}}_{\text{Euclidean Algorithm}} \operatorname{adj} A$

$$\tau_g = \begin{pmatrix} 6+3\xi & 4+2\xi & -2-\xi \\ 4+2\xi & 4+4\xi & -2\xi \\ -2-\xi & -2\xi & 2+3\xi \end{pmatrix}$$

where $\zeta = \zeta_7$ and $\xi = \zeta + \zeta^2 + \zeta^4 = (-1+\sqrt{-7})/2$.

Summary

- We explicitly determine τ_g by the affine equation $w^2 = z^{2g+1} - 1$, its entries being elements of the $\mathbb{Q}(\zeta_{2g+1})$
- We made a program which computes

$$(p, l, m) \longrightarrow \text{period matrix of } X$$

Summary

- We explicitly determine τ_g by the affine equation $w^2 = z^{2g+1} - 1$, its entries being elements of the $\mathbb{Q}(\zeta_{2g+1})$
- We made a program which computes

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Find an explicit expression of period matrices of other curves for generic genus!!

Summary

Thank you very much!
ありがとうございました!

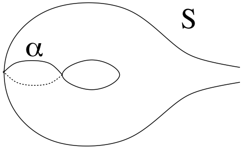
Exotic components in linear slices of quasi-Fuchsian groups

Yuichi Kabaya
(Kyoto University)
https://www.math.kyoto-u.ac.jp/~kabaya/
(These slides are available.)

Osaka, February 14 2015

Outline

S : once punctured torus
$\alpha \subset S$: essential simple closed curve



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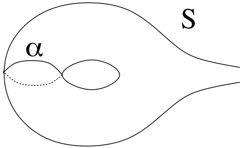

$$QF(S) = \{\rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \mid \text{injective, } \rho(\pi_1(S)) \text{ quasi-Fuchsian}\} / \sim_{conj.}$$

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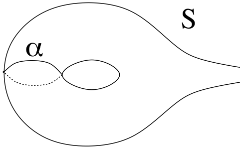

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$\lambda_\alpha : QF(S) \to \mathbb{C}/2\pi\sqrt{-1}\mathbb{Z}$: the (complex) length of α

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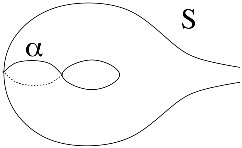

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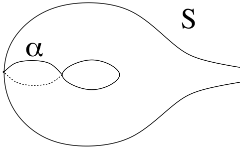
For $\ell > 0$, consider a slice of $QF(S)$

$$QF(\ell) = \{\rho \in QF(S) \mid \lambda_\alpha(\rho) = \ell\}.$$

Outline



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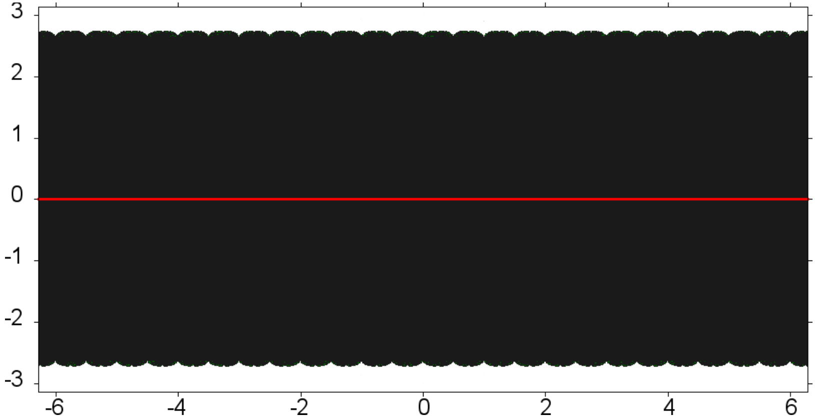
$$QF(\ell) = \{\rho \in QF(S) \mid \lambda_\alpha(\rho) = \ell\}.$$

This can be regarded as a subset of

$$\{\tau \in \mathbb{C} \mid -\pi < \mathrm{Im}(\tau) \leq \pi\}.$$

Outline

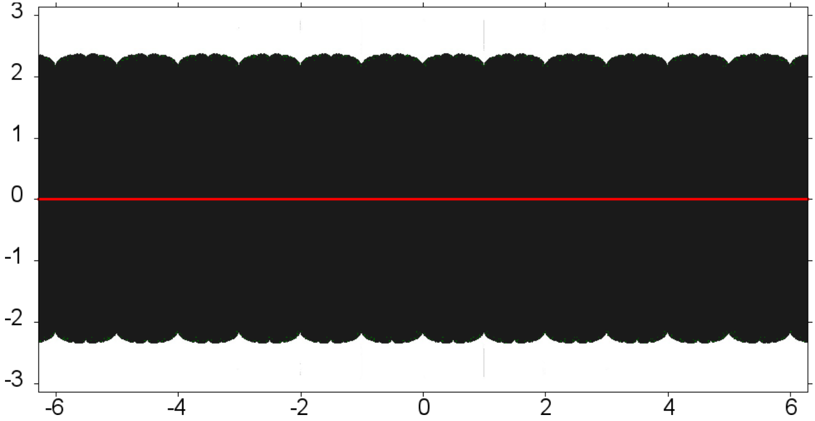
$QF(\ell)$ looks like this:



$\ell = 0.5$

Outline

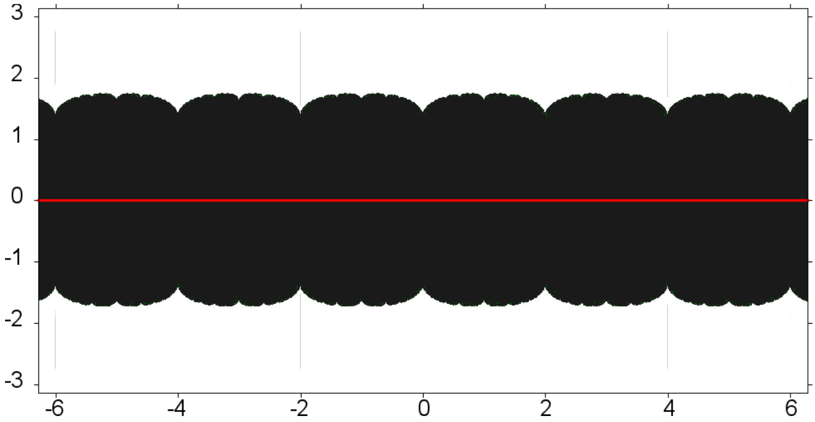
$QF(\ell)$ looks like this:



$\ell = 1.0$

Outline

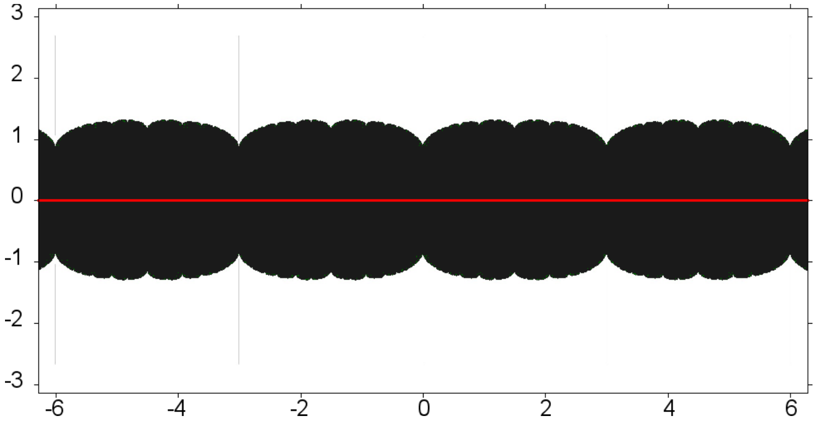
$QF(\ell)$ looks like this:



$\ell = 2.0$

Outline

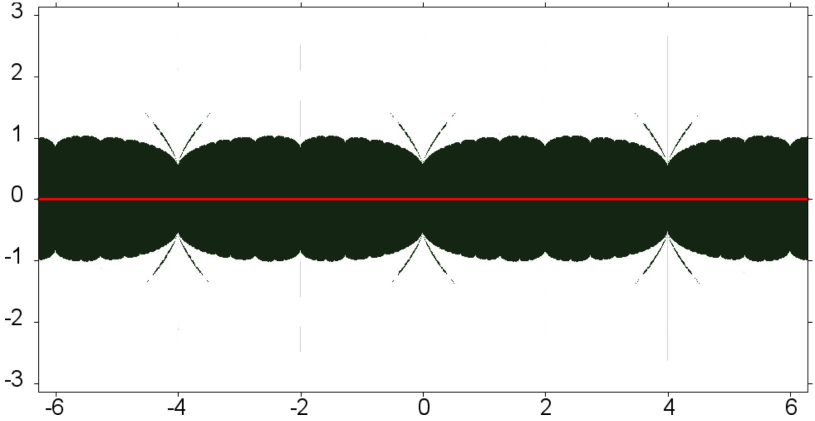
$QF(\ell)$ looks like this:



$\ell = 3.0$

Outline

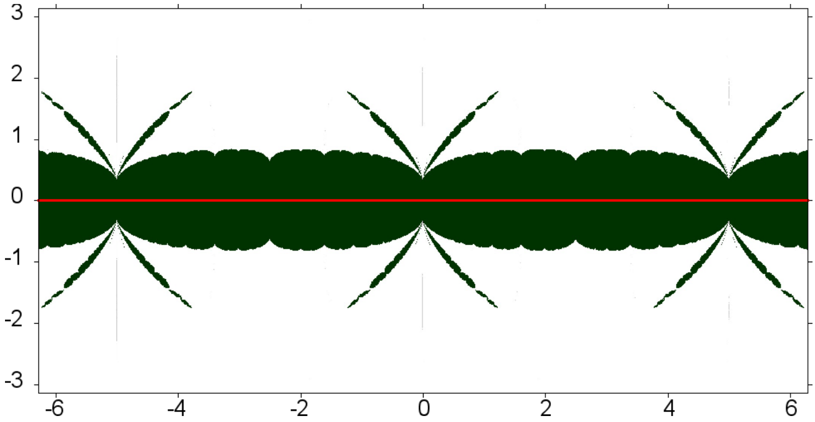
$QF(\ell)$ looks like this:



$\ell = 4.0$

Outline

$QF(\ell)$ looks like this:



$\ell = 5.0$

Outline

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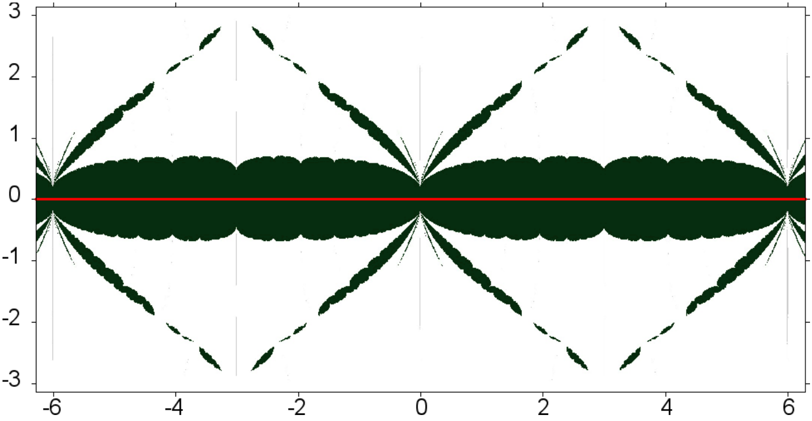

$\ell = 6.0$

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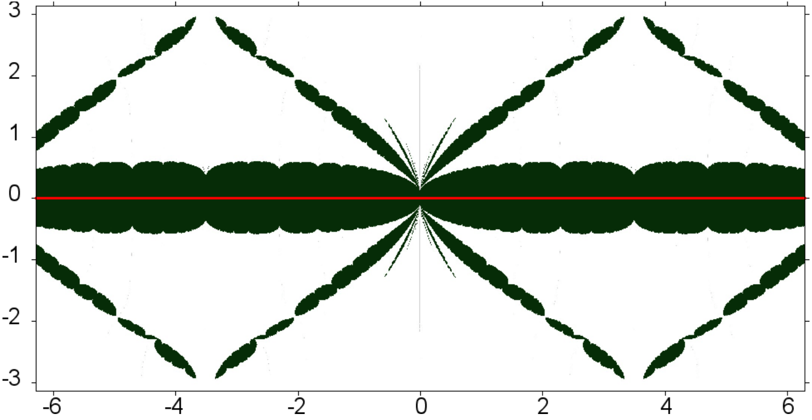

$\ell = 7.0$

Outline

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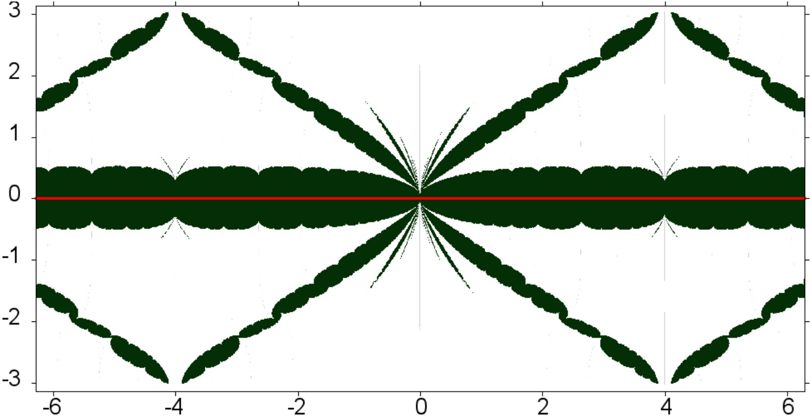

$\ell = 8.0$

Outline

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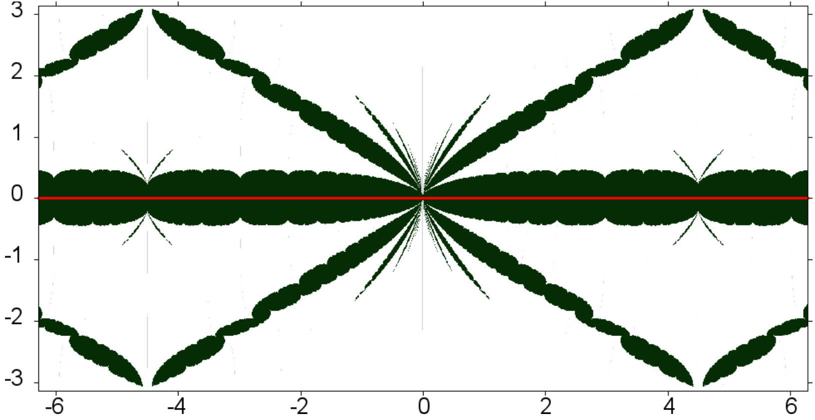
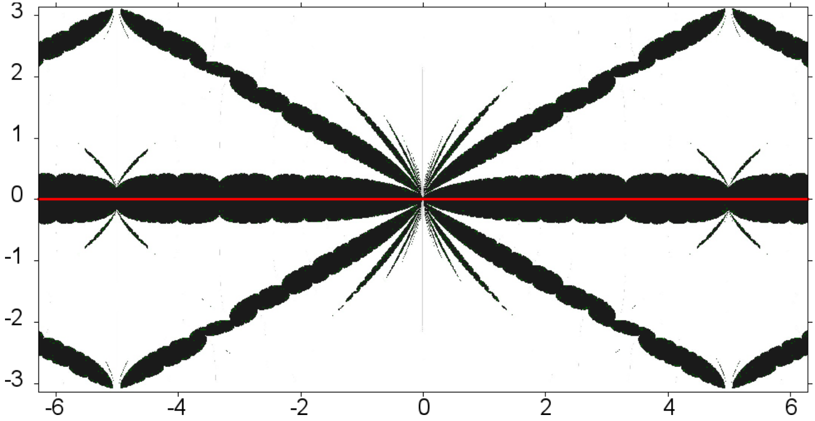


$\ell = 9.0$

Outline

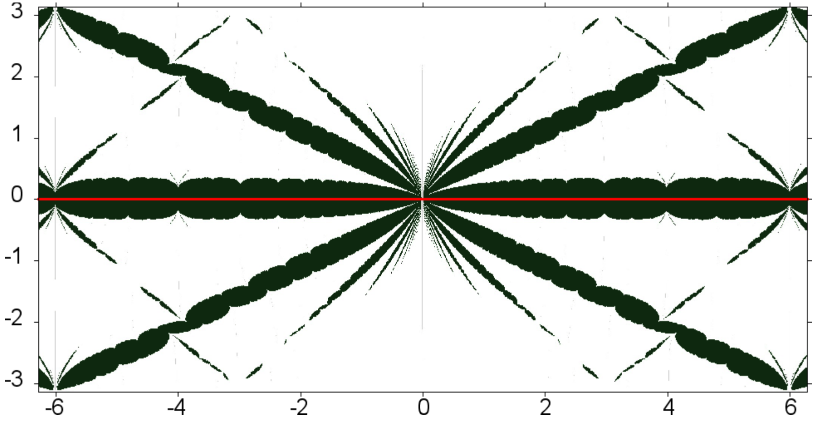
$QF(\ell)$ looks like this:



$\ell = 10.0$

Outline

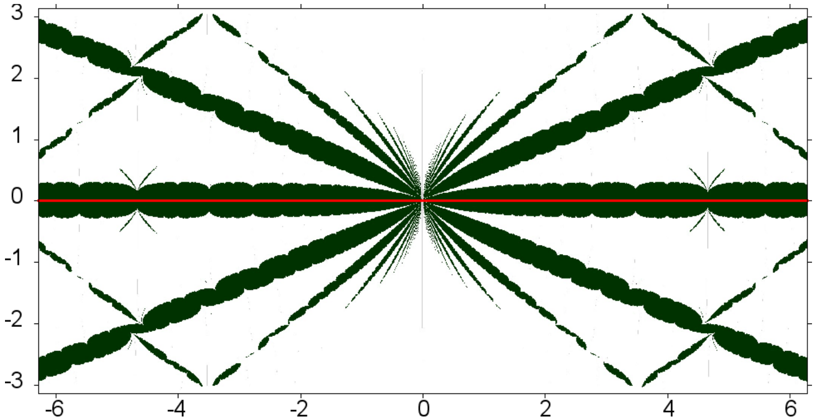
$QF(\ell)$ looks like this:



$$\ell = 12.0$$

Outline

$QF(\ell)$ looks like this:



$\ell = 14.0$

Outline

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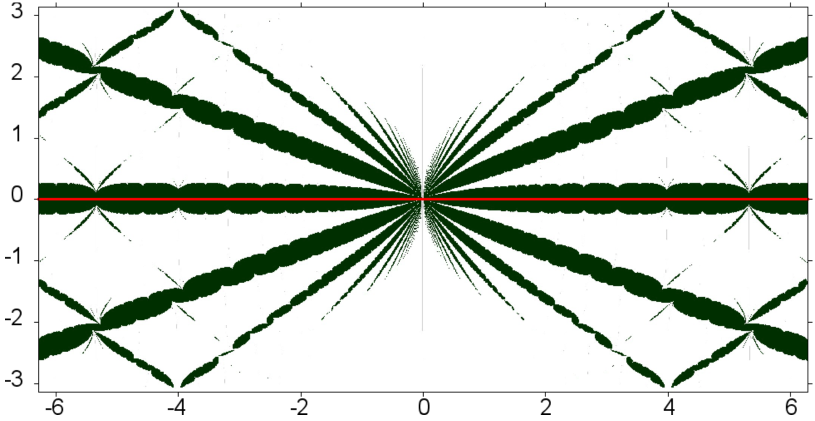
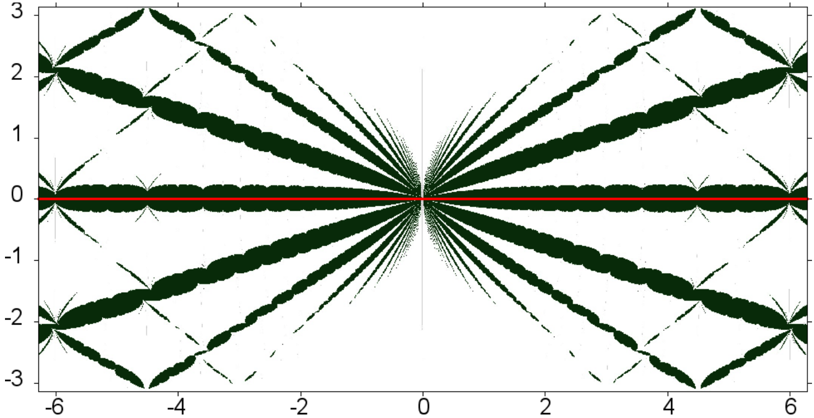


$\ell = 16.0$

Outline

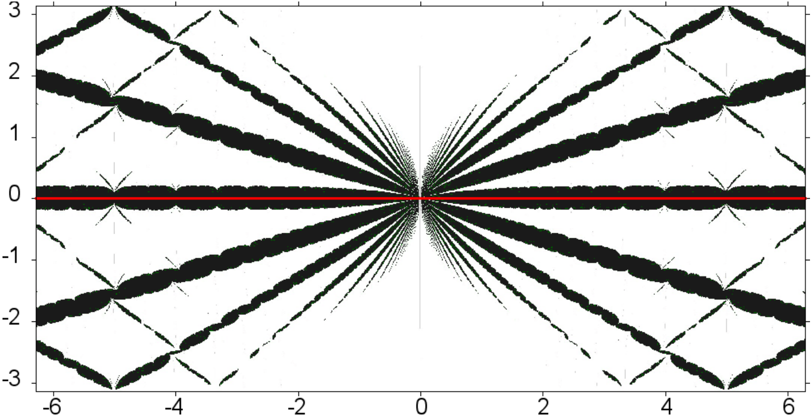
$QF(\ell)$ looks like this:



$\ell = 18.0$

Outline

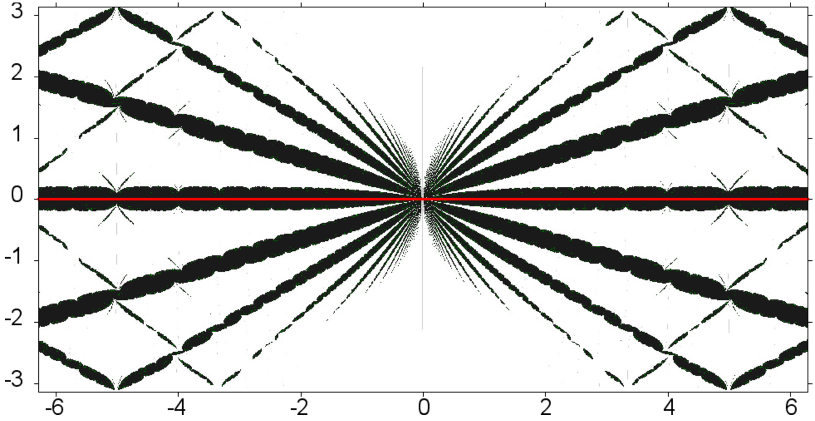
$QF(\ell)$ looks like this:



$\ell = 20.0$

Outline

$QF(\ell)$ looks like this:



$\ell = 20.0$

Interested in the shape of $QF(\ell)$ as ℓ getting longer.

Outline

1. Basics on Kleinian (once punctured torus) groups
2. Linear slices & Main theorem
3. Complex projective structures and complex earthquake
4. Proof of the main theorem

With many pictures ...

Basics (Hyperbolic space)

$\mathbb{H}^3 = \{(z, t) \mid z \in \mathbb{C},\ t \in \mathbb{R}_{>0}\}$: 3-dim hyperbolic space

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This action extends to the interior $\mathbb{H}^3$ isometrically.

$\Gamma < \mathrm{PSL}_2\mathbb{C}$: torsion free discrete subgroup

$\Rightarrow M = \mathbb{H}^3/\Gamma$ is a complete hyperbolic 3-manifold
s.t. $\pi_1(M) \cong \Gamma$

Basics (Deformation space)

$S = S_{g,n}$: genus g, n punctured surface ($\chi(S) < 0$)

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$$AH(S) = \{[\rho] \in X(S) \mid \text{faithful, discrete image}\}$$

If $\rho \in AH(S)$, then $\mathbb{H}^3/\rho(\pi_1(S))$ is a complete hyperbolic 3-manifold homotopy equivalent to S.

$AH(S)$ is the deformation space of such structures.

Basics (Limit sets)

$\Gamma < \mathrm{PSL}_2\mathbb{C}$: discrete subgroup

Fix a point $p \in \mathbb{H}^3$. The limit set of Γ is defined by

$$\Lambda(\Gamma) = \{\text{accumulation points of } \Gamma \cdot p \text{ on } \mathbb{C}P^1\}.$$

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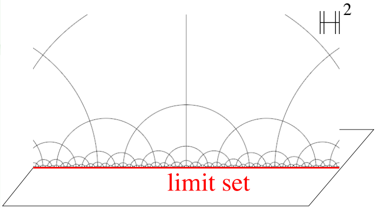
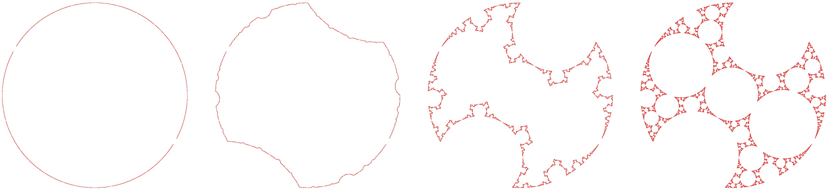
Example (Fuchsian groups)

If $\Gamma < \mathrm{PSL}_2(\mathbb{R})$, Γ preserves $\mathbb{H}^2 (\subset \mathbb{H}^3)$, thus $\Lambda(\Gamma)$ is a subset of $\mathbb{R} \cup \{\infty\}$ (a 'round circle' in $\mathbb{C}P^1$).



Basics (Quasi-Fuchsian representations)

We can deform a Fuchsian rep a little in $PSL_2\mathbb{C}$. The limit set is no longer a round circle, but may be $\cong S^1$.



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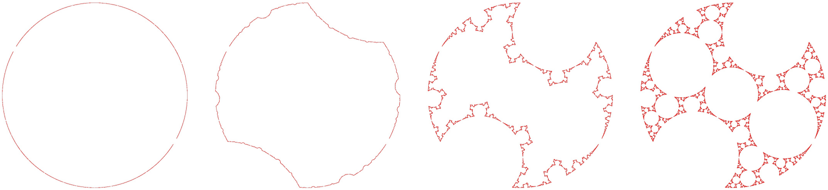

Definition

Let $\rho \in AH(S)$. If the limit set $\Lambda(\rho(\pi_1(S)))$ is homeomorphic to S^1, ρ is called quasi-Fuchsian.

$$QF(S) = \{\rho \in AH(S) \mid \rho \text{ is quasi-Fuchsian.}\}$$

Basics (Known properties)

Known facts

- $QF(S) \subset X(S)$ open subset

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- $QF(S) \subset X(S)$ open subset
- $QF(S) \cong \mathcal{T}(S) \times \mathcal{T}(\overline{S})$, where $\mathcal{T}(S)$ is the Teichmüller space of S. ($\mathcal{T}(S_{g,n}) \cong \mathbb{R}^{6g-6+2n}$)

Basics (Known properties)

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But the shape of $QF(S)$ in $X(S)$ is very complicated! (e.g. self-bumping, $AH(S)$ is not locally connected.)

Basics (Complex length)

For $\gamma \in \pi_1(S)$, $\rho \in X(S)$, $\rho(\gamma)$ acts on $\mathbb{H}^3$.

Define the (complex) length by

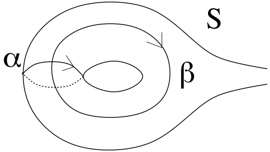
$$\lambda_\gamma(\rho) = (\text{translation length of } \rho(\gamma)) + \sqrt{-1}\,(\text{rotation angle of } \rho(\gamma))$$

mod $2\pi\sqrt{-1}\mathbb{Z}$. This is characterized by

$$\mathrm{tr}(\rho(\gamma)) = 2\cosh(\frac{\lambda_\gamma(\rho)}{2}).$$

Character variety

$S = S_{1,1}$: once punctured torus
Fix $\alpha, \beta \in \pi_1(S)$ so that $[\alpha, \beta]$ is peripheral.



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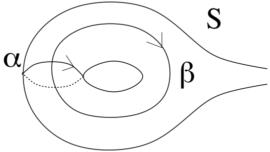

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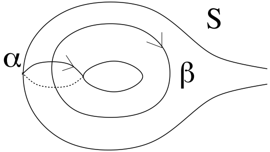

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$$X_{SL}(S) \cong \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^2 + z^2 = xyz\}$$

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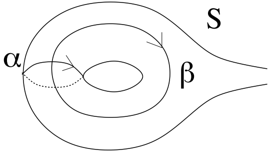

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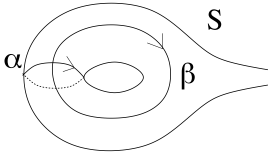
$X(S)$ is obtained as a quotient of $X_{SL}(S)$ by the action of $\mathbb{Z}/2\mathbb{Z}$ generated by

$$(x, y, z) = (-x, -y, z), \quad (x, y, z) = (x, -y, -z).$$

Linear slices

Any essential simple closed curve on $S = S_{1,1}$ is represented by a primitive element $p[\alpha] + q[\beta] \in H_1(S;\mathbb{Z})$. Regard it as $p/q \in \mathbb{Q} \cup \{\infty\}$.



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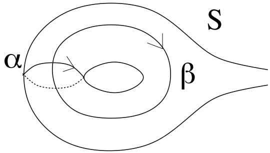

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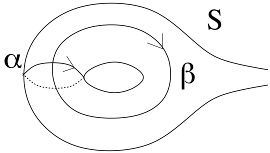

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Definition

For $\ell > 0$, let

$$X(\ell) = \{\rho \in X(S) \mid \lambda_{1/0}(\rho) = \ell\}$$

$X(\ell)$ is a slice of $X(S)$ on which (cpx length of α) $\equiv \ell$.

Complex Fenchel-Nielsen coordinates

For $\ell > 0$, define a map

$$\{\tau \in \mathbb{C} \mid -\pi < \mathrm{Im}(\tau) \leq \pi\} \xrightarrow{\cong} X(\ell)$$

by

$$\tau \mapsto \left(2\cosh(\ell/2),\ \frac{2\cosh(\tau/2)}{\tanh(\ell/2)},\ \frac{2\cosh((\tau+\ell)/2)}{\tanh(\ell/2)}\right).$$

This gives a bijection. (Recall $\mathrm{tr}\,\rho(\alpha) = 2\cosh(\lambda_{1/0}/2)$.)

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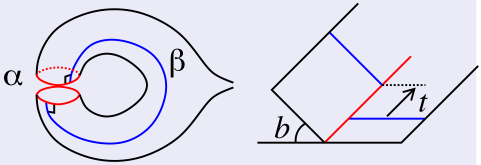
Note

If we let $\tau = t + \sqrt{-1}b$, t is the twisting distance and b is the bending angle along α.



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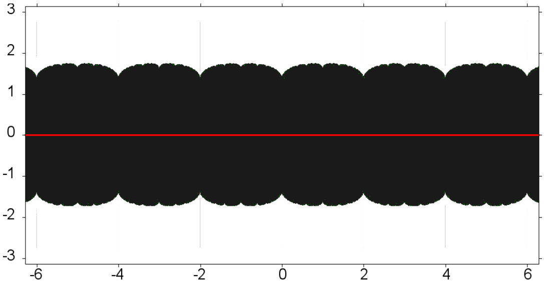

$QF(2.0) \subset X(2.0)$ (black region)

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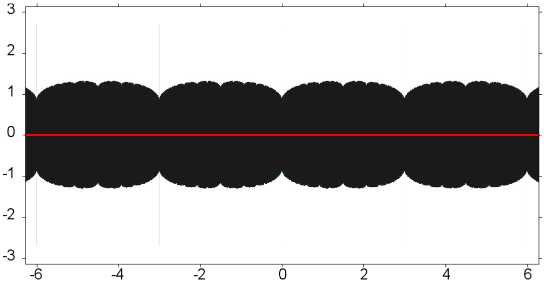

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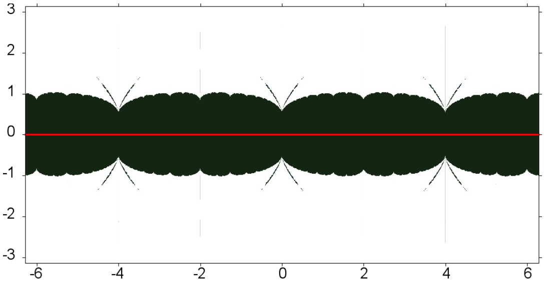

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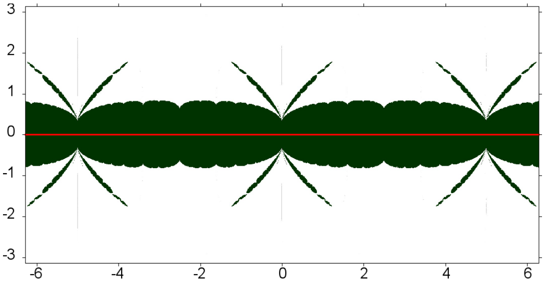

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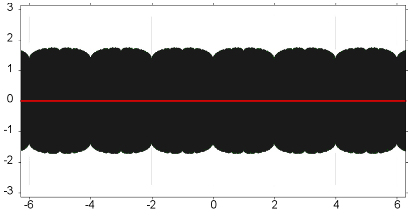

$QF(2.0)$

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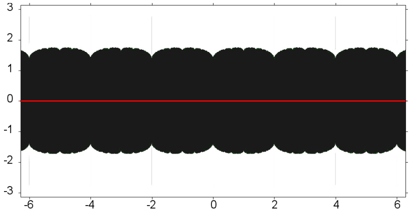
- The Dehn twist along α acts on $X(\ell)$ as

$$\tau \mapsto \tau + \ell. \quad \text{(translation)}$$

Linear slices of $QF(S)$



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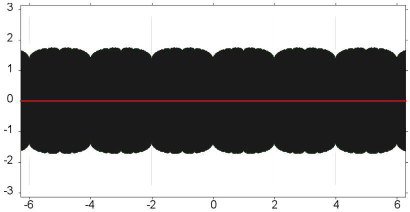

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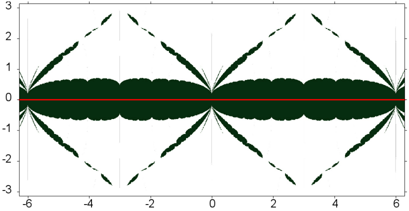

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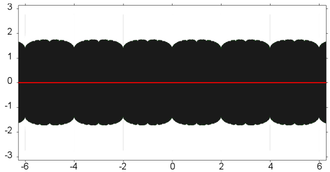
$QF(\ell)$ has only one component if ℓ is sufficiently small, has more than one component if ℓ is sufficiently large.



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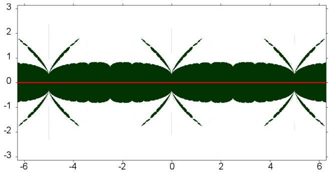

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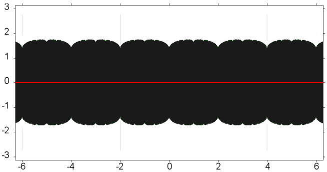

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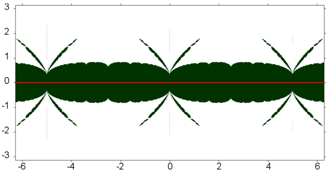

$QF(5.0)$

Today, we will give another proof for the latter part, and give refined results.

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Roughly, a representation in BM is obtained from a Fuchsian one by bending along α continuously.

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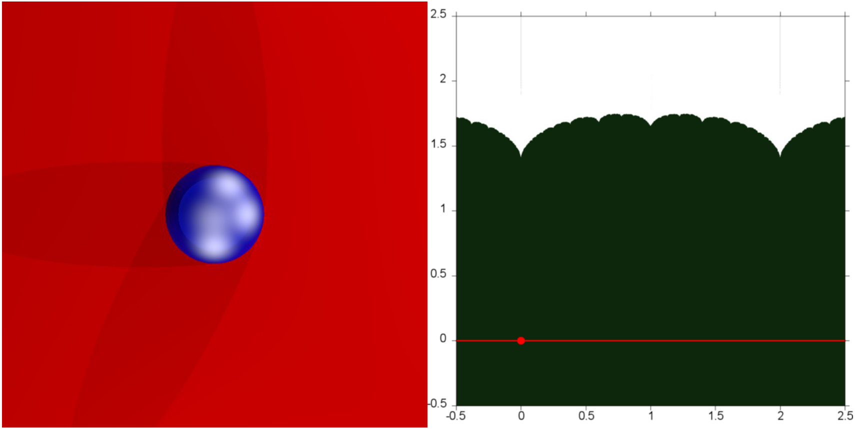
Recall $\quad \tau = \text{(twisting dist.)} + \sqrt{-1}\text{(bending angle)}.$



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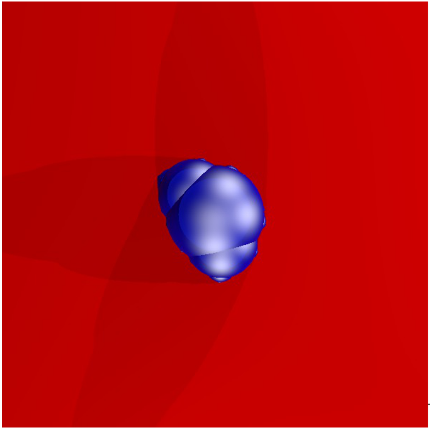

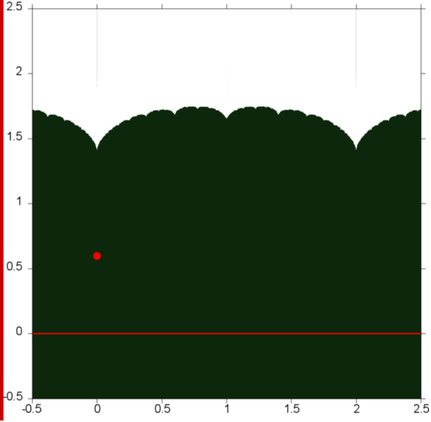

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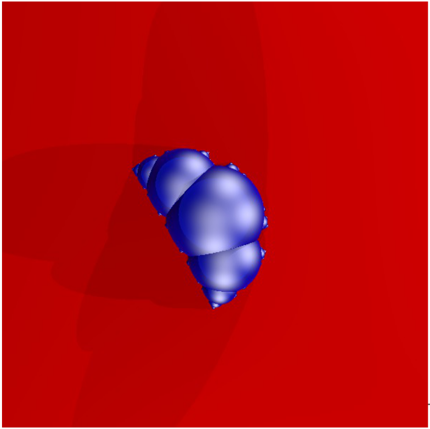

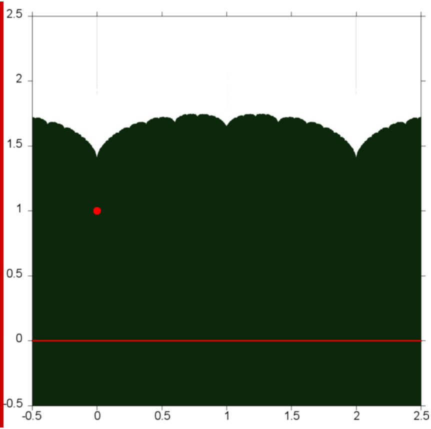

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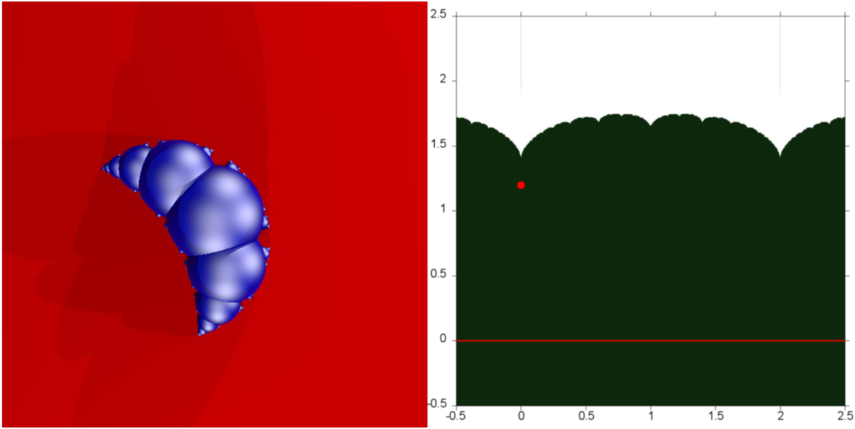

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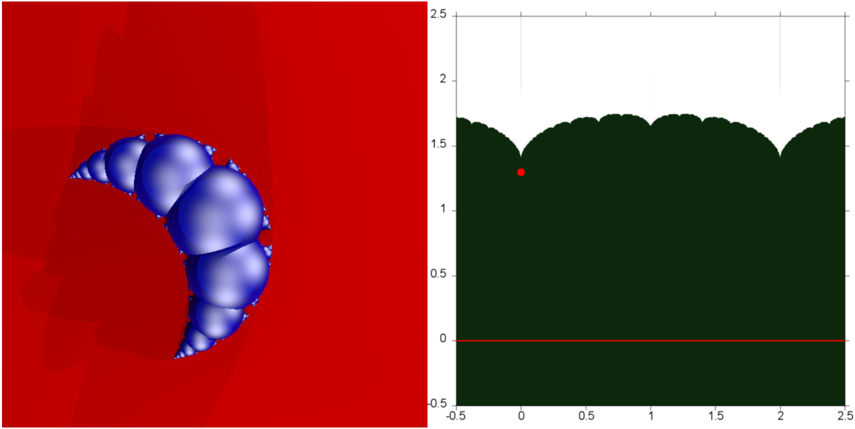

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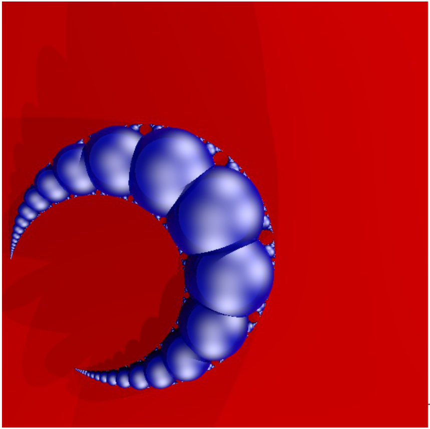

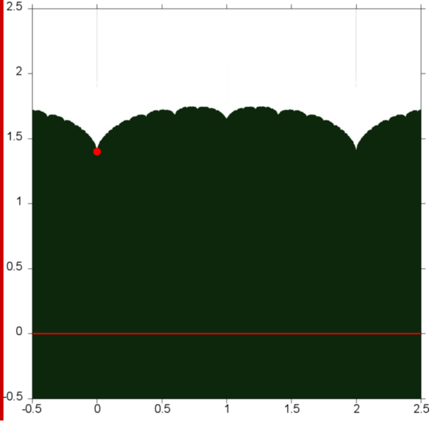

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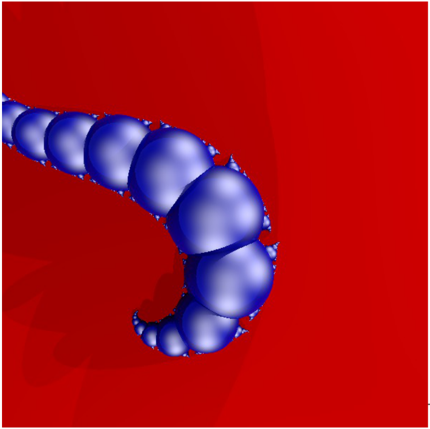

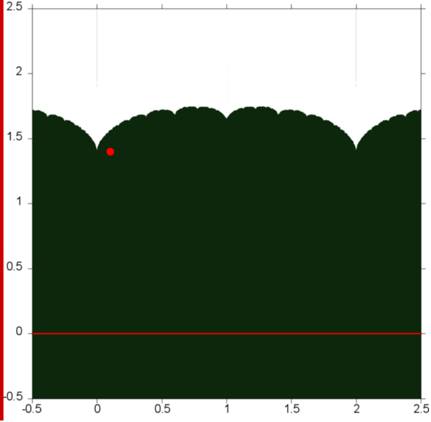

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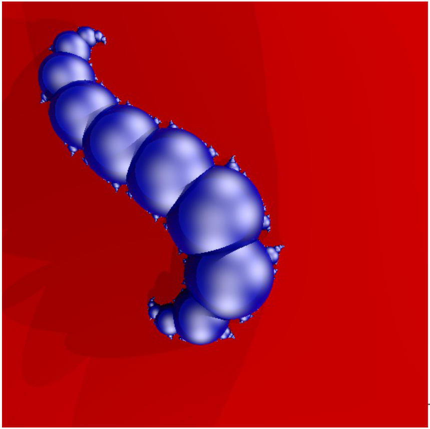

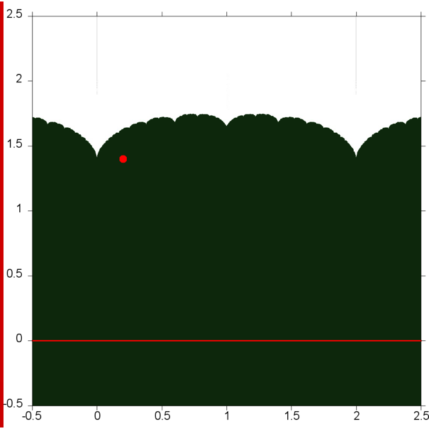

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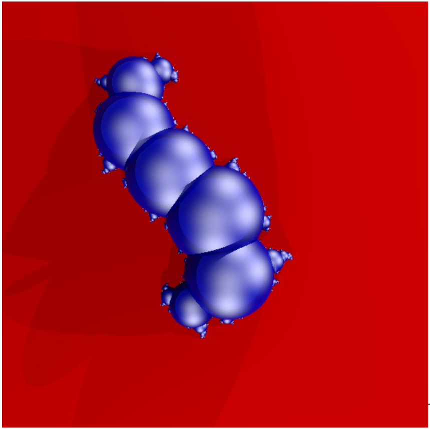

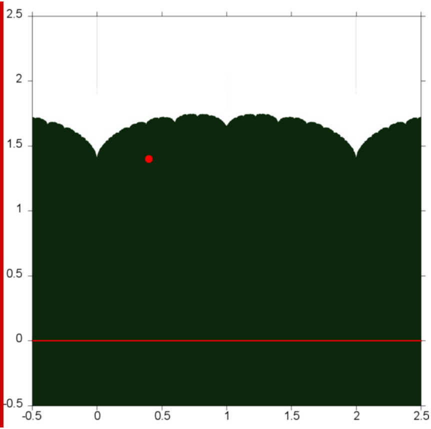

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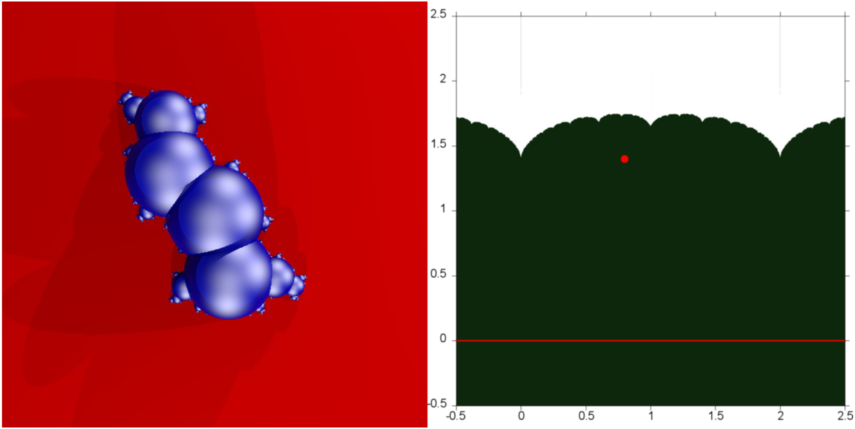

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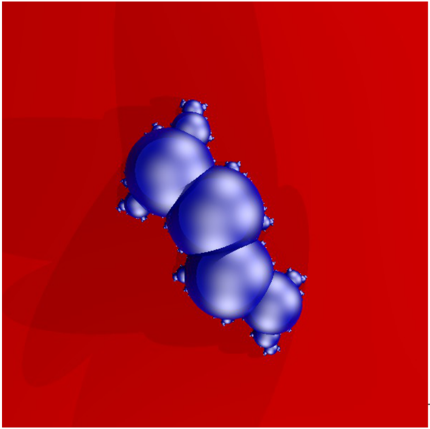

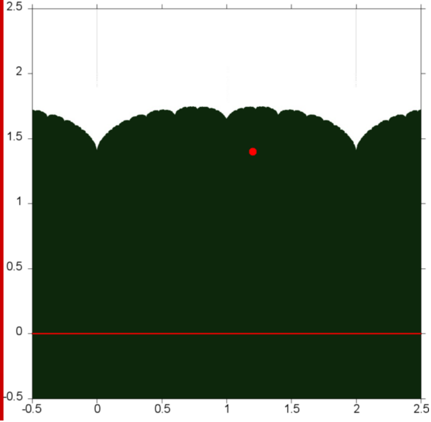

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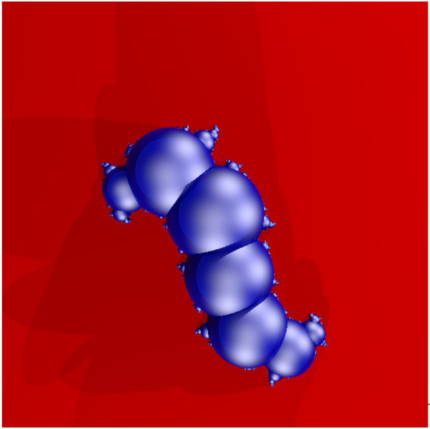

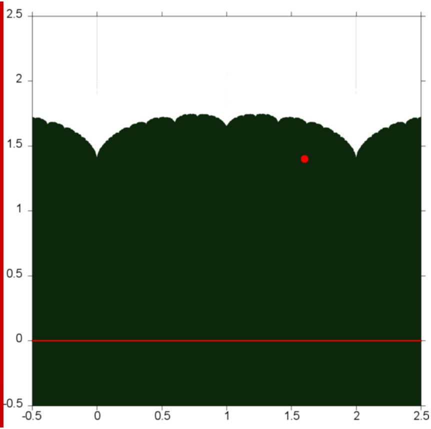

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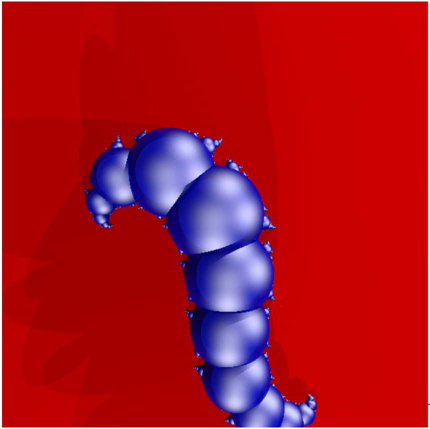

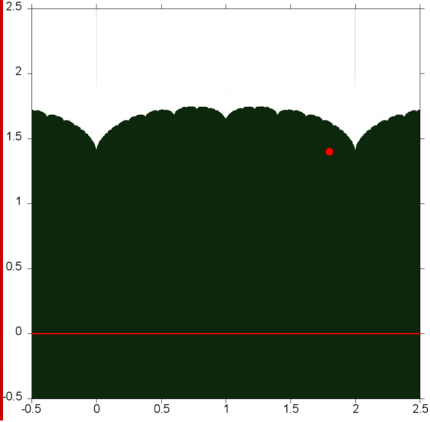

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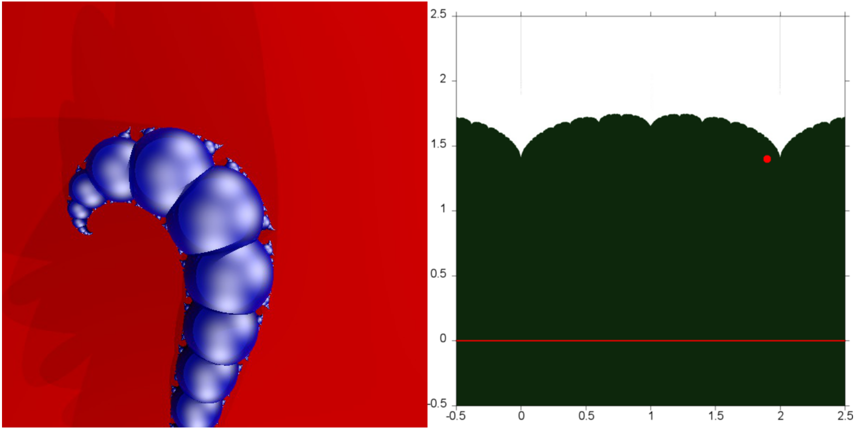

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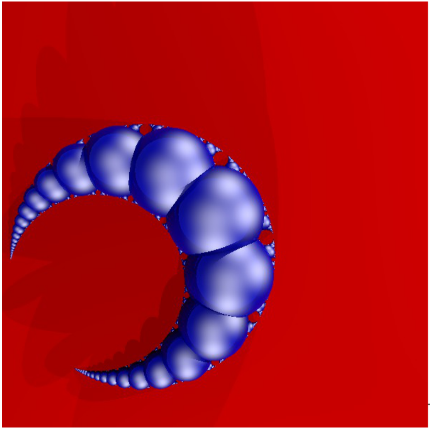

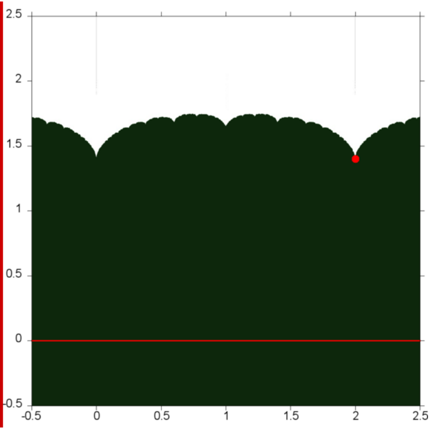

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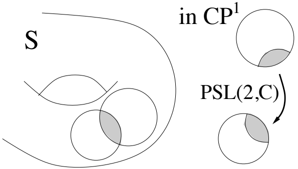
Complex projective structures

S : surface ($\chi(S) < 0$)

Definition

A complex projective structure or $\mathbb{C}P^1$-structure on S is a geometric structure locally modelled on $\mathbb{C}P^1$ with transition functions in $\mathrm{PSL}_2\mathbb{C}$.



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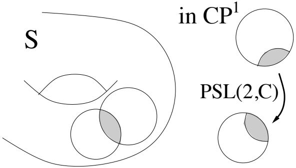

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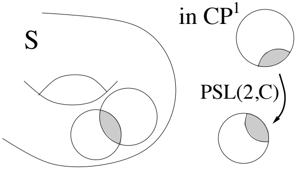

(If S has punctures, assume some boundary conditions.)

Example (Fuchsian uniformization)

A hyperbolic str on S gives an identification $\widetilde{S} \cong \mathbb{H}^2$. Since $\mathbb{H}^2 \subset \mathbb{C}P^1$, this gives a $\mathbb{C}P^1$-str.

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Similarly as Teichmüller space, we can define $P(S)$ the set of marked $\mathbb{C}P^1$-structures on S.

Complex projective structures

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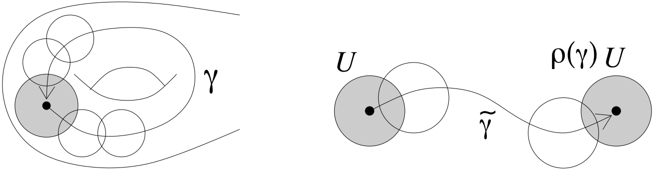
By analytic continuation, we have a holonomy map

$$\text{hol} : P(S) \to X(S).$$



This is known to be a local homeomorphism.

Grafting

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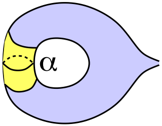

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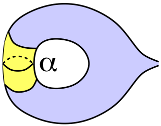
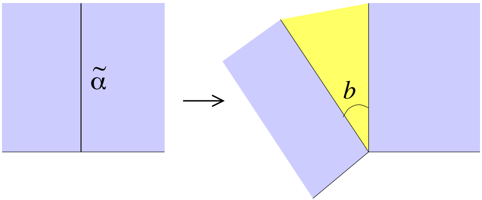


In the universal cover $\widetilde{X}$, the local picture looks like:



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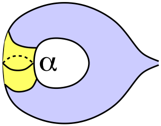

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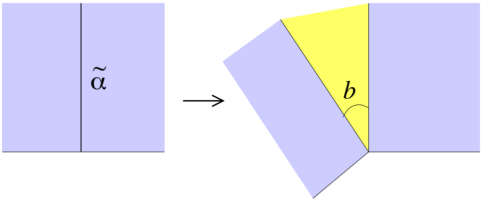
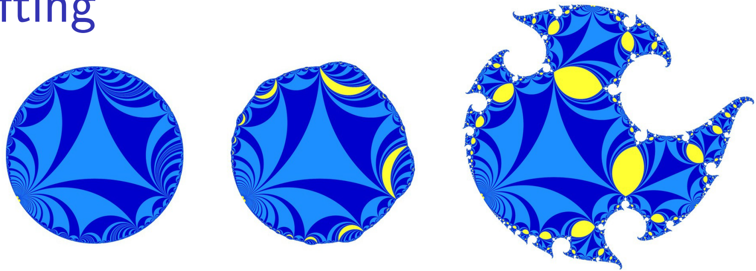


But there are infinitely many lifts of α $\cdots$

Grafting



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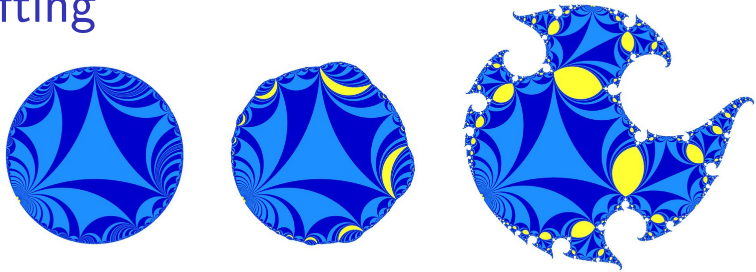

The grafting operation $\mathrm{Gr}_{b\cdot\alpha} : \mathcal{T}(S) \to P(S)$ can be generalized for measured laminations.

Theorem (Thurston, Kamishima-Tan)

$$\begin{aligned} \mathrm{Gr} : \mathcal{ML}(S) \times \mathcal{T}(S) &\to P(S) \\ (\mu, X) &\mapsto \mathrm{Gr}_\mu(X) \end{aligned}$$

is a homeomorphism (Thurston coordinates).

$\mathbb{C}P^1$-structures with q-F holonomy

$$Q_0 = \{ \text{ marked } \mathbb{C}P^1\text{-strs with q-F holonomy and injective developing maps } \} \subset P(S)$$

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For $\mu \in \mathcal{ML}_{\mathbb{Z}}(S)$, let Q_μ be the set of $\mathbb{C}P^1$-strs obtained from Q_0 by $2\pi\mu$-grafting. (**Remark** $Q_\mu \cong Q_0$.)

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Theorem (Goldman)

$$\mathrm{hol}^{-1}(QF(S)) = \bigsqcup_{\mu \in \mathcal{ML}_{\mathbb{Z}}(S)} Q_\mu$$

The component Q_0 is called standard, Q_μ $(\mu \neq 0)$ exotic.

Complex Earthquake

Let $\overline{\mathbb{H}} = \{\tau = t + \sqrt{-1}b \in \mathbb{C} \mid b \geq 0\}$.

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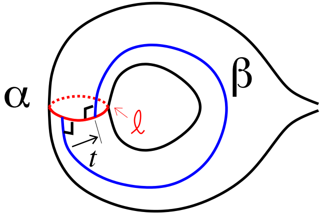
Let $\mathrm{tw}_{t \cdot \alpha}(X_\ell) = \Big($  $\Big) \in \mathcal{T}(S)$.

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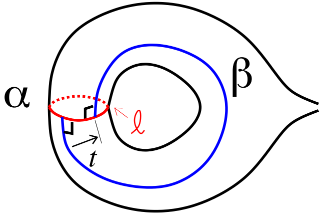

$\Bigg) \in \mathcal{T}(S)$.

Define Eq : $\overline{\mathbb{H}} \to P(S)$ by

$$\mathrm{Eq}(t + \sqrt{-1}b) = \mathrm{Gr}_{b\cdot\alpha}(\mathrm{tw}_{t\cdot\alpha}(X_\ell)) \in P(S)$$

By Thurston coords, we can regard $\overline{\mathbb{H}} \subset P(S)$.

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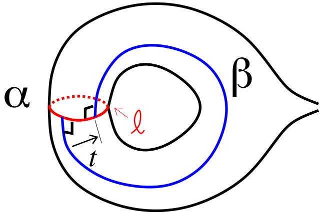

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By Thurston coords, we can regard $\overline{\mathbb{H}} \subset P(S)$.

Simply denote the image of $\overline{\mathbb{H}}$ by $\mathrm{Eq}(\ell)$.

Complex Earthquake

By construction, hol is the natural projection:

$$\begin{array}{ccc}
P(S) & \xrightarrow{\text{hol}} & X(S) \\
\cup & & \cup \\
\mathrm{Eq}(\ell) & \to & X(\ell) \\
\| & & \| \\
\{\tau \mid \mathrm{Im}(\tau) \geq 0\} & & \{\tau \mid -\pi < \mathrm{Im}(\tau) \leq \pi\} \\
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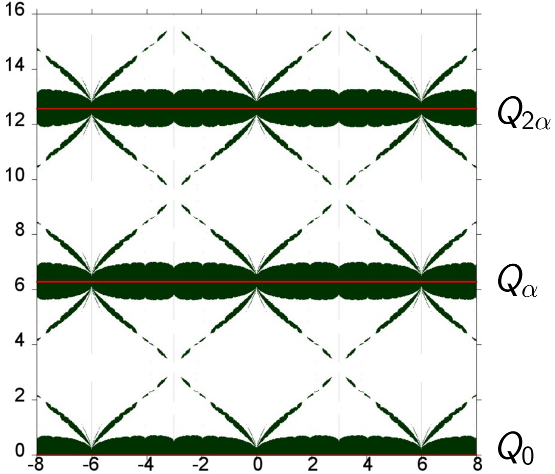
We are interested in $QF(\ell) \subset X(\ell)$, so consider

$$\begin{aligned} \mathrm{hol}^{-1}(QF(\ell)) &= \mathrm{hol}^{-1}(X(\ell) \cap QF(S)) \\ &= \mathrm{Eq}(\ell) \cap \mathrm{hol}^{-1}(QF(S)). \end{aligned}$$

Complex Earthquake

By Goldman's Theorem, we have

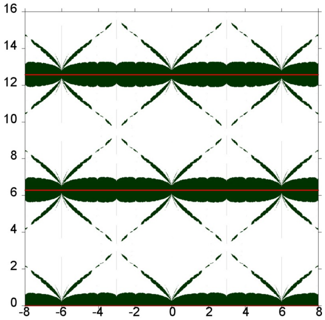
$$\mathrm{Eq}(\ell) \cap \mathrm{hol}^{-1}(QF(S)) = \bigsqcup_{\mu \in \mathcal{ML}_{\mathbb{Z}}(S)} \mathrm{Eq}(\ell) \cap Q_\mu .$$



Eq(6.0)

Complex Earthquake

hol maps each component of $\mathrm{Eq}(\ell) \cap Q_\mu$ into a comp of $QF(\ell)$.



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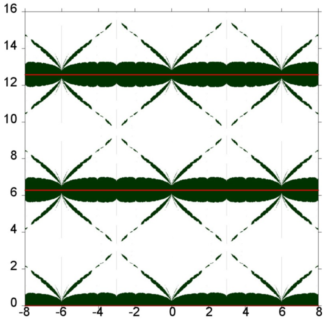

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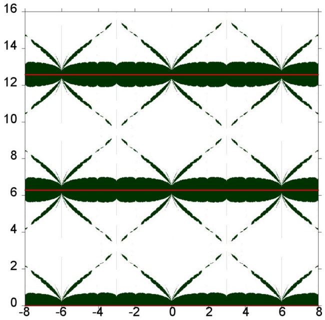

Prop (K.)

$$Eq(\ell) \cap \mathrm{hol}^{-1}(BM) = \bigsqcup_{k \geq 0} Eq(\ell) \cap Q_{k \cdot \alpha}$$

for any $\ell > 0$.

Existence of exotic components in Eq(ℓ)

We need to find $\mu \notin \{0, \alpha, 2\alpha, \cdots\}$ s.t. $\mathrm{Eq}(\ell) \cap Q_\mu \neq \emptyset$ for sufficiently large $\ell > 0$. Consider the case $\mu = \beta$.

Existence of exotic components in $\mathrm{Eq}(\ell)$

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Let D_β be the Dehn twist along β. Fix $X \in \mathcal{T}(S)$.

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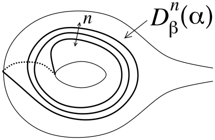
Consider a sequence in $P(S) \cong \mathcal{ML}(S) \times \mathcal{T}(S)$

$$\left(\frac{2\pi}{n} D_\beta^n(\alpha),\, X\right)$$



which converges to $(2\pi\beta,\, X) \in Q_\beta$ as $n \to \infty$.

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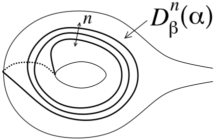

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Thus $(\frac{2\pi}{n} D_\beta^n(\alpha),\, X) \in Q_\beta$ for large n.

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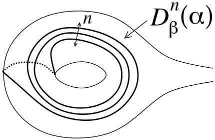

which converges to $(2\pi\beta,\, X) \in Q_\beta$ as $n \to \infty$.

Thus $(\frac{2\pi}{n} D_\beta^n(\alpha),\, X) \in Q_\beta$ for large n.

Apply D_β^{-n}, then $(\frac{2\pi}{n}\alpha,\, D_\beta^{-n}(X)) \in Q_\beta$ for large n.

Existence of exotic components in $\mathrm{Eq}(\ell)$

We need to find $\mu \notin \{0, \alpha, 2\alpha, \cdots\}$ s.t. $\mathrm{Eq}(\ell) \cap Q_\mu \neq \emptyset$ for sufficiently large $\ell > 0$. Consider the case $\mu = \beta$.

Let D_β be the Dehn twist along β. Fix $X \in \mathcal{T}(S)$.

Consider a sequence in $P(S) \cong \mathcal{ML}(S) \times \mathcal{T}(S)$

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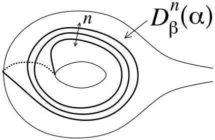

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Apply D_β^{-n}, then $(\frac{2\pi}{n}\alpha,\, D_\beta^{-n}(X)) \in Q_\beta$ for large n.

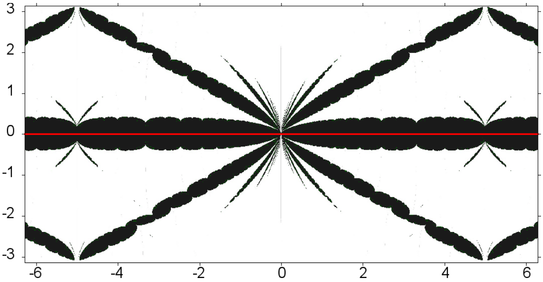
But if we let $\ell = \ell_\alpha(D_\beta^{-n}(X))$, $(\frac{2\pi}{n}\alpha,\, D_\beta^{-n}(X)) \in \mathrm{Eq}(\ell)$.

Final Remarks

- For $k \in \mathbb{N}$, we can show $\mathrm{Eq}(\ell) \cap Q_{k\cdot\beta} \neq \emptyset$ similarly for large ℓ by considering

$$\left(\frac{2\pi k}{n} D_\beta^n(\alpha),\ X\right) \xrightarrow{n\to\infty} (2\pi k\beta,\ X) \in Q_{k\cdot\beta}.$$

$\ell = 10.0$



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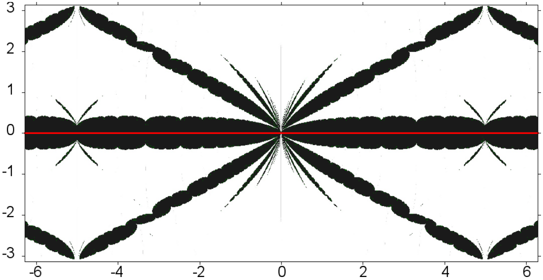


- Moreover we can use $\mu \in \mathcal{ML}(S)_{\mathbb{Z}}$ instead of β provided $i(\mu, \alpha) \neq 0$.

Some new problems in the theory of conformal mappings of an open Riemann surface of finite genus

M. Shiba
— partially with S. Hamano and H. Yamaguchi —

February 14, 2015 — Osaka U. Nakanoshima Center

Abstract

Let R be an open (=noncompact) Riemann surface of finite genus g.
If a closed (=compact) Riemann surface R' of genus g contains R as a subregion, R' is historically called a

"**compact continuation of the same genus**"

of R, but we prefer to use a shorter term:

"**closing**."

We give a precise definition in modern terminology and construct a closing of R with a remarkable hydrodynamic property.
These closings are used to comprehend

the totality $\mathcal{C}$ of the closings of R;

if $g = 1$ in particular, we use the modulus of a torus to describe $\mathcal{C}$ as a closed disk $\mathfrak{M}$ in $\mathbb{H}$. We generalize this result to $g > 1$.
The hyperbolic diameter $\sigma_H(R)$ of $\mathfrak{M}$ is called the hyperbolic span of R.
If $R = R_t$ moves holomorphically so that the set $\{(R_t, t) \mid t \in \mathbb{D}\}$ is pseudoconvex, $\sigma_H(R_t)$ is a subharmonic function.

Classical Results

Riemann's Theorem

$\forall\, G\,(\subset \hat{\mathbb{C}})$ **: simply connected domain whose boundary** ∂G
consists of more than one point

$\exists\, f : G \to \mathbb{D}$**, conformal [bijection]**

(or : $\exists\, f : G \to \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$**, conformal [bijection])**

Koebe's Theorem

$\forall\, R$ **: planar (= of genus 0) Riemann surface**

$\exists\,\Sigma$**: horizontal slit plane**
(correctly: [extremal] horizontal slit sphere)

$\exists\, f : R \to \Sigma$**, conformal embedding**

Remarks:

(1) ∂R **is subject to NO conditions**

(2) some of the slits may reduce to a point

An elementary "important" example

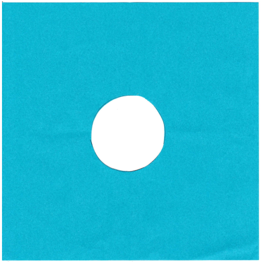
$$\Sigma := \hat{\mathbb{C}}_w \setminus \{w \in \hat{\mathbb{C}} \mid |\mathrm{Re}\, w| \leq 2, \mathrm{Im}\, w = 0\}$$

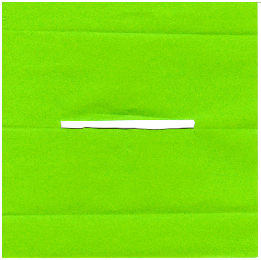
Joukowski map $J : z \to w = z + 1/z$

J is **NOT** conformal at $z = \pm 1\,(w = \pm 2)$

$$\xrightarrow{J}$$

$$\xleftarrow{f}$$





The Joukowski transformation J is a meromorphic function on

$$\hat{\mathbb{C}} = G \cup \overline{\mathbb{D}};$$

and has the same range on $G = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and its mirror image $\mathbb{D}$

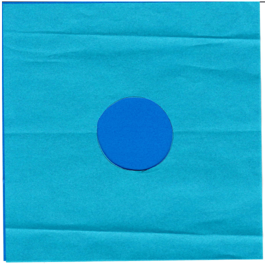
$\cdots$ and therefore one gets a "healthy" idea

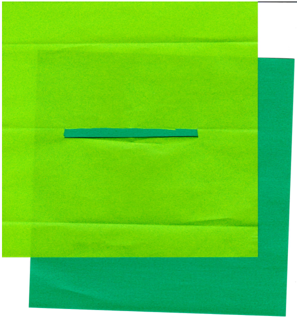
Consider

the Schottky double $\hat{G}$ of G: $\hat{G} = G \cup \overline{\mathbb{D}} = \hat{\mathbb{C}}$, and
J on $\hat{G} = \hat{\mathbb{C}}$

The image is the doule cover of $\hat{\mathbb{C}}$

$$\xrightarrow{J}$$

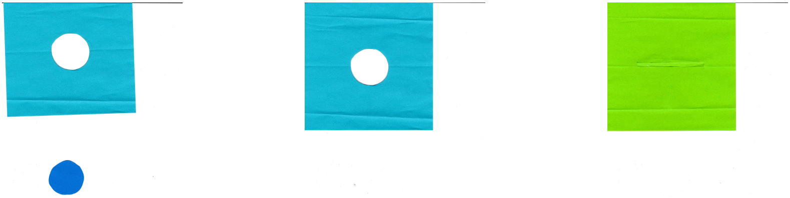




Less healthy but more “natural” observation shall be

(1) Throw away the mirror image of G.
(2) consider the given G **only**, and
(3) identify the points on ∂G if they have the same **value** under J

J extends holomorphically onto the new “closed” Riemann surface — as the identity function on $\hat{\mathbb{C}}$!



Conformal Mappings of a "Riemann Surface"

R: a fixed open Riemann surface of genus g $(0 \leq g < \infty)$

While in case of $g = 0$ a conformal mapping is always supposed to be

$f : R \to \hat{\mathbb{C}}$, injection

in case $g > 0$ we have to consider

$f : R \to \boxed{\quad ? \quad}$, conformal embedding

The most natural candidate**s are**

closed Riemann surface**s** of **the same** genus

(compact **continuations** of the same genus)

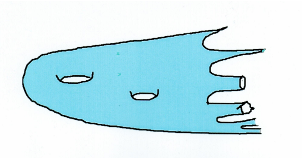
for short:

"closing"

Problem: Characterize the closings of R

Classical "Cut & Paste Method"

$$\begin{cases} g = 0: & \text{Koebe's generalized Uniformization Thm} \\ g > 0: & \text{Cut \& Paste Method} \end{cases}$$



Hydrodynamic differentials

φ is called a *t*-hydrodynamic differential (for short: an S_t-differential) if it is

- a meromorphaic differential on R
- outside a compact set
  - exact: $\varphi = d\Phi$
  - Dirichlet finite
- on each component of ∂R — in an **intuitive** sense –

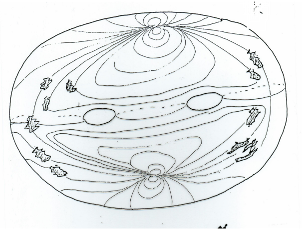
$$\mathrm{Im}\left[e^{-\frac{\pi}{2}it}\,\Phi\right] = \text{const.}$$

Historical Notes

$\varphi = d\Phi$ is an S_0-dfrtl $\iff$

- $i\varphi$ is a canonical semiexact dfrtl of Kusunoki
- $\operatorname{Im}\varphi$ is a distinguished dfrtl of Ahlfors
- $\operatorname{Im}\Phi$ is a (multi-valued) $(Q)L_1$-principal fn of Sario

"S_t-differential" $\longleftarrow$ *<u>St</u>römungsfunktion* (Klein) with parameter <u>t</u>
$\neq$ Stream function φ describes an ideal fluid flow on R



Realization Theorem or Embedding Theorem — as a generalized Riemann Mapping Theorem

$\forall \varphi$: S_t-differential on R $(-1 < t \leq 1)$

$\exists\ \tilde{R}$: compact, of genus g

$\exists\, \tilde{\iota} : R \to \tilde{R}$: conformal injection

$\exists\ \tilde{\varphi}$: a meromorphic dfrtl on $\tilde{R}$

- $\tilde{\iota}^*(\tilde{\varphi}) = \varphi$ on R (pull back)
- $\tilde{B} := \tilde{R} \setminus \tilde{\iota}(R)$ is
  - a closed null set
  - $\tilde{\varphi}$ is holomorphic on $\tilde{B}$
  - on each component of $\tilde{B}$ — in a **strict** sense ! —

$$\mathrm{Im}\left[e^{-\frac{\pi}{2}it}\,\Phi\right] = \text{const.}$$

Precision of definition — topological property

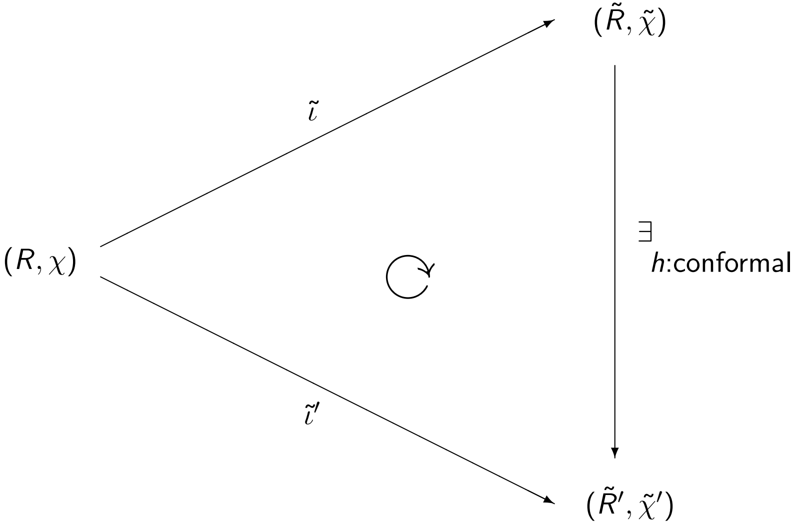
$\chi := \{A_j, B_j\}_{j=1}^{g}$: Canonical homology basis of R <u>mod ∂R</u>
$\tilde{\chi} := \{\tilde{A}_j, \tilde{B}_j\}_{j=1}^{g}$: Canonical homology basis of $\tilde{R}$

$$\text{Condition : } \tilde{\iota}(A_j) \sim \tilde{A}_j, \quad \tilde{\iota}(B_j) \sim \tilde{B}_j, \quad j = 1, 2, \ldots, g$$

Object of study :

$$\tilde{R}, \qquad \tilde{\chi}, \quad \text{and} \quad \tilde{\iota} : (R, \chi) \longrightarrow (\tilde{R}, \tilde{\chi})$$

Precision — Closing as an equiv. class



Problem setting

Definition

$$\mathcal{C}(R,\chi)\ :\ \text{the set of closings of } (R,\chi)$$

Problem:

Find/describe/characterize $\mathcal{C}(R,\chi)$

Label of an element of $C(R,\chi)$

$$\mathcal{C}(R,\chi) \ni (\tilde{R},\tilde{\chi},\tilde{\iota}) \to \tilde{T} = \tilde{T}(\tilde{R},\tilde{\chi})\ :\ \text{Riemann's period matrix}$$

Period set

$$\mathfrak{M}(R,\chi) := \left\{ T(\tilde{R},\tilde{\chi}) \right\}_{\in\mathcal{C}(R,\chi)}$$

Riemann's period matrices

For a compact Riemann surface $(\tilde{R}, \tilde{\chi})$ of genus $g > 0$

there exist g holomorphic differentials $\tilde{\varphi}_j$ with $\displaystyle\int_{A_k} \tilde{\varphi}_j = \delta_{jk}$

$(j, k = 1, 2, \ldots, g)$

$$\tilde{T} = T(\tilde{R}, \tilde{\chi}) := \begin{pmatrix} \tilde{\tau}_{11} & \tilde{\tau}_{12} & \ldots\ldots & \tilde{\tau}_{1g} \\ \tilde{\tau}_{21} & \tilde{\tau}_{22} & \ldots\ldots & \tilde{\tau}_{2g} \\ \ldots & \ldots & \ldots & \ldots \\ \tilde{\tau}_{g1} & \tilde{\tau}_{g2} & \ldots\ldots & \tilde{\tau}_{gg} \end{pmatrix}, \quad \tilde{\tau}_{jk} := \int_{\tilde{B}_k} \tilde{\varphi}_j$$

is symmetric and

its imaginary part is positive definite

$\tilde{T}$ is a point of Siegel upper half space $\mathfrak{S}_g$

Some of the results for $g = 1$

Theorem

(1) $\mathfrak{M}(R,\chi) = \{\tau \in \mathbb{C} \mid \tau$ is the modulus of $(\tilde{R},\tilde{\chi},\tilde{\iota}) \in \mathcal{C}(R,\chi)\}$

(i) is a closed disk in $\mathbb{H} := \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$

(ii) reduces to a point iff $R \in O_{AD}$

(2) $\forall\, \tau \in \partial\mathfrak{M}(R,\chi)\ \exists_1\, (\tilde{R},\tilde{\chi},\tilde{\iota}) \in \mathcal{C}(R,\chi)$ with modulus τ

(3) The area α of $\tilde{R} \setminus \tilde{\iota}(R)$

(i) vanishes on $\partial\mathfrak{M}(R,\chi)$

(ii) attains its maximum at the center of $\mathfrak{M}(R,\chi)$

$\cdots$ and so on

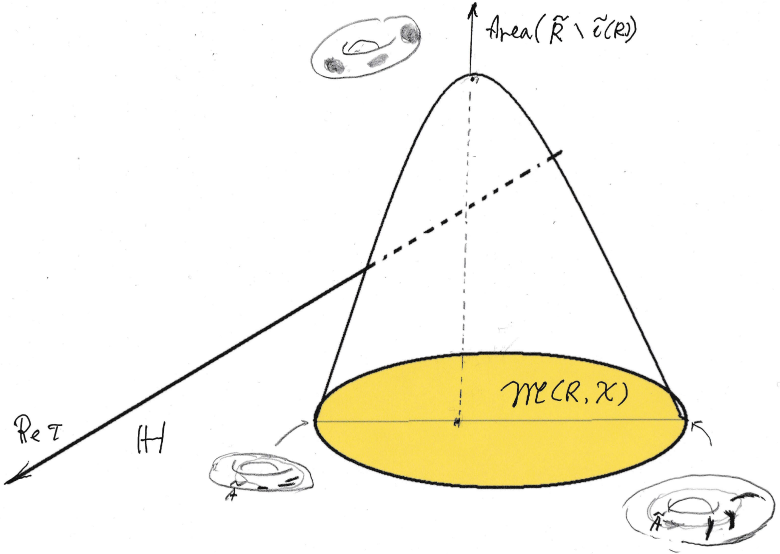
Remarks

(a) (R,χ) is realized as a "parallel slit torus" on the $(\tilde{R},\tilde{\chi},\tilde{\iota})$ in (2)

(b) More precisely: $\mathcal{C}(R,\chi)$ is described in $\mathbb{H} \times \mathbb{R}^+$ as

$$\left\{(\tau,\alpha) \;\middle|\; |\tau - \tau^*| \le \rho,\ 0 \le \alpha \le \frac{\rho^2 - |\tau - \tau^*|^2}{2\rho}\right\}$$

A sketch of $\mathcal{C}(R, \chi)$ $(g = 1)$



Euclidean Span and Hyperbolic Span

Definition & Notation

$$\begin{aligned}\text{Euclidean span} \quad \sigma_E &:= \text{the euclidean diameter of } \mathfrak{M}(R,\chi)\\ \text{Hyperbolic span} \quad \sigma_H &:= \text{the hyperbolic diameter of } \mathfrak{M}(R,\chi)\end{aligned}$$

Remark

σ_E and σ_H describe the behavior of $\tilde{R} \setminus \tilde{\iota}(R)$

Proposition

σ_H is independent of χ

$\boldsymbol{a}$-modulus of $(\tilde{R}, \tilde{\chi}, \tilde{\iota}) \in \mathcal{C}(R, \chi)$

For $\boldsymbol{a} := (a_1, a_2, \ldots, a_g) \in \mathbb{R}^g \setminus \{(0, 0, \ldots, 0)\}$ define

"the $\boldsymbol{a}$-modulus" of $(\tilde{R}, \tilde{\chi})$: $\tau_{\boldsymbol{a}} := \tau_{\boldsymbol{a}}(\tilde{R}, \tilde{\chi}, \tilde{\iota}) = \boldsymbol{a}\, \tilde{T}\, \boldsymbol{a}'$

and

"the $\boldsymbol{a}$-moduli set" of $(\tilde{R}, \tilde{\chi})$:

$$\mathfrak{M}_{\boldsymbol{a}} := \{\tau \in \mathbb{C} \mid \tau = \tau_{\boldsymbol{a}}(\tilde{R}, \tilde{\chi}, \tilde{\iota}) \text{ for } (\tilde{R}, \tilde{\chi}, \tilde{\iota}) \in \mathcal{C}(R, \chi)\}$$

respectively

$$\tilde{T} = \tilde{T}(\tilde{R}, \tilde{\chi}, \tilde{\iota})$$

$$\boldsymbol{a}' \quad : \quad \text{transpose of } \boldsymbol{a}$$

Generalization to $g > 1$

Theorem (with Yamaguchi)

$\forall\ \boldsymbol{a} := (a_1, a_2, \ldots, a_g) \in \mathbb{R}^g \setminus \{(0, 0, \ldots, 0)\}$

$\exists\ \tau^*_{\boldsymbol{a}} \in \mathbb{H}\,(= \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\})$

$\exists\ \rho_{\boldsymbol{a}} \in \mathbb{R}$ with $\quad 0 \le \rho_{\boldsymbol{a}} < \operatorname{Im}\tau^*_{\boldsymbol{a}}$

such that the closed disk

$$\mathfrak{M}^*_{\boldsymbol{a}} := \{|\tau - \tau^*_{\boldsymbol{a}}| \le \rho_{\boldsymbol{a}}\}$$

satisfies

(1) $\mathfrak{M}_{\boldsymbol{a}} \subset \mathfrak{M}^*_{\boldsymbol{a}} \subset \mathbb{H}$

(2) $\partial\mathfrak{M}^*_{\boldsymbol{a}} \subset \mathfrak{M}_{\boldsymbol{a}}$

(2') Each point of $\partial\mathfrak{M}^*_{\boldsymbol{a}}$ is realized by a **hydrodynamic closing** — a closing constructed by an S_t-differential
— Extremality of S_t-differentials

$\cdots$ and, in particular,

$\mathfrak{M}(R, \chi)$ is bounded

(I) Each element of the diagonal of $\tilde{T} \in \mathfrak{M}(R, \chi)$ is inside a closed disk
(II) $\exists\ T^* \in \mathfrak{S}_g$

(1) $\forall\ \boldsymbol{a} := (a_1, a_2, \ldots, a_g) \in \mathbb{R}^g \setminus \{(0, 0, \ldots, 0)\}$
$\exists\ \rho_{\boldsymbol{a}} > 0$ such that

$$\left| \boldsymbol{a}\,(\tilde{T} - T^*)\,\boldsymbol{a}' \right| \le \rho_{\boldsymbol{a}} \qquad (\forall\ \tilde{T} \in \mathfrak{M}(R, \chi))$$

(2) There exists a hydrodynamic closing
whose period matrix $\tilde{T}$ yields the equality sign

Hyperbolic Span and Pseudoconvexity

Theorem (with Yamaguchi and Hamano)

Suppose that

$$\mathcal{R} := \{(R_\zeta, \chi_\zeta)\}_{\zeta \in \mathbb{D}}$$

is a pseudo convex family of finite Riemann surfaces of genus one

Then, the hyperbolic span $\sigma_H(R_\zeta)$ is

(1) a subharmonic function of $\zeta \in \mathbb{D}$

(2) harmonic iff

$$\mathcal{R} \text{ is a product: } \mathcal{R} = R_0 \times \mathbb{D}$$

Remark.

A more general result will be reported at the annual meeting of Math. Soc. Japan, March 24, 2015

Numerical quasiconformal mappings by certain linear systems

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リーマン面・不連続群 合同研究集会

2015.2.15

大阪大学

This talk is based on a joint work
with Prof. R. Michael Porter (CINVESTAV).

Introduction

Motivation

Applications of quasiconformal mappings

- Lipman, Y., Kim, V. G., and Funkhouser, T. A. (2012). Simple formulas for quasiconformal plane deformations. ACM Transactions on Graphics (TOG), 31(5), 124.
- Astala, K., Mueller, J. L., Päivärinta, L., Perämäki, A., and Siltanen, S. (2011). Direct electrical impedance tomography for nonsmooth conductivities. Inverse Probl. Imaging, 5(3), 531-549.
- Gaidashev, D., and Yampolsky, M. (2007). Cylinder renormalization of Siegel disks. Experimental Mathematics, 16(2), 215-226.
- etc.

With this increasing use of computer applications it has become of great interest to know how construct the quasiconformal mappings numerically.

Quasiconformal mapping

Definition

Let $K > 1$ and D, D' be the domains in the complex plane $\mathbb{C}$. An orientation-preserving homeomorphism $f : D \to D'$ is a K-quasiconformal mapping if f satisfies the following:

1. For any closed rectangle $R := \{z = x + iy \,|\, a \leq x \leq b, c \leq y \leq d\}$ in D, f is absolutely continuous on almost every horizontal and vertical line in R.
2. The dilatation condition

$$|f_{\overline{z}}(z)| \leq \frac{K-1}{K+1}|f_z(z)| \qquad (1)$$

holds almost everywhere in D, where $f_z = (f_x - if_y)/2, f_{\overline{z}} = (f_x + if_y)/2$ and $z = x + iy$.

It follows from the definition that the quasiconformal mapping $f: D \to D'$ has partial derivatives $f_z, f_{\overline{z}}$ almost everywhere in D. Further f is differentiable a.e. in D, i.e. the real-linear approximation

$$f(z) - f(z_0) = f_z(z_0)(z - z_0) + f_{\overline{z}}(z_0)(\overline{z - z_0}) + o(|z - z_0|)$$

holds a.e. in $z_0 \in D$.

Definition

The Beltrami coefficients can be defined as

$$\mu_f(z) := \frac{f_{\overline{z}}(z)}{f_z(z)} \tag{2}$$

a.e. in D for a qausiconformal mapping f.

If $\mu_f(z_0) = 0$ at $z_0 \in D$, f is conformal at z_0.

$L^\infty(D)_1 := \{\mu : D \to \mathbb{C} \mid \mu$ is measurable on D with $\|\mu\|_\infty < 1\}$.

Theorem (Measurable Riemann mapping theorem)

For given $\mu \in L^\infty(\mathbb{C})_1$, there exists a quasiconformal mappings $f : \mathbb{C} \to \mathbb{C}$ whose Beltrami coefficient coincides with μ almost everywhere in $\mathbb{C}$. This mapping is uniquely determined up to a conformal mapping of $\mathbb{C}$ onto itself.

Corollary

Let D, D' be bounded simply connected domains in $\mathbb{C}$ and $\mu \in L^\infty(D)_1$. Then there exists a quasiconformal mapping $f : D \to D'$ whose Beltrami coefficient coincides with μ almost everywhere in D. This mapping is uniquely determined up to a conformal mapping of D' onto it self.

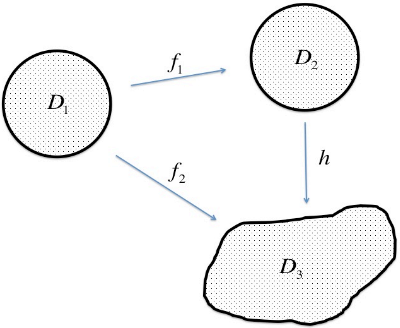
Definition

We say a quasiconformal mapping of D is μ-comformal if its Beltrami coefficients coincide with μ almost everywhere in D.

A quasiconformal mapping $f : D \to D'$ is a homeomorphism which satisfies the Beltrami equation $f_{\bar{z}} = \mu f_z$ almost everywhere in D.

Proposition (Composition with conformal mapping)

Let μ be a measurable function on a domain D_1 with $\|\mu\|_\infty < 1$. Assume that $f_1 : D_1 \to D_2$ is a μ-conformal mapping and $h : D_2 \to D_3$ a conformal mapping. Then $f_2 = h \circ f_1$ is μ-conformal.



<u>Remark</u> If we have self μ-conformal mappings of the unit disk, then we can obtain μ-conformal mapping from the unit disk to arbitrary simply connected domains by the classical Riemann mapping. Further there are many efficient methods for the numerical conformal mappings.

WANT

For given $\mu \in L^\infty(\mathbb{D})_1$, let f^μ is self μ-conformal mapping $f^\mu : \mathbb{D} \to \mathbb{D}$ of the unit disk which fixes 0 and 1 (self μ-conformal mapping of $\mathbb{D}$ can be extended to a self homeomorphism of $\overline{\mathbb{D}}$). We want to obtain either

(A) calculation method of a point w_k which reduce $|w_k - f^\mu(z_k)|$ for a given point $z_k \in \mathbb{D}$, or

(B) construction method of a function $g : D \to \mathbb{C}$ which reduce $\sup_{z \in D} |g(z) - f^\mu(z)|$ where $D \subset \mathbb{D}$.

Known methods

1. Methods based on finite difference based method and finite elements method (1978-)
2. Method based on Circle packing Riemann mapping theorem which suggested by Thurston, proved by Rodin and Sullivan (1990-)
3. Methods based on the proof of the existence theorem for quasiconformal mapping by Ahlfors-Bers (1993-)
4. Methods based on the Beltrami holomorphic flow (2005-)

- Computational cost
- Discretization error
- Guarantee for convergence and quasiconformality
- Limitation of Beltrami coefficient

Aim of this study

To obtain a practical method of numerical quasiconformal mapping which has reasonable guarantee for convergence.

Formulation of problem

Definition (Triangulation of the unit disk)

We say a Euclidian simplicial complex T, which consist of finite closed 2-simplices $\{\tau_i\}$ in $\mathbb{C}$, form a triangulation of $\mathbb{D}$ if:

1. $P := |T|$ is a closed simple jordan polygon whose vertices lies on the boundary of the unit disk $\partial\mathbb{D}$ where $|T|$ is the union of all 2-simplices in T,
2. each 1-face l_k of any 2-simplex τ_i of T is either:
   - an edge of P, or
   - there exists unique $j (j \neq i)$ such that l_k is an edge of a 2-simplex τ_j in T.

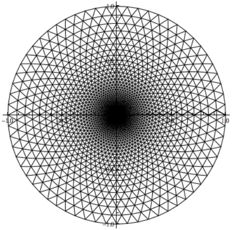
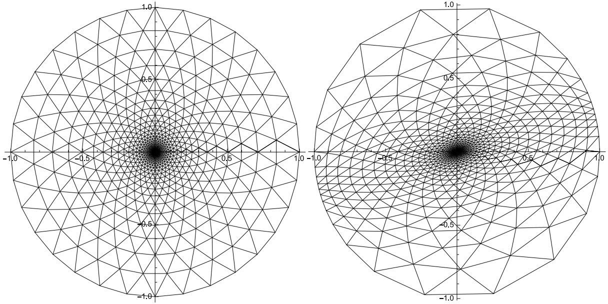


Figure. An triangulation of $\mathbb{D}$ which consists of 4096 2-simplices.



- Let T_z, T_w be triangulations of $\mathbb{D}$. If T_z and T_w are simplicially equivalent, then the piecewise linear mapping $f : |T_z| \to |T_w|$ which sent 2-simplex in T_z to the corresponding 2-simplex in T_w linearly, is a homeomorphism between $|T_z|$ and $|T_w|$. We say f is induced piecewise mapping by T_z and T_w.

Definition (PL space of T_z)

For given triangulation of the unit disk T_z, we say $f : |T_z| \to \mathbb{C}$ is in $PL(T_z)$ if f is continuous on $|T_z|$, and is linear on each 2-simplex in $|T_z|$.

- The Beltrami coefficients μ_f of $f : |T_z| \to |T_w|$ is defined on each interior of 2-simplex.

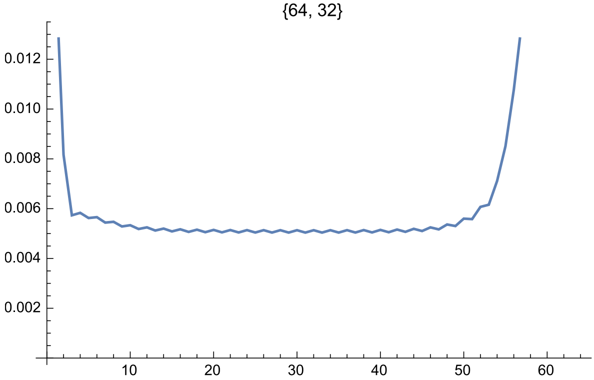


Figure. Plot of $|\mu_f|_{\tau_j} - 0.3i|$ where $\tau_j \in T_z$.

Lemma (Good approximation lemma)

Let $\{\mu_n \in L^\infty(\mathbb{D})_1\}_{n\in\mathbb{N}}$ *be measurable functions which satisfies*

$$\|\mu_n\|_\infty \leq k < 1$$

for all $n \in \mathbb{N}$*, and such that the pointwise limit*

$$L^\infty(\mathbb{D})_1 \ni \mu(z) := \lim_{n\to\infty} \mu_n(z)$$

exists almost everywhere. Let $f_n : \mathbb{D} \to \mathbb{D}$ *be the* μ*-conformal mappings with* $f_n(0) = f_n(1) - 1 = 0$*. Then* $f_n(z)$ *converges to the* μ*-conformal mapping* $f(z)$ *with* $f(0) = f(1) - 1 = 0$ *uniformly on compact subsets of* $\mathbb{D}$*.*

Formulation of our problem

We want to obtain an algorithm as the following.

Input:

- $\mu \in L^\infty(\mathbb{D})_1$.
- A triangulations of the unit disk T_z whose vertices include 0 and 1.

Output:

- A triangulations of the unit disk $T_w \cong T_z$ whose vertices include 0 and 1 in suitable position, so that the Beltrami coefficient μ_g of the induced piecewise linear mapping $g : |T_z| \to |T_w| \in PL(T_z)$, reduce $\|\mu - \mu_g\|_\infty$ on each $\tau \in T_z$.

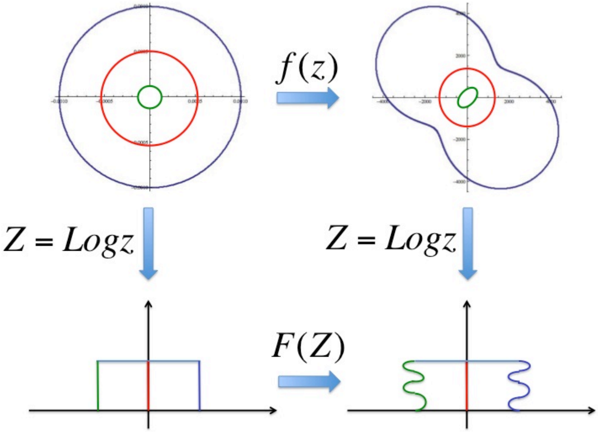
Algorithm

Logarithmic cordinates

- Let μ be a measurable function with $\|\mu\|_\infty < 1$.
- Set $\mu(z) := \frac{z^2}{\bar{z}^2}\overline{\mu(\frac{1}{\bar{z}})}$ for $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$.
- Then, there exists self μ-conformal mapping $f : \mathbb{C} \to \mathbb{C}$ of the complex plane which fix 0 and 1.
- Actually $f|_{\mathbb{D}}$ is desired quasiconformal mapping.

Corollary

Let $\mu \in L^\infty(\mathbb{C})_1$ and $f : \mathbb{C} \to \mathbb{C}$ be the μ-conformal mapping with $f(0) = f(1) - 1 = 0$. If $\overline{\mu(z)} = \mu(1/\bar{z})\bar{z}^2/z^2$, then the restriction $f|_{\mathbb{D}} : \mathbb{D} \to \mathbb{D}$ is a self $\mu|_{\mathbb{D}}$-conformal mapping of the unit disk with $f|_{\mathbb{D}}(0) = f|_{\mathbb{D}}(1) - 1 = 0$.



- Take the logarithmic coordinates $Z = \log z$ and set $F(Z) := \log f(e^Z)$.
- Then F have the symmetry with respect to the imaginary axis.
- First we approximate $F(Z)$ on a finite rectangle.

Triangulation

Take $M, N \in \mathbb{N}$. We define $(M+1)N$ vertices

$$Z_{j,k} = \frac{\sqrt{3}\,\pi\, j}{N} + \frac{2\pi(k + (j \bmod 2)/2)}{N} i \tag{3}$$

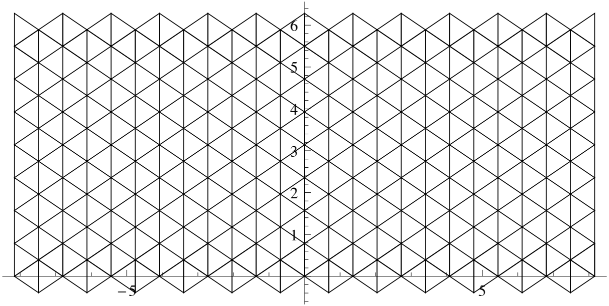
for $-M \leq j \leq 0$ and $0 \leq k \leq N-1$. The $M \times N$ *rightward pointing* 2*-simplexes* are defined by

$$\tau^{+}_{j,k} = \begin{cases} \text{Conv}(Z_{j-1,k-1},\ Z_{j-1,k},\ Z_{j,k}), & j \text{ even}, \\ \text{Conv}(Z_{j-1,k},\ Z_{j-1,k+1},\ Z_{j,k}), & j \text{ odd}, \end{cases} \tag{4}$$

for $-M+1 \leq j \leq 0$ where Conv(Z_1, Z_2, Z_3) is the 2-simplex which vertices are Z_1, Z_2, Z_3. We also define $M \times N$ *leftward pointing* 2*-simplexes*

$$\tau^{-}_{j,k} = \begin{cases} \text{Conv}(Z_{j+1,k-1},\ Z_{j+1,k},\ Z_{j,k}), & j \text{ even}, \\ \text{Conv}(Z_{j+1,k},\ Z_{j+1,k+1},\ Z_{j,k}), & j \text{ odd}, \end{cases} \tag{5}$$

for $-M \leq j \leq -1$.



We extend this mesh symmetrically to the right half-plane as

$$Z_{j,k} = \varrho(Z_{-j,k})$$

where ϱ is the reflection of the imaginary axis

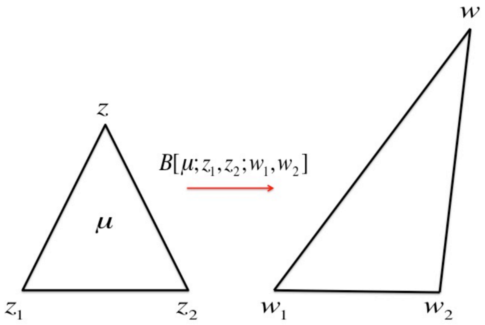
$$\varrho(Z) = -\overline{Z}. \tag{6}$$

Now we have $(2M+1)N$ vertices and $4MN$ 2-simplexes. In the case the simplexes $\tau_{jk}^{\pm}$ are equilateral. We say this the basic mesh in the logarithmic coordinates.

Observation

Proposition

Let $z_1, z_2, w_1, w_2 \in \mathbb{C}$ with $z_1 \neq z_2$ and $w_1 \neq w_2$. For given complex constant $\mu \in \mathbb{D}$, there is a unique μ-conformal affine linear mapping $B(z) = B[\mu; z_1, z_2; w_1, w_2](z)$ which sends z_i to w_i $(i = 1, 2)$.



Let μ, a, b be complex constants with $a \neq 0$, $|\mu| < 1$. We consider a μ-conformal real-linear mapping

$$L_\mu(z) \quad := \quad \frac{z + \mu\overline{z}}{1+\mu}. \tag{7}$$

Proposition

$B(z)$ is given by

$$\begin{aligned} B(z) &= w_1 + \frac{w_2 - w_1}{L_\mu(z_2 - z_1)} L_\mu(z - z_1) \\ &= \frac{L_\mu(z_2 - z)}{L_\mu(z_2 - z_1)} w_1 + \frac{L_\mu(z_1 - z)}{L_\mu(z_1 - z_2)} w_2. \end{aligned}$$

Remark We note that the coefficients of w_1, w_2 in the last expression are never 0, 1, or ∞ if z_1, z_2, z_3 are distinct.

Corollary

Let $z_i \in \mathbb{C}$ $(i = 1, 2, 3)$ *noncollinear,* $w_i \in \mathbb{C}$ $(i = 1, 2, 3)$ *noncollinear and* $\mu \in \mathbb{D}$. *If an* μ*-conformal affine linear map* B *takes* z_1, z_2, z_3 *to* w_1, w_2, w_3 *respectively, then the following holds:*

$$w_3 = B(z_3) = \frac{L_\mu(z_2 - z_3)}{L_\mu(z_2 - z_1)} w_1 + \frac{L_\mu(z_1 - z_3)}{L_\mu(z_1 - z_2)} w_2$$

$$\iff \quad L_\mu(z_2 - z_3)\, w_1 + L_\mu(z_3 - z_1)\, w_2 + L_\mu(z_1 - z_2)\, w_3 = 0. \quad (8)$$

Corollary

Let $z_i \in \mathbb{C}$ $(i = 1, 2, 3)$ *noncollinear and* $w_i \in \mathbb{C}$ $(i = 1, 2, 3)$ *noncollinear. There is a unique affine linear mapping which sends* z_i *to* w_i $(i = 1, 2, 3)$. *Further its Beltrami coefficient is equal to*

$$\mu = -\frac{(z_2 - z_1)(w_3 - w_1) - (z_3 - z_1)(w_2 - w_1)}{(\overline{z_2} - \overline{z_1})(w_3 - w_1) - (\overline{z_3} - \overline{z_1})(w_2 - w_1)}. \quad (9)$$

Beltrami coefficient of F

The Beltrami coefficient of $F(Z)$ are given as follows:

$$\nu(Z) = \mu(e^Z)\frac{e^{\overline{Z}}}{e^Z} = \mu(e^Z)e^{-2i\,\mathrm{Im}\,Z}, \mathrm{Re}\ Z < 0. \tag{10}$$

Using ν, we set the Beltrami coefficients as $\nu(Z) = \overline{\nu(\varrho(Z))}$ for Re $Z > 0$.

We will write $\nu^{\pm}_{j,k}$ for the average value of $\nu(Z)$ on the 2-simplexes $\tau^{\pm}_{j,k}$ (or, the average of $\nu(Z)$ over the three vertices as an approximation of this average, at least when ν is continuous). Let us note that

$$\nu_{jk} = \overline{\nu_{-j,k}}, \quad j > 0. \tag{11}$$

Triangle equations

For all rightward pointing 2-simplicies $\tau_{jk}^{+} \in T_{M,N} := \{\tau_{j,k}^{\pm}\}$, we construct following MN linear equations by Corollary 3:

$$a_{jk}^{+} W_{jk} + b_{jk}^{+} W_{j-1,k} + c_{jk}^{+} W_{j-1,k+1} = 0 \tag{12}$$

where

$$\begin{aligned}
a_{jk}^{+} &= \begin{cases} L_{\nu_{jk}}(Z_{j-1,k-1} - Z_{j-1,k}), & j \text{ even}, \\ L_{\nu_{jk}}(Z_{j-1,k} - Z_{j-1,k+1}), & j \text{ odd}, \end{cases} \\
b_{jk}^{+} &= \begin{cases} L_{\nu_{jk}}(Z_{j-1,k} - Z_{j,k}), & j \text{ even}, \\ L_{\nu_{jk}}(Z_{j-1,k+1} - Z_{jk}), & j \text{ odd}, \end{cases} \\
c_{jk}^{+} &= \begin{cases} L_{\nu_{jk}}(Z_{j,k} - Z_{j-1,k-1}), & j \text{ even}, \\ L_{\nu_{jk}}(Z_{jk} - Z_{j-1,k}), & j \text{ odd}. \end{cases}
\end{aligned} \tag{13}$$

Further MN linear equations for the leftward pointing 2-simplexes τ_{jk}^{-} are constructed Corollary 3,

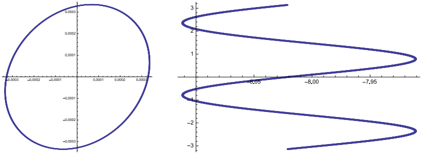
$$a_{jk}^{-}W_{jk} + b_{jk}^{-}W_{j+1,k-1} + c_{jk}^{-}W_{j+1,k} = 0 \tag{14}$$

where

$$\begin{aligned} a_{jk}^{-} &= \begin{cases} L_{\nu_{jk}}(Z_{j+1,k-1} - Z_{j+1,k}), & j \text{ even}, \\ L_{\nu_{jk}}(Z_{j+1,k} - Z_{j+1,k+1}), & j \text{ odd}, \end{cases} \\ b_{jk}^{-} &= \begin{cases} L_{\nu_{jk}}(Z_{j+1,k} - Z_{j,k}), & j \text{ even}, \\ L_{\nu_{jk}}(Z_{j+1,k+1} - Z_{j,k}), & j \text{ odd}, \end{cases} \\ c_{jk}^{-} &= \begin{cases} L_{\nu_{jk}}(Z_{j,k} - Z_{j+1,k-1}). & j \text{ even}, \\ L_{\nu_{jk}}(Z_{j,k} - Z_{j+1,k}), & j \text{ odd}. \end{cases} \end{aligned} \tag{15}$$

<u>Remark</u> We have totally $4MN$ triangle equations.

Boundary equations



Originally, the image of the inner boundary circle by a quasiconformal mapping is approximately an ellipse. The shape of this ellipse is depend on the Beltrami coefficients at the origin. Under this situation, we will add the following equations.

Let e_k be the images of these points under the real-linear mapping L_{μ_0}, i.e.

$$e_k = L_{\mu_0}(\log Z_{-M,k}) = r_{-M} L_{\mu_0}(e^{2\pi i k/N}), \quad 0 \le k \le N-1,$$

where μ_0 denotes the average value of $\mu(z)$ inside of the inner boundary circle. We want a condition that the images $\{W_k\}$ lie on unknown complex nonzero constant multiple of the ellipse which include $\{e_k\}$. Hence the boundary equations which achieve above condition are the following $2(N-1)$ equations

$$\begin{aligned} W_{-M,k} - W_{-M,k-1} &= D_k, \\ W_{M,k} - W_{M,k-1} &= \overline{D_k}, \end{aligned} \qquad (16)$$

where $D_k = \log e_k - \log e_{k-1}$ and $1 \le k \le N-1$. The magnitude of r_{-M} does not influence the value of D_k.

Nomalization

Finally, for normalization of the solution we add one more equation,

$$W_{0,0} = 0. \tag{17}$$

This says that $F(0) = 0$, or equivalently, $f(1) = 1$.

Associated linear system

In argument above, we construct $n_{\mathrm{e}} = 4MN + 2(N-1) + 1$ complex linear equations for the $n_{\mathrm{v}} = (2M+1)N$ unknown variables W_{jk}, $-M \leq j \leq M$, $0 \leq k \leq N-1$. Let $p = p(j,k)$ be an fixed bijection from the set of index pairs $\{(j,k)\}$ to $\{p \in \mathbb{N} : 1 \leq p \leq n_{\mathrm{v}}\}$. Using this bijection p, we will rename the variables in a single vector $\boldsymbol{W}$ with

$$\boldsymbol{W} := \{W_p\} = \{W_{j,k}\} \tag{18}$$

for the convenience. The linear system now takes the form:

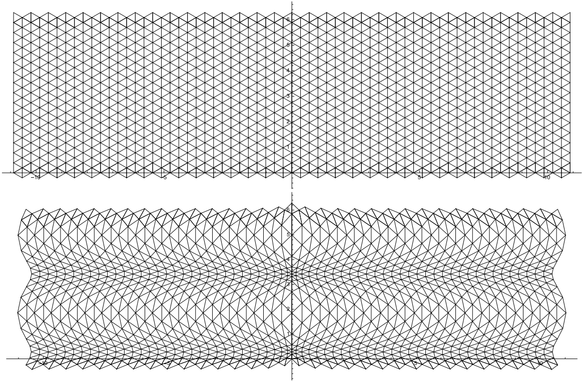
$$\boldsymbol{A}\boldsymbol{W} = \boldsymbol{B} \tag{19}$$

where $\boldsymbol{A} = (A_{j,k})$ is the $n_{\mathrm{e}} \times n_{\mathrm{v}}$-type complex matrix and $\boldsymbol{B} = (B_k)$ is a complex vector of length n_{e}. When we take a pair of N, M, the mesh $\{Z_{jk}\}$ is fixed, and linear system above is defined. We will say that this linear system $(\boldsymbol{A}, \boldsymbol{B})$ is the *associated linear system* to the collection of ν-values $\{\nu_{jk}\}$.

Since our linear system is over determined, we chose the standard least squares method for the approximation.

Definition (Least squares solution)

Let $m, n \in \mathbb{N}$ with $m > n$. Let $\boldsymbol{AW} = \boldsymbol{B}$ an overdetermined linear system where $\boldsymbol{A} \in \boldsymbol{M}_{m,n}(\mathbb{C})$, $\boldsymbol{B} \in \mathbb{C}^m$ and unknown vector $\boldsymbol{W} \in \mathbb{C}^n$. We call $\boldsymbol{W}$ is the least squares solution of $(\boldsymbol{A}, \boldsymbol{B})$ if $\boldsymbol{W}$ minimize the residual vector $\|\boldsymbol{AW} - \boldsymbol{B}\|_2$.



Lemma

For given $\mu \in L^\infty(\mathbb{D})_1$ and $M, N \in \mathbb{N}$, the least squares solution $\boldsymbol{W} = \{W_{j,k}\}$ $(-M \leq j \leq M,\ 0 \leq k \leq N-1)$ of the associated linear system $(\boldsymbol{A}, \boldsymbol{B})$ exists uniquely. Furthermore $\boldsymbol{W}$ satisfies the following symmetric relation: $\boldsymbol{A}\overleftrightarrow{\boldsymbol{W}} = \boldsymbol{A}\varrho(\boldsymbol{W})$ where $\overleftrightarrow{W}_{j,k} = W_{-j,k}$, $\varrho(\boldsymbol{W}) = \{\varrho(W_{j,k})\}$ and ρ is defined by (6), i.e. the entries of $\boldsymbol{W}$ satisfy the symmetry $W_{-j,k} = \rho(W_{j,k})$. In particular, the values $\{W_{0,k}\}$ are purely imaginary.

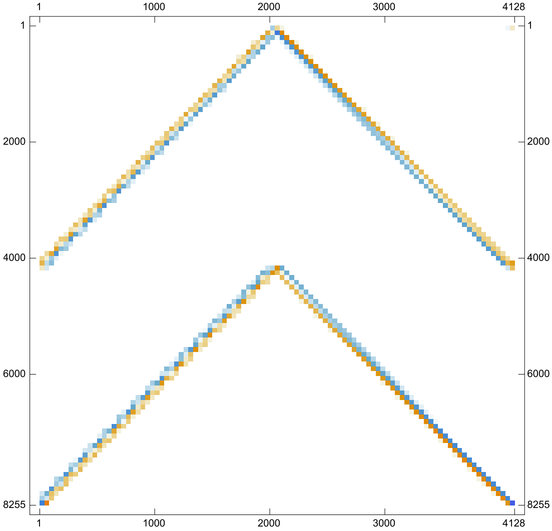


Figure. An exmaple of $\boldsymbol{A}$ $(M = 64, N = 32, \mu(z) = 0.3)$.

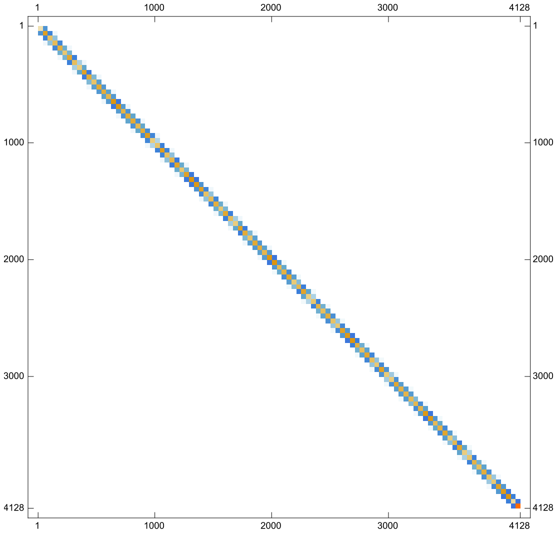
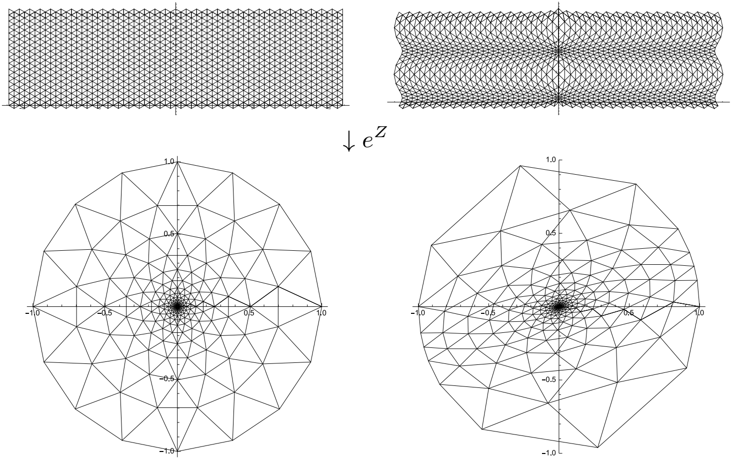


Figure. ${}^tAA$ $(M = 64, N = 32, \mu(z) = 0.3)$.



Finally, we apply the exponential mapping to the vertices of $\{Z_{j,k}\}$ and $\{W_{j,k}\}$. Then we take the piecewise linear mapping which is induced by the correspondence of the simplices.

The algorithm is summarized as follows.

Algorithm

Input: The measurable function $\mu : \mathbb{D} \to \mathbb{C}$ with $\|\mu\|_\infty < 1$ and the dimensions M, N for a simplicial complex $\{Z_{j,k}\}$ in the Z-plane.

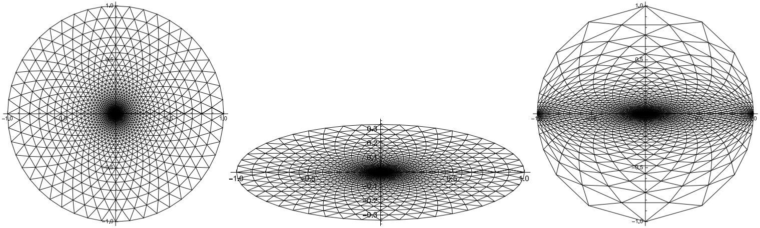
1. Calculate the averages of the Beltrami coefficients $\nu_{j,k}$ on each triangle in the logarithmic coodrinates via (10) (or we use averages on the 3 vertices if μ is continuous on $\overline{\mathbb{D}}$).
2. Calculate the coefficients of the associated linear system $(\boldsymbol{A}, \boldsymbol{B})$ of $\{\nu_{jk}\}$ and $T_{M,N}$ as prescribed by equations (12), (14), (16), and (17).
3. Calculate the least squares solution $\boldsymbol{W}$ to the associated linear system $(\boldsymbol{A}, \boldsymbol{B})$, and arrange the entries of $\boldsymbol{W}$ to form the mesh $\{W_{jk}\}$.
4. 4. Calculate $w_{jk} = \exp W_{jk}$ for $-M \le j \le 0$ and $0 \le k \le N-1$.

Output: The piecewise linear mapping such that $z_{jk} \mapsto w_{jk}$ where $z_{jk} = \exp Z_{jk}$.

Numerical experiments

Constant Beltrami coefficients

The image of the circle $|z| = 1$ under the mapping L_μ is an ellipse with semiaxes 1, $(1 - |\mu|)/(1 + |\mu|)$ slanted in the directions $(1/2)\arg\mu$, $(1/2)(\arg\mu + \pi)$ respectively, modulo π. This ellipse is sent by the conformal linear mapping $H_{1/(2\sqrt{\mu}),0}$ to the ellipse with semiaxes a, b. Then the ellipse is transformed conformally to the unit disk, by an explicit formula for the conformal mapping to $\mathbb{D}$ from this ellipse.



The algorithm was applied for the constant Beltrami derivatives $\mu = 0.1,\ 0.3,\ 0.5,\ 0.7$, and meshes defined by $N = 16$, 32, 48, 64, 72, 84, with M equal to the least multiple of 4 no less than $N \log N/(\pi\sqrt{3})$. In the last case there are 24359 equations in 14196 variables. It took about 1.5 seconds to calculate the part of the matrix in the left half-plane, and about 10 seconds to solve the full set of equations.

(M, N)	(12,16)	(24,32)	(36,48)	(52,64)	(60,72)	(72,84)
$\mu =$0.1	0.012	0.0031	0.0014	0.0008	0.0006	0.0004
$\mu =$0.3	0.0274	0.007	0.0031	0.0018	0.0014	0.001
$\mu =$0.5	0.0615	0.0205	0.0109	0.0065	0.0051	0.0038
$\mu =$0.7	0.2439	0.1201	0.0856	0.0627	0.053	0.0412

Table: The maximum of the absolute errors between the solutions and the real values of some constant Beltrami derivative and $M \approx N \log N/(\pi\sqrt{3})$.

Radial quasiconformal mappings

Let $\varphi\colon [0,1] \to [0,1]$ be an increasing diffeomorphism of the unit interval. Then the radially symmetric function

$$f(z) = \varphi(|z|)e^{i \arg z} = \varphi(|z|)\frac{z}{|z|} \tag{20}$$

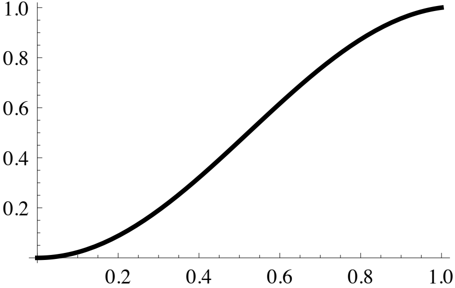
has Beltrami derivative equal to

$$\mu(z) = \frac{|z|\varphi'(z)/\varphi(z) - 1}{|z|\varphi'(z)/\varphi(z) + 1}\frac{z}{\overline{z}} \tag{21}$$

when $z \neq 0$. As an illustration we will take

$$\varphi(r) = (1 - \cos 3r)/(1 - \cos 3).$$

The resulting Beltrami derivative satisfies $\|\mu\|_\infty = 0.65$ approximately.



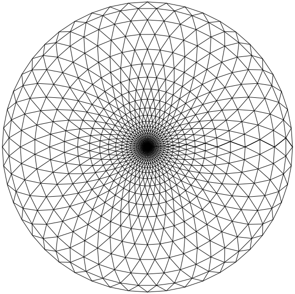


Figure. φ and T_w

(M,N)	$(12,16)$	$(24,32)$	$(36,48)$	$(52,64)$	$(60,72)$	$(72,84)$
Error	0.0398	0.0135	0.0058	0.0034	0.0027	0.0020

Table. $|\phi(|z|) - f(z)|$

Sectrial quasiconformal mappings

In a similar spirit, we let $\psi\colon [0, 2\pi] \to [0, 2\pi]$ be an increasing diffeomorphism. Write $\widetilde{\psi}(e^{i\theta}) = e^{i\psi(\theta)}$. Then the sectorially symmetric function

$$f(z) = |z|\,\widetilde{\psi}\left(\frac{z}{|z|}\right) \tag{22}$$

has Beltrami derivative equal to

$$\mu(z) = \frac{1-\psi'(\theta)}{1+\psi'(\theta)}\,\frac{z}{\overline{z}} \tag{23}$$

when $z \neq 0$. As an example we will take

$$\psi(\theta) = \begin{cases} \frac{\theta}{2}, & 0 \le \theta \le \pi, \\ \frac{\pi}{2} + \frac{3(\theta-\pi)}{2}, & \pi \le \theta \le 2\pi. \end{cases}$$

The arguments of the final boundary values on the unit circle were compared with the true values $\psi(\theta)$.

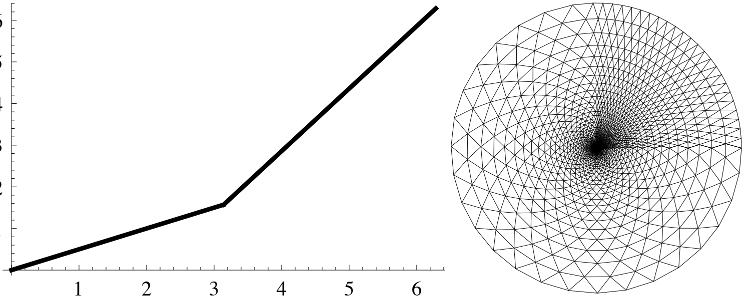


Figure. $\psi(\theta)$ and T_w

(M,N)	$(12, 16)$	$(24, 32)$	$(36, 48)$	$(52, 64)$	$(60, 72)$	$(72, 84)$
Error	0.0712	0.0362	0.0251	0.0193	0.0173	0.0150

Table. $|\psi(\theta) - f(e^{i\theta})|$

Trivial Beltrami coefficients

Let $\mu \in L^{\infty}(\mathbb{D})$ with $\|\mu\|_{\infty} < 1$. If the corresponding normalized solution f^{μ} satisfies $f^{\mu}(z) = z$ on the unit circle, μ called a trivial Beltrami coefficient. Trivial Beltrami coefficients play an important role in the theory of Teichmüller space. Prof. Sugawa showed a criterion for the triviality of the Beltrami coefficients, and gave an example for a trivial Beltrami coefficient. Let N be a non-negative integer and $a_j(t)$ $(1 \leq j \leq N)$ be essentially bounded measurable functions in $t \geq 0$ so that

$$\mu(z) := \sum_{j=0}^{N} a_j\left(-\log|z|\right)\left(\frac{z}{|z|}\right)^{j+2}$$

satisfies $\|\mu\|_{\infty} < 1$. Then his results implies that μ is a trivial Beltrami coefficient. For the experiment, we chose

$$a_j(z) := \frac{2}{3}\left(\frac{\sin 10z}{2}\right)^{j+1}.$$

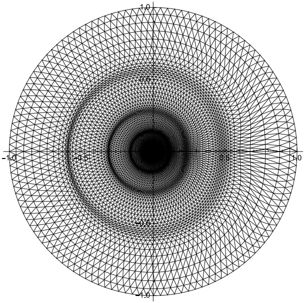
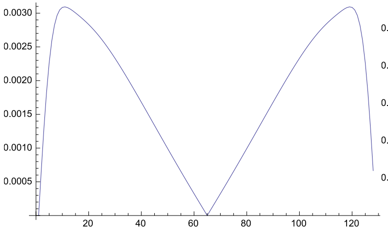


Figure. The result made by our algorithm with trivial coefficient μ.



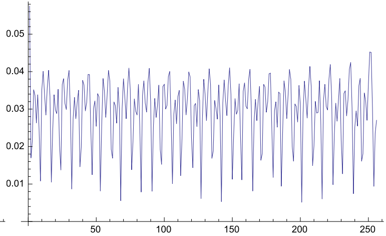


Figure. The errors of the boundary values (left), the difference between the induced Beltrami coefficients to μ (right).

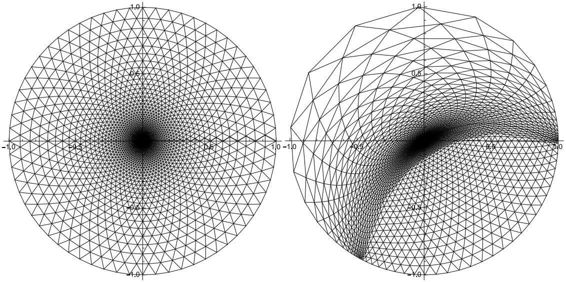


Figure. $\mu(z) = (0.5 \text{ if } \operatorname{Im} z > -0.2, 0 \text{ else})$.

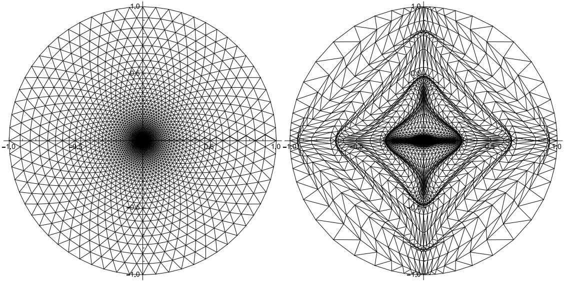


Figure. $\mu(z) = 0.9 \sin(20|z|)$.

Convergence

Theorem (Porter, S ,2014)

Let $s \in \mathbb{N}$ and $M_s, N_s \in \mathbb{N}$ be strictly increasing sequences which satisfy

$$c_1 N_s \log N_s \leq M_s \leq c_2 N_s \log N_s \tag{24}$$

for constants c_1, c_2 where $c_1 > 1/(\pi\sqrt{3})$. If $\mu \in L_\infty(\mathbb{D})_1 \cup C^1(\mathbb{D})$, then the following holds.

i. *If s is large enough, the points $\{z_{j,k}^{(s)}\}$ and the points $\{w_{j,k}^{(s)}\}$ produced by the algorithm form the vertex sets of triangulations $T_z^{(s)}$ and $T_w^{(s)}$ of the unit disk $\mathbb{D}$. Furthermore, for any fixed compact set $K \subset \text{int}\,\mathbb{D}$, $K \subset |T_z^{(s)}|$ and $K \subset |T_w^{(s)}|$ hold when s is large enough.*
ii. *The mappings $f^{(s)}$ converge to the μ-conformal mapping f normalized by $f(0) = f(1) - 1 = 0$ uniformly on compact subsets of $\mathbb{D}$ as $s \to \infty$.*

Input:

- $\mu \in C^1(\mathbb{D}) \cap L^\infty(\mathbb{D})_1$.
- $M_s, N_s \to \infty$ as $s \to \infty$ with $c_1 N_s \log N_s \leq M_s \leq c_2 N_s \log N_s$.

Output:

- $\{g^{(s)} \in PL(T_z^{(s)})\}$ s.t. $g^{(s)} \to f^\mu$ as $s \to \infty$.

Remark We conjecture that the condition $\mu \in C^1$ is overly restrictive by the numerical experiments.

The following Lemma is important for proof of main theorem.

Lemma

Let $T_z := \{\tau_j\}$ be a triangulation of the unit disk $\mathbb{D}$ with $P_z := |T_z|$ is a simple jordan polygon of k sides. Suppose $f\colon |T_z| \to \mathbb{D} \in PL(T_z)$ preserve the orientation on each $\tau \in T_z$ and maps $\partial|T_z|$ homeomorphically to a boundary of a simple Jordan polygon P_w with k sides. Then f is a orientation preserving homeomorphism from P_z to P_w.

Proof:

We will replace $\mu(z)$ with $\mu(rz)$ for $r < 1$ arbitrarily close to 1, and then apply the standard approximation arguments. Thus we assume that $\mu(z)$ belongs to the class C^1 on a neighborhood of the closed unit disk. This condition implies that the corresponding normalized μ-conformal mapping f is in the class C^2. Let $\{Z^{(s)}_{j,k}\}$ be the vertices of the simplicial complex $T^{(s)}_Z$ in the logarithmic coordinates which are produced by our algorithm. For fixed s, we take the least squares solution for the associated linear system $(\boldsymbol{A}_s, \boldsymbol{B}_s)$ which we will call $\boldsymbol{W}'_s$. $\boldsymbol{W}'_s$ minimizes the L_2-norm $\|\boldsymbol{R}_s\|_2$ of the residual vector

$$\boldsymbol{R}_s = \boldsymbol{A}_s \boldsymbol{W}'_s - \boldsymbol{B}_s. \tag{25}$$

Let $\nu_s := \{\nu^{\pm}_{s,j,k}\}$ denote the collection of average values of the function ν which are defined by (10) on the triangles of $T^{(s)}_Z$.
Now we consider another linear system. Let $\boldsymbol{W}_s$ be defined by

$$\boldsymbol{W}_s := \{W^{(s)}_{j,k}\} = \{F(Z^{(s)}_{j,k})\} \tag{26}$$

which contains the images of the vertices under the true ν-conformal mapping $F(Z) = \log f(e^Z)$ in the logarithmic coordinates.

Let F_s^* be the secant map which is induced by the correspondence $T_Z^{(s)}$ and F, and let $\nu_s^* := \{\nu_{s,j,k}^*\}$ be the Beltrami coefficient of F_s^*. By the construction, F_s^* coincides with F on the vertices of $\{Z_{j,k}^{(s)}\}$. However the Beltrami coefficient of F_s^* is constant on each 2-simplex. Let $(\boldsymbol{A}_s^*, \boldsymbol{B}_s^*)$ be the associated linears system which is induced by $T_Z^{(s)}$ and ν_s^*. We will consider the following:

$$\boldsymbol{A}_s \boldsymbol{W}'_s - \boldsymbol{B}_s = \boldsymbol{R}'_s, \quad (27)$$

$$\boldsymbol{A}_s^* \boldsymbol{W}_s - \boldsymbol{B}_s^* = \boldsymbol{\varepsilon}_s. \quad (28)$$

where

$$\|\boldsymbol{\varepsilon}_s\|_2 < O\left(\left((2N_s)\frac{1}{(N_s^{\pi\sqrt{3}c_1})^2}\right)^{(1/2)}\right) < O(1/N_s) \to 0. \quad (29)$$

Proposition

$$\max_{\tau_{j,k}\in T_Z^{(s)}} \left| \nu_s^* \left|_{\tau_{j,k}} - \nu_s \right|_{\tau_{j,k}} \right| = O\left(\frac{1}{N_s}\right). \tag{30}$$

We have

$$\|\boldsymbol{R}_s'\|_\infty < \|\boldsymbol{R}_s'\|_2 = \|\boldsymbol{A}_s\boldsymbol{W}_s' - \boldsymbol{B}_s\|_2 \le \|\boldsymbol{A}_s\boldsymbol{W}_s - \boldsymbol{B}_s\|_2$$

by minimality of $\|\boldsymbol{A}_s\boldsymbol{W}_s' - \boldsymbol{B}_s\|_2 = \|\boldsymbol{R}_s'\|_2$ (recall (25) and (29)).

$$\begin{aligned} \|\boldsymbol{A}_s\boldsymbol{W}_s - \boldsymbol{B}_s\|_2 &\le \|\boldsymbol{A}_s\boldsymbol{W}_s - \boldsymbol{A}_s^*\boldsymbol{W}_s\|_2 + \|\boldsymbol{A}_s^*\boldsymbol{W}_s - \boldsymbol{B}_s^*\|_2 \\ &\quad + \|\boldsymbol{B}_s^* - \boldsymbol{B}_s\|_2 \\ &= \|(\boldsymbol{A}_s - \boldsymbol{A}_s^*)\boldsymbol{W}_s\|_2 + \|\boldsymbol{\varepsilon}_s\|_2 + \|\boldsymbol{B}_s^* - \boldsymbol{B}_s\|_2 \\ &\to 0 \end{aligned} \tag{31}$$

by the regularity of the Beltrami coefficient, sparseness of the associated linear systems and (30).

By (27) and (28), we obtain

$$\begin{aligned}\|\boldsymbol{A}_s(\boldsymbol{W}'_s-\boldsymbol{W}_s)\|_\infty &\le \|\boldsymbol{A}_s\boldsymbol{W}'_s-\boldsymbol{B}_s\|_\infty+\|\boldsymbol{B}_s-\boldsymbol{B}^*_s\|_\infty\\ &\quad +\|\boldsymbol{B}^*_s-\boldsymbol{A}^*_s\boldsymbol{W}_s\|_\infty+\|\boldsymbol{A}^*_s\boldsymbol{W}_s-\boldsymbol{A}_s\boldsymbol{W}_s\|_\infty\\ &= \|\boldsymbol{R}_s\|_\infty+\|\boldsymbol{B}^*_s-\boldsymbol{B}_s\|_\infty+\|\boldsymbol{\varepsilon}_s\|_\infty\\ &\quad +\|(\boldsymbol{A}^*_s-\boldsymbol{A}_s)\boldsymbol{W}_s\|_\infty. \end{aligned} \tag{32}$$

Further we can proved: If $\{\boldsymbol{X}_s\}\in\mathbb{C}^{n_v}$ is such that $\boldsymbol{A}_s\boldsymbol{X}_s\to 0$, then $\boldsymbol{X}_s\to 0$. Hence we have

$$\|\boldsymbol{W}'_s-\boldsymbol{W}_s\|_\infty\to 0. \tag{33}$$

We apply the exponential mapping to obtain the sequence of PL-mappings $f^{(s)}\in PL(|T_z^{(s)}|)$ which send each 2-simplices (z_1,z_2,z_3) $\in T_z^{(s)}$ to the 2-simplices (w_1,w_2,w_3) where $w_j:=\exp\circ F_s\circ\log(z_j)$. Let f^*_s be the secant map which is induced by f and $T_z^{(s)}$, and let μ^*_s be the Beltrami coefficients of f^*_s. Since quasiconformal mapping $f\in C^2(\mathbb{D})$, we see that f^*_s is a quasiconformal mapping if s is large enough, and it converges to the true solution f locally uniformly on $\mathbb{D}$ by good approximation lemma. We obtain that f_s is also a quasiconformal mapping if s is large enough and $f_s\to f$ locally uniformly in $\mathbb{D}$. □

- We propose an algorithm for numerical quasiconformal mappings which converge to the true solution at least in the case the Beltrami coefficients are in C^1. Our algorithm behave numerically well for some difficult cases.
- We use characteristic properties of quasiconformal mappings for our piecewise linear mapping.
- The computational cost of our algorithm is $O(M^3N)$ for one approximation (not for a point).

Thank you very much for your attention.

Notes on complex hyperbolic triangle groups of type (m, n, ∞)

Lijie Sun

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Osaka University. Feb. 15, 2015

Contents

- Background
  — Complex hyperbolic space, Complex reflections, Complex hyperbolic triangle groups
- Main tools for non-discrete results
  — Jørgensen's inequality, Shimizu's lemma
- Thought about discrete cases
  — Cygan ball, Ford domain, Klein's combination theorem

Background

Complex hyperbolic space

Let $\mathbb{C}^{2,1}$ denote the vector space $\mathbb{C}^3$ equipped with the Hermitian form

$$\langle z, w\rangle = z_1\overline{w}_1 + z_2\overline{w}_2 - z_3\overline{w}_3$$

of signature (2,1). Let $\mathbb{P} : \mathbb{C}^{2,1} - \{0\} \longrightarrow \mathbb{CP}^2$ be the projection map.

- $V_- = \{z \in \mathbb{C}^{2,1} : \langle z, z\rangle < 0\}$, $\quad V_0 = \{z \in \mathbb{C}^{2,1} - \{0\} : \langle z, z\rangle = 0\}$, $V_+ = \{z \in \mathbb{C}^{2,1} : \langle z, z\rangle > 0\}$.
- $\mathbb{H}^2_{\mathbb{C}} = \mathbb{P}V_- = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ Unit ball model $\mathfrak{U}$. $\partial\mathbb{H}^2_{\mathbb{C}} = \mathbb{P}V_0 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ homeo. to $\mathbb{S}^3$.
- The Bergman metric on $\mathbb{H}^2_{\mathbb{C}}$ is given by

$$\cosh^2\Big(\frac{\rho(x, y)}{2}\Big) = \frac{\langle \tilde{x}, \tilde{y}\rangle\langle \tilde{y}, \tilde{x}\rangle}{\langle \tilde{x}, \tilde{x}\rangle\langle \tilde{y}, \tilde{y}\rangle} \quad x, y \in \mathbb{H}^2_{\mathbb{C}},$$

where $\tilde{x}$, $\tilde{y}$ are standard lifts in $\mathbb{C}^{2,1}$ of x, y respectively.

Choose a line spanned by the null vector Q_∞ representing a point q_∞ in $\partial \mathbb{H}^2_{\mathbb{C}}$. There is unique complex projective hyperplanes $H_\infty \subset \mathbb{C}\mathbb{P}^2$ that is tangent to $\partial \mathbb{H}^2_{\mathbb{C}}$ at q_∞. Using affine coordinates on $\mathbb{C}\mathbb{P}^2 - H_\infty$ complex hyperbolic space is realised as a Siegel domain model $\mathfrak{S}$ with horospherical coordinates. In these coordinates $z \in \mathfrak{S}$ is given by $z = (\zeta, v, u) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+$.

In order to see how $\mathfrak{S}$ relates to $\mathbb{P}(V_-)$, we define the map $\psi : \overline{\mathfrak{S}} \longrightarrow \mathbb{C}\mathbb{P}^2$ by

$$\psi : (\xi, v, u) \longmapsto \begin{pmatrix} \xi \\ \frac{1}{2}(1 - |\xi|^2 - u + iv) \\ \frac{1}{2}(1 + |\xi|^2 + u - iv) \end{pmatrix},$$

$$q_\infty \longmapsto \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Heisenberg group and Cygan metric

The 3-dimensional Heisenberg group $\mathcal{H}$ is the set $\mathbb{C} \times \mathbb{R}$ with the group law

$$(\xi_1, v_1) * (\xi_2, v_2) = (\xi_1 + \xi_2, v_1 + v_2 + 2\,\mathrm{Re}\,(\xi_1 \overline{\xi_2})).$$

Note that $(\xi_1, v_1)^{-1} = (-\xi_1, -v_1)$, $|(\xi, v)| = \left||\xi|^2 - iv\right|^{\frac{1}{2}}$.
The Cygan metric ρ_0 on the Heisenberg group is

$$\begin{aligned}\rho_0((\xi_1, v_1), (\xi_2, v_2)) &= \left|(\xi_1, v_1)^{-1} * (\xi_2, v_2)\right| \\ &= \left||\xi_1 - \xi_2|^2 - iv_1 + iv_2 - 2i\,\mathrm{Re}\,(\xi_1 \overline{\xi_2})\right|^{\frac{1}{2}}.\end{aligned}$$

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We can extend the Cygan metric to $\overline{\mathbb{H}^2_{\mathbb{C}}} - q_\infty$ as follows

$$\rho_0((\xi_1, v_1, u_1), (\xi_2, v_2, u_2)) = \left| |\xi_1 - \xi_2|^2 + |u_1 - u_2| - iv_1 + iv_2 - 2i\,\mathrm{Re}\,(\xi_1 \overline{\xi_2}) \right|^{\frac{1}{2}}.$$

Complex geodesic and complex reflection

- Give $x,\ y \in \overline{\mathbb{H}^2_{\mathbb{C}}}$. Take $\widetilde{C} = \mathrm{span}_{\mathbb{C}}\{\tilde{x},\ \tilde{y}\}$, $\tilde{x},\ \tilde{y} \in \mathbb{C}^{2,1}$ are lifts of $x,\ y$ respectively. We define the complex geodesic $C = \mathbb{P}(\widetilde{C})$, which can be uniquely determined by a positive vector $p \in \mathbb{C}^{2,1}$, i.e.

$$C = \mathbb{P}(\{z \in \mathbb{C}^{2,1} : \langle z, p\rangle = 0\}).$$

  We call p a polar vector to C.

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  We call p a polar vector to C.

- The complex reflection in C is represented by an element $I_C \in \text{SU}(2,1)$ that is given by

$$I_C = -z + 2\frac{\langle z, p\rangle}{\langle p, p\rangle}p,$$

  where p is a polar vector of C.

Two kinds of chain

- z – chain ($z \in \mathbb{C}$) is the chain having polar vector

$$\begin{pmatrix} 1 \\ -\bar{z} \\ \bar{z} \end{pmatrix}.$$

The z – chain is the vertical chain in $\mathcal{H}$ through the point $(z, 0)$.

- $(z,\ r)$ – chain ($z,\ r \in \mathbb{R}$) is the chain having polar vector

$$\begin{pmatrix} 0 \\ 1 + r^2 + iz \\ 1 - r^2 - iz \end{pmatrix}.$$

The $(z,\ r)$ – chain is the cicle of radius r centered at the origin in $\mathbb{C} \times \{z\} \subset \mathcal{H}$.

Classification of complex hyperbolic isometries

Let PU(2, 1) be the projectivisation of the group U(2, 1), which preserves $\langle \cdot, \cdot \rangle$. We pass between matrix groups and isometries without comment. An isometry g of $\mathbb{H}^2_{\mathbb{C}}$ is

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NOTE:

- An elliptic element g is called regular elliptic if all its eigenvalues are distinct.
- Define the discriminant polynomial

$$f(z) = |z|^4 - 8\mathrm{Re}(z^3) + 18|z|^2 - 27.$$

An element $g \in \mathrm{SU}(2,1)$ is regular elliptic if and only if $f(\tau(g)) < 0$, where $\tau(g)$ is the trace of g.

Complex triangle groups

Assume that integers p, q, r, with $p,\ q,\ r \in \mathbb{N}_+$.
A complex hyperbolic triangle is a triple (C_1, C_2, C_3) of complex geodesics in $\mathbb{H}^2_{\mathbb{C}}$.

Definition of the angle of two complex geodesics

Let $C_1,\ C_2$ be two complex geodesics with two polar vectors $p_1,\ p_2$ respectively. Define the angle between C_1 and C_2 as follows

$$\measuredangle(C_1,\ C_2) = \measuredangle(p_1,\ p_2) = \min_{x_1,\ x_2} \{\angle(x_1,\ x_2) :\ x_i \in span_{\mathbb{R}}(p_i)\},$$

where $\angle$ denotes the angle between the two vectors measured normally.

Note that $0 \le \measuredangle(C_1,\ C_2) \le \pi/2$.

Complex triangle groups

Complex triangle groups

- If the complex geodesics C_{k-1} and C_k meet at the angle $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ $(p, q, r \in \mathbb{N}_+)$, where the indices are taken mod 3, we call the triangle (C_1, C_2, C_3) a (p, q, r) – triangle.

Complex triangle groups

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- We call Γ a (p, q, r) – triangle group, if Γ is generated by three complex reflections I_1, I_2, I_3 in the sides C_1, C_2, C_3 of a (p, q, r) – triangle.

Complex triangle groups

- If the complex geodesics C_{k-1} and C_k meet at the angle $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ $(p, q, r \in \mathbb{N}_+)$, where the indices are taken mod 3, we call the triangle (C_1, C_2, C_3) a (p, q, r) – triangle.
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In this talk, we consider (m, n, ∞) – triangle groups. In this case $\mathrm{ord}(I_1I_3) = m$, $\mathrm{ord}(I_1I_2) = n$ and I_2I_3 is a Heisenberg translation.

History

- 1992, W. M. Goldman, J. R. Parker Groups of type (∞, ∞, ∞)
- 2007, Kamiya Groups of type (n, n, ∞)
- 2008, J. R. Parker Groups of type (n, n, n)
- 2010, Kamiya, Parker, Thompson Groups of type $(p, q, r; n)$
- 2012, Kamiya, Parker, Thompson Groups of type $(n, n, \infty; k)$

Proposition (2000 Justin Wyss-Gallifent)

Any (m, n, ∞) – triangle group is PU(2, 1) – equivalent to one generated by inversions in the (0, 1) – chain and in two vertical chains.

By conjugation in PU(2, 1), we can take three involutions I_j in C_j such that $\partial C_1,\ \partial C_2,\ \partial C_3$ are (0,1) – chain, z_1 – chain, z_2 – chain resp., where $z_1 = \cos(\pi/n)$, $z_2 = e^{i\theta}\cos(\pi/m)$.
The three polar vectors correspondingly are:

$$p_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 1 \\ -z_1 \\ z_1 \end{pmatrix}, \quad p_3 = \begin{pmatrix} 1 \\ -\bar{z}_2 \\ \bar{z}_2 \end{pmatrix}.$$

Define the parameter of the (m, n, ∞) – triangle angular invariant α by

$$\alpha = \arg\Big(\prod_{i=1}^{3}\langle p_{i-1}, p_{i+1}\rangle\Big) = \arg(z_1 z_2) = \theta$$

Main tools and non-discrete results

Let $g \in \mathrm{PU}(2,1)$ be a parabolic element. Define the translation length $t_g(z)$ of g at $z \in \mathcal{H}$ by $t_g(z) = \rho_0(g(z), z)$.

Definition of Ford isometric sphere (Goldman)

Ford isometric sphere of a map h in $\mathrm{SU}(2,1)$ that does not projectively fix q_∞ is the spinal hypersurface given by

$$I(h) = \{z \in \mathbb{H}^2_{\mathbb{C}} : |\langle \tilde{z}, \infty \rangle| = |\langle \tilde{z}, h^{-1}\infty) \rangle|\},$$

where $\tilde{z}$ and ∞ are standard lifts in $\mathbb{C}^{2,1}$ of z and q_∞, respectively.

Complex hyperbolic version of Shimizu's lemma (1997 Parker)

Let G be a discrete subgroup of $\mathrm{PU}(2,1)$ that contains the Heisenberg translation $g = (\xi, t)$. Let h be any element of G not fixing ∞ and with isometric sphere of radius r_h. Then

$$r_h^2 \leq t_g(h^{-1}(\infty))\, t_g(h(\infty)) + 4|\xi|^2.$$

Complex hyperbolic version of Jørgensen's inequality (2012 K., P., T.)

Let $A \in \mathrm{SU}(2,1)$ be a regular elliptic element of order $n \geq 7$ that preserves a Lagrangian plane (i.e. $\mathrm{tr}(A)$ is real). Suppose that A fixes a point $\mathrm{z} \in \mathbb{H}^2_{\mathbb{C}}$. Let B be any element of $\mathrm{PU}(2,1)$ with $B(z) \neq z$. If

$$\cosh\left(\frac{\rho(Bz,z)}{2}\right)\sin\left(\frac{\pi}{n}\right) < \frac{1}{2},$$

then $\langle A, B\rangle$ is not discrete and consequently any group containing A and B is not discrete.

Theorem (2014 S.)

Γ of type (m, n, ∞) is not discrete if m, n, θ satisfy one of the two following conditions

(1) $7 \le n < \infty$ and

$$\left|\cos^2\left(\frac{\pi}{n}\right) + 2\cos^2\left(\frac{\pi}{m}\right) - 4\cos\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{m}\right)\cos\theta + 1\right| < \frac{1}{2}\sin\left(\frac{\pi}{n}\right);$$

(2) Let $u = \cos^2(\frac{\pi}{m}) + \cos^2(\frac{\pi}{n}) - 2\cos(\frac{\pi}{m})\cos(\frac{\pi}{n})\cos\theta$,
$v = \cos(\frac{\pi}{m})\cos(\frac{\pi}{n})\sin\theta$,

$$|u - 2iv| + 4u < \frac{1}{4}.$$

Example 1: Let $m = 8$. Show the interval of a corresponding to the non-discrete Γ when $a \in (c_n, 1)$ or $a \in (d_n, 1)$, where $a = \cos\theta$.

Table: Approximations of c_n, d_n.

n	c_n	d_n
4	—	0.99961
5	—	0.99419
6	—	0.99289
7	0.99170	0.99279
8	0.98685	0.99299
20	0.98750	0.99442
30	0.99147	0.99464
100	0.99911	0.99480
200	—	0.99481

Lemma (2014 S.)

Γ of type (m, n, ∞) $(m \neq n)$ is not discrete if $I_1 I_2 I_3$ is regular elliptic.

- Example 2: Show the interval of a corresponding to the non-discrete Γ of type $(8, n, \infty)$ when $a_n \leq a \leq b_n$.

Table: Approximations of a_n, b_n.

n	a_n	b_n
11	0.93067	0.93114
15	0.93437	0.93512
20	0.93575	0.93654
30	0.93662	0.93733
40	0.93690	0.93757
100	0.93719	0.93780

NOTE: There are no solutions for a when $n \leq 10$.

Thought about discrete triangle groups

Definition of generalised isometric sphere (2003 Kamiya)

Let y be a point of $\partial \mathbb{H}^2_{\mathbb{C}}$. For an element $h \in \mathrm{PU}(2,1)$ with $h(y) \neq y$, we define the *generalized isometric sphere* $I_y(h)$ of h at y as

$$I_y(h) = \{z \in \mathbb{H}^2_{\mathbb{C}} : |\langle \tilde{z}, \tilde{y} \rangle| = |\langle \tilde{z}, h^{-1}(\tilde{y}) \rangle|\}.$$

Correspondingly, $\mathrm{Ext}\, I_y(h) = \{z \in \mathbb{H}^2_{\mathbb{C}} : |\langle \tilde{z}, \tilde{y} \rangle| < |\langle \tilde{z}, h^{-1}(\tilde{y}) \rangle|\}$.
$\mathrm{Int}\, I_y(h) = \{z \in \mathbb{H}^2_{\mathbb{C}} : |\langle \tilde{z}, \tilde{y} \rangle| > |\langle \tilde{z}, h^{-1}(\tilde{y}) \rangle|\}$.

Theorem (2003 Kamiya)

Let G be a discrete subgroup of $\mathrm{PU}(2,1)$. Let ∞ be a point of $\Omega(G)$ and the stabilizer of ∞ only consist of identity. If $y \in \Omega(G) \cap \partial \mathbb{H}^2_{\mathbb{C}}$ such that $G_y = \{\mathrm{id}\}$, then

$$P_y(G) = \bigcap_{f \in G - \{id\}} \mathrm{Ext}\, I_y(f)$$

is a fundamental domain for G.

F. D. of $\langle f \rangle$

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A Cygan ball S with center at $x_0 \in \partial\mathbb{H}^2_{\mathbb{C}}$ and radius r is defined as $S = \{z \in \mathbb{H}^2_{\mathbb{C}} : \rho_0(\tilde{z}, \widetilde{x_0}) < r\}$.

F. D. of $\langle f \rangle$

A Cygan ball S with center at $x_0 \in \partial \mathbb{H}^2_{\mathbb{C}}$ and radius r is defined as $S = \{z \in \mathbb{H}^2_{\mathbb{C}} : \rho_0(\tilde{z}, \widetilde{x_0}) < r\}$.

Assume that $f = I_1 I_2$, $g = I_1 I_3$. Find elements $A,\ B \in \mathrm{PU}(2,1)$ which conjugate $f,\ g$ to normalized form $f_0,\ g_0$ respectively. We take the normalised form $A^{-1} f A$ for example,

$$f_0 = \begin{pmatrix} \frac{1}{2}(e^{\frac{2i\pi}{n}} + 1) & 0 & \frac{1}{2}(e^{\frac{2i\pi}{n}} - 1) \\ 0 & e^{-\frac{2i\pi}{n}} & 0 \\ \frac{1}{2}(e^{\frac{2i\pi}{n}} - 1) & 0 & \frac{1}{2}(e^{\frac{2i\pi}{n}} + 1) \end{pmatrix}.$$

By considering the the construction of Ford domain, we shall know that the exterior of a F. D. of $\langle f_0 \rangle$, $\bigcup_{k=1}^{n-1} \mathrm{Int}\, I(f_0^k)$ is contained in a Cygan ball S_1 with center at the origin and radius $\sqrt{\frac{1+\cos(\pi/n)}{\sin(\pi/n)}}$.

We show the relation between the Ford isometric sphere $I(f_0)$ w. r. t infinity and $I_y(f)$, where $y = A(\infty)$.

$$\begin{aligned} z \in \operatorname{Int} I(f_0^k) &\Leftrightarrow |\langle z, \infty\rangle| > |\langle z, f_0^{-k}(\infty)\rangle| \\ &\Leftrightarrow |\langle z, \infty\rangle| > |\langle z, A^{-1} f^{-k} A(\infty)\rangle \\ &\Leftrightarrow |\langle Az, y\rangle| > |\langle Az, f^{-k} y)\rangle| \\ &\Leftrightarrow Az \in \operatorname{Int} I_y(f^k). \end{aligned}$$

Therefore $\bigcup_{k=1}^{n-1} \operatorname{Int} I_y(f^k) = A(\bigcup_{k=1}^{n-1} \operatorname{Int} I(f_0^k)) \subseteq A(S_1)$.

Similarly, the exterior of the F. D of $\langle g\rangle$ will be contained in $B(S_2)$, where S_2 is a Cygan ball containing $\bigcup_{j=1}^{m-1} \operatorname{Int} I(g_0^j)$.

Klein's combination theorem (1992 Goldman, Parker)

Let G_1, G_2 be discrete subgroups of $\mathrm{PU}(2,1)$ with connected fundamental domains D_1 and D_2. Let E_1 and E_2 be the interior of the complement of D_1 and D_2 in $\mathbb{H}^2_{\mathbb{C}}$ respectively. Suppose that $E_1 \cap E_2 = \emptyset$ and $D_1 \cap D_2 \neq \emptyset$. Then $G = \langle G_1, G_2 \rangle$ is discrete.

If $A(S_1) \bigcap B(S_2) = \emptyset$, then from the above theorem we know $\langle I_1 I_2, I_1 I_3 \rangle$ is discrete. It follows that the triangle group is discrete because $\langle I_1 I_2, I_1 I_3 \rangle$ is of index two of $\Gamma = \langle I_1, I_2, I_3 \rangle$.

Thank you for your attention!

Projective embeddings of the Teichmüller spaces

Yohei Komori (Waseda Univ.)

Riemann surfaces and Discontinuous groups 2015

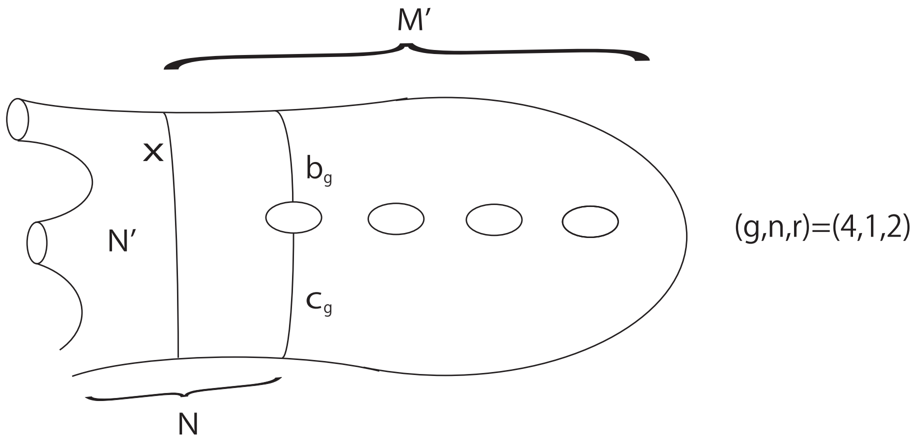
OSAKA

15.02.2015

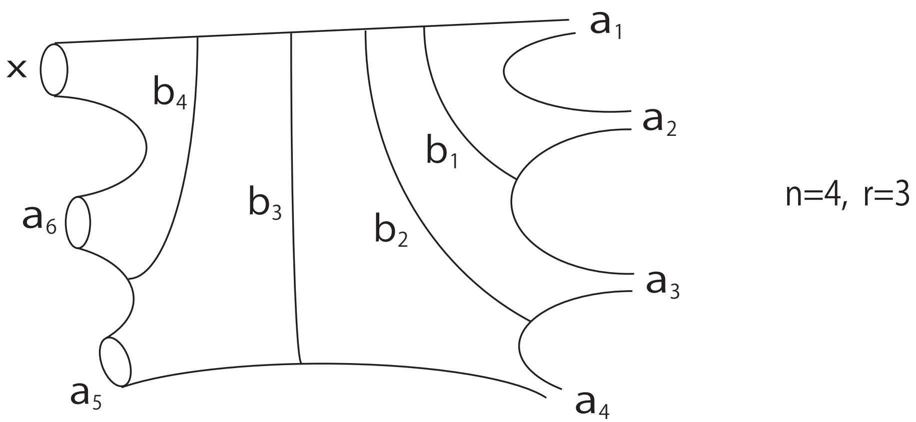
Contents

1 Introduction

Let M be a hyperbolic Riemann surface of genus g with n punctures and r holes. The Teichmüller space $\mathcal{T}_{g,n,r}$ is the space of isotopy classes of hyperbolic metrics on M which is homeomorphic to the real affine space of dimension $6g-6+2n+3r$.



By using hyperbolic lengths of simple closed geodesics we can embed $\mathcal{T}_{g,n,r}$ into the real affine space of dimension, for example $9g-9+3n+4r$ $> 6g-6+2n+3r$: Fix a pants decomposition $\mathcal{P}$ on M, i.e. a multicurve such that $M\backslash\mathcal{P}$ is homeomorphic to the disjoint union of thrice punctured spheres. $\mathcal{P}$ consists of $3g-3+n+r$ numbers of disjoint simple close curves.



The **Fenchel-Nielsen coordinates** associate to each $m \in \mathcal{T}_{g,n,r}$ the **length** of each components of $\mathcal{P}$ and boundary geodesics, and the **twist** of each components of $\mathcal{P}$, which is a diffeomorphism from $\mathcal{T}_{g,n,r}$ onto $\mathbb{R}_+^{3g-3+n+2r} \times \mathbb{R}^{3g-3+n+r}$. On the other hand the twist of each components of $\mathcal{P}$ can be determined by the lengths of two more curves for each components so that $\mathcal{T}_{g,n,r}$ can be embedded into the affine space of dimension

$$(3g-3+n+r+r)+2\times(3g-3+n+r)=9g-9+3n+4r$$

by length functions of simple closed geodesics. Schmutz showed that the minimal number of simple closed geodesics whose hyperbolic lengths globally parametrize $\mathcal{T}_{g,n,r}$ for $r \geqq 1$ is equal to $dim_{\mathbb{R}}\mathcal{T}_{g,n,r} = 6g-6+2n+3r$, so that the image of $\mathcal{T}_{g,n,r}$ becomes an unbounded domain in $\mathbb{R}^{dim_{\mathbb{R}}\mathcal{T}_{g,n,r}}$.

Moreover Okumura and Schmutz showed that the minimal number of simple closed geodesics whose hyperbolic lengths globally parametrize $\mathcal{T}_{g,n}$ is equal to $dim_{\mathbb{R}}\mathcal{T}_{g,n} + 1 = 6g - 6 + 2n + 1$.

Now we have the following natural question:

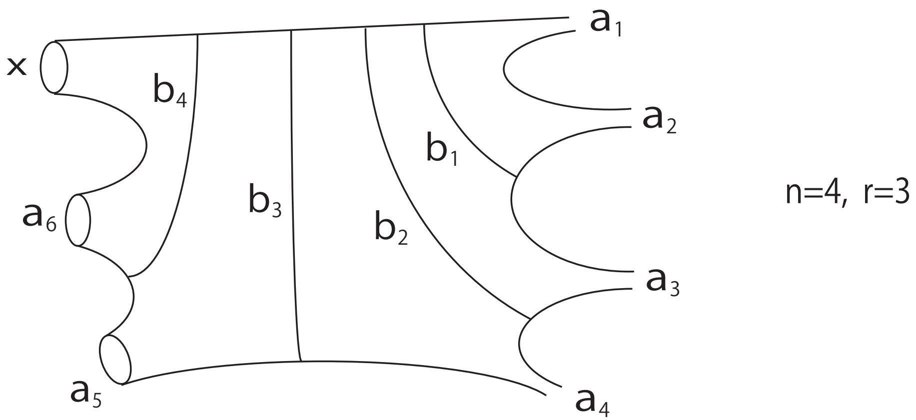
> Can we find $dim_{\mathbb{R}}\mathcal{T}_{g,n,r} + 1$ simple closed geodesics whose hyperbolic lengths embed $\mathcal{T}_{g,n,r}$ into the real projective space of dimension equal to $dim_{\mathbb{R}}\mathcal{T}_{g,n,r}$?

Because of the PL-Structure of the Thurston boundary, we might expect that the image of $\mathcal{T}_{g,n,r}$ should be the interior of some convex polyhedron in the projective space.

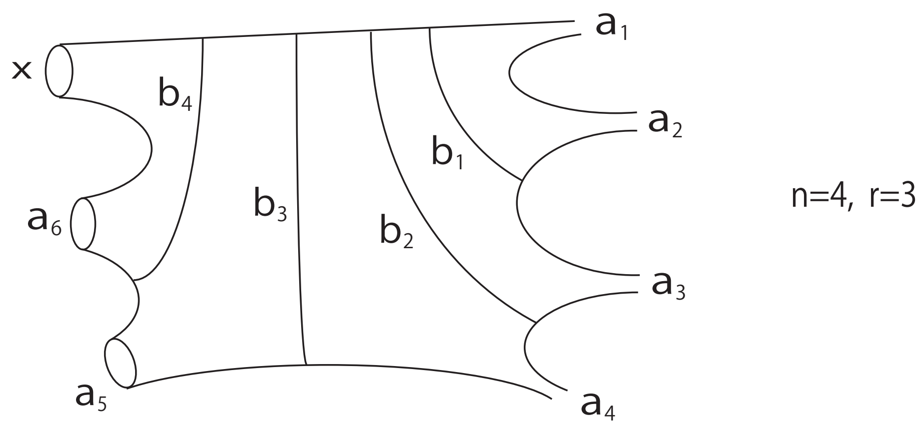
2 Results of Schmutz on bordered surfaces

2.1 Bordered surfaces with no handles

Let M be a bordered hyperbolic Riemann surface of type $(0, n, r)$. We denote the boundary geodesics $x, a_1, a_2, \cdots, a_{n+r-1}$ and dividing geodesics $b_1, b_2, \cdots, b_{n+r-3}$ which decompose M into disjoint union of (degenerate) pair of pants.



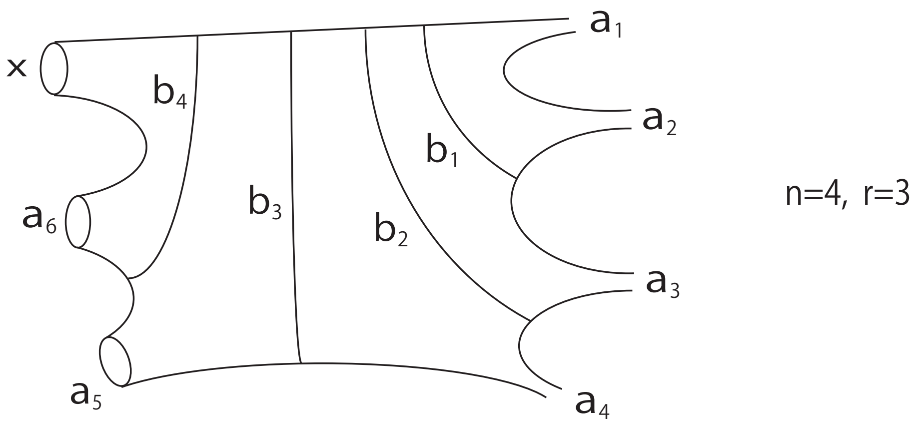
For each $i = 1, 2, \cdots, n+r-3$, let X_i be the subsurface of type $(0, n_i, r_i)$ where $n_i + r_i = 4$ with boundary geodesics $a_{i+1}, a_{i+2}, b_{i-1}, b_{i+1}$. Choose geodesics c_i and d_i in X_i so that the triple $\{b_i, c_i, d_i\}$ mutually intersect exactly twice.



Proposition 1. (cf. Proposition2 [S1])
The hyperbolic lengths of $2n + 3r - 6$ *geodesics*

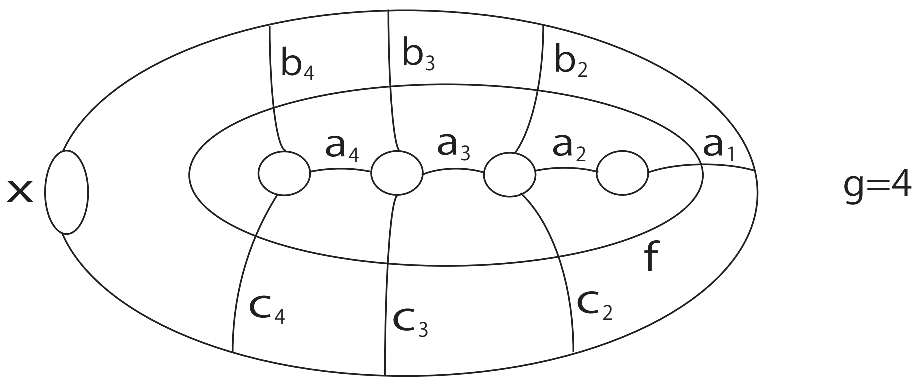
$$a_1, a_2, \cdots, a_{n+r-1}, b_1, c_1, c_2, c_{n+r-3}, , d_1, d_2, d_{n+r-3}$$

embeds $T_{0,n,r}$ *into* $\mathbb{R}^{2n+3r-6}$*. Here we remark that the length of* a_k *is equal to* 0 *when* a_k *corresponds to a puncture.*

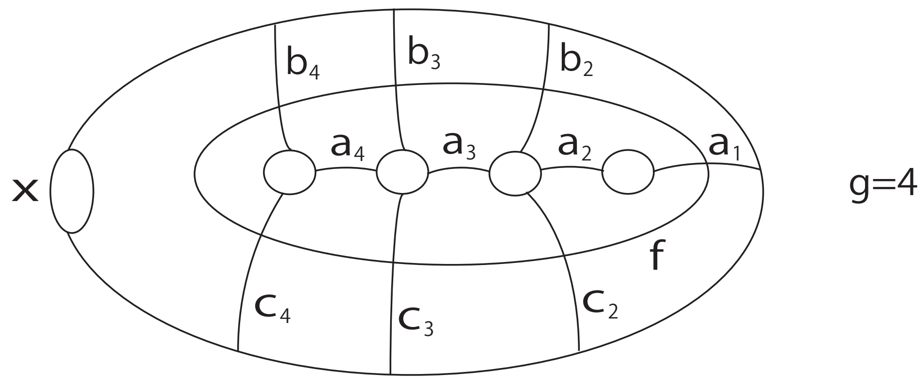


2.2 Bordered surfaces of type $(g, 0, 1)$ with $g \geqq 1$

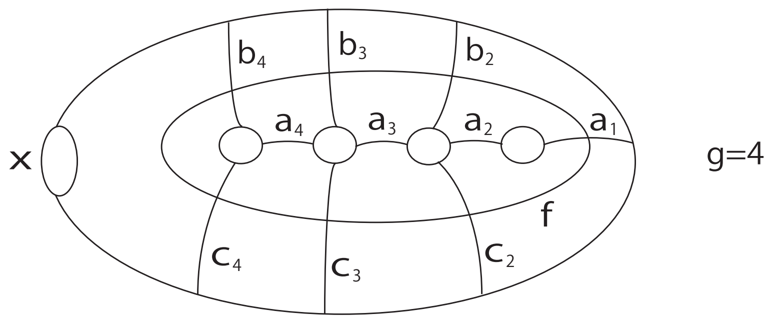
Next we consider a hyperbolic Riemann surface M of type $(g, 0, 1)$. We denote the boundary geodesic by x. Choose non-dividing geodesics $a_1, a_2, \cdots, a_g, b_2, b_3, \cdots, b_g, c_2, c_3, \cdots, c_g$ which decompose M into disjoint union of pair of pants.



For each $i = 2, \cdots, g-1$, let X_i be the subsurface of type $(0,0,4)$ with boundary geodesics $b_i, c_i, b_{i+1}, c_{i+1}$, Choose geodesics d_{i+1} and e_{i+1} in X_i so that the triple $\{a_{i+1}, d_{i+1}, e_{i+1}\}$ mutually intersect exactly twice.



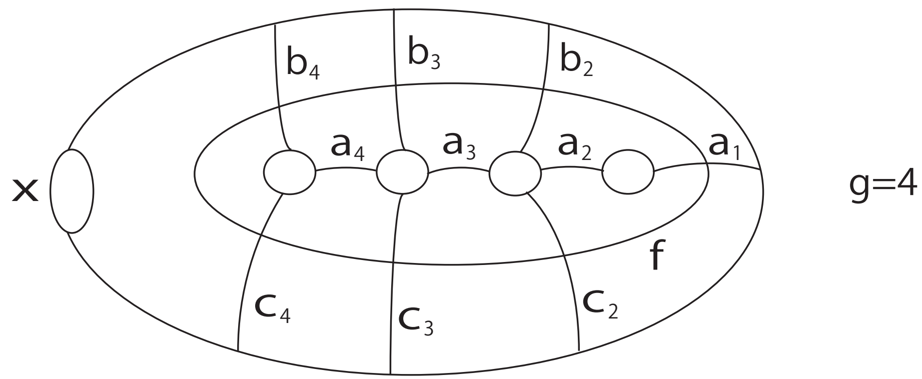
Let X_1 be the subsurface of M of type $(0,0,4)$ with boundary geodesics a_1, a_1, b_2, c_2, and choose d_2 and e_2 on X_1 so that the triple $\{a_2, d_2, e_2\}$ mutually intersect exactly twice. Moreover let f be a geodesic intersecting with $a_1, b_2, b_3, \cdots, b_g, c_2, c_3, \cdots, c_g$ exactly once. Then for $i = 2, \cdots, g$, we can find geodesics $r_1, s_2, s_3, \cdots, s_g, t_2, t_3, \cdots .t_g$ so that $\{a_1, r_1, f\}, \{b_i, s_i, f\}$ and $\{c_i, t_i, f\}$ mutually intersect exactly once.



Proposition 2. (cf. Proposition3 [S1])
The hyperbolic lengths of $6g-3$ geodesics

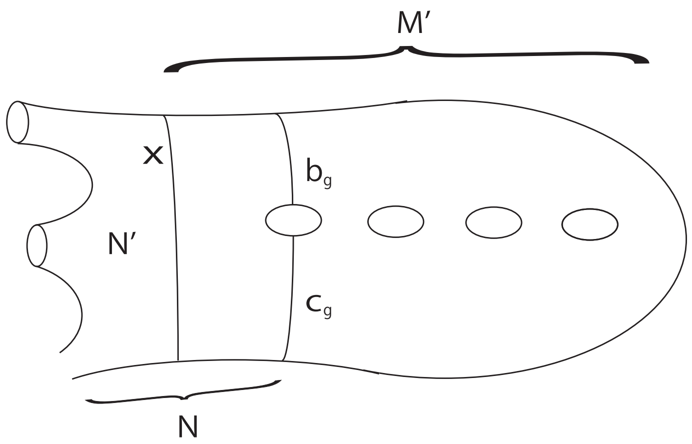
$$a_1, a_2, \cdots, a_g, b_2, \cdots, b_g, d_2, \cdots, d_g, e_2, \cdots, e_g, f, r_1, s_2, \cdots, s_g, t_2, \cdots, t_g$$

embeds $T_{g,0,1}$ into $\mathbb{R}^{6g-3}$.



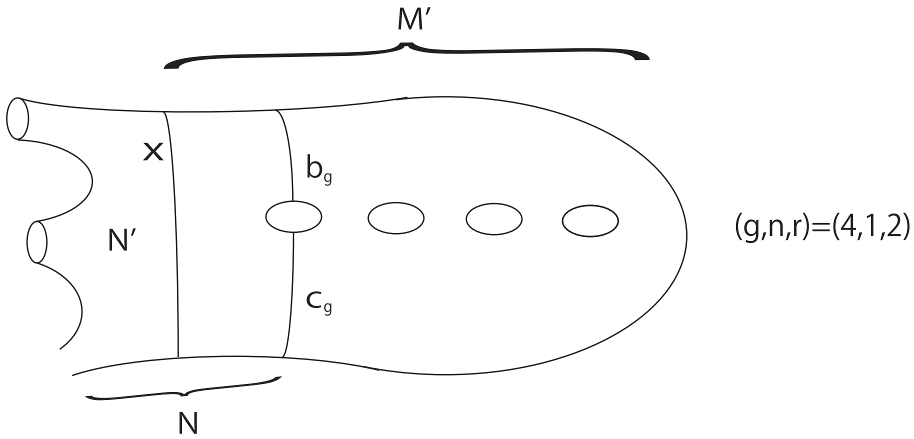
2.3 Bordered surfaces of type (g, n, r) with $g \geqq 1$ in general

Finally we consider a Riemann surface M of type (g, n, r) where $g \geqq 1$ $r \geqq 1$ in general. First we choose a dividing geodesic x to decompose M into subsurfaces M' of type $(g, 0, 1)$ and N' of type $(0, n, r+1)$.



(g,n,r)=(4,1,2)

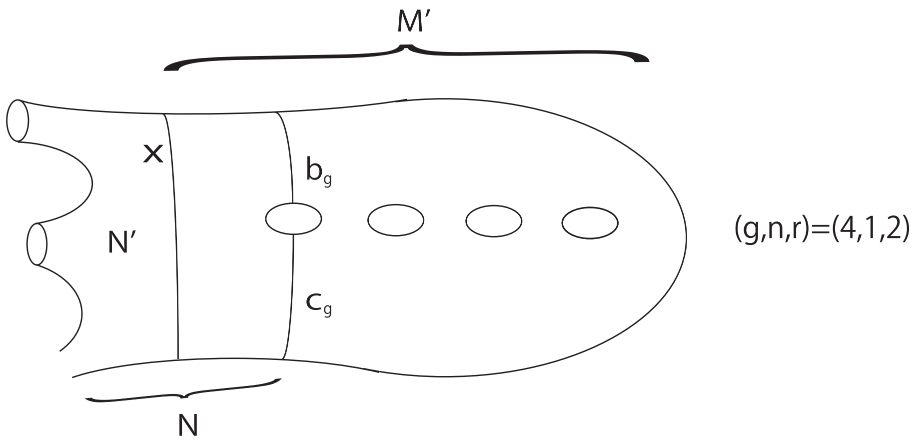
Let N be the subsurface of M consisting of N' and the pair of pants whose boundary curves are x, b_g and c_g. Then from the above argument we can choose $6g-3$ curves from M' and $2n+3(r+2)-6$ curves from N which determines M' and N in $T_{g,0,1}$ and $T_{0,n,r+2}$ respectively.



On the other hand the lengths of curves x, b_g and c_g are counted twice in M' and N so that we can find

$$(6g-3)+(2n+3(r+2)-6)-3=6g+2n+3r-6$$

geodesics whose hyperbolic lengths embed $T_{g,n,r}$ into $\mathbb{R}^{6g+2n+3r-6}$.



3 A lemma of Kerckhoff on the Thurston compactification of T_g

Let $\mathcal{S}$ be the non-trivial and non-peripheral free homotopy classes of simple closed curves on M. For any hyperbolic structure $m \in T_{g,n}$ and any free homotopy class $\alpha \in \mathcal{S}$, we denote the hyperbolic length of a unique simple closed geodesic belonging to α by $l(m, \alpha)$. Then the mapping $l_* : T_{g,n} \to \mathbb{R}_+^{\mathcal{S}}$ defined by $l_*(m) = (l(m, \alpha))_{\alpha \in \mathcal{S}}$ is injective.

Let π be the projection from $\mathbb{R}^{\mathcal{S}} \setminus \{0\}$ to the infinite-dimensional real projective space $P(\mathbb{R}^{\mathcal{S}})$. In Proposition 6 of Exposé 7 [FLP] Kerckhoff showed that the composition map $\pi \circ l_* : T_{g,n} \to P(\mathbb{R}^{\mathcal{S}})$ is also injective: In his argument, it is essential that the surface M has at least one handle, because he used the fact that for the case $g \geq 1$ we can find two simple closed curves γ_1 and γ_2 whose intersection number is equal to one.

Then simple closed curves γ_3 and γ_4 which are freely homotopic to $\gamma_1 \cdot \gamma_2$ and $\gamma_1^{-1} \cdot \gamma_2$ respectively satisfy the key identity for his proof:

$$\cosh(\frac{l_1 + l_2}{2}) + \cosh(\frac{l_1 - l_2}{2}) = \cosh(\frac{l_3}{2}) + \cosh(\frac{l_4}{2}).$$

where $l_i := l(m, [\gamma_i])$ for $m \in \mathcal{T}(X)$ and $i = 1, 2, 3, 4$.

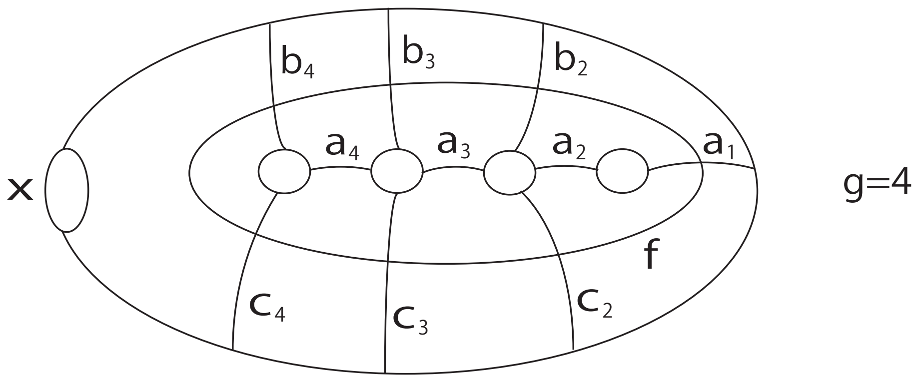
Lemma 1. *Let α, β, γ and δ be four nonnegative numbers and let $k \neq 1$ be a positive number. If*

$$\cosh \alpha + \cosh \beta = \cosh \gamma + \cosh \delta$$
$$\cosh k\alpha + \cosh k\beta = \cosh k\gamma + \cosh k\delta,$$

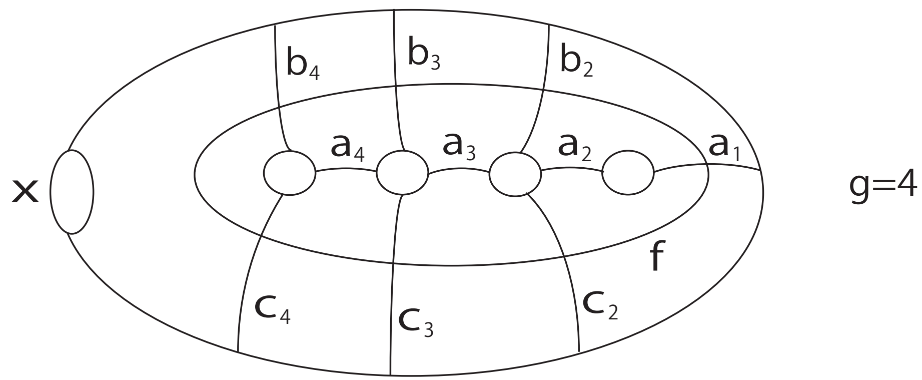
then $\{\alpha, \beta\} = \{\gamma, \delta\}$.

4 Projective embeddings of $T_{g,n,r}$ for $g \geqq 1$

Theorem 1. *Assume that $g \geqq 1$. Then the Teichmüller space $T_{g,n,r}$ of a bordered Riemann surface can be embedded into the real projective space of $dim_{\mathbb{R}} \mathcal{T}_{g,n,r} = 6g + 2n + 3r - 6$ by the hyperbolic length functions of $dim_{\mathbb{R}} \mathcal{T}_{g,n,r} + 1$ simple closed geodesics.*



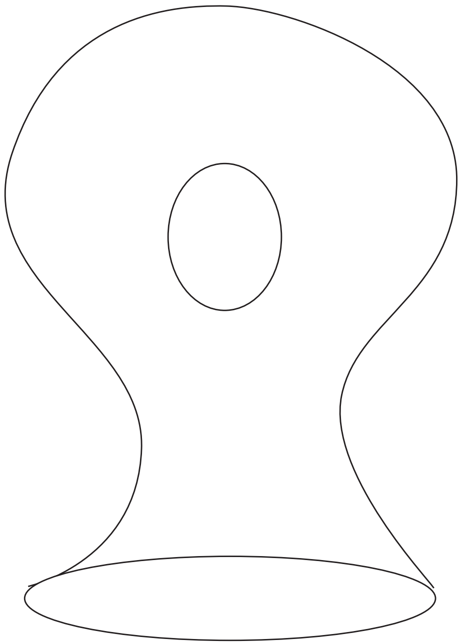
(Proof.) Suppose M is a bordered (i.e. $r \geqq 1$) Riemann surface of type (g, n, r) with $g \geqq 1$. Then there is a subsurface X of M with a geodesic boundary, which is a tubular neighborhood of the union of geodesics a_1 and f. X is homeomorphic to a torus with a hole on which the triple $\{a_1, r_1, f\}$ mutually intersect exactly once. Then the proportion of the hyperbolic lengths of $6g+2n+3r-6+1 = 6g+2n+3r-5$ geodesics embeds $T_{g,n,r}$ into $P(\mathbb{R}^{6g+2n+3r-5})$.



5 Cook hats and Crowns

5.1 Cook hats

We call a hyperbolic torus with a hole a **cook hat**.



Three simple closed geodesics (α, β, γ) on a cook hat is called a **canonical triple** if each pair of them has the intersection number equal to **one**. We remark that the hyperbolic lengths of a canonical triple (α, β, γ) satisfy **triangle inequalities**. For the lengths of a canonical triple (α, β, γ) and the boundary geodesic δ on a cook-hat, we have the following equality and inequality.

Proposition 3. *For any cook-hat with the boundary geodesic δ and a canonical triple (α, β, γ), their hyperbolic lengths $l(\alpha), l(\beta), l(\gamma)$ and $l(\delta)$ satisfy the following equality and inequality:*

$$\cosh^2 \frac{l(\delta)}{4} = (\cosh \frac{l(\beta)+l(\gamma)}{2} - \cosh \frac{l(\alpha)}{2})(\cosh \frac{l(\alpha)}{2} - \cosh \frac{l(\beta)-l(\gamma)}{2}). \tag{1}$$

$$l(\alpha) + l(\beta) + l(\gamma) > l(\delta). \tag{2}$$

By means of the equality (1) in Proposition 3, we can embed the Teichmüller space $\mathcal{T}(T)$ of a cook hat T into the 3-dimensional real projective space $P(\mathbb{R}^4)$.

Theorem 2. *For a cook hat with a canonical triple (α, β, γ) and the boundary geodesic δ, their hyperbolic lengths $l(\alpha), l(\beta), l(\gamma)$ and $l(\delta)$ satisfy*

$$\cosh^2 \frac{sl(\delta)}{4} < (\cosh \frac{sl(\beta)+sl(\gamma)}{2} - \cosh \frac{sl(\alpha)}{2})(\cosh \frac{sl(\alpha)}{2} - \cosh \frac{sl(\beta)-sl(\gamma)}{2})$$

for any $s > 1$. In particular the system of length functions $L := (l(\alpha), l(\beta), l(\gamma), l(\delta))$ gives homogeneous coordinates of the Teichmüller space $\mathcal{T}(T)$ of a cook hat T into $P(\mathbb{R}^4)$.

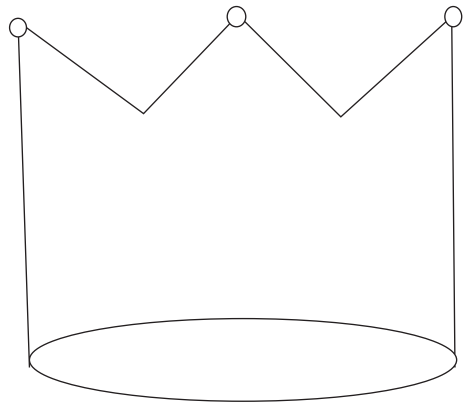
By means of the triangle inequalities of $l(\alpha), l(\beta), l(\gamma)$ and the inequality (2) in Proposition 3, we can determine the image of $\mathcal{T}(T)$ in $\mathcal{P}(\mathbb{R}^4)$ as follows.

Theorem 3. *The image of $\mathcal{T}(T)$ the Teichmüller space of a cook-hat T under the map $L := (l(\alpha) : l(\beta) : l(\gamma) : l(\delta))$ is the convex polyhedron Δ in $\mathcal{P}(\mathbb{R}^4)$ defined by*

$$\Delta := \{(a : b : c : d) \in \mathcal{P}(\mathbb{R}^4) \mid a > 0,\ b > 0,\ c > 0,\ d > 0,$$
$$a < b + c,\ b < c + a,\ c < a + b,\ d < a + b + c\}.$$

5.2 Crowns

We call a hyperbolic thrice-punctured sphere with a hole a **crown**.



Definition 1. *Three simple closed geodesics* (α, β, γ) *on a crown is called a* **canonical triple** *if each pair of them has the intersection number equal to two.*

We will show that similar results of cook hats also hold for crowns with the help of an "algebraic" bijection between the Teichmüller space of cook hats $\mathcal{T}(T)$ and that of crowns $\mathcal{T}(S)$ explained below. For this purpose we realize $\mathcal{T}(T)$ and $\mathcal{T}(S)$ as hypersurfaces in $\mathbb{R}^4$ in terms of trace functions:

Theorem 4. (Theorem 2 of [L] and Proposition 3.1 of [NN])

1. *We uniformize a cook-hat $m \in \mathcal{T}(T)$ by a Fuchsian group and denote the traces of elements representing a canonical triple α, β, γ and boundary geodesic δ by $t_\alpha(m), t_\beta(m), t_\gamma(m)$ and $t_\delta(m)$. Then the map $\varphi_T : \mathcal{T}(T) \to \mathbb{R}^4$ defined by $\varphi_T(m) := (t_\alpha(m), t_\beta(m), t_\gamma(m), t_\delta(m))$ is injective and the image*

$\varphi_T(\mathcal{T}(T))$ *is described as follows:*

$$\{(a,b,c,d) \in \mathbb{R}^4 | a > 2, b > 2, c > 2, d > 2,$$
$$abc - a^2 - b^2 - c^2 + 2 = d\}.$$

2. *We uniformize a crown* $m \in \mathcal{T}(S)$ *by a Fuchsian group and denote the traces of elements representing a canonical triple* α, β, γ *and boundary geodesic* δ *by* $t_\alpha(m), t_\beta(m), t_\gamma(m)$ *and* $t_\delta(m)$. *Then the map* $\varphi_S : \mathcal{T}(S) \to \mathbb{R}^4$ *defined by* $\varphi_S(m) := (t_\alpha(m), t_\beta(m), t_\gamma(m), t_\delta(m))$ *is injective and the image* $\varphi_S(\mathcal{T}(S))$ *is described as follows:*

$$\{(p,q,r,s) \in \mathbb{R}^4 \mid p > 2, q > 2, r > 2, s > 2,$$
$$s^2 + 2(p+q+r+4)s + 4(p+q+r) + p^2 + q^2 + r^2 - pqr + 8 = 0\}.$$

Then by means of trace functions, we have the following "algebraic" bijection between $\mathcal{T}(T)$ and $\mathcal{T}(S)$:

Theorem 5. *There is a bijection from $\mathcal{T}(T)$ to $\mathcal{T}(S)$ which sends a cook-hat T with the lengths of a canonical triple and the boundary geodesic equal to (l_1, l_2, l_3, l_4) to a crown S with the lengths of a canonical triple and the boundary geodesic equal to $(2l_1, 2l_2, 2l_3, l_4)$.*

Proof. When we substitute (a^2-2, b^2-2, c^2-2, d) for (p, q, r, s), the equation $s^2 + 2(p+q+r+4)s + 4(p+q+r) + p^2 + q^2 + r^2 - pqr + 8$ factorizes as

$$\begin{aligned} & d^2 + 2(p+q+r+4)d + 4(p+q+r) + p^2 + q^2 + r^2 - pqr + 8 \\ = & (d - (abc - a^2 - b^2 - c^2 + 2))(d - (-abc - a^2 - b^2 - c^2 + 2)). \end{aligned}$$

Hence the map $\Psi : \varphi_T(\mathcal{T}(T)) \to \varphi_S(\mathcal{T}(S))$ defined by $\Psi(a, b, c, d) := (a^2 - 2, b^2 - 2, c^2 - 2, d)$ is bijective. Also the relation between trace functions and length functions

$$|t(\alpha)| = 2\cosh\frac{l(\alpha)}{2}$$

tells us the length relations between $m \in \mathcal{T}(T)$ and $\varphi_S^{-1} \circ \Psi \circ \varphi_T(m) \in \mathcal{T}(S)$. $\square$

Remark 1. *For the limiting case $l(\delta) = 0$, this bijection reduces to the well-known correspondence between punctured tori and forth-punctured spheres, which follows from the commensurability of uniformizing Fuchsian groups (see [ASWY]).*

Corollary 1. *For any crown with the boundary geodesic δ and a canonical triple (α, β, γ), their hyperbolic lengths $l(\alpha), l(\beta), l(\gamma)$ and $l(\delta)$ satisfy the following inequality:*

$$l(\alpha) + l(\beta) + l(\gamma) > 2l(\delta).$$

Next result is the counterpart of Theorem 2 and 3 for crowns.

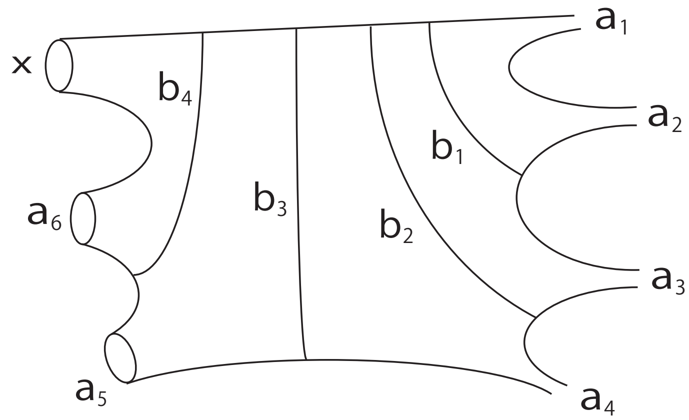
Corollary 2. *For a crown with a canonical triple (α, β, γ) and the boundary geodesic δ, the system of length functions $(l(\alpha), l(\beta), l(\gamma), l(\delta))$ gives a homogeneous coordinate of the Teichmüller space $\mathcal{T}(S)$ into $P(\mathbb{R}^4)$. The image of $\mathcal{T}(S)$ is the convex polyhedron in $\mathcal{P}(\mathbb{R}^4)$ defined by*

$$\{(a : b : c : d) \in P(\mathbb{R}^4) \mid a > 0,\ b > 0,\ c > 0,\ d > 0,$$
$$a < b + c,\ b < c + a,\ c < a + b,\ 2d < a + b + c\}.$$

Corollary 3. *The composition map* $\pi \circ l_* : T_{0,n} \to P(\mathbb{R}^{\mathcal{S}})$ *is also injective. (Hence we can talk about the Thurston compactification of* $T_{0,n}$ *without worrying now.)*

6 Projective embeddings of $T_{0,n,r}$ for $n \geqq 3$

Suppose M is a bordered hyperbolic Riemann surface of type $(0, n, r)$. with $n \geq 3$ and a_1, a_2, a_3 are punctures. Then the subsurface X_1 bounded by a_1, a_2, a_3 and b_2 is a thrice-punctured sphere with a hole, on which the triple $\{b_1, c_1, d_1\}$ mutually intersect exactly twice.

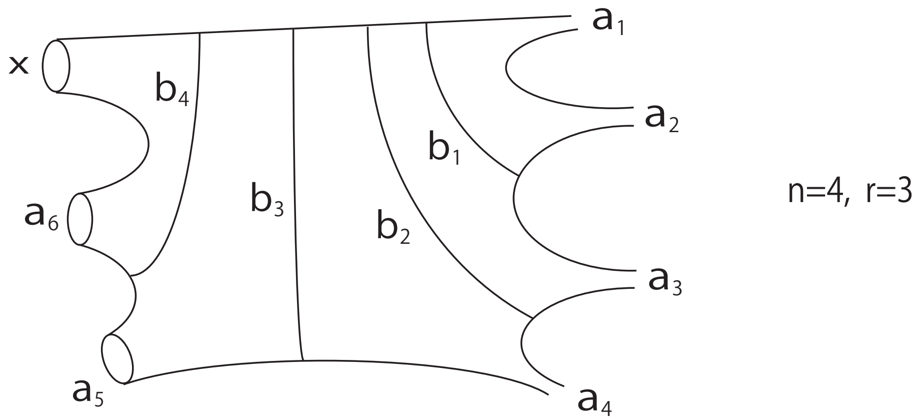


n=4, r=3

Therefore the hyperbolic lengths of $2n + 3r - 5$ geodesics

$$a_1, a_2, \cdots, a_{n+r-1}, b_1, c_1, c_2, c_{n+r-3},, d_1, d_2, d_{n+r-3}, \mathbf{b_2}$$

embeds $T_{0,n,r}$ into $P(\mathbb{R}^{2n+3r-5})$.



7 Problems

1. What is a polyhedral shape of $T_{g,n,r}$? How does the Thurston boundary of $T_{g,n,r}$ collapse when $T_{g,n,r}$ is realized as a projective polyhedron?

2. What is a geometric meaning of the "algebraic" correspondence between cook hats and crowns?

3. How about projective embeddings for $T_{0,n,r}$ with $n = 0, 1, 2$?

4. Also how about projective embeddings for $T_{g,n}$?
   Only cases of $(g, n) = (1, 1), (0, 4)$ and $(2, 0)$ are known.

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On quasiconformality and some properties of harmonic mappings in the unit disk

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Lewy's Theorem

Let $U \subset \mathbb{C}$ be a domain and $F : U \to \mathbb{C}$ be a harmonic mapping.

Then

F is locally injective in U

$$\Longleftrightarrow \mathrm{J(F)(z)} := |\partial \mathrm{F(z)}|^2 - |\bar{\partial} \mathrm{F(z)}|^2 \neq 0 \text{ for } \forall z \in U.$$

$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$

If $F : \mathbb{D} \to \mathbb{C}$ is a harmonic mapping, then F has the unique representation

$F = H + \overline{G}$, where H and G are holomorphic in $\mathbb{D}$ and $G(0) = 0$.

$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$
$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$

Radó-Kneser-Choquet Theorem

Let $\Omega \subset \mathbb{C}$ be a bounded convex domain whose boundary is a Jordan curve Γ. Let f be a homeomorphism of $\mathbb{T}$ onto Γ.

$$\Rightarrow \mathrm{P}(f)(z) := \frac{1}{2\pi} \int_{\mathbb{T}} f(u) \,\mathrm{Re}\, \frac{u+z}{u-z} |\,\mathrm{d}\, u| \ , \quad z \in \mathbb{D}$$

is an injective harmonic mapping of $\mathbb{D}$ onto Ω.

Theorem (Bshouty and Hengartner [BH1994, Th.2.7])

Let $\Omega \subset \mathbb{C}$ be a bounded Jordan domain,let $f : \mathbb{T} \to \partial\Omega$ be a homeomorphic surjection and $F := \mathrm{P}(f)$. Then F is injective in $\mathbb{D}$ if and only if $F(\mathbb{D}) = \Omega$.

Under the weaker assumption that $f : \mathbb{T} \to \Gamma$ is a continuous surjection such that $f(e^{\mathrm{i}\,t})$ runs once around Γ monotonically as $e^{\mathrm{i}\,t}$ runs around $\mathbb{T}$ (that is, for $\forall w \in \Gamma$, $f^{-1}(w)$ is connected), the same result holds. f is not required to be a homeomorphism ; f may have arcs of constancy. The Radó-Kneser-Choquet theorem has a partial converse that every injective harmonic mapping F of $\mathbb{D}$ onto a bounded strict convex domain can be extended continuously to the boundary. And on the boundary F is a continuous surjection such that $F(e^{\mathrm{i}\,t})$ runs once around Γ monotonically as $e^{\mathrm{i}\,t}$ runs around $\mathbb{T}$ (see [HS1986]). To say that a bounded convex domain is strictly convex means that its boundary contains no line segments. The boundary function need not be a homeomorphism; it can be constant on some arcs of the circle.

$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$

Lemma A (Schwarz's lemma for harmonic mappings).
Let $F : \mathbb{D} \to \mathbb{C}$ be a harmonic function with $F(0) = 0$ and $|F(z)| < 1$. Then $|F(z)| \leq \dfrac{4}{\pi} \arctan |z|$, and this inequality is sharp for each point z in $\mathbb{D}$. Furthermore, the bound is sharp everywhere (but is attained only at the origin) for injective harmonic mappings F of $\mathbb{D}$ onto itself with $F(0) = 0$. (Note that $|z| < \dfrac{4}{\pi} \arctan |z|$ for $0 < |z| < 1$.)

[Heinz 1959] showed the lemma by using Schwarz's lemma for analytic functions (see [Duren 2004]). In [Axler-Bourdon-Ramey 2001], $\mathbb{R}^n$ ($n \geq 2$) version of the lemma was shown, where the cases such that the equality holds for some nonzero z were determined.
Remark. In [Duren2004], an injective harmonic function defined in a domain Ω in $\mathbb{C}$ is said to be "simply" a harmonic mapping in Ω.

In 1959, **by using Lemma A**, E. Heinz proved the following.

Lemma B (Heinz's inequality for harmonic mappings ([Heinz 1959])).
Assume that F is a sense-preserving injective harmonic mapping of the unit disk $\mathbb{D}$ onto itself and normalized by $F(0) = 0$.

$$|\partial F(z)| \geq \frac{1}{\pi}$$

$$|\partial_x F(z)|^2 + |\partial_y F(z)|^2 \geq \frac{2}{\pi^2}\,, \quad z \in \mathbb{D}\,.$$

Let $f : \mathbb{T} \to \mathbb{C}$ be a function of bounded variation. Then we put

$$\begin{aligned} \dot{f}(z) &:= \frac{df}{d\theta}(e^{i\theta}) \quad \text{a.e.} \quad z = e^{i\theta} \in \mathbb{T}, \\ f'(z) &:= \lim_{u \to z} \frac{f(u) - f(z)}{u - z} \quad \text{a.e.} \quad z \in \mathbb{T}, \\ d_f &:= \operatorname*{ess\,inf}_{z \in \mathbb{T}} |\dot{f}(z)|, \\ e_f &:= \operatorname*{ess\,sup}_{z \in \mathbb{T}} |\dot{f}(z)| \ . \end{aligned}$$

For $h \in L^1(\mathbb{T})$ and $z \in \mathbb{T}$

$$\mathrm{A}(h)(z) := \frac{1}{2\pi} \lim_{r \to 1^-} \int_0^{2\pi} h(\mathrm{e}^{\mathrm{i}\,t}) \, \mathrm{Im} \, \frac{\mathrm{e}^{\mathrm{i}\,t} + rz}{\mathrm{e}^{\mathrm{i}\,t} - rz} \, \mathrm{d}\,t \ ,$$

whenever the limit exists and $\mathrm{A}(h)(z) := 0$ otherwise. It is known that for a.e. $z \in \mathbb{T}$ the limit exists and that

$$\mathrm{A}(h)(z) := \frac{1}{2\pi} \lim_{\varepsilon \to 0+} \int_{\varepsilon < |t-x| \le \pi} h(e^{it}) \cot \frac{x-t}{2} dt \ ,$$

$$\text{a.e.} \quad z = e^{ix} \in \mathbf{T} \ .$$

$$\mathrm{Hom}^{+}(\mathbb{T}) := \left\{ f : \mathbb{T} \to \mathbb{T} \;\middle|\; \begin{array}{l} \text{sense-preserving} \\ \text{homeomorphisms} \end{array} \right\}$$

Theorem C ([Pavlović 2002]).
Let $f \in \mathrm{Hom}^{+}(\mathbb{T})$ and $F = \mathrm{P}(f)$. Then

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

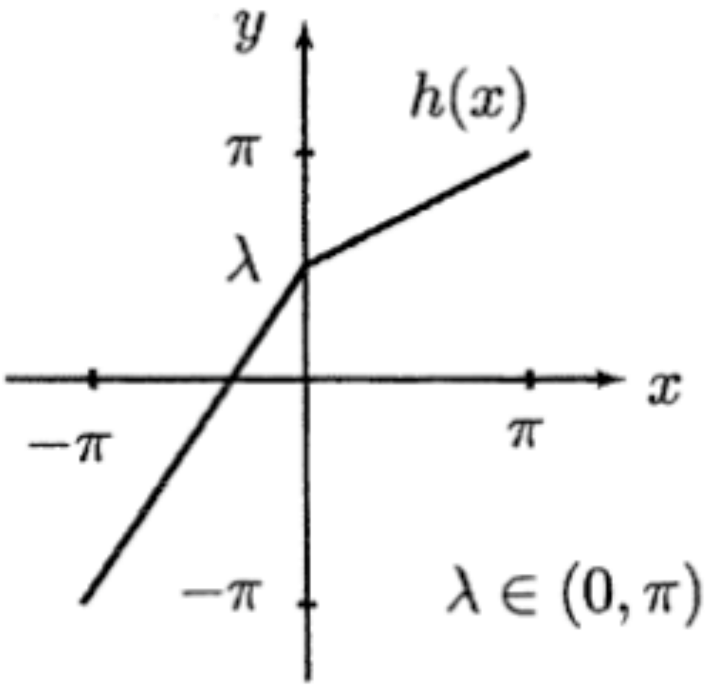
(i) F is a quasiconformal self-mapping of $\mathbb{D}$,

(ii) F is a bi-Lipschitz self-mapping of $\mathbb{D}$,

(iii) f is absolutely continuous on $\mathbb{T}$,

$$0 < d_f \leq e_f < \infty \quad \text{and } A(\dot{f}) \in L^{\infty}(\mathbb{T}).$$

Example



$$h : \mathbb{R} \to \mathbb{R},\ h(x + 2\pi) = h(x) + 2\pi$$

$$f(e^{ix}) = e^{ih(x)}$$

$P(f)$ is not quasiconformal.

The Schwarz lemma applies naturally to the theory of subordination. First recall that for any holomorphic functions F and G on the unit disk $\mathbb{D}$, F is *subordinated* to G on $\mathbb{D}$ — which is denoted by $F \prec G$ — provided $F = G \circ \omega$ for some holomorphic function $\omega : \mathbb{D} \to \mathbb{D}$ such that $\omega(0) = 0$. Such a function ω is usually called a Schwarz function.

Lemma 1 ([PS2006, Lemma 1.1]).
Given a harmonic function $u : \mathbb{D} \to \mathbb{R}$ let $\tilde{F}$ be a holomorphic function on $\mathbb{D}$ such that

$$\operatorname{Re} \tilde{F}(z) = u(z) \ , \quad z \in \mathbb{D} \ .$$

If $\tilde{F} \prec G$ for some holomorphic function G on $\mathbb{D}$, then

$$u(z) \leq \max\{\operatorname{Re} G(\zeta) : |\zeta| = |z|\} \ , \quad z \in \mathbb{D}.$$

In particular, the above estimate holds provided G is an injective holomorphic function on $\mathbb{D}$ such that

$$\tilde{F}(0) = G(0) \quad \text{and} \quad \tilde{F}(\mathbb{D}) \subset G(\mathbb{D}) \ .$$

Example D ([PS2006, Example 1.3]).

Let $F : \mathbb{D} \to \mathbb{D}$ be a harmonic function normalized by $F(0) = 0$. Then $\tilde{F} := \mathsf{Re}\, F + \mathsf{i}(\mathsf{Re}\, F)^\dagger$ is a holomorphic function on $\mathbb{D}$ satisfying

$$\mathsf{Re}\,\tilde{F}(z) = \mathsf{Re}\, F(z) \ , \quad z \in \mathbb{D} \ .$$

and

$$\tilde{F}(\mathbb{D}) \subset \{w \in \mathbb{C} : |\,\mathsf{Re}\, w| < 1\} \ .$$

It is easy to show that the function G defined by the formula

$$G(z) := \frac{2\,\mathsf{i}}{\pi} \log \frac{1+z}{1-z} \ , \quad z \in \mathbb{D} \ ,$$

maps conformally $\mathbb{D}$ onto the strip $\{w \in \mathbb{C} : |\,\mathsf{Re}\, w| < 1\}$. Then by

$$\tilde{F}(\mathbb{D}) \subset \{w \in \mathbb{C} : |\,\mathsf{Re}\, w| < 1\} \ .$$

, the condition

$$\tilde{F}(0) = G(0) \quad \text{and} \quad \tilde{F}(\mathbb{D}) \subset G(\mathbb{D}) \ .$$

holds. Thus Lemma 1 implies that

$$\begin{aligned}\operatorname{Re} F(z) &\le \max\left\{\operatorname{Re}\left(\frac{2\,\mathrm{i}}{\pi}\log\frac{1+\zeta}{1-\zeta}\right) : |\zeta| = |z|\right\}\\ &= \max\left\{-\frac{2}{\pi}\arctan\frac{2\operatorname{Im}\zeta}{1-|\zeta|^2} : |\zeta| = |z|\right\}\\ &\le \frac{2}{\pi}\arctan\frac{2|z|}{1-|z|^2} = \frac{4}{\pi}\arctan|z| \ , \qquad z \in \mathbb{D} \ .\end{aligned}$$

Since for each $a \in \mathbb{T}$ the function aF is also harmonic on $\mathbb{D}$ as well as $aF(\mathbb{D}) \subset \mathbb{D}$ and $aF(0) = 0$, we deduce from the above inequalities that

$$\text{(i)} \qquad |\mathrm{F}(\mathrm{z})| \le \frac{4}{\pi}\arctan|\mathrm{z}| \ , \qquad \mathrm{z} \in \mathbb{D} \ .$$

Moreover, the above estimate is sharp. The equality in (i) holds if $F = \operatorname{Re} G$ and if a point $z \in \mathbb{D}$ satisfies $\operatorname{Re} z = 0$.

By applying Lemma 1 to a linear function G we obtain the following.

Lemma 2 ([PS2006, Lemma 1.4]).
Suppose that F is a harmonic function on $\mathbb{D}$ which has a continuous extension F^* to the closure $\overline{\mathbb{D}}$. If $F(\mathbb{D}) \subset \mathbb{D}$ and $F(0) = 0$, then

$$|F(z)| \leq (1 + L_F)|z| \ , \quad z \in \mathbb{D} \ ,$$

provided

$$L_F := \frac{1}{\pi} \sup_{-\pi \leq \theta \leq \pi} \int_{-\pi}^{\pi} \left| \frac{F^*(\mathrm{e}^{\mathrm{i}(t+\theta)}) - F^*(\mathrm{e}^{\mathrm{i}\,\theta})\,\mathrm{e}^{\mathrm{i}\,t}}{\mathrm{e}^{\mathrm{i}\,t} - 1} \right| \mathrm{d}\,t < +\infty \ .$$

Theorem 1 ([PS2006, Theorem 1.5]).
If $K \geq 1$, , $F = \mathrm{P}(f)$ is the Poisson integral of some sense-preserving homeomorphism f of the unit circle which admits a K-quasiconformal extension $\tilde{F}$ of $\mathbb{D}$ satisfying $\tilde{F}(0) = 0$ and $F(0) = 0$, then

$$|F(z)| \leq \left(1 + \frac{8}{\pi}\int_0^{1/\sqrt{2}} \frac{\Phi_K(x) - \Phi_{1/K}(x)}{x\sqrt{1-x^2}}\,\mathrm{d}\,x\right)|z| \ , \quad z \in \mathbb{D} \ ,$$

where Φ_K denotes the Hersch-Pfluger distortion function defined for any $K > 0$ by the equalities

$$\Phi_K(r) := \mu^{-1}(\mu(r)/K) \ , \quad 0 < r < 1 \ ; \quad \Phi_K(0) := 0\,, \Phi_K(1) := 1 \ ,$$

where μ stands for the module of the Grötzsch extremal domain $\mathbb{D}\setminus[0;r]$.

Note that

$$\frac{8}{\pi}\int_0^{1/\sqrt{2}} \frac{\Phi_K(x) - \Phi_{1/K}(x)}{x\sqrt{1-x^2}}\,\mathrm{d}\,x \leq \frac{32\sqrt{2}}{\pi}(K2^{-5/(2K)} - \frac{1}{K}2^{-5K/2}) \ , \quad z \in \mathbb{D} \ .$$

Theorem 2 ([PS2006, Theorem 2.1]).
If $K \geq 1$ and $F = \mathsf{P}(f)$ is the Poisson integral of some sense-preserving homeomorphism f of the unit circle which admits a K-quasiconformal extension $\tilde{F}$ of $\mathbb{D}$ satisfying $\tilde{F}(0) = 0$, then

$$|F(z)| \leq \mathsf{P}[\Psi_K](|z|) \ , \quad z \in \mathbb{D} \ ,$$

where

$$\Psi_K(\mathrm{e}^{\mathrm{i}\,t}) := \begin{cases} 2\Phi_K\left(\cos\frac{t}{2}\right)^2 - 1 & , \ 0 \leq |t| \leq \frac{\pi}{2} \ , \\ 2\Phi_{1/K}\left(\cos\frac{t}{2}\right)^2 + 4\Phi_K\left(\frac{1}{\sqrt{2}}\right)^2 - 3 & , \ \frac{\pi}{2} \leq |t| \leq \pi \ . \end{cases}$$

Note that

$$\mathsf{P}[\Psi_1](|z|) = |z| \ , \quad z \in \mathbb{D} \ .$$

Theorem 3 ([PS2006, Theorem 3.3]).
If $K \geq 1$, $F = \mathrm{P}(f)$ is the Poisson integral of some sense-preserving homeomorphism f of the unit circle which admits a K-quasiconformal extension $\tilde{F}$ of $\mathbb{D}$ satisfying $\tilde{F}(0) = 0$ and $F(0) = 0$, then for every $z \in \mathbb{D}$,

$$\begin{aligned}
&|F(z)| \leq \Lambda_K(|z|) \\
&:= \frac{4}{\pi} \arctan |z| - \frac{8|z|(1-|z|^2)}{\pi} \int_0^{\pi/2} \frac{(\cos t)\Phi_{1/K}(\sin(t/2))^2}{(1+|z|^2)^2 - 4|z|^2(\cos t)^2} \,\mathrm{d}\, t \\
&= |z| + \frac{8|z|(1-|z|^2)}{\pi} \int_0^{\pi/2} \frac{(\cos t)\left[(\sin(t/2))^2 - \Phi_{1/K}(\sin(t/2))^2\right]}{(1+|z|^2)^2 - 4|z|^2(\cos t)^2} \,\mathrm{d}\, t.
\end{aligned}$$

Moreover, $\Lambda_1(|z|) = |z|$ and the following equalities hold:

$$\lim_{K \to 1^+} \limsup_{r \to 0^+} \frac{\Lambda_K(r)}{r} = 1 \quad , \qquad \lim_{K \to 1^+} \liminf_{r \to 1^-} \frac{1 - \Lambda_K(r)}{1 - r} = 1 \ .$$

Lemma B (Heinz's inequality for harmonic mappings ([Heinz 1959])).

Assume that F is a sense-preserving injective harmonic mapping of the unit disk $\mathbb{D}$ onto itself and normalized by $F(0) = 0$.

$$|\partial F(z)| \geq \frac{1}{\pi}$$

$$|\partial_x F(z)|^2 + |\partial_y F(z)|^2 \geq \frac{2}{\pi^2}\,, \quad z \in \mathbb{D}\,.$$

Theorem C ([Pavlović 2002]).

Let $f \in \mathrm{Hom}^+(\mathbb{T})$ and $F = \mathrm{P}(f)$. Then

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

(i) F is a quasiconformal self-mapping of $\mathbb{D}$,

(ii) F is a bi-Lipschitz self-mapping of $\mathbb{D}$,

(iii) f is absolutely continuous on $\mathbb{T}$,

$0 < d_f \leq e_f < \infty$ and $A(\dot{f}) \in L^\infty(\mathbb{T})$.

Theorem 4 ([PS2002, Theorem 0.4]).
If $f \in \mathrm{Hom}^+(\mathbb{T})$ and if $F = \mathrm{P}(f)$ satisfies $F(0) = 0$, then

$$\inf_{z\in\mathbb{D}} |\partial F(z)|^2 \geq \frac{1}{\pi^2} + \frac{1}{4}d_f^2 + \frac{1}{4}\max\{d_f \ , \ 2d_f^3\}$$

and

$$\inf_{z\in\mathbb{D}} (|\partial_x F(z)|^2 + |\partial_y F(z)|^2) \geq \frac{2}{\pi^2} + \frac{1}{2}d_f^2 + \frac{1}{2}\max\{d_f \ , \ 2d_f^3\}.$$

Recall that Φ_K denotes the Hersch-Pfluger distortion function defined for any $K > 0$ by the equalities

$$\Phi_K(r) := \mu^{-1}(\mu(r)/K)\ , \quad 0 < r < 1\ ; \quad \Phi_K(0) := 0\,,\, \Phi_K(1) := 1\ ,$$

where μ stands for the module of the Grötzsch extremal domain $\mathbb{D}\backslash[0;r]$.
For $K \geq 1$ let

$$L_K^* := \frac{2}{\pi}\int_0^{\Phi_{1/K}(1/\sqrt{2})^2} \frac{\mathrm{d}\,t}{\Phi_K\left(\sqrt{t}\right)\Phi_{1/K}\left(\sqrt{1-t}\right)}\ .$$

Then

Lemma 3 ([PS2005, Lemma 1.4]).

$$L_K^* = \frac{2}{\pi}\int_0^{1/\sqrt{2}} \frac{\mathrm{d}\,\Phi_{1/K}(s)^2}{s\sqrt{1-s^2}}$$
$$= \frac{4}{\pi}\Phi_{1/K}\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{2}{\pi}\int_0^{1/\sqrt{2}} \frac{1-2s^2}{s^2(1-s^2)^{3/2}}\Phi_{1/K}(s)^2\,\mathrm{d}\,s\ .$$

Moreover, L_K^* is a strictly decreasing function of $K \geq 1$ such that

$$\lim_{K\to 1} L_K^* = L_1 = 1 \quad \text{and} \quad \lim_{K\to+\infty} L_K^* = 0$$

as well as

$$|L_{K_2}^* - L_{K_1}^*| \leq L|K_2 - K_1|\ , \quad K_1, K_2 \geq 1\ ,$$

where

$$L := \frac{4}{\pi}(1 + 65\ln 2)\ .$$

Theorem 5 ([PS2005, Theorem 2.1]). Given $K \geq 1$ let F be a K-quasiconformal and harmonic self-mapping of $\mathbb{D}$ satisfying $F(0) = 0$. If f is the boundary limiting valued function of F, then

$$d_f \geq \frac{1}{K} \max\left\{\frac{2}{\pi}, L_K^*\right\} \ .$$

Moreover, the right hand side in the above inequality is a decreasing and continuous function of $K \geq 1$ with values in $(0; 1]$.

Theorem 6 ([PS2005, Theorem 2.2]).
Given $K \geq 1$ let F be a K-quasiconformal and harmonic self-mapping of $\mathbb{D}$ satisfying $F(0) = 0$. Then the inequalities

$$|\partial F(z)| \geq \frac{K+1}{2K} \max\left\{\frac{2}{\pi}, L_K^*\right\}$$

and

$$|\partial_x F(z)|^2 + |\partial_y F(z)|^2 \geq \frac{1}{2}\left(1 + \frac{1}{K}\right)^2 \max\left\{\frac{4}{\pi^2}, {L_K^*}^2\right\}$$

hold for every $z \in \mathbb{D}$. Moreover, the right hand sides in the above inequalities are decreasing and continuous functions of $K \geq 1$ with values in $(1/\pi; 1]$ and $(2/\pi^2; 2]$, respectively.

Theorem 7 ([PS2007, Theorem 1.2]).
If $f \in \mathsf{Hom}^{+}(\mathbb{T})$ is absolutely continuous on $\mathbb{T}$, then for a.e. $z \in \mathbb{T}$, the following limits exist and

$$2\,\mathsf{C}_{\mathbb{T}}[f'](z) = \overline{z} f(z)\Big(\mathsf{V}[f](z) + \mathsf{i}\,\mathsf{V}^{*}[f](z)\Big) \ ,$$

$$\mathsf{C}_{\mathbb{T}}[f](z) := \mathsf{PV}\,\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(u)}{u-z}\,\mathsf{d}\,u := \lim_{\varepsilon \to 0+} \frac{1}{2\pi i}\int_{\mathbb{T}\setminus\mathbb{T}(z,\varepsilon)} \frac{f(u)}{u-z}\,\mathsf{d}\,u$$

$$\mathsf{V}[f](z) := \lim_{\varepsilon \to 0+} \frac{1}{2\pi}\int_{\mathbb{T}\setminus\mathbb{T}(z,\varepsilon)} \frac{|f(u)-f(z)|^2}{|u-z|^2}|\,\mathsf{d}\,u| \ ,$$

$$\mathsf{V}^{*}[f](z) := -\lim_{\varepsilon \to 0+} \frac{1}{\pi}\int_{\mathbb{T}\setminus\mathbb{T}(z,\varepsilon)} \frac{\mathsf{Im}[f(u)\overline{f(z)}]}{|u-z|^2}|\,\mathsf{d}\,u| \ .$$

Lemma 4 ([PS2007, Lemma 1.3]).
For every $K \geq 1$ the following inequalities hold:

$$1 \leq M_K := \frac{4}{\pi}\int_0^{1/\sqrt{2}} \left(\frac{\Phi_K(r)}{r}\right)^{1+1/K} \frac{\mathrm{d}\,r}{\sqrt{1-r^2}} \leq K^2 2^{5(1-1/K^2)/2}$$

and

$$1 \geq L_K := \frac{4}{\pi}\int_0^{1/\sqrt{2}} \left(\frac{\Phi_{1/K}(r)}{r}\right)^{1+1/K} \frac{\mathrm{d}\,r}{\sqrt{1-r^2}} \geq \frac{K 2^{5(1-K^2)/(2K)}}{K^2+K-1} \ .$$

In particular, $L_K \to 1$ and $M_K \to 1$ as $K \to 1^+$.

Theorem 8 ([PS2007, Lemma 1.4]).

$G : \mathbb{D} \to \mathbb{D}$ $\quad K$-qc, $\quad f = G|_{\mathbb{T}}$, $G(0) = 0$

$$\Rightarrow L_K(f^-(z))^{1-1/K} \leq \mathsf{V}[f](z) \leq M_K(f^+(z))^{1-1/K} \ , \quad z \in \mathbb{T} \ .$$

where

$$f^+(z) := \sup_{u \in \mathbb{T} \setminus \{z\}} \left| \frac{f(u) - f(z)}{u - z} \right| \in [0, +\infty] \ ,$$

$$f^-(z) := \inf_{u \in \mathbb{T} \setminus \{z\}} \left| \frac{f(u) - f(z)}{u - z} \right| \in [0, +\infty) \ .$$

Lemma 5 ([PS2007, Lemma 1.5]).
Suppose that $f \in \mathrm{Hom}^+(\mathbb{T})$ is absolutely continuous on $\mathbb{T}$.

$$\Rightarrow \sup_{z\in\mathbb{T}} f^+(z) = e_f := \operatorname*{ess\,sup}_{z\in\mathbb{T}} |f'(z)|$$

$$\inf_{z\in\mathbb{T}} f^-(z) = d_f$$

Lemma 6 ([PS2007, Corollary 1.6]).
$G : \mathbb{D} \to \mathbb{D}$ $\quad K$-qc, $f = G|_{\mathbb{T}}$, $G(0) = 0$, f is absolutely continuous on $\mathbb{T}$

$$\Rightarrow L_K d_f^{1-1/K} \le \mathsf{V}[f](z) = 2\,\mathrm{Re}\left[z\overline{f(z)}\,\mathsf{C}_{\mathbb{T}}[f'](z)\right] \le M_K e_f^{1-1/K}$$

for a.e. $z \in \mathbb{T}$.

Lemma 7 ([PS2007, Lemma 2.1).

$F : \mathbb{D} \to \Omega(\subset \mathbb{C})$ K-qc, harmonic, $\Gamma := \partial\Omega$ a rectifiable Jordan curve

$$\Rightarrow \sup_{0<r<1} \int_{\mathbb{T}_r} |\partial F(z)||\,\mathrm{d}\,z| \leq \frac{K+1}{2}|\Gamma|_1$$

$$\sup_{0<r<1} \int_{\mathbb{T}_r} |\bar{\partial} F(z)||\,\mathrm{d}\,z| \leq \frac{K-1}{2}|\Gamma|_1$$

where $\mathbb{T}_r := \{z \in \mathbb{C} : |z| = r\}$, $|\Gamma|_1$ the length of Γ. In particular, ∂F, $\overline{\bar{\partial} F} \in \mathrm{H}^1(\mathbb{D})$.

Corollary 1 ([PS2007, Corollary 2.2).

Under the hypotheses of Lemma 7

$f = F|_{\mathbb{T}}$ is absolutely continuous.

Lemma 8 ([PS2007, Lemma 2.3). Under the hypotheses of Lemma 7

$$|f(u) - f(v)| \leq L|u - v| \ , \quad u, v \in \mathbb{T} \ ,$$

for some $L > 0$ where $f = F|_{\mathbb{T}}$

$$\Rightarrow \sup_{z \in \mathbb{D}} |\partial F(z)| \leq \frac{K+1}{2} L$$

$$\sup_{z \in \mathbb{D}} |\bar{\partial} F(z)| \leq \frac{K-1}{2} L \ .$$

In particular, ∂F, $\overline{\bar{\partial} F} \in \mathsf{H}^\infty(\mathbb{D})$.

The bi-Lipschitz property for quasiconformal harmonic self-mappings of the unit disk

Theorem 9 ([PS2007, Theorem 3.1).
$f \in \mathsf{Hom}^+(\mathbb{T}), F = \mathsf{P}(f)$ $\quad K$-qc
$\Rightarrow$ for a.e. $z \in \mathbb{T}$,

$$\left| \mathsf{V}[f](z) + \mathrm{i}\,\mathsf{V}^*[f](z) - \frac{1}{2}(K + \frac{1}{K})|f'(z)| \right| \le \frac{1}{2}(K - \frac{1}{K})|f'(z)| \ .$$

In particular, for a.e. $z \in \mathbb{T}$,

$$\frac{1}{K}|f'(z)| \le \mathsf{V}[f](z) \le K|f'(z)|, \quad |\mathsf{V}^*[f](z)| \le \frac{1}{2}(K - \frac{1}{K})|f'(z)| \ .$$

Theorem 10 ([PS2007, Theorem 3.2).

$f \in \mathsf{Hom}^+(\mathbb{T}), F = \mathsf{P}(f) \qquad K\text{-qc},\ F(0) = 0$

$\Rightarrow$ for a.e. $z \in \mathbb{T}$,

$$\frac{2^{5(1-K^2)/2}}{(K^2+K-1)^K} \leq (L_K/K)^K$$
$$\leq |f'(z)| \leq (M_K K)^K \leq K^{3K} 2^{5(K-1/K)/2}\ .$$

Theorem 11 ([PS2007, Theorem 3.3).

$F : \mathbb{D} \to \mathbb{D}$ K-qc, harmonic, $F(0) = 0$

$\Rightarrow$ for all $z, w \in \mathbb{D}$,

$$|F(z) - F(w)| \le K(M_K K)^K |z - w|$$
$$\le K^{3K+1} 2^{5(K-1/K)/2} |z - w| \ ,$$
$$|F(z) - F(w)| \ge \frac{L_K^{3K}}{K^{4K+1} M_K^K} |z - w|$$
$$\ge \frac{2^{5(1-K^2)(3+1/K)/2}}{K^{3K+1}(K^2 + K - 1)^{3K}} |z - w| \ .$$

Theorem 12 ([PS2007, Theorem 3.3).

$F : \mathbb{D} \to \mathbb{D}$ $\quad$ K-qc, harmonic, $F(0) = 0$

$\Rightarrow$ for all $z, w \in \mathbb{D}$,

$$|F(z) - F(w)| \geq \frac{1}{K} \max\{\frac{2}{\pi}, L_K^*\}|z - w|$$

Set $\mathbb{D}(R) := \mathbb{D}(0, R) := \{z \in \mathbb{C} : |z| < R\}$ for $R > 0$.

($\mathbb{D}(a, R) := \{z \in \mathbb{C} : |z - a| < R\}$ for $a \in \mathbb{C}$ and $R > 0$.)

Assume that F is a sense-preserving injective harmonic mapping of the unit disk $\mathbb{D} := \mathbb{D}(1)$ onto itself and normalized by $F(0) = 0$.

(Recall that in 1959,) E. Heinz proved that

$$|\partial F(z)| \geq \frac{1}{\pi}$$

$$|\partial_x F(z)|^2 + |\partial_y F(z)|^2 \geq \frac{2}{\pi^2}\ , \qquad z \in \mathbb{D}\ .$$

Theorem E ([Kalaj 2003]).

If F is a sense-preserving injective harmonic mapping of $\mathbb{D}$ onto a convex domain Ω satisfying $\Omega \supset \mathbb{D}(R_1)$ and $F(0) = 0$, then

$$|\partial_x F(z)|^2 + |\partial_y F(z)|^2 \geq \frac{1}{8} R_1^2\ , \qquad z \in \mathbb{D}\ .$$

Lemma 9 ([PS2009, Lemma 1.1]).
Given a harmonic function $u : \mathbb{D} \to \mathbb{R}$ and $a, b \in \mathbb{R}$ satisfying $a < b$, assume that $u(0) = 0$ and

$$a \le u(z) \le b \ , \qquad z \in \mathbb{D} \ .$$

Then

$$u(z) \le 2\frac{b-a}{\pi}\,\mathrm{arctan}\,\frac{|z|+|p|}{1+|p||z|} + \frac{b+a}{2} \ , \qquad z \in \mathbb{D} \ ,$$

where

$$p := -\mathrm{i}\tan\frac{\pi}{4}\frac{b+a}{b-a} \ .$$

By using Lemma 9, we showed the following Theorem 13 and Theorem 14.

Theorem 13 (Partyka-Sakan [PS2009,Theorem 2.2]).
If F is a sense-preserving injective harmonic mapping of $\mathbb{D}$ onto a bounded convex domain Ω including 0 and $F(0) = 0$, then for all $R_1, R_2 > 0$ satisfying $\mathbb{D}(R_1) \subset \Omega \subset \mathbb{D}(R_2)$ the following inequalities hold

$$|\partial F(w)| \geq \frac{R_1 + R_2}{2\pi} \tan\left(\frac{\pi}{2}\frac{R_1}{R_1 + R_2}\right) \ , \quad w \in \mathbb{D} \ ,$$

as well as

$$|\partial_x F(w)|^2 + |\partial_y F(w)|^2 \geq 2\left(\frac{R_1 + R_2}{2\pi} \tan\left(\frac{\pi}{2}\frac{R_1}{R_1 + R_2}\right)\right)^2 \ , \quad w \in \mathbb{D} \ .$$

Let $f:\mathbb{T}\to\mathbb{C}$ be a function of bounded variation. Then (recall that) we put

$$\begin{aligned}\dot f(z) &:= \frac{df}{d\theta}(e^{i\theta}) \quad \text{a.e.} \quad z=e^{i\theta}\in\mathbb{T},\\ f'(z) &:= \lim_{u\to z}\frac{f(u)-f(z)}{u-z} \quad \text{a.e.} \quad z\in\mathbb{T}.\\ d_f &:= \operatorname*{ess\,inf}_{z\in\mathbb{T}}|\dot f(z)|,\\ e_f &:= \operatorname*{ess\,sup}_{z\in\mathbb{T}}|\dot f(z)|\ .\end{aligned}$$

A function $f:\mathbb{T}\to\mathbb{C}$ is called *Dini-smooth* if f is differentiable on $\mathbb{T}$ and the derivative $\dot f$ is not vanishing, and *Dini-continuous* on $\mathbb{T}$, i.e. its modulus of continuity

$$\omega(\delta) := \sup\{|\dot f(\mathrm{e}^{\mathrm{i}\,t})-\dot f(\mathrm{e}^{\mathrm{i}\,s})| : t,s\in\mathbb{R}\,,\ |t-s|\le\delta\}\ , \quad \delta\in[0;2\pi]\ ,$$

satisfies the following condition

$$\int_0^{2\pi}\frac{\omega(t)}{t}\,\mathrm{d}\,t<+\infty\ .$$

Theorem 14 ([PS2009, Theorem 2.1]).

Given a Dini-smooth function $f : \mathbb{T} \to \mathbb{C}$ assume that $F := \mathrm{P}(f)$ is a sense-preserving injective harmonic mapping of $\mathbb{D}$ onto a convex domain Ω including 0 such that $F(0) = 0$. Then the following inequalities

$$|\partial F(\zeta)| \geq \frac{R_1 + R_2}{2\pi} \tan\left(\frac{\pi}{2}\frac{R_1}{R_1 + R_2}\right) + \frac{1}{2}\min_{z\in\mathbb{T}} |f'(z)|$$

as well as

$$|\partial_x F(\zeta)|^2 + |\partial_y F(\zeta)|^2 \geq 2\left(\frac{R_1 + R_2}{2\pi} \tan\left(\frac{\pi}{2}\frac{R_1}{R_1 + R_2}\right) + \frac{1}{2}\min_{z\in\mathbb{T}} |f'(z)|\right)^2$$

hold for every $\zeta \in \mathbb{D}$ and all $R_1, R_2 > 0$ satisfying $\mathbb{D}(0, R_1) \subset \Omega \subset \mathbb{D}(0, R_2)$.

In [PS2014b], Theorem 14 was generalized into the following Theorem 15. By using Theorem 13 above and Theorem F and Lemma G below we obtained Theorem 15.

Theorem 15 ([PS2014b, Th.4.4]).

Given a function $f : \mathbb{T} \to \mathbb{C}$ of bounded variation assume that $F := \mathsf{P}(f)$ is a sense-preserving injective harmonic mapping of $\mathbb{D}$ onto a bounded convex domain Ω, then

$$|\partial F(\zeta)| \geq \frac{R_1 + R_2}{2\pi} \tan\left(\frac{\pi}{2}\frac{R_1}{R_1 + R_2}\right) + \frac{1}{2}d_f \geq \frac{R_1}{4} + \frac{1}{2}d_f \ , \quad \zeta \in \mathbb{D} \ ,$$

for all $R_1, R_2 > 0$ satisfying

$$\mathbb{D}(F(0), R_1) \subset \Omega \subset \mathbb{D}(F(0), R_2) \ .$$

If $f : \mathbb{T} \to \mathbb{C}$ is a function of bounded variation, then we write

$$\mathsf{P}(\mathsf{d}\, f)(\zeta) := \frac{1}{2\pi} \int_{\mathbb{T}} \mathsf{Re}\, \frac{u+\zeta}{u-\zeta}\, \mathsf{d}\, f(u) \ , \quad \zeta \in \mathbb{D} \ .$$

$$\mathsf{A}(\mathsf{d}\, f)(z) := \frac{1}{2\pi} \lim_{r \to 1^-} \int_0^{2\pi} \mathsf{Im}\, \frac{\mathsf{e}^{\mathsf{i}\, t} + rz}{\mathsf{e}^{\mathsf{i}\, t} - rz}\, \mathsf{d}\, f(\mathsf{e}^{\mathsf{i}\, t}) \ ,$$

whenever the limit exists and $\mathsf{A}(\mathsf{d}\, f)(z) := 0$ otherwise. Note that

$$\partial_\theta\, \mathsf{P}(f)(\zeta) = \mathsf{P}(\mathsf{d}\, f)(\zeta) \ , \quad \zeta = re^{\mathsf{i}\,\theta} \in \mathbb{D}$$

$$\partial_\theta\, \mathsf{P}(f)(\zeta) + \mathsf{i}\, r\partial_r\, \mathsf{P}(f)(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathsf{e}^{\mathsf{i}\, t} + \zeta}{\mathsf{e}^{\mathsf{i}\, t} - \zeta}\, \mathsf{d}\, f(\mathsf{e}^{\mathsf{i}\, t}) \ , \quad \zeta = re^{\mathsf{i}\,\theta} \in \mathbb{D} \ .$$

Note that

$$\mathsf{A}(\mathsf{d}\, f)(z) = \mathsf{A}(\dot{f})(z) \ , \quad z \in \mathbb{T} \ ,$$

provided f is an absolutely continuous function.

Theorem F ([PS2014b, Corollary 2.4]).
Given a function $f : \mathbb{T} \to \mathbb{C}$ of bounded variation assume that $F := \mathsf{P}(f)$ is a locally injective mapping in $\mathbb{D}$, $\mathsf{J(F)(0)} > 0$ and

$$\sup_{0<r<1} \int_0^{2\pi} |\partial F(r\,\mathrm{e}^{\mathrm{i}\,\theta})|^{-p}\,\mathrm{d}\,\theta < +\infty$$

for a certain $p > 0$, where

$$\mathsf{J(F)(z)} := |\partial \mathsf{F(z)}|^2 - |\bar\partial \mathsf{F(z)}|^2 \ , \quad \mathsf{z} \in \mathbb{D} \ .$$

Then for every $\zeta \in \mathbb{D}$,

$$\begin{aligned} |\partial F(\zeta)| &\geq \frac{1}{2} \operatorname*{ess\,inf}_{z\in\mathbb{T}} \Big(|\,\mathsf{A}(\mathrm{d}\,f)(z)|^2 + |f'(z)|^2 + 2\,\mathrm{Re}(\overline{z f'(z)}\,\mathsf{A}(\mathrm{d}\,f)(z)) \Big)^{1/2} \\ &\geq \frac{1}{2} \operatorname*{ess\,inf}_{z\in\mathbb{T}} \Big(|\,\mathsf{A}(\mathrm{d}\,f)(z)|^2 + |f'(z)|^2 \Big)^{1/2} \\ &\geq \frac{1}{2} \operatorname*{ess\,inf}_{z\in\mathbb{T}} |f'(z)| \ . \end{aligned}$$

Lemma G ([PS2014b, Lemma 4.3]).

Let $f : \mathbb{T} \to \mathbb{C}$ be a function of bounded variation and differentiable at a point $z \in \mathbb{T}$ such that $\mathsf{P}(f)(0) = 0 \neq \mathsf{J}(\mathsf{P}(\mathsf{f}))(0)$, the limit $\lim_{r\to 1^-} \frac{\mathrm{d}}{\mathrm{d}\,r} \mathsf{P}(f)(rz)$ exists and $\liminf_{r\to 1^-} \mathsf{J}(\mathsf{P}(\mathsf{f}))(\mathsf{rz}) \geq 0$. holds.

If there exists $\xi \in \mathbb{T}$ satisfying

$$\mathrm{Re}(\xi\, \mathsf{P}(f)(u)) \leq \mathrm{Re}(\xi f(z)) \ , \quad u \in \mathbb{D} \ ,$$

then

$$|\,\mathsf{A}(\mathrm{d}\, f)(z)| \geq \frac{a+b}{\pi} \tan\left(\frac{\pi}{2}\frac{\min(a,b)}{a+b}\right) \geq \frac{\min(a,b)}{2}$$

and

$$\mathrm{Re}(\overline{z f'(z)}\, \mathsf{A}(\mathrm{d}\, f)(z)) \geq |f'(z)| \frac{a+b}{\pi} \tan\left(\frac{\pi}{2}\frac{\min(a,b)}{a+b}\right) \geq |f'(z)| \frac{\min(a,b)}{2} \ ,$$

Let us recall that a function $f : \mathbb{T} \to \mathbb{C}$ is said to be *Hölder-smooth* if f is differentiable on $\mathbb{T}$ and the derivative $\dot{f}$ is not vanishing and *Hölder-continuous* on $\mathbb{T}$, i.e., there exist $L \geq 0$ and $\alpha \in (0;1]$ such that

$$|\dot{f}(\mathrm{e}^{\mathrm{i}\,t}) - \dot{f}(\mathrm{e}^{\mathrm{i}\,s})| \leq L|t-s|^{\alpha} \ , \quad t, s \in \mathbb{R} \ .$$

Theorem H.
Suppose that Ω is a Jordan domain in $\mathbb{C}$ and that there exists a Hölder-smooth and injective function $h : \mathbb{T} \to \mathbb{C}$ such that $h(\mathbb{T})$ is the boundary curve of a Ω. Let $F : \mathbb{D} \to \Omega$ be a sense-preserving injective (onto) harmonic mapping.
(i) (Kalaj '08) If Ω is convex, then
F is quasiconformal $\Longleftrightarrow \quad F$ is bi-Lipschitz
(ii) (Božin and Mateljević, to appear ?)
F is quasiconformal $\Longleftrightarrow \quad F$ is bi-Lipschitz

Theorem I ([Clunie and Sheil-Small 1984,Cor.5.8]).
If $F = H + \overline{G}$ is a sense-preserving injective harmonic mapping of $\mathbb{D}$ onto a convex domain, then

$$|G(z_2) - G(z_1)| < |H(z_2) - H(z_1)| \ , \quad z_1, z_2 \in \mathbb{D}\, , \, z_1 \neq z_2 \ ,$$

where H and G are holomorphic mappings in $\mathbb{D}$ satisfying $G(0) = 0$.

Theorem 16 ([Partyka-Sakan [PS2012, Th.3.8]).

Let F be a sense-preserving injective harmonic mapping of $\mathbb{D}$ onto a convex domain in $\mathbb{C}$.Then the following five conditions are equivalent to each other:

(i) F is a quasiconformal mapping;

(ii) $\exists$ a constant L_1 such that $1 \le L_1 < 2$ and
$$|F(z_2) - F(z_1)| \le L_1|H(z_2) - H(z_1)| \ , \quad z_1, z_2 \in \mathbb{D} \ ;$$

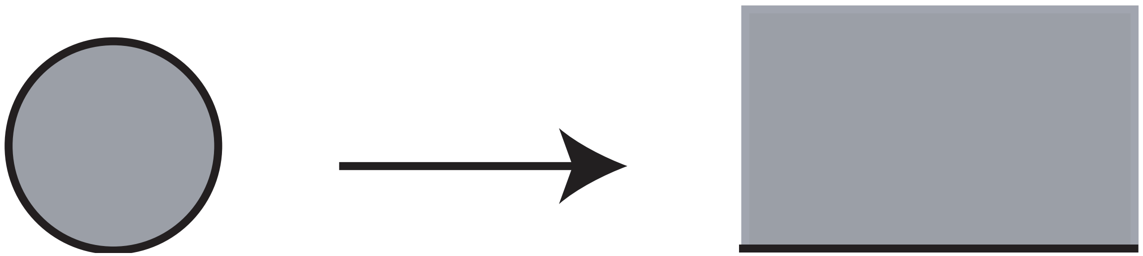
(iii) $\exists$ a constant l_1 such that $0 \le l_1 < 1$ and
$$|G(z_2) - G(z_1)| \le l_1|H(z_2) - H(z_1)| \ , \quad z_1, z_2 \in \mathbb{D} \ ;$$

(iv) $\exists$ a constant $L_2 \ge 1$ such that
$$|H(z_2) - H(z_1)| \le L_2|F(z_2) - F(z_1)| \ , \quad z_1, z_2 \in \mathbb{D} \ ;$$

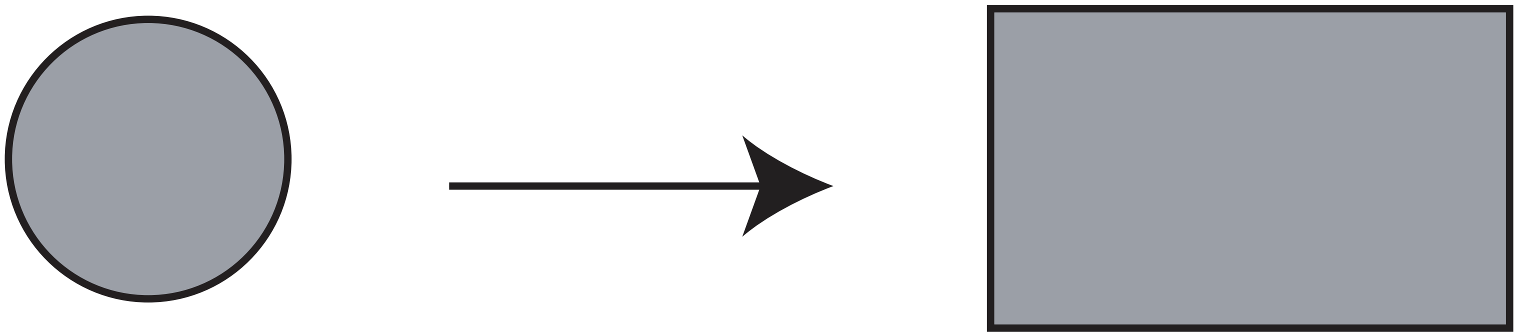
(v) $H \circ F^{-1}$ and $F \circ H^{-1}$ are bi-Lipschtz mappings.

Moreover, (ii) $\Rightarrow \|\mu_{\mathsf{F}}\|_\infty \le \mathsf{L}_1 - 1$, (iii) $\Rightarrow \|\mu_{\mathsf{F}}\|_\infty \le l_1$,

(iv) $\Rightarrow \|\mu_{\mathsf{F}}\|_\infty \le 1 - \frac{1}{L_2}$, where
$$\mu_F(z) := \frac{\bar{\partial} F(z)}{\partial F(z)} = \frac{\overline{G'(z)}}{H'(z)} \ , \quad z \in \mathbb{D}.$$



$H : \mathbb{D} \to \{z \in \mathbb{C} |\ \text{Im} z > 0\}, \quad H(z) = \frac{i(z+1)}{1-z}$, conformal, not Lipschitz



the Schwarz-Christoffel mapping $H(z)$, conformal, not Lipschitz

$F(z) = H(z) + t\overline{H(z)}, \quad |t| < 1,$ quasiconformal, not Lipschitz

Theorem 17 ([Partyka-Sakan [PS2014a, Th.3.4]).

Let $F : \mathbb{D} \to \mathbb{C}$ be a sense-preserving injective harmonic mapping such that $F(\mathbb{D})$ is a bounded convex domain in $\mathbb{C}$. Then the following conditions are equivalent to each other:

(i) F is a quasiconformal and Lipschitz mapping;

(ii) F is a quasiconformal mapping and its boundary limiting valued function f is a Lipschitz mapping;

(iii) F is a quasiconformal mapping and a holomorphic part H of F is a bi-Lipschitz mapping;

(iv) F is a bi-Lipschitz mapping;

(v) F has a continuous extension to the closure $\overline{\mathbb{D}}$ and its boundary limiting valued function f is absolutely continuous and satisfies the following condition

$$0 < d_f \ , \quad \|\dot{f}\|_\infty < +\infty \quad \text{and} \quad \|\,\mathsf{A}(\dot{f})\|_\infty < +\infty \ .$$

Corollary 2 ([Partyka-Sakan [PS2014a, Cor.4.1]).
Let $f : \mathbb{T} \to \mathbb{C}$ be a Dini-smooth and injective function. If $f(\mathbb{T})$ is the boundary curve of a convex domain Ω in $\mathbb{C}$, then $d_f > 0$, $\|\dot{f}\|_\infty < +\infty$ and $F := \mathsf{P}[f]$ is a bi-Lipschitz mapping of $\mathbb{D}$ onto Ω with

$$\mathsf{L}(F) \leq \sqrt{\frac{4}{\pi^2} D_f^2 + \|\dot{f}\|_\infty^2}$$

and

$$\frac{1}{\mathsf{L}(F^{-1})} \geq \frac{d_f}{\mathsf{L}(F)} \cdot \frac{R_1 + R_2}{\pi} \tan\left(\frac{\pi}{2} \frac{R_1}{R_1 + R_2}\right) \geq \frac{d_f R_1}{2\mathsf{L}(F)} ,$$

provided $R_1, R_2 > 0$ satisfy

$$\mathbb{D}(F(0), R_1) \subset \Omega \subset \mathbb{D}(F(0), R_2) \ . \tag{1}$$

In particular, if additionally $\mathsf{J}[F](0) > 0$, then F is $\mathsf{L}(F)\mathsf{L}(F^{-1})$-quasiconformal.

($\mathbb{D}(a, R) := \{z \in \mathbb{C} : |z - a| < R\}$ for $a \in \mathbb{C}$ and $R > 0$.)

Corollary 3 ([Partyka-Sakan [PS2014a, Cor.4.3]).

Let $h : \mathbb{T} \to \mathbb{C}$ be a Hölder-smooth and injective function such that $h(\mathbb{T})$ is the boundary curve of a convex domain Ω in $\mathbb{C}$. Then for every sense-preserving injective harmonic mapping F of $\mathbb{D}$ onto Ω the following conditions are equivalent to each other:

(i) F is a quasiconformal mapping;

(ii) F is a bi-Lipschitz mapping;

(iii) F has a continuous extension to the closure $\overline{\mathbb{D}}$ and its boundary limiting valued function f is absolutely continuous and satisfies the condition

$$0 < d_f \ , \quad \|\dot{f}\|_\infty < +\infty \quad \text{and} \quad \|\,\mathsf{A}(\dot{f})\|_\infty < +\infty \ .$$

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Complex analytic structure on the p-integrable Teichmüller space

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Riemann surfaces and Discontinuous groups 2014
February 16th, 2015

Teichmüller space

$\longrightarrow$ Deformation space of Fuchsian groups (Riemann surfaces)

Teichmüller space of Γ

$$T(\Gamma) = \mathrm{Bel}(\Delta, \Gamma)/\sim_T$$

- Γ: Fuchsian group acting on the unit disk Δ
- $\mathrm{Bel}(\Delta, \Gamma)$: Beltrami coefficients

$$\|\mu\|_\infty = \operatorname*{ess\,sup}_{z \in \Delta} |\mu(z)| < 1 \quad (\forall \mu \in \mathrm{Bel}(\Delta, \Gamma))$$

$$(\mu \circ \gamma)\frac{\overline{\gamma'}}{\gamma'} = \mu \quad (\forall \gamma \in \Gamma)$$

Teichmüller space

Teichmüller space of Γ

$$T(\Gamma) = \mathrm{Bel}(\Delta, \Gamma)/\sim_T$$

- $\mu \sim_T \nu \overset{\text{def}}{\iff} f^\mu|_{\partial\Delta} = f^\nu|_{\partial\Delta}$
  f^μ: the quasiconformal self-mapping on Δ with Beltrami coeff. μ fixing $1, i, -1$

$[\mu]$: Teichmüller equivalence class represented by $\mu \in \mathrm{Bel}(\Delta, \Gamma)$
$0 := [0]$: the base point of $T(\Gamma)$

Basic fact

Teichmüller distance $\cdots$ difference between two conformal structures

$$d_{T(\Gamma)}(p,q) = \frac{1}{2}\inf\{\log K(f \circ g^{-1}) | f \in p, g \in q\} \quad (p,q \in T(\Gamma))$$

$$K(f) = \frac{1+\|\mu_f\|_\infty}{1-\|\mu_f\|_\infty} : \quad \text{maximal dilatation of q.c. } f$$

$$f_0 \in \tau \text{ is extremal} \Leftrightarrow K(f_0) \leq K(f) \quad (\forall f \in \tau)$$

Fact

The metric space $(T(\Gamma), d_{T(\Gamma)})$ is complete and contractible.

p-integrable Teichmüller space

p-integrable Teichmüller space ($p \geq 1$)

$$T^p(\Gamma) = \{\tau \in T(\Gamma) | \exists \mu \in \tau \text{ s.t. } \mu \in \mathrm{Ael}^p(\Delta, \Gamma)\}$$

- $\mathrm{Ael}^p(\Delta, \Gamma)$: p-integrable Beltrami coefficients

$$\|\mu\|_p = \left(\iint_N |\mu(z)|^p \rho(z)^2 d^2 z \right)^{\frac{1}{p}} < \infty$$

N: fundamental region of Δ for Γ, $\quad d^2 z = dxdy \; (z = x + iy)$
$\rho_\Delta(z) = (1 - |z|^2)^{-1}$: the Poinca r é metric on Δ

- The space $\mathrm{Ael}^p(\Delta, \Gamma)$ is open with norm

$$\|\mu\|_{p,\infty} = \|\mu\|_p + \|\mu\|_\infty.$$

p-integrable Teichmüller space

Fact

If Γ is cofinite (i.e. Δ/Γ is of analytically finite type),

$$T^p(\Gamma) = T(\Gamma).$$

$\downarrow$ Hence

This study is significant for coinfinite Fuchsian groups.

p-integrable Teichmüller space

Fact

If Γ is cofinite (i.e. Δ/Γ is of analytically finite type),

$$T^p(\Gamma) = T(\Gamma).$$

$\downarrow$ Hence

This study is significant for coinfinite Fuchsian groups.

Background

2000 G. Cui considered $T^2(1)$.

- Introduction of a complex structure on $T^2(1)$
- Completeness of Weil-Petersson metric on $T^2(1)$

2000 L. A. Takhtajan, L.-P. Teo also considered $T^2(1)$.

- Kählerity of Weil-Petersson metric on $T^2(1)$
- Curvatures of Weil-Petersson metric on $T^2(1)$

2013 S. Tang extended the arguments of Cui to $p \geq 2$.

Purpose of this study

We extend their arguments to non-trivial Fuchsian groups.

Main results

Let $p \geq 2$.

(1) For every Fuchsian group Γ, $T^p(\Gamma)$ has a complex structure modeled on a Banach space.

(2) For every Fuchsian group Γ with Lehner's condition, the Teichmüller distance on $T^p(\Gamma)$ coincides with the Kobayashi distance.

Notations

- $\mathcal{B}(\Delta^*,\Gamma)$: bounded holomorphic quadratic differentials

$$\|\varphi\|_\infty = \sup_{z\in\Delta^*} |\varphi(z)|\rho_{\Delta^*}(z)^{-2} < \infty \quad (\forall\varphi \in \mathcal{B}(\Delta^*,\Gamma))$$

$$\rho_{\Delta^*}(z) = (|z|^2-1)^{-1} \text{ : the Poincaré metric on } \Delta^*$$

$$(\varphi\circ\gamma)\gamma'^2 = \varphi \quad (\forall\gamma\in\Gamma)$$

- $A^p(\Delta^*,\Gamma)$: p-integrable holomorphic quadratic differentials

$$\|\varphi\|_p = \left(\iint_{N^*} |\varphi(z)|^p \rho_{\Delta^*}(z)^{2-2p} d^2z\right)^{\frac{1}{p}} < \infty \quad (\forall\varphi\in A^p(\Delta^*,\Gamma))$$

N^*: fundamental region of Δ^* for Γ

- $A^p_b(\Delta^*,\Gamma) = \mathcal{B}(\Delta^*,\Gamma)\cap A^p(\Delta^*,\Gamma)$: Banach sp. with norm

$$\|\varphi\|_{p,\infty} = \|\varphi\|_p + \|\varphi\|_\infty$$

Notations

- $\mathcal{B}(\Delta^*, \Gamma)$: bounded holomorphic quadratic differentials

$$\|\varphi\|_\infty = \sup_{z \in \Delta^*} |\varphi(z)| \rho_{\Delta^*}(z)^{-2} < \infty \quad (\forall \varphi \in \mathcal{B}(\Delta^*, \Gamma))$$

$$\rho_{\Delta^*}(z) = (|z|^2 - 1)^{-1} : \text{the Poincaré metric on } \Delta^*$$

$$(\varphi \circ \gamma)\gamma'^2 = \varphi \quad (\forall \gamma \in \Gamma)$$

- $A^p(\Delta^*, \Gamma)$: p-integrable holomorphic quadratic differentials

$$\|\varphi\|_p = \left(\iint_{N^*} |\varphi(z)|^p \rho_{\Delta^*}(z)^{2-2p} d^2 z \right)^{\frac{1}{p}} < \infty \quad (\forall \varphi \in A^p(\Delta^*, \Gamma))$$

N^*: fundamental region of Δ^* for Γ

- $A_b^p(\Delta^*, \Gamma) = \mathcal{B}(\Delta^*, \Gamma) \cap A^p(\Delta^*, \Gamma)$: Banach sp. with norm

$$\|\varphi\|_{p,\infty} = \|\varphi\|_p + \|\varphi\|_\infty$$

Notations

- $\mathcal{B}(\Delta^*, \Gamma)$: bounded holomorphic quadratic differentials

$$\|\varphi\|_\infty = \sup_{z \in \Delta^*} |\varphi(z)| \rho_{\Delta^*}(z)^{-2} < \infty \quad (\forall \varphi \in \mathcal{B}(\Delta^*, \Gamma))$$

$$\rho_{\Delta^*}(z) = (|z|^2 - 1)^{-1} : \text{the Poincaré metric on } \Delta^*$$

$$(\varphi \circ \gamma)\gamma'^2 = \varphi \quad (\forall \gamma \in \Gamma)$$

- $A^p(\Delta^*, \Gamma)$: p-integrable holomorphic quadratic differentials

$$\|\varphi\|_p = \left(\iint_{N^*} |\varphi(z)|^p \rho_{\Delta^*}(z)^{2-2p} d^2 z \right)^{\frac{1}{p}} < \infty \quad (\forall \varphi \in A^p(\Delta^*, \Gamma))$$

N^*: fundamental region of Δ^* for Γ

- $A^p_b(\Delta^*, \Gamma) = \mathcal{B}(\Delta^*, \Gamma) \cap A^p(\Delta^*, \Gamma)$: Banach sp. with norm

$$\|\varphi\|_{p,\infty} = \|\varphi\|_p + \|\varphi\|_\infty$$

Background

Theorem

For every Fuchsian group Γ, $T(\Gamma)$ has a complex structure modeled on $\mathcal{B}(\Delta^*, \Gamma)$.

This follows from the following proposition.

Proposition (Bers 1965)

The Bers embedding β is a homeomorphism of $(T(\Gamma), d_{T(\Gamma)})$ into $(\mathcal{B}(\Delta^*, \Gamma), \|\cdot\|_\infty)$.

Indeed, $\{(T(\Gamma), \beta)\}$ is an atlas of $T(\Gamma)$.

Background

Bers embedding

$$\beta : T(\Gamma) \ni [\mu] \mapsto S_{f_\mu|_{\Delta^*}} \in \mathcal{B}(\Delta^*, \Gamma)$$

- f_μ: self-q.c. of $\hat{\mathbb{C}}$ with $\hat{\mu}$, $\lim_{z\to\infty}(f(z) - z) = 0$

$$\hat{\mu}(z) = \begin{cases} \mu(z) & (z \in \Delta) \\ 0 & (z \in \Delta^*). \end{cases}$$

- $S_f = (f''/f')' - 1/2(f''/f')^2$: Schwarzian derivative of hol. func. f

Our purpose

$\longrightarrow$ To introduce a complex structure on $T^p(\Gamma)$

What we have to show?

The Bers embedding β is a homeomorphism of $(T^p(\Gamma), \ell_{p,\infty})$ into $(A_b^p(\Delta^*, \Gamma), \|\cdot\|_{p,\infty})$.

- $\ell_{p,\infty}$: quotient topology induced by $\mathrm{Ael}^p(\Delta, \Gamma)$

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How to achieve our purpose?

The estimation of hyperbolic L^p-norm (Cui, Tang)

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$$(1)\quad \iint_{\Delta} \cdots d^2z = \sum_{\gamma\in\Gamma} \iint_{\gamma(N)} \cdots d^2z$$

$$(2)\quad \frac{\gamma'(z)\gamma'(w)}{(\gamma(z)-\gamma(w))^2} = \frac{1}{(z-w)^2} \quad (\gamma: \text{ Möbius transformation})$$

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Outline of the proof

Step 1. Continuity of β

Proposition (Cui 2000, Y. 2014)

$\sigma : (T^p(\Gamma), \ell_{p,\infty}) \ni \tau \mapsto \mu_{E(\tau)} \in (\mathrm{Ael}^p(\Delta, \Gamma), \| \cdot \|_\infty)$ is continuous.

- $E(\tau)$: Douady-Earle extension for τ

Proposition (Cui 2000, Tang 2013, Y. 2014)

For every $\tau, \eta \in T^p(\Gamma)$, there exists a constant $C(\tau, \eta) > 0$ such that

$$\|\beta(\tau) - \beta(\eta)\|_p \leq C(\tau, \eta) \left\| \frac{\sigma(\tau) - \sigma(\eta)}{1 - \sigma(\tau)\overline{\sigma(\eta)}} \right\|_p .$$

Outline of the proof

Step 2. Continuity of β^{-1}

Fact

For every $\varphi \in \beta(T(\Gamma))$, there exists a continuous section $s_\varphi : U_\varphi \to \mathrm{Bel}(\Delta, \Gamma)$, where $U_\varphi \subset \mathcal{B}(\Delta^*, \Gamma)$ is a neighborhood at φ.

$\longrightarrow$ As the case of $\varphi \in \beta(T^p(\Gamma))$, we can show the continuity of $s_\varphi : U'_\varphi \to \mathrm{Ael}^p(\Delta, \Gamma)$ ($U'_\varphi \subset A_b^p(\Delta^*, \Gamma)$: a nbd. at φ).

Conclusion 1

Theorem (Cui 2000, Takhtajan-Teo 2000, Tang 2013, Y. 2014)

Let $p \geq 2$. For every Fuchsian group Γ, $T^p(\Gamma)$ has a complex structure modeled on $A_b^p(\Delta^*, \Gamma)$.

By the continuous of σ, we have

Proposition (Douady-Earle 1986, Y. 2014)

Let $p \geq 2$. For every Fuchsian group Γ, $(T^p(\Gamma), \ell_{p,\infty})$ is contractible.

1. p-integrable Teichmüller space
2. Introduction of complex structure
3. Teichmüller distance and Kobayashi distance

Teichmüller distance and Kobayashi distance

Kobayashi distance: the distance defined on complex manifolds

Contractivity of Kobayashi distance

M, N: complex manifolds, $d_{K,M}, d_{K,N}$: Kobayashi distances,
$F : M \to N$ holomorphic

$$d_{K,N}(F(p), F(q)) \leq d_{K,M}(p, q) \quad (\forall p, q \in M)$$

Background

Theorem (Royden 1971, Gardiner 1984)

For every Fuchsian group Γ, the Teichmüller distance $d_{T(\Gamma)}$ on $T(\Gamma)$ coincides with the Kobayashi distance $d_{K(\Gamma)}$.

Theorem (Earle-Gardiner-Lakic 2004, Hu-Jiang-Wang 2011)

For every Fuchsian group Γ, the Teichmüller distance on $T_0(\Gamma)$ coincides with the Kobayashi distance.

A q.c. f on the Riemann surface $\mathcal{R} = \Delta/\Gamma$ is asymptotically conformal if $\forall \varepsilon > 0$, $\exists E \subset \mathcal{R}$: compact s.t.

$$K(f|_{\mathcal{R}\setminus E}) < 1 + \varepsilon.$$

$$T_0(\Gamma) = \{\tau \in T(\Gamma) | \exists f \in \tau : \text{ asymptotically conformal}\}$$

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$\longrightarrow$ The coincidence of the Teichmüller distance and Kobayashi distance on $T^p(\Gamma)$

$d_{T^p(\Gamma)} = d_{T(\Gamma)}|_{T^p(\Gamma)}$, $d_{K^p(\Gamma)}$: Kobayashi distance on $T^p(\Gamma)$

What point is difficult to show?

$$d_{T^p(\Gamma)} \geq d_{K^p(\Gamma)}$$

Since $\iota : T^p(\Gamma) \to T(\Gamma)$ is holomorphic,

$$d_{T^p(\Gamma)}(\tau,\eta) = d_{K(\Gamma)}(\iota(\tau),\iota(\eta)) \leq d_{K^p(\Gamma)}(\tau,\eta)$$

for $\forall \tau, \eta \in T^p(\Gamma)$.

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for $\forall \tau, \eta \in T^p(\Gamma)$.

How to overcome the difficulty?

The argument in the proof by Hu-Jiang-Wang

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Existence of an exhaustion

There exists a sequence $\{E_n\}$ of relatively compact domains for every non-compact Riemann surface:

$$(1) \quad \overline{E}_n \subset E_{n+1}; \quad (2) \quad \bigcup_{n=1}^{\infty} E_n = \mathcal{R};$$

$$(3) \quad \partial E_n = \bigsqcup \{\text{analytic Jordan curves of } \mathcal{R}\};$$

(4) Each connected component of $\mathcal{R} \setminus E_n$ is non-compact.

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More precisely . . .

$\longrightarrow$ We modify an extremal quasiconformal mapping by using an exhaustion.

Key theorem (Strebel's frame mapping theorem)

For every frame mapping f, there exists the unique extremal quasiconformal mapping $f_0 \in [f]$ such that

$$\mu_{f_0} = \|\mu_{f_0}\|_\infty \frac{\overline{\varphi_0}}{|\varphi_0|}.$$

Here, φ_0 is a holomorphic quadratic differential with $\iint_{\mathcal{R}} |\varphi_0| = 1$.

A q.c. f is a frame mapping if $\exists E \subset \mathcal{R}$ s.t.

$$K(f|_{\mathcal{R}\setminus E}) < K(f_0) \quad (f_0 : \text{ extremal for } [f]).$$

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A geometric condition for Riemann surfaces

Lehner's condition

Γ: Fuchsian group, $\mathcal{R} = \Delta/\Gamma$

$$\inf\{\text{lengths of simple closed geodesics on } \mathcal{R}\} > 0$$

Proposition A

Γ: Fuchsian group with Lehner's condition
Then for every $\tau \in T^p(\Gamma)$, the projection of the Douady-Earle extension $E(\tau)$ to $\mathcal{R}$ is asymptotically conformal.

Outline of the proof $d_{T^p(\Gamma)} \geq d_{K^p(\Gamma)}$

It is sufficient to show

$$d_{K^p(\Gamma)}(0,\tau) \leq d_{T^p(\Gamma)}(0,\tau) \quad (\forall \tau \in T^p(\Gamma)).$$

f: proj. of $E(\tau)$ to $\mathcal{R} = \Delta/\Gamma$, $\{E_n\}$: exhaustion of $\mathcal{R}$

$$f_n(z) = \begin{cases} \tilde{f}_n(z) & (z \in \overline{E}_n) \\ \\ f(z) & (z \in \mathcal{R} \setminus \overline{E}_n). \end{cases}$$

$\tilde{f}_n$: extremal mapping for $[f|_{E_n}] \in T(E_n)$

Outline of the proof $d_{T^p(\Gamma)} \geq d_{K^p(\Gamma)}$

Since

$$g_n : \Delta \ni t \mapsto [t\mu_{f_n}/\|\mu_{f_n}\|_\infty] \in T^p(\Gamma)$$

is holomorphic, we have

$$d_{K^p(\Gamma)}(0,\tau) = d_{K^p(\Gamma)}(g_n(0), g_n(\|\mu_{f_n}\|_\infty)) \leq \frac{1}{2}\log\frac{1+\|\mu_{f_n}\|_\infty}{1-\|\mu_{f_n}\|_\infty}.$$

Let

$$K_n = \frac{1+\|\mu_{f_n}\|_\infty}{1-\|\mu_{f_n}\|_\infty}.$$

Outline of the proof $d_{T^p(\Gamma)} \geq d_{K^p(\Gamma)}$

Since f is a frame mapping for τ (Proposition A), $\exists_1 f_0 \in \tau$ s.t.

$$\mu_{f_0} = \|\mu_{f_0}\|_\infty \frac{\overline{\varphi_0}}{|\varphi_0|} \quad (\iint_{\mathcal{R}} |\varphi_0| = 1).$$

Because $f|_{E_n}$ is a frame mapping for $[f|_{E_n}] \in T(E_n)$,

$$\mu_{\tilde{f}_n} = \|\mu_{\tilde{f}_n}\|_\infty \frac{\overline{\varphi_n}}{|\varphi_n|} \quad (\iint_{E_n} |\varphi_n| = 1).$$

Outline of the proof $d_{T^p(\Gamma)} \geq d_{K^p(\Gamma)}$

By taking a subsequence,

$$\|\mu_{\tilde{f}_n}\|_\infty \to \exists k^*, \quad \varphi_n \to \exists \varphi^* \ (\iint_{\mathcal{R}} |\varphi^*| \leq 1).$$

From the uniqueness for extremal q.c.,

$$k^* = \|\mu_{f_0}\|_\infty, \quad \varphi^* = \varphi_0.$$

$$\frac{1}{2}\log K_n \to \frac{1}{2}\log K(f_0) = d_{T^p(\Gamma)}(0,\tau).$$

Conclusion 2

Theorem (Y. 2014)

For $p \geq 2$ and a Fuchsian group with Lehner's condition Γ, the Teichmüller distance on $T^p(\Gamma)$ coincides with the Kobayashi distance.

Noting the red colored points, we have

Corollary (Earle-Gardiner-Lakic 2004, Hu-Jiang-Wang 2011, Y. 2014)

For every Fuchsian group Γ, the Teichmüller distance on $T_0(\Gamma)$ coincides with the Kobayashi distance.

What is so special about the following integer quadruples?

$$\begin{array}{cccc} (1,5,24,30), & (1,6,14,21), & (1,8,9,18), & (1,9,10,10), \\ (2,3,10,15), & (2,5,5,8), & (3,3,6,6), & (4,4,4,4) \end{array}$$

Flipping numbers and curves

Yi Huang

The University of Melbourne
(soon to be Tsinghua University)

February 16th, 2015

Markoff triples

A *Markoff triple* is a triple of numbers (x, y, z) satisfying:

$$x^2 + y^2 + z^2 = xyz.$$

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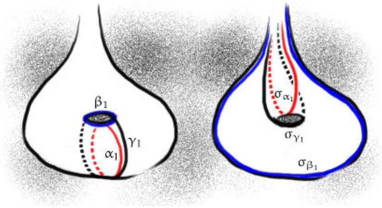
Any $(x, y, z) \in \mathbb{R}^3_+$ arises as $(2\cosh\frac{\ell_{\alpha_1}}{2}, 2\cosh\frac{\ell_{\beta_1}}{2}, 2\cosh\frac{\ell_{\gamma_1}}{2})$,



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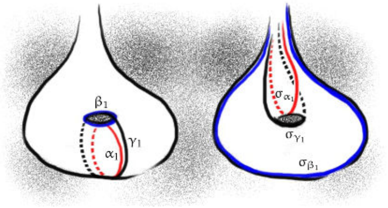

or as λ-lengths of an ideal triangulation $(\sigma_{\alpha_1}, \sigma_{\beta_2}, \sigma_{\gamma_1})$.

Representations

Given a 1-cusped torus $S_{1,1}$, $\pi_1(S_{1,1}) \cong \langle \xi, \eta \mid - \rangle$.

Any non-zero Markoff triple (x, y, z) arises as the traces of the following representation $\rho : \pi_1(S_{1,1}) \to \mathrm{SL}(2, \mathbb{C})$:

$$\rho(\xi) = \frac{1}{z} \begin{bmatrix} xz - y & xz^{-1} \\ xz & y \end{bmatrix}, \ \rho(\eta) = \frac{1}{z} \begin{bmatrix} yz - x & -yz^{-1} \\ -yz & x \end{bmatrix},$$

$$\rho(\xi\eta) = \begin{bmatrix} z & -z^{-1} \\ z & 0 \end{bmatrix}.$$

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$\Rightarrow$ Markoff triples arise as the characters of (type-preserving) representations.

Character varieties

- The traces of $\rho(\xi)$, $\rho(\eta)$ and $\rho(\xi\eta)$ of any $\rho : \pi_1(S_{1,1}) \to \mathrm{SL}(2, \mathbb{C})$ satisfy the Markoff triples relation.

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Character varieties

- The traces of $\rho(\xi)$, $\rho(\eta)$ and $\rho(\xi\eta)$ of any $\rho : \pi_1(S_{1,1}) \to \mathrm{SL}(2, \mathbb{C})$ satisfy the Markoff triples relation.
- The set of Markoff triples is the character variety for $S_{1,1}$.
- Any maximal dimensional component of the real character subvariety is the Teichmüller space $\mathcal{T}(S_{1,1})$, e.g.:

$$\mathcal{T}(S_{1,1}) = \left\{(x, y, z) \in \mathbb{R}^+ \mid x^2 + y^2 + z^2 = xyz\right\}$$

Flippin' out!

We can generate new Markoff triples using *flips*:

$$(x, y, z) \mapsto (x, y, xy - z).$$

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- Flips can be thought of as extended mapping classes — (potentially non-orientable) homeomorphisms of $S_{1,1}$ up to isotopy.
- Flips and coordinate permutations generate the entire extended mapping class group of $S_{1,1}$.

Systolic geometry

Flips give us a simple geodesic length generating algorithm for any 1-cusped hyperbolic torus.

- Algorithm for working out the shortest geodesic (systole).

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Systolic geometry

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- Algorithm for working out the shortest geodesic (systole).
- The maximum of the systole function over Teichmüller space (and moduli space) is the "$(3, 3, 3)$" 1-cusped torus.
- The shortest geodesic for any 1-cusped hyperbolic torus is at most $2\text{arccosh}(\frac{3}{2})$.

Geodesic growth rates

Let (x, y, z) and $(x' = yz - x, y, z)$ be flips of each other where $x \leq x'$. Generically, $yz \gg x$, thus:

$$\log(x') \approx \log(y) + \log(z).$$

$\Rightarrow$ *Fibonacci growth.*

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$$\log(x') \approx \log(y) + \log(z).$$

$\Rightarrow$ *Fibonacci growth.*

The length of a simple closed geodesics is roughly $2\log(\cdot)$ of its corresponding trace. Consider:

$$N_S(L) := \{ \text{ simple closed geodesics on } S \text{ shorter than } L \},$$

Fibonacci growth $\Rightarrow N_S(L)$ is asymptotically $\alpha \cdot L^2$.

McShane identity

Rewriting the Markoff triples equation and the flipping relation:

$$1 = \frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} \text{ and } 1 = \frac{x}{yz} + \frac{x'}{yz} \Rightarrow \frac{x'}{yz} = \frac{y}{xz} + \frac{z}{xy}.$$

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In the limit, we obtain *McShane identities*:

$$1 = \sum_{\gamma \in \mathrm{Sim}\pi_1(S)} \frac{2}{1 + \exp \ell_\gamma},$$

where $\mathrm{Sim}\pi_1(S)$ is the set of simple closed geodesics on S.

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where $\mathrm{Sim}\pi_1(S)$ is the set of simple closed geodesics on S.

Each term is the chance a geodesic shot out from the cusp on S won't hit γ before self-intersecting.

Markoff quads

A *Markoff quad* is a 4-tuple of numbers (a, b, c, d) satisfying:

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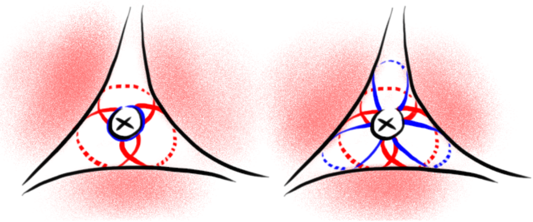
Any $(a, b, c, d) \in \mathbb{R}^4_+$ arises as $2\sinh(\frac{1}{2}\cdot)$ of geodesic lengths:



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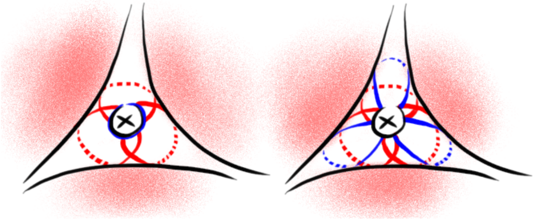

$\{Markoffquads\}$ = character variety for 3-cusped projective planes.
Maximal dimension components of real character subvariety $\leftrightarrow$ Teichmüller space.

Representations

Given a 3-cusped projective plane S, $\pi_1(S) \cong \langle \alpha, \beta, \gamma \mid - \rangle$.

Any non-zero Markoff quad (a, b, c, d) arises as the traces of the following representation $\rho : \pi_1(S) \to \mathrm{SL}^{\pm}(2, \mathbb{C})$:

$$
\begin{aligned}
\rho : F_3 = \langle \alpha, \beta, \gamma \rangle &\to \mathrm{SL}^{\pm}(2, \mathbb{C}) \\
\alpha &\mapsto \frac{1}{a+b+c+d} \begin{bmatrix} ab & b(a+c) \\ a(a+d) & a(a+c+d) \end{bmatrix}, \\
\beta &\mapsto \frac{1}{a+b+c+d} \begin{bmatrix} ab & -b(b+d) \\ -a(b+c) & b(b+c+d) \end{bmatrix}, \\
\gamma &\mapsto \frac{1}{a+b+c+d} \begin{bmatrix} ab + c(a+b+c+d) & b(a+c) \\ -a(b+c) & -ab \end{bmatrix}.
\end{aligned}
$$

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- A quad of such geodesics can be flipped to any other quad $\Rightarrow$ geodesic length generating algorithm.
- Flips can be thought of as extended mapping classes, and flips+even permutations generate the (extended) mapping class group.

Systolic geometry

Theorem

The maximum of the systole function over the moduli space of 3-cusped projective plane is $2\mathrm{arcsinh}(2)$*, and uniquely attained by the* $(4,4,4,4)$ *surface.*

Geodesic growth rates

Let (a, b, c, d) and (a', b, c, d) be flips of each other where $a \leq a'$, generically:

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Fibonacci growth $\Rightarrow N_S(L)$ is between $O(L^{2.430})$ and $O(L^{2.477})$.

McShane identity

We similarly obtain the following sum refinement:

$$1 = \frac{a+b+c+d}{bcd} + \frac{a+b+c+d}{acd} + \frac{a+b+c+d}{abd} + \frac{a+b+c+d}{abc}$$

$$\text{and } 1 = \frac{a+b+c+d}{bcd} + \frac{a'+b+c+d}{bcd}.$$

McShane identity

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$$\text{and } 1 = \frac{a+b+c+d}{bcd} + \frac{a'+b+c+d}{bcd}.$$

Theorem

Given any 3-cusped projective plane S,

$$1 = \sum_{\gamma \in \mathrm{Sim}\pi_1^1(S)} \frac{2}{1+\exp \ell_\gamma},$$

where $\mathrm{Sim}\pi_1^1(S)$ is the set of 2-sided simple closed geodesics on S.

Future research - Geometry

- Are BQ-conditions trace-based characterisations of quasi-Fuchsian representations?
- Are there geometric interpretations for more general Markoff-Hurwitz numbers (to do with Hengnan Hu's thesis work with Ser Peow Tan)?
- Mapping class group equivariant map from the Teichmüller space of 3-bordered projective planes to the Teichmüller space of 3-cusped projective planes.

Future research - Number theory

Integer Markoff triples flip to integer Markoff triples, and integer Markoff triples are central in number theory:

- rational approximation;
- Markoff's theorem for indefinite binary quadratic forms;
- the unicity conjecture.

Do they have Markoff quad equivalents?

(And just before we finish...)

What is so special about the following integer quadruples?

(1,5,24,30), (1,6,14,21), (1,8,9,18), (1,9,10,10),
(2,3,10,15), (2,5,5,8), (3,3,6,6), (4,4,4,4)

(And just before we finish...)

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(1,5,24,30), (1,6,14,21), (1,8,9,18), (1,9,10,10),
(2,3,10,15), (2,5,5,8), (3,3,6,6), (4,4,4,4)

Answer: every (no-where zero) integral solution to the Markoff quad equation can be obtained from a sequence of flips and coordinate swaps of these eight Markoff quads.

Diophantine approximation via Gaussian integers

Ryuji ABE

2015.2.16

(joint work with I. R. Aitchison)

Plan

1. Introduction
   Lagrange spectrum, Markoff spectrum

2. Geometric problem
   hyperbolic upper half-space, geodesic, horosphere, action of $\mathrm{SL}(2, \mathbb{Z}[i])$

3. Ford's method

4. Two-color Markoff graph

5. Outline of proof

1. Introduction

Definition :

- For an irrational number ξ, we define
$k(\xi) := \sup \left\{ k \;\middle|\; \left| \xi - \frac{p}{q} \right| < \frac{1}{kq^2} \text{ has infinitely many co-prime integer solutions } (p,q) \in \mathbb{Z}^2 \right\}$
- $\mathcal{L} := \{k(\xi) \mid \xi \in \mathbb{R} - \mathbb{Q}\}$: the Lagrange spectrum for the rational number field $\mathbb{Q}$
- $HC := \inf \mathcal{L}$: the Hurwitz constant

Well-known facts :

- $HC = \sqrt{5}$ (the first proof by Hurwitz in 1891 using continued fractions, a geometric proof by Ford in 1917)
- The discrete part of $\mathcal{L}$ coincides with the discrete part of the Markoff spectrum for $\mathbb{Q}$.

Theorem 1 (Markoff)**.** *For any irrational number ξ, $k(\xi) \geq \sqrt{5}$, the sign of equality being necessary if ξ is equivalent under $\mathrm{SL}(2,\mathbb{Z})$ to $\xi_1 = (1 \pm \sqrt{5})/2$. If ξ is not equivalent to ξ_1 then $k(\xi) \geq \sqrt{8}$, the sign of equality being necessary if ξ is equivalent to $\xi_2 = 1 \pm \sqrt{2}$. If ξ is not equivalent to ξ_1, ξ_2 then $k(\xi) \geq \sqrt{221}/5$. And so on indefinitely, there exists the sequence of numbers*

$$\sqrt{9 - \frac{4}{k^2}}, \quad k = 1, 2, 5, 13, 29, \cdots$$

converges to 3.

Two points ξ, η are said to be equivalent under $\mathrm{SL}(2,\mathbb{Z})$ if there exists $M \in \mathrm{SL}(2,\mathbb{Z})$ with $M(\xi) = \eta$.

- A geometric proof of Theorem 1 is given by Nicolls in 1978.

Definition :

- For an irrational complex number[†] ξ, we define
$k_1(\xi) := \sup\left\{k \;\middle|\; \left|\xi - \frac{p}{q}\right| < \frac{1}{k|q|^2} \text{ has infinitely many co-prime integer solutions } (p,q) \in \mathbb{Z}[i]^2\right\}$
- $\mathcal{L}_1 := \{k_1(\xi) \mid \xi \in \mathbb{C} - \mathbb{Q}(i)\}$: the Lagrange spectrum for the imaginary quadratic number field $\mathbb{Q}(i)$
- $HC_1 := \inf \mathcal{L}_1$: the Hurwitz constant for $\mathbb{Q}(i)$

Well-known facts :

- $HC_1 = \sqrt{3}$ (a geometric proof by Ford in 1925)
- The discrete part of $\mathcal{L}_1$ coincides with the discrete part of the Markoff spectrum for $\mathbb{Q}(i)$.

Theorem 2. *For any irrational complex number $\xi \in \mathbb{C} - \mathbb{Q}(i)$, $k_1(\xi) \geq \sqrt{3}$, the sign of equality being necessary if ξ is equivalent under $\mathrm{SL}(2, \mathbb{Z}[i])$ to $\xi_1 = (1 \pm \sqrt{3}i)/2$. If ξ is not equivalent to ξ_1 then $k_1(\xi) \geq \sqrt{99}/5$, the sign of equality being necessary if ξ is equivalent to $\xi_5 = 1/2 + i(1 \pm \sqrt{99}/5)/2$. If ξ is not equivalent to ξ_1, ξ_5 then $k_1(\xi) \geq \sqrt{3363}/29$. And so on indefinitely, there exists the sequence of numbers*

$$\sqrt{4 - \frac{1}{\lambda^2}}, \quad \lambda = 1, 5, 29, 65, 169 \cdots$$

converges to 2 [††].

Two points ξ, η are said to be equivalent under $\mathrm{SL}(2, \mathbb{Z}[i])$ if there exists $M \in \mathrm{SL}(2, \mathbb{Z}[i])$ with $M(\xi) = \eta$.

Aim : Give a geometric proof of Theorem 2.

[†] Here an irrational complex number is a number not expressible as a rational complex fraction whose numerator and denominator are complex integers.

[††] This is another description of Theorem 3 analogous to Theorem 1. In this talk, we do not consider the second smallest value $\sqrt{3\sqrt{41}/5}$ in Theorem 3, so it is not complete in this point.

Definition :

- For a binary infinite quadratic form $f(x,y) = ax^2 + bxy + cy^2$ with complex coefficients and with discriminant $D(f) = b^2 - 4ac$, we define

$$m_1(f) = \inf_{(x,y)\in\mathbb{Z}[i]^2-\{(0,0)\}} |f(x,y)|.$$

- $\mathcal{M}_1 = \left\{ \sqrt{|D(f)|}/m_1(f) \,\middle|\, D(f) \neq 0 \right\}$: the Markoff spectrum for $\mathbb{Q}(i)$

For Vulakh's equation $\begin{cases} x_1 + x_2 &= 2y_1y_2 \\ 2x_1x_2 &= y_1^2 + y_2^2 \end{cases}$ we define

- $\mathcal{N}(\Lambda) = \{1, 5, 29, 65, 169, \cdots\}$: the set of integer solutions as x_1, x_2,
- $\mathcal{N}(M) = \{1, 3, 11, 17, 41, \cdots\}$: the set of integer solutions as y_1, y_2.

Note that Vulakh's equation is equivalent to $2x^2 + y_1^2 + y_2^2 = 4xy_1y_2$.

Theorem 3. *The discrete part of the Markoff spectrum for $\mathbb{Q}(i)$ is described as*

$$\left\{ \sqrt{4 - \frac{1}{\lambda^2}} \,\middle|\, \lambda \in \mathcal{N}(\Lambda) \right\} \bigcup \left\{ \sqrt{\frac{3}{5}\sqrt{41}} \right\}.$$

- This is first proved by Vulakh in 1971. Schmidt gave another proof depending on continued fractional expansion in 1975.

Theorem 4 (A.-A. 2013)**.** *For each $\lambda \in \mathcal{N}(\Lambda)$ we can get an element Λ_λ of the Picard group* $\mathrm{SL}(2,\mathbb{Z}[i])$ *such that the Euclidean height of its axis is equal to $\frac{1}{2}\sqrt{4 - 1/\lambda^2}$ and the axis of Λ_λ projects to a simple closed geodesic on a twice punctured torus immersed in the Borromean rings complement.*

- The matrix Λ_λ is defined by using the two-color Markoff graph which will be shown later and plays a crucial role in a proof of Theorem 2.

2. Geometric problem

$\mathbb{H}^3 = \{z + jt \mid z = x + iy \in \mathbb{C}, t > 0\}$: the upper half-space endowed with the hyperbolic metric $ds^2 = \dfrac{dx^2 + dy^2 + dt^2}{t^2}$

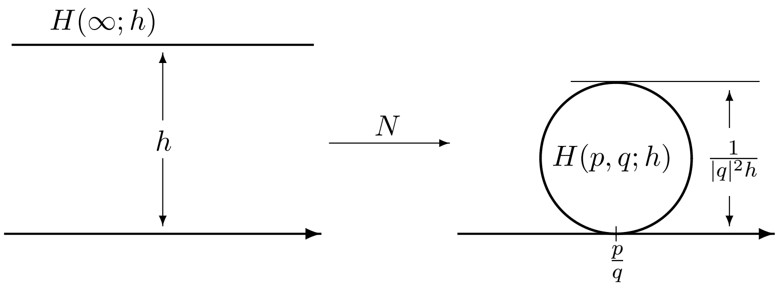
- A geodesic is a semicircle or a ray perpendicular to the complex plane $\mathbb{C}$.
- A horosphere is a sphere in $\mathbb{H}^3$ tangent to $\mathbb{C}$ or a plane in $\mathbb{H}^3$ parallel to $\mathbb{C}$.

The action of a matrix $N = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$ on $\mathbb{C}$ and in $\mathbb{H}^3$:

$$N(z) = \frac{pz + r}{qz + s} \quad \text{for } z \in \mathbb{C},$$
$$N(z + tj) = \frac{(pz + r)\overline{(qz + s)} + p\overline{q}t^2 + tj}{|qz + s|^2 + |q|^2 t^2} \quad \text{for } z + it \in \mathbb{H}^3.$$

Proposition 1. *The image of a horosphere $t = h$ by $N = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ is a horosphere tangent to $\mathbb{C}$ at p/q of radius $1/(2|q|^2 h)$.*

The height of a plane parallel to $\mathbb{C}$ and a radius of a horosphere is measured by the Euclidean metric.



Notation :

- $H(\infty; h)$: a horosphere parallel to $\mathbb{C}$ of height h
- $H(p, q; h)$: a horosphere tangent to $\mathbb{C}$ at p/q of radius $1/(2|q|^2 h)$; the image of $H(\infty; h)$ by N

Basic fact
Let C be a semicircle perpendicular to the complex plane. If C properly intersects some horosphere $H(p, q; h)$ then C has an image of larger radius than h.

If ξ is an irrational complex number then ξ is a point of approximation for $\mathrm{SL}(2,\mathbb{Z}[i])$: there exists a sequence $\{V_n\}$ of distinct transforms of $\mathrm{SL}(2,\mathbb{Z}[i])$ such that $V_n(\infty) \to \alpha$ and $V_n(\xi) \to \beta$ with $\alpha \neq \beta$.

Lemma 1.

$$k_1(\xi) = \sup_{(\alpha,\beta)\in E(\xi)} |\alpha - \beta|,$$

where $E(\xi) = \{(\alpha,\beta) \mid V_n(\infty) \to \alpha,\ V_n(\xi) \to \beta$ *for a sequence of distinct transforms*$\}$.

Recall
$k_1(\xi) := \sup \left\{ k \;\middle|\; \left|\xi - \frac{p}{q}\right| < \frac{1}{k|q|^2} \text{ has infinitely many co-prime integer solutions } (p,q) \in \mathbb{Z}[i]^2 \right\}$

The first assertion of Theorem 2, $HC_1 = \sqrt{3}$, can be described in the following way:

Lemma 2. *If C is a semicircle perpendicular to the complex plane then $V(C)$ has radius at least $\sqrt{3}/2$ for some $V \in \mathrm{SL}(2,\mathbb{Z}[i])$.*

We will give a geometric proof of this lemma following Ford's method.

3. Ford's method

A fundamental region of the Picard group $\mathrm{SL}(2, \mathbb{Z}[i])$:

$$\left\{ z + jt = x + iy + jt \in \mathbb{H}^3 \,\middle|\, x^2 + y^2 + t^2 > 1,\ |x| < \frac{1}{2},\ 0 < y < \frac{1}{2} \right\}.$$

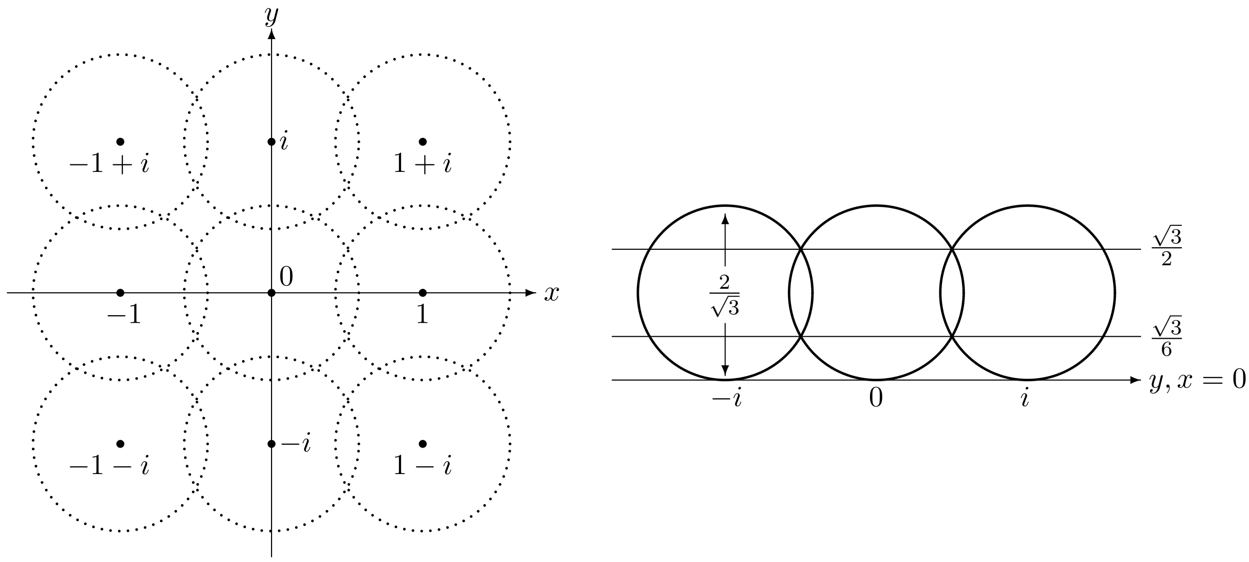
Since the Euclidean height of the lowest point of this region is $\sqrt{2}/2$, we only to show that when the radius r of a semicircle C perpendicular to $\mathbb{C}$ satisfies

$$\frac{\sqrt{2}}{2} < r < \frac{\sqrt{3}}{2},$$

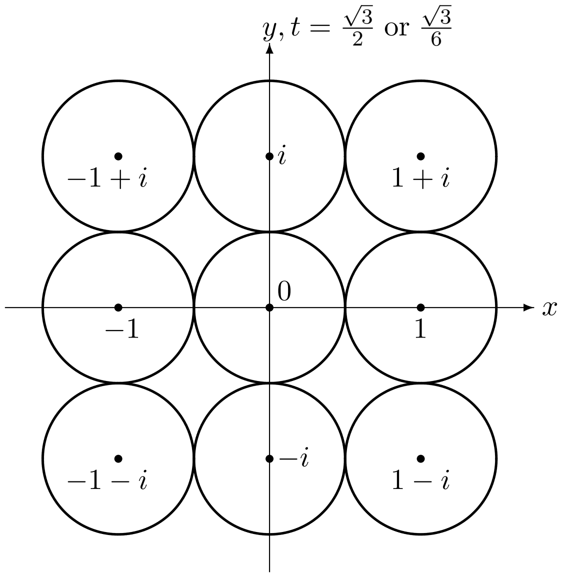
C properly intersects some images of $H(\infty; \frac{\sqrt{3}}{2})$.

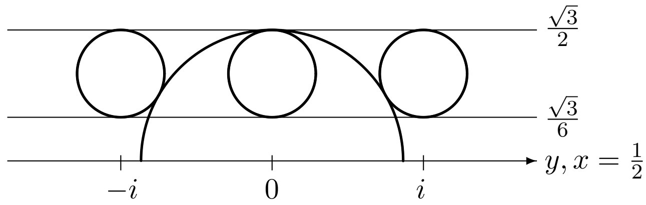
In the images of $H(\infty; \frac{\sqrt{3}}{2})$ by the action of $\mathrm{SL}(2, \mathbb{Z}[i])$,

- the radius of the horospheres tangent to $\mathbb{C}$ at the Gaussian integer points $\{m + in \,|\, m, n \in \mathbb{Z}\}$ is $\frac{1}{\sqrt{3}}$,
- the radius of the horospheres tangent to $\mathbb{C}$ at $\{\frac{1+i}{2} + m + in \,|\, m, n \in \mathbb{Z}\}$ is $\frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$.



The left is the projection of horospheres tangent to $\mathbb{C}$ at $m+ni$ $(m, n = -1, 0, 1)$ of radius $2/\sqrt{3}$. The right is the vertical slice of them by the plane $x = 0$.



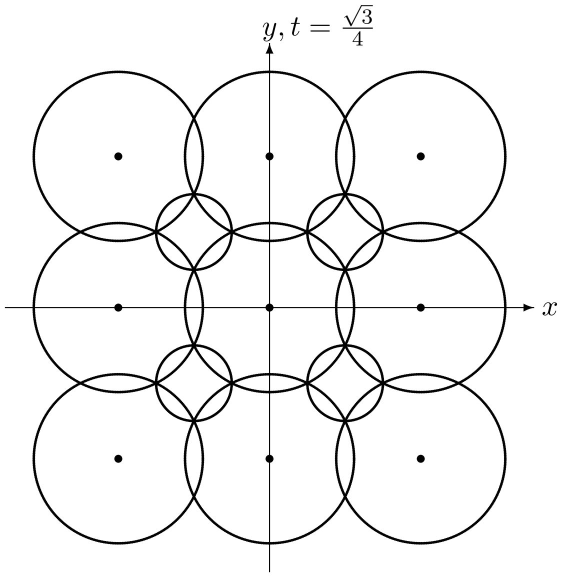


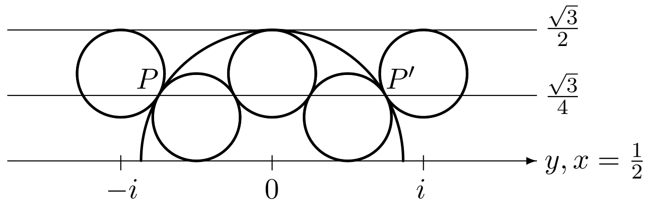
The left is the horizontal slice of horospheres tangent to $\mathbb{C}$ at $m + ni$ $(m, n = -1, 0, 1)$ of radius $2/\sqrt{3}$ by the plane $t = \sqrt{3}/2$ or $t = \sqrt{3}/6$. The right is the vertical slice of them by the plane $x = 1/2$.

We find that a geodesic on the plane $x = 1/2$, which is the axis of the matrix

$$\Lambda_1^{-1} = \begin{pmatrix} 2-i & 2i \\ -2i & 2+i \end{pmatrix},$$

is tangent to $H(\infty; \frac{\sqrt{3}}{2})$ and its images $H(m+ni, 1; \frac{\sqrt{3}}{2})$, $(m = 0, 1; n = -1, 0, 1)$.



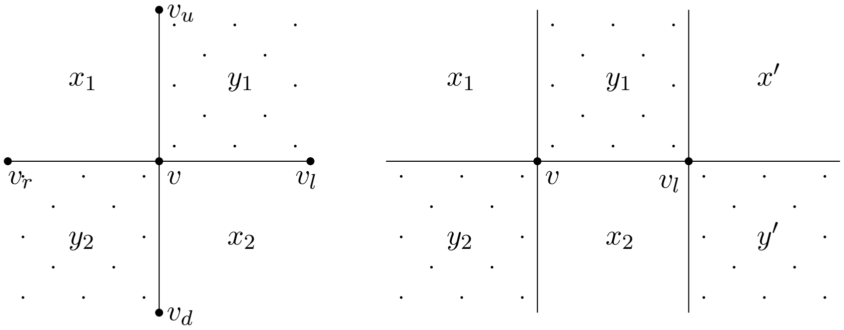


The left is the horizontal slice by the plane $t = \sqrt{3}/4$ of horospheres tangent at $m + ni$ $(m, n = -1, 0, 1)$ of radius $2/\sqrt{3}$ and at $m + ni$ $(m, n = -1/2, 1/2)$ of radius $\sqrt{3}/6$. The right is the vertical slice by the plane $x = 1/2$.

The plane $t = \sqrt{3}/4$, except for a set of points, is covered by the disks determined by the intersection with the horospheres tangent to the complex plane at $\{m + in \mid m, n \in \mathbb{Z}\}$ and $\{\frac{1+i}{2} + m + in \mid m, n \in \mathbb{Z}\}$. The exceptional points are on the lines which are the intersection of $t = \sqrt{3}/4$ with the planes $x = 1/2 + m$, $m \in \mathbb{Z}$ and $y = (1/2 + n)i$, $n \in \mathbb{Z}$. Geodesics through exceptional points, except for ones such as the axis of Λ_1^{-1}, properly intersects some images of $H(\infty; \frac{\sqrt{3}}{2})$. The action of Λ_1^{-1} maps the tangent point P to P', so there are infinitely many horospheres tangent to the axis. This is the case of the largest radius of a geodesic which does not properly intersect the images of $H(\infty; \frac{\sqrt{3}}{2})$. Lemma 2 is thus proved.

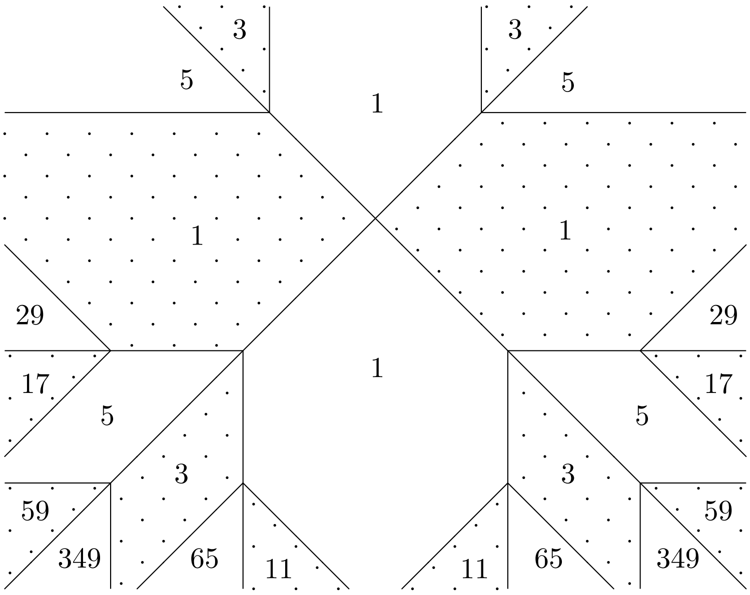
4. Two-color Markoff graph

Two-color Markoff graph of numbers: a graph obtained from a noncommutative infinite checkerboard whose faces are labeled by the following rule



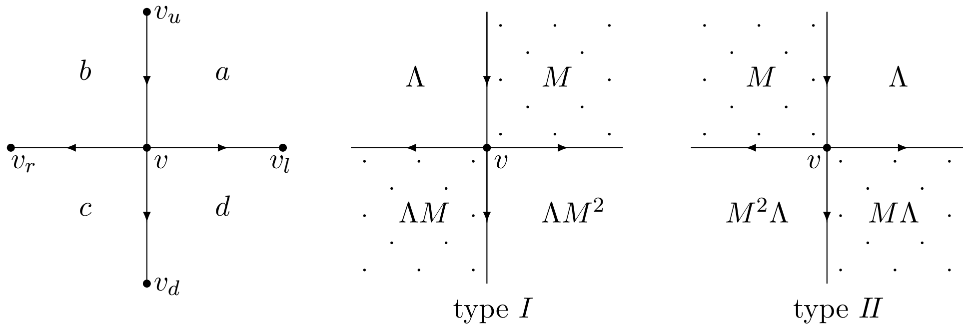
- Vertex relation $x_1 + x_2 = 2y_1y_2$, $2x_1x_2 = y_1^2 + y_2^2$.
- Edge relation $y' = 4x_2y_1 - y_2$. Besides $y' = 4x_1y_2 - y_1$, $y' = 4x_1y_1 - y_2$, $y' = 4x_2y_2 - y_1$.

Two-color Markoff graph of numbers built from a vertex $(1,1;1,1)$

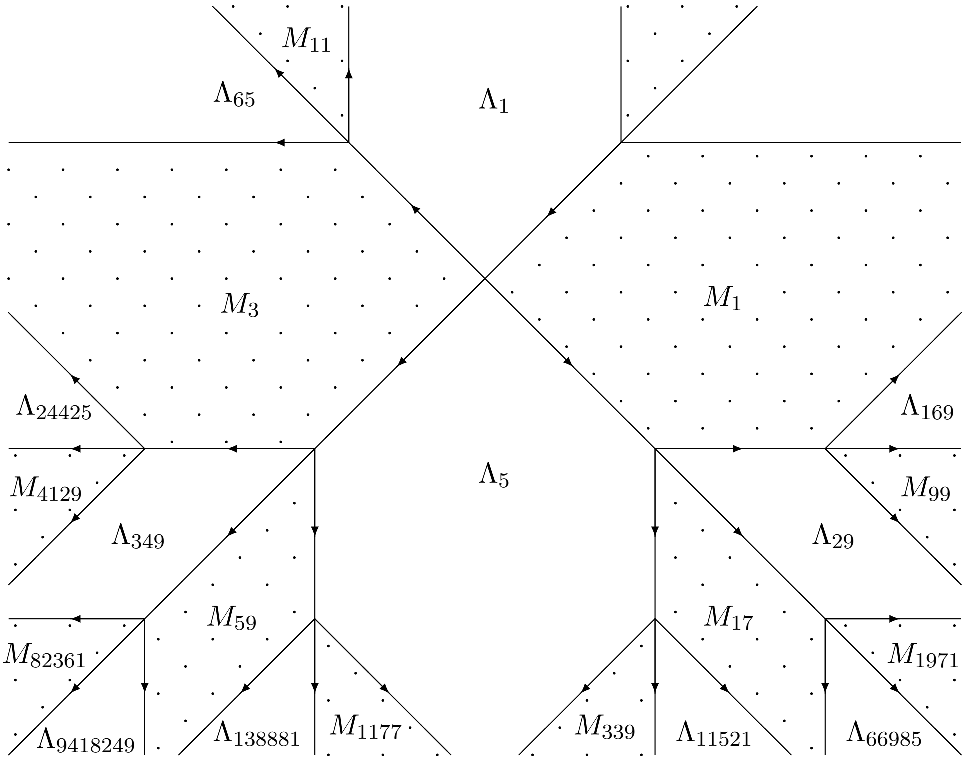


- $\mathcal{N}(\Lambda) = \{1, 5, 29, 65, 169, \cdots\}$: the set of integers labelling the white faces
- $\mathcal{N}(M) = \{1, 3, 11, 17, 41, \cdots\}$: the set of integers labelling the black faces

Two-color Markoff graph of matrices: a graph obtained from a noncommutative infinite checkerboard with directed edges whose faces labeled by the following rule



Two-color Markoff graph of matrices built from a vertex $(\Lambda_1, \Lambda_5; M_1, M_3)$



$$\Lambda_1 = \begin{pmatrix} 2+i & -2i \\ 2i & 2-i \end{pmatrix}, \quad M_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1+i & 1-i \\ 2i & 3-i \end{pmatrix}$$

$$\Lambda_5 = M_3 M_1 = \begin{pmatrix} 8+5i & 2-12i \\ 10i & 12-5i \end{pmatrix}, \quad M_3 = \Lambda_1 M_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 5+3i & 1-7i \\ 6i & 7-3i \end{pmatrix}$$

Recall

- $\mathcal{N}(\Lambda) = \{1, 5, 29, 65, 169, \cdots\}$: the set of integer solutions as x_1, x_2,
- $\mathcal{N}(M) = \{1, 3, 11, 17, 41, \cdots\}$: the set of integer solutions as y_1, y_2,

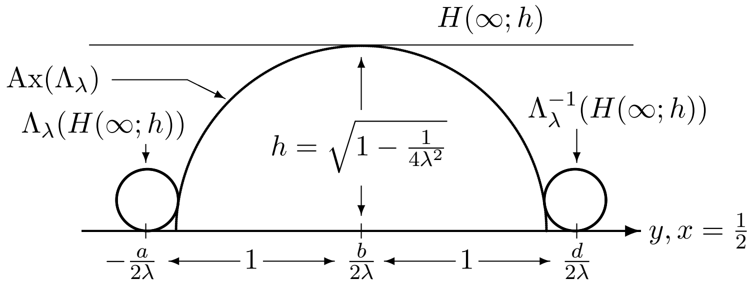
for $x_1 + x_2 = 2y_1y_2$, $2x_1x_2 = y_1^2 + y_2^2$. Each element of $\mathcal{N}(\Lambda)$ (or $\mathcal{N}(M)$) labels a white (or black) face.

Theorem 5. *Matrices Λ_λ, $\lambda \in \mathcal{N}(\Lambda)$ and M_m, $m \in \mathcal{N}(M)$ defined by the two-color Markoff graph of matrices built from a vertex $(\Lambda_1, \Lambda_5; M_1, M_3)$ have the following form:*

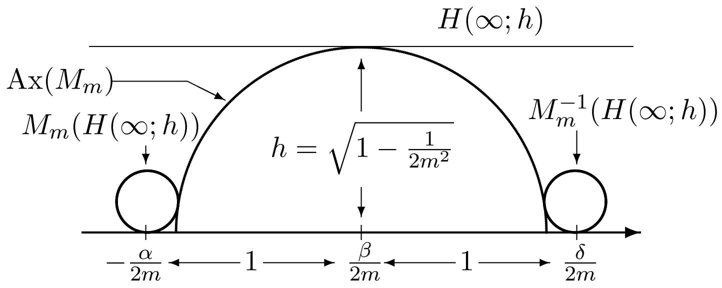
$$\Lambda_\lambda = \begin{pmatrix} a + \lambda i & b + ci \\ 2\lambda i & d - \lambda i \end{pmatrix}, \ \ (a, b, c, d) \in \mathbb{Z}^4, \ \ \mathrm{tr}(\Lambda_\lambda) = 4\lambda, \ \ \det(\Lambda_\lambda) = 1,$$

$$M_m = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha + mi & \beta + \gamma i \\ 2mi & \delta - mi \end{pmatrix}, \ \ (\alpha, \beta, \gamma, \delta) \in \mathbb{Z}^4, \ \ \mathrm{tr}(M_m) = 2\sqrt{2}m, \ \ \det(M_m) = 1.$$

The axis of Λ_λ, $\lambda \in \mathcal{N}(\Lambda)$ is on the plane $x = 1/2$ and is depicted as follows:



The axis of M_m, $m \in \mathcal{N}(M)$ is on the plane $x = 1/2$ and is depicted as follows:

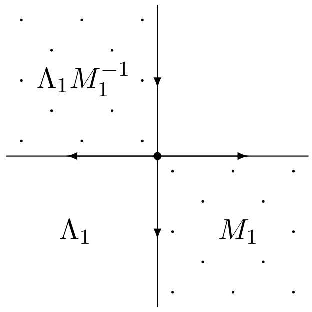


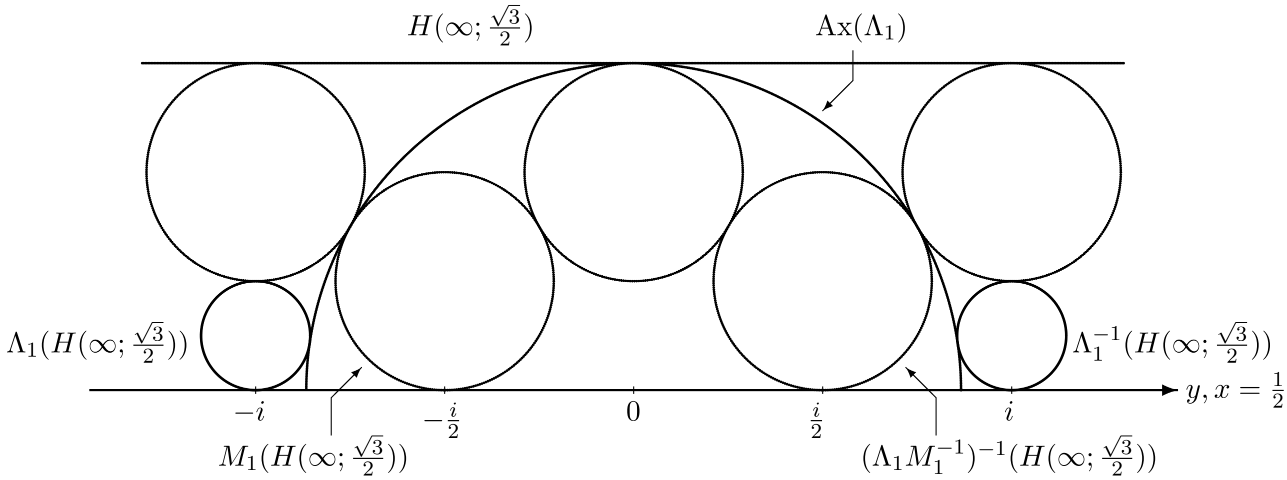
5. Outline of proof

The picture used in Ford's method is obtained from a vertex of the two-color Markoff graph of matrices:

$$\Lambda_1 M_1^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 3+i & -1-i \\ 2i & 1-i \end{pmatrix}$$

$$\Lambda_1 = \begin{pmatrix} 2+i & -2i \\ 2i & 2-i \end{pmatrix} \qquad M_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1+i & 1-i \\ 2i & 3-i \end{pmatrix}$$

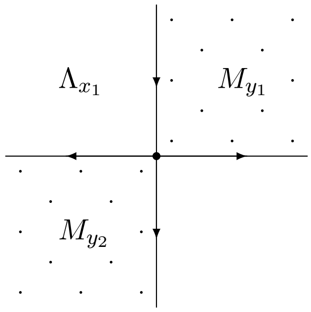




For each vertex of the two-color Markoff graph of matrices, we get a geodesic and horospheres which satisfy the similar property.

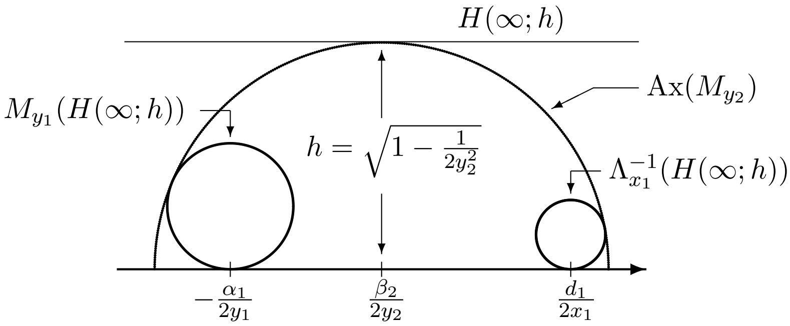
(A) For $M_{y_2} = \Lambda_{x_1} M_{y_1}$ in a vertex of type I, we get the axis of M_{y_2} to which three special horospheres are tangent.

$$\Lambda_{x_1} = \begin{pmatrix} a_1 + x_1 i & b_1 + c_1 i \\ 2x_1 i & d_1 - x_1 i \end{pmatrix}$$



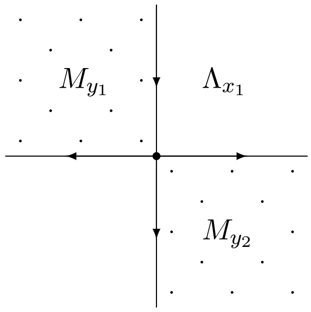
$$M_{y_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha_1 + y_1 i & \beta_1 + \gamma_1 i \\ 2y_1 i & \delta_1 - y_1 i \end{pmatrix}$$

$$M_{y_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha_2 + y_2 i & \beta_2 + \gamma_2 i \\ 2y_2 i & \delta_2 - y_2 i \end{pmatrix}$$



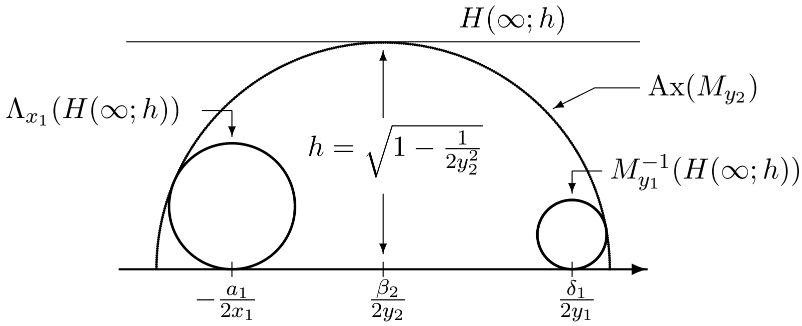
(B) For $M_{y_2} = M_{y_1}\Lambda_{x_1}$ in a vertex of type II, we get the axis of M_{y_2} to which three special horospheres are tangent.

$$M_{y_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha_1 + y_1 i & \beta_1 + \gamma_1 i \\ 2y_1 i & \delta_1 - y_1 i \end{pmatrix}$$

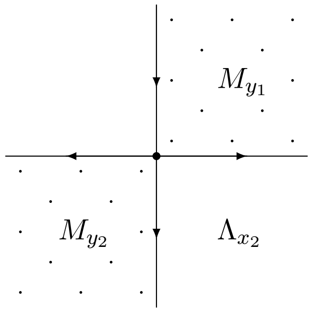


$$\Lambda_{x_1} = \begin{pmatrix} a_1 + x_1 i & b_1 + c_1 i \\ 2x_1 i & d_1 - x_1 i \end{pmatrix}$$

$$M_{y_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha_2 + y_2 i & \beta_2 + \gamma_2 i \\ 2y_2 i & \delta_2 - y_2 i \end{pmatrix}$$



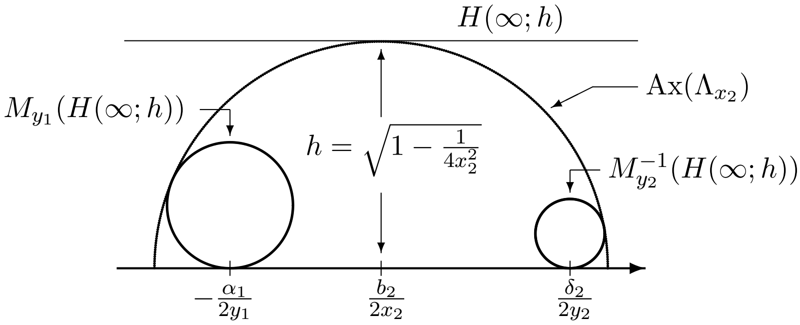
(C) For $\Lambda_{x_2} = M_{y_2} M_{y_1}$ in a vertex of type I, we get the axis of Λ_{x_2} to which three special horospheres are tangent.



$$M_{y_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha_1 + y_1 i & \beta_1 + \gamma_1 i \\ 2y_1 i & \delta_1 - y_1 i \end{pmatrix}$$

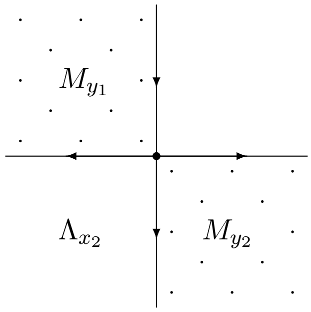
$$M_{y_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha_2 + y_2 i & \beta_2 + \gamma_2 i \\ 2y_2 i & \delta_2 - y_2 i \end{pmatrix}$$

$$\Lambda_{x_2} = \begin{pmatrix} a_2 + x_2 i & b_2 + c_2 i \\ 2x_2 i & d_2 - x_2 i \end{pmatrix}$$

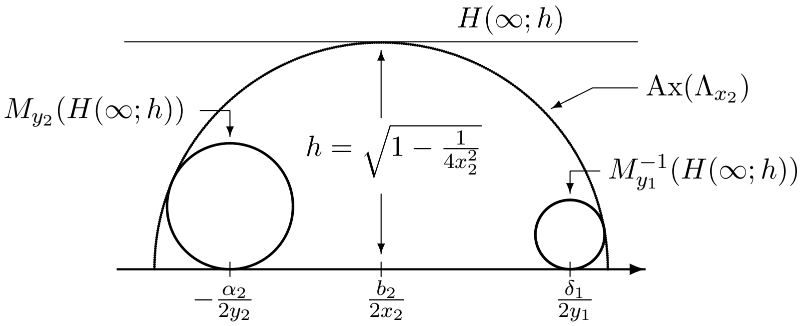


(D) For $\Lambda_{x_2} = M_{y_1} M_{y_2}$ in a vertex of type II, we get the axis of Λ_{x_2} to which three special horospheres are tangent.

$$M_{y_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha_1 + y_1 i & \beta_1 + \gamma_1 i \\ 2y_1 i & \delta_1 - y_1 i \end{pmatrix}$$



$$\Lambda_{x_2} = \begin{pmatrix} a_2 + x_2 i & b_2 + c_2 i \\ 2x_2 i & d_2 - x_2 i \end{pmatrix} \qquad M_{y_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha_2 + y_2 i & \beta_2 + \gamma_2 i \\ 2y_2 i & \delta_2 - y_2 i \end{pmatrix}$$

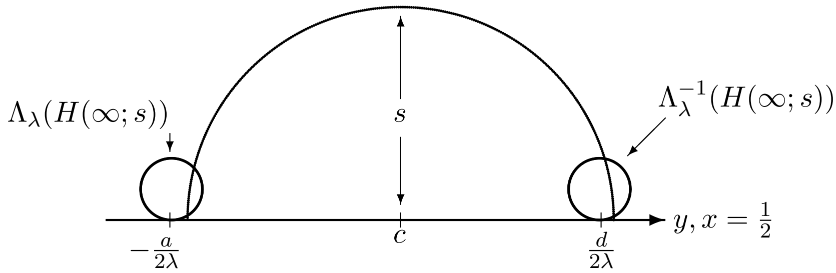


Thus the two-color Markoff graph of matrices determines a system of geodesics on $x = 1/2$ which have special kissing horospheres.

Now let us examine geodesics on $x = 1/2$ which intersect horospheres and are close to a geodesic in the system.

Lemma 3. *A geodesic centered on a point $c \in \mathbb{C}$ of radius s properly intersects a horosphere $H(p, q; s)$ if and only if*

$$s^2 - \frac{1}{|q^2|} < \left| c - \frac{p}{q} \right|^2 < s^2 + \frac{1}{|q^2|}.$$



For the image of $H(\infty; s)$ by Λ_λ we define

$$C_\lambda^{i-} : s = \sqrt{\left(c + \frac{a}{2\lambda}\right)^2 - \frac{1}{4\lambda^2}}, \quad C_\lambda^{i+} : s = \sqrt{\left(c + \frac{a}{2\lambda}\right)^2 + \frac{1}{4\lambda^2}},$$

which are curves on (c, s)-plane.

For the image by Λ_λ^{-1} we define

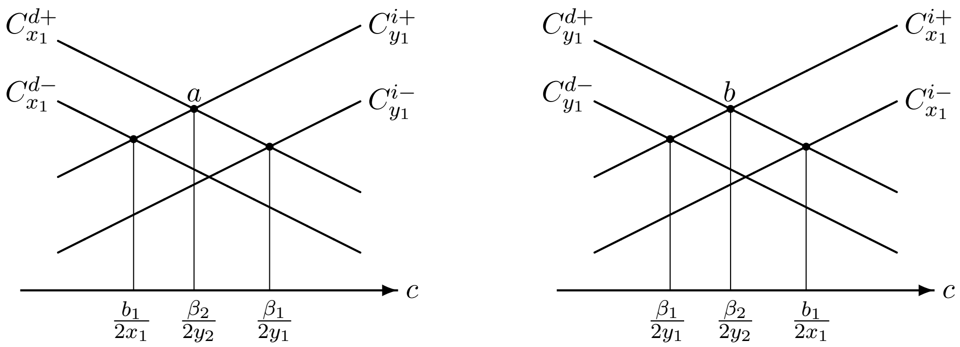
$$C_\lambda^{d-} : s = \sqrt{\left(c - \frac{d}{2\lambda}\right)^2 - \frac{1}{4\lambda^2}}, \quad C_\lambda^{d+} : s = \sqrt{\left(c - \frac{d}{2\lambda}\right)^2 + \frac{1}{4\lambda^2}}.$$

Corollary 1. *A geodesic on the plane $x = 1/2$ centered on a point c of radius s properly intersects the image of $H(\infty; s)$*

(i) by Λ_λ if and only if (c, s) is between the curves C_λ^{i-} and C_λ^{i+}.

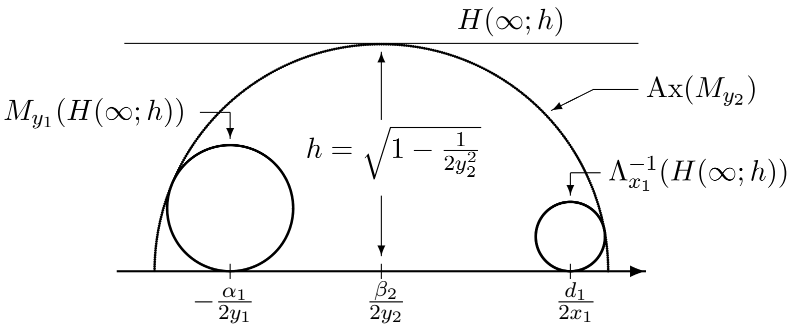
(ii) by Λ_λ^{-1} if and only if (c, s) is between the curves C_λ^{d-} and C_λ^{d+}.

Lemma 4. *If a vertex v is of type I, the shape of the curves $C_{x_1}^{d+}$, $C_{x_1}^{d-}$, $C_{y_1}^{i+}$, and $C_{y_1}^{i-}$ around their intersection points is depicted as in (A). If v is of type II, that of $C_{x_1}^{i+}$, $C_{x_1}^{i-}$, $C_{y_1}^{d+}$, and $C_{y_1}^{d-}$ is depicted as in (B).*

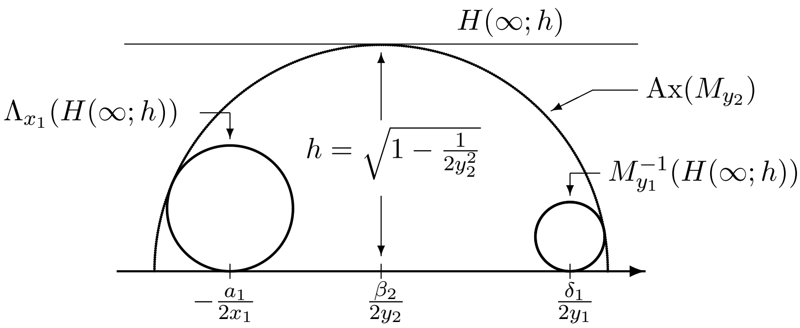


(A) type I (B) type II

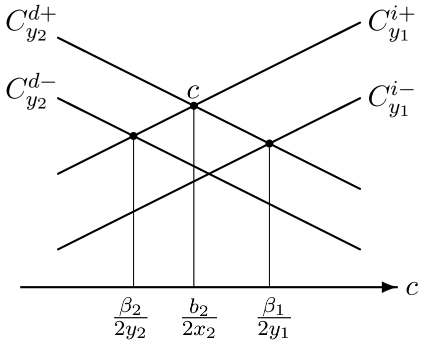
- the point a is attained by the following geodesic and horospheres.



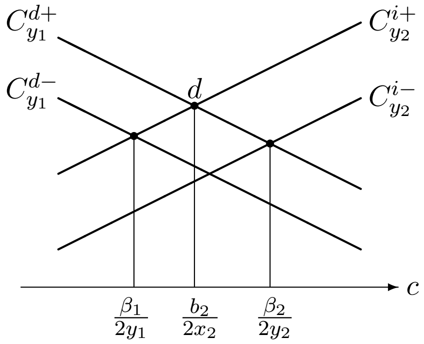
- the point b is attained by the following geodesic and horospheres.



Lemma 5. *If a vertex v is of type I, the shape of the curves $C^{d+}_{y_2}$, $C^{d-}_{y_2}$, $C^{i+}_{y_1}$, and $C^{i-}_{y_1}$ around their intersection points is depicted as in (C). If v is of type II, that of $C^{i+}_{y_2}$, $C^{i-}_{y_2}$, $C^{d+}_{y_1}$, and $C^{d-}_{y_1}$ is depicted as in (D).*

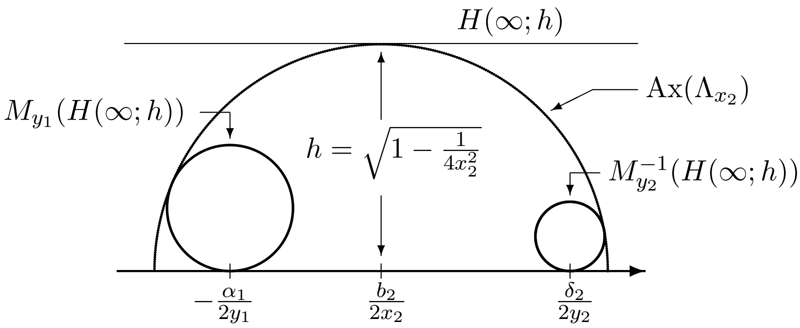


(C) type I

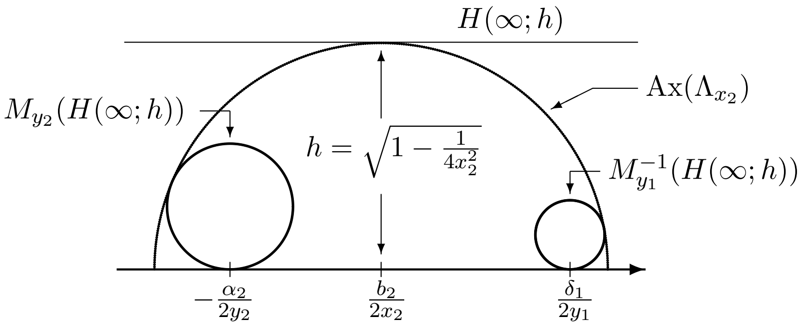


(D) type II

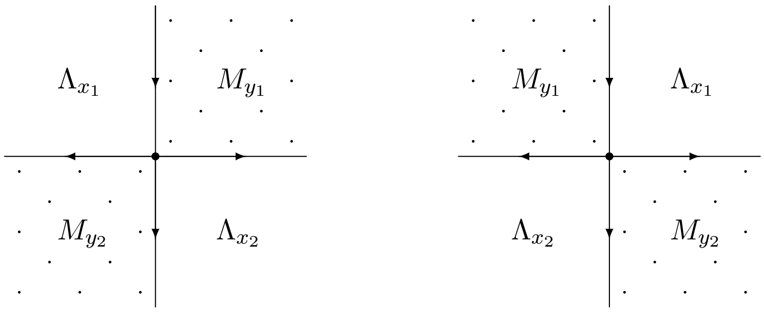
- the point c is attained by the following geodesic and horospheres.

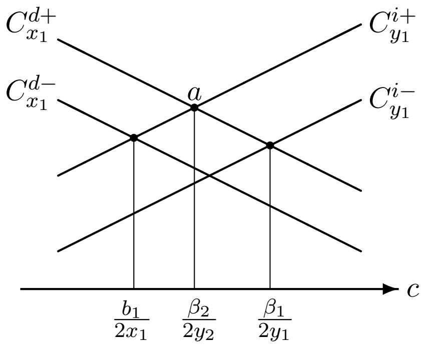


- the point d is attained by the following geodesic and horospheres.

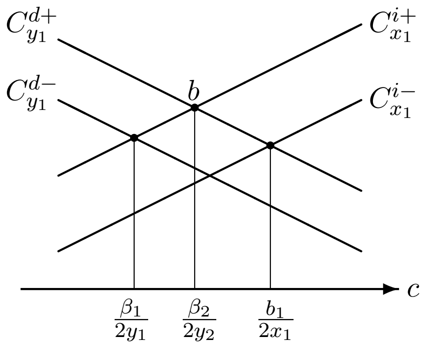


Using Lemmas 4 and 5 repeatedly, a neighborhood of the valley point a (or b, c, d) is covered by two bands between four new curves, and we get two new valley points. The height of them is larger than that of a. Thus we obtain successively values of the Lagrange spectrum.

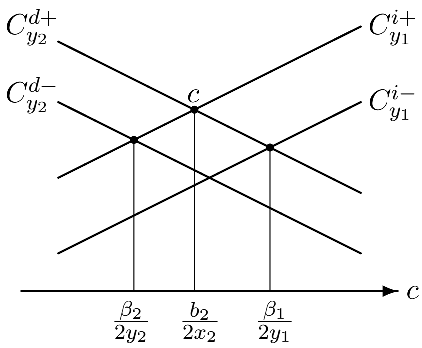




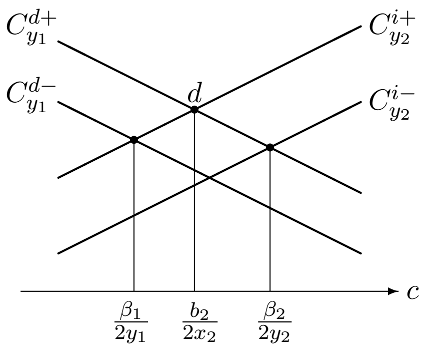
(A) type I



(B) type II

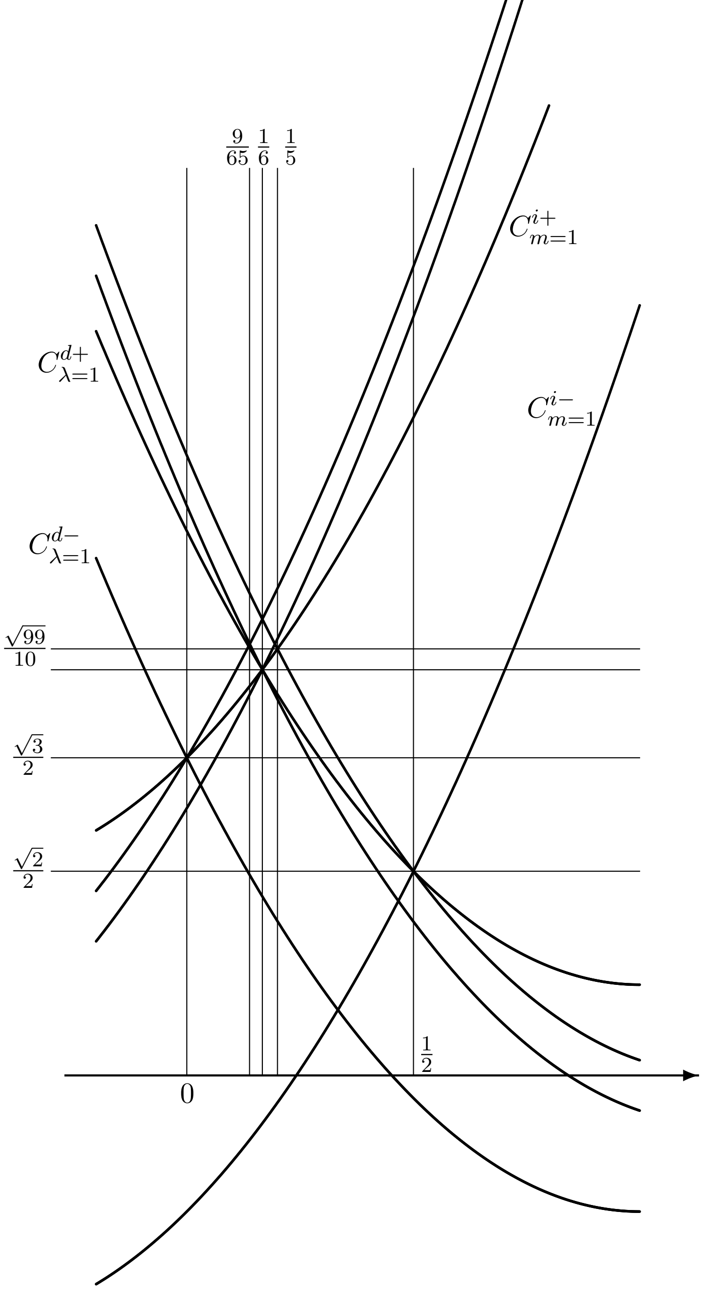


(C) type I



(D) type II

Example.



$\frac{9}{65}$ $\frac{1}{6}$ $\frac{1}{5}$
$C^{i+}_{m=1}$
$C^{d+}_{\lambda=1}$
$C^{i-}_{m=1}$
$C^{d-}_{\lambda=1}$
$\frac{\sqrt{99}}{10}$
$\frac{\sqrt{3}}{2}$
$\frac{\sqrt{2}}{2}$
$\frac{1}{2}$
0

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ON THE REPRODUCING KERNEL FOR THE SPACE OF SEMI-EXACT ANALYTIC DIFFERENTIALS

SACHIKO HAMANO

We discuss here some analytic invariants associated with a holomorphic family of Riemann surfaces.

Let R be a bordered Riemann surface of genus $g(\geq 0)$ with a finite number of C^ω-smooth contours C_j $(j = 1, \ldots, \nu)$ in a larger Riemann surface $\widetilde{R}$. Let $S(R)$ be the space of *semi-exact* L^2-analytic differentials on R. Let $\widetilde{K}(z,\zeta)$ denote the reproducing kernel function for $S(R)$.

Definition. Let R be as above. Fix two points $a,\ b \in R$ with local coordinates $U_a : |z-\zeta| < r_a$ and $U_b : |z| < r_b$, where a and b correspond to ζ and 0, respectively (where U_a and U_b have no relations). Among all harmonic functions u on $R \setminus \{a,b\}$ with two logarithmic poles of $\log|z-\zeta|$ at a and $-\log|z|$ at b normalized so that $\lim_{z\to 0}(u(z)+\log|z|) = 0$, we have uniquely determined functions h_i $(i = 1,0)$ with the L_i-boundary conditions $(i = 1,0)$: (L_1) for each C_j, h_1 satisfies $h_1(z) = c_j$ (constant) on C_j and $\int_{C_j} \frac{\partial h_1}{\partial n_z} ds_z = 0$; (L_0) h_0 satisfies $\frac{\partial h_0(z)}{\partial n_z} = 0$ on C_j. We call $h_i(z)$ the *L_i-principal function* and $\mu_i := \lim_{z\to\zeta}(h_i(z) - \log|z-\zeta|)$ the *L_i-constant* for (R,b,a) with respect to the local coordinates U_a and U_b (simply, for $(R,0,\zeta)$).

Theorem 1 ([3])**.** *Let the notation be as above. We have*

$$\widetilde{K}(z,\zeta) = \frac{2}{\pi}\frac{\partial^2 h_1(z,\zeta)}{\partial z \partial\bar{\zeta}}, \qquad \widetilde{K}(\zeta,\zeta) = \frac{1}{\pi}\frac{\partial^2 \mu_1(\zeta)}{\partial\zeta\partial\bar{\zeta}}.$$

Let $\pi : \widetilde{\mathcal{R}} \to B$ be a holomorphic family such that $\widetilde{\mathcal{R}}$ is a complex 2-dimensional manifold, π is a holomorphic projection from $\widetilde{\mathcal{R}}$ onto a disk B in $\mathbb{C}_t$, and each fiber $\widetilde{R}(t) = \pi^{-1}(t)$, $t \in B$ is irreducible and non-singular in $\widetilde{\mathcal{R}}$. We set $\widetilde{\mathcal{R}} = \cup_{t\in B}(t, \widetilde{R}(t))$. Let $\mathcal{R} = \cup_{t\in B}(t, R(t))$ be a subdomain with C^ω smooth boundary $\partial\mathcal{R} = \cup_{t\in B}(t,\partial R(t))$ in $\widetilde{\mathcal{R}}$ such that $\widetilde{R}(t) \Supset R(t) \neq \emptyset$ for $t \in B$, $R(t)$ is a bordered Riemann surface of genus $g(\geq 0)$ in $\widetilde{R}(t)$, and $\partial R(t)$ in $\widetilde{R}(t)$ consists of a finite number of C^ω smooth contours $C_j(t)$ $(j = 1, \ldots, \nu)$.

Theorem 2 ([3])**.** *We assume that the total space $\mathcal{R} = \cup_{t\in B}(t, R(t))$ is 2-dimensional pseudoconvex in $\widetilde{\mathcal{R}}$. Then $\log\widetilde{K}(t,\zeta,\zeta)$ is a plurisubharmonic function on $\mathcal{R}$.*

This phenomenon is the same as the Bergman metrics (see [5]).

Here we recall the definition of Schiffer spans for *planar* Riemann surfaces. Let R be a finite bordered planar Riemann surface. Let $\mathcal{P}(R)$ be the set of

Partly supported by JSPS Grant-in-Aid for Young Scientists (B), 23740098.

all univalent functions P on R with the expression

$$P(z) = \frac{1}{z-\zeta} + 0 + A_1(z-\zeta) + A_2(z-\zeta)^2 + \cdots$$

at a given point $\zeta \in R$. The *Schiffer span* s for (R,ζ) is defined by

$$s := \frac{2}{\pi} \sup_{P\in\mathcal{P}(R)} \{\text{the Euclidean area } \mathcal{E}_P \text{ of } E_P := \mathbb{C}\setminus P(R)\}.$$

The Schiffer span $s(\zeta)$ induces the metric $s(\zeta)|d\zeta|^2$ on R.

Theorem 3 ([3])**.** *Let R be a finite bordered* planar *Riemann surface. Then we have the following:*

(i) *The metrics* $\widetilde{K}(\zeta,\zeta)|d\zeta|^2$, $\frac{1}{\pi}\frac{\partial^2\mu_1(\zeta)}{\partial\zeta\partial\bar\zeta}|d\zeta|^2$, $-\frac{1}{\pi}\frac{\partial^2\mu_0(\zeta)}{\partial\zeta\partial\bar\zeta}|d\zeta|^2$, *and* $s(\zeta)|d\zeta|^2$ *are all identical on R;*

(ii) $\widetilde{K}(\zeta,\zeta)|d\zeta|^2$ *is of negative curvature at every point $\zeta\in R$;*

(iii) $\widetilde{K}(\zeta,\zeta)|d\zeta|^2$ *is complete on R.*

For the proofs of the above theorems, we use the following variational formulas for principal functions and the plurisubharmonic variation of the Schiffer span under pseudoconvexity ([2]).

Lemma 4 ([1], [4])**.** *Let $R(t)$ is of genus $g(\geq 0)$, and let $\{A_l(t),B_l(t)\}_{l=1}^g$ be a canonical homology basis on $R(t)$ such that each $A_l(t)$ and $B_l(t)$ varies continuously with $t\in B$. Then we have*

$$\frac{\partial^2\mu_1(t)}{\partial t\partial\bar t} = \frac{1}{\pi}\int_{\partial R(t)} k_2(t,z)\left|\frac{\partial h_1(t,z)}{\partial z}\right|^2 ds_z + \frac{4}{\pi}\iint_{R(t)}\left|\frac{\partial^2 h_1(t,z)}{\partial\bar t\partial z}\right|^2 dxdy;$$

$$\frac{\partial^2\mu_0(t)}{\partial t\partial\bar t} = -\left(\frac{1}{\pi}\int_{\partial R(t)} k_2(t,z)\left|\frac{\partial h_0(t,z)}{\partial z}\right|^2 ds_z + \frac{4}{\pi}\iint_{R(t)}\left|\frac{\partial^2 h_0(t,z)}{\partial\bar t\partial z}\right|^2 dxdy\right) - \frac{2}{\pi}\mathrm{Im}\left\{\sum_{l=1}^g \frac{\partial}{\partial t}\left(\int_{A_l(t)} *dh_0(t,z)\right)\frac{\partial}{\partial\bar t}\left(\int_{B_l(t)} *dh_0(t,z)\right)\right\}.$$

Here, for the defining function $\varphi(t,z)$ of $\partial\mathcal{R}$,

$$k_2(t,z) = \left(\frac{\partial^2\varphi}{\partial t\partial\bar t}\left|\frac{\partial\varphi}{\partial z}\right|^2 - 2\mathrm{Re}\left\{\frac{\partial^2\varphi}{\partial\bar t\partial z}\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial\bar z}\right\} + \left|\frac{\partial\varphi}{\partial t}\right|^2\frac{\partial^2\varphi}{\partial z\partial\bar z}\right)\left|\frac{\partial\varphi}{\partial z}\right|^{-3}.$$

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Analytic Study of Singular Curves

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Feb. 16, 2015

Introduction

Generalized Jacobi varieties for singular curves were algebraically defined by Rosenlicht in 1954. Since then, the theory has been developed extremely. A genaralized Jacobi variety is analytically considered as a complex Lie group. We generalize the analytic theory for compact Riemann surfaces to singular curves. We expect to get some analytic properties of generalized Jacobi varieties from our treatment.

1 Construction of singular curves

X: an irreducible non-singular complex projective algebraic curve (i.e. a compact Riemann surface)

$\mathcal{O}_X$: the structure sheaf on X

$S \subset X$: a finite subset

R : an equivalent relation on S

$\overline{S} := S/R$

$\overline{X} := (X \setminus S) \cup \overline{S}$

$\rho : X \longrightarrow \overline{X}$ the canonical projection

We use notations according to

J. -P. Serre, Groupes algébriques et corps de classes, Hermann, Paris, 1959.

Definition 1. $\mathfrak{m}$: a modulus with support S

$\underset{\text{def}}{\iff} \quad \forall P \in S$, $\mathfrak{m}(P) > 0$ integer

We may assume $\deg \mathfrak{m} \geqq 2$.

$\mathrm{Mer}(X)$: the field of meromorphic functions on X

$\forall f \in \mathrm{Mer}(X)$, $\forall P \in X$, $\mathrm{ord}_P(f)$: the order of f at P

Definition 2. $f, g \in \mathrm{Mer}(X)$

$$f \equiv g \mod \mathfrak{m}$$

if $\mathrm{ord}_P(f-g) \geq \mathfrak{m}(P)$ for any $P \in S$.

$\rho_*\mathcal{O}_X$: the direct image of $\mathcal{O}_X$ by ρ

$\forall Q \in \overline{S}$

$\mathcal{I}_Q$: the ideal of $(\rho_*\mathcal{O}_X)_Q$ formed by the function f with

$\mathrm{ord}_P(f) \geq \mathfrak{m}(P)$, $\forall P \in \rho^{-1}(Q)$

We define a sheaf $\mathcal{O}_\mathfrak{m}$ on $\overline{X}$ by

$$\mathcal{O}_{\mathfrak{m}\,,Q} := \begin{cases} (\rho_*\mathcal{O}_X)_Q = \mathcal{O}_{X,Q} & \text{if } Q \in X \setminus S \\ \mathbb{C} + \mathcal{I}_Q & \text{if } Q \in \overline{S}. \end{cases}$$

$(\overline{X}, \mathcal{O}_\mathfrak{m})$: 1-dimensional compact reduced complex space

We denote it by $X_\mathfrak{m}$.

Conversely, any reduced and irreducible singular curve is obtained as above.

2 Genus of $X_\mathfrak{m}$

$\forall Q \in X_\mathfrak{m}$

$$\delta_Q := \dim((\rho_*\mathcal{O}_X)_Q/\mathcal{O}_{\mathfrak{m}\,,Q})$$

$$\delta := \sum_{Q \in X_\mathfrak{m}} \delta_Q = \deg \mathfrak{m} - \#\overline{S}.$$

g: the genus of X

$\pi := g + \delta$: the genus of $X_{\mathfrak{m}}$

$$\dim H^1(X_{\mathfrak{m}}, \mathcal{O}_{\mathfrak{m}}) = \pi$$

3 Riemann-Roch Theorem

Definition 3. A divisor D on X is said to be prime to S if $D(P) = 0$ for $P \in S$.

$\mathrm{Div}(X_{\mathfrak{m}})$: the group of divisors prime to S

$\mathrm{Mer}(X_{\mathfrak{m}})$: the field of meromorphic functions on $X_{\mathfrak{m}}$

$$\rho^* \mathrm{Mer}(X_{\mathfrak{m}}) \subset \mathrm{Mer}(X)$$

$f \in \mathrm{Mer}(X_{\mathfrak{m}})$

$$(f) = \sum_{Q \in X_{\mathfrak{m}}} \mathrm{ord}_Q(f) Q,$$

where $\mathrm{ord}_Q(f) = \sum_{P \in \rho^{-1}(Q)} \mathrm{ord}_P(f \circ \rho)$.

Definition 4. $D_1, D_2 \in \mathrm{Div}(X_{\mathfrak{m}})$

$$D_1 \sim D_2 \quad \underset{\text{def}}{\Longleftrightarrow} \quad \exists f \in \mathrm{Mer}(X_{\mathfrak{m}}) \; s.t. \; D_1 - D_2 = (f)$$

$$\overline{\mathrm{Div}(X_{\mathfrak{m}})} := \mathrm{Div}(X_{\mathfrak{m}})/\sim, \quad \overline{\mathrm{Div}^0(X_{\mathfrak{m}})} := \mathrm{Div}^0(X_{\mathfrak{m}})/\sim$$

$D \in \mathrm{Div}(X_{\mathfrak{m}}) \subset \mathrm{Div}(X)$

$L(D) := \{f \in \mathrm{Mer}(X); (f) \geqq -D\}$

$\mathcal{L}(D)$: sheafication of $L(D)$

$$\mathcal{L}_{\mathfrak{m}}(D)_Q := \begin{cases} \mathcal{O}_{\mathfrak{m},Q} & \text{if } Q \in \overline{S} \\ \mathcal{L}(D)_Q & \text{if } Q \in X \setminus S. \end{cases}$$

Theorem 1 (Riemann-Roch Theorem)**.** *Let $X, S, \mathfrak{m}, X_{\mathfrak{m}}$ be as above. Let $D \in \mathrm{Div}(X_{\mathfrak{m}})$. Then, $H^0(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}(D))$ and $H^1(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}(D))$ are finite dimensional, and we have*

$$\dim H^0(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}(D)) - \dim H^1(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}(D)) = \deg D + 1 - \pi.$$

4 Serre duality

$U \subset X_{\mathfrak{m}}$: an open set

$\Omega_{\mathfrak{m}}(U) := \{$a mero. 1-form ω on $\rho^{-1}(U)$ satisfying the condition $(*)\}$

The condition $(*)$:

$\forall Q \in U, \forall f \in \mathcal{O}_{\mathfrak{m},Q}$

$$\sum_{P \in \rho^{-1}(Q)} \mathrm{Res}_P(\rho^* f \omega) = 0.$$

$\Omega_{\mathfrak{m}}$: the sheaf defined by $\{\Omega_{\mathfrak{m}}(U), r_V^U\}$ (the duality sheaf on $X_{\mathfrak{m}}$)

Ω : the sheaf of germs of hol. 1-forms on X

$D \in \mathrm{Div}(X_{\mathfrak{m}}) \subset \mathrm{Div}(X)$

$W \subset X$: an open subset

$\Omega(D)(W) :=$ { a meromorphic 1-form η on W with $(\eta) \geq -D$ on W}.

$\Omega(D)$: the sheaf on X defined by $\{\Omega(D)(W), r^W_{W'}\}$

We define a sheaf $\Omega_{\mathfrak{m}}(D)$ on $X_{\mathfrak{m}}$ by

$$\Omega_{\mathfrak{m}}(D)_Q := \begin{cases} \Omega_{\mathfrak{m},Q} & \text{if } Q \in \overline{S} \\ \Omega(D)_Q & \text{if } Q \in X \setminus S. \end{cases}$$

Theorem 2 (Serre duality)**.** *For any $D \in \mathrm{Div}(X_{\mathfrak{m}})$ we have*

$$H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}}(-D)) \cong H^1(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}(D))^*,$$

where $H^1(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}(D))^*$ is the dual space of $H^1(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}(D))$.

For a completely analytic proof of Theorem 2, we need special sheaves $\mathcal{E}^{(1,0)}_{\mathfrak{m}}$ and $\mathcal{E}^{(2)}_{\mathfrak{m}}$, some modifications of the proof of non-singular case. However we omit details.

Using Theorem 2, we can rewrite the Riemann-Roch Theorem as follows

Theorem 3 (Riemann-Roch Theorem (second version))**.** *For any $D \in \mathrm{Div}(X_{\mathfrak{m}})$ we have*

$$\dim H^0(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}(D)) - \dim H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}}(-D)) = \deg D + 1 - \pi.$$

5 Generalized Abel's Theorem

Rosenlich first formulated and proved a generalized Abel's theorem for a singular curve which was considered algebraically.

Jambois tried to treat it analytically. However, we think Jambois' argument was imcomplete.

(T. Jambois, The theorem of Torelli for singular curves, Trans. Amer. Math. Soc., **239** (1978),123–146)

Rosenlicht and Jambois considered functions f satisfying

$$\boxed{f \equiv 1 \bmod \mathfrak{m}}.$$

This means that f takes the common value 1 at all singular points. Then it is a special function for the number of singular points $\neq 1$ in general.

We assign a non-zero constant c_Q to each point Q in $\overline{S}$. We call

$$c(\overline{S}) := (c_Q)_{Q \in \overline{S}}$$

a multiconstant on $\overline{S}$.

Definition 5. $f \in \mathrm{Mer}(X)$, $c(\overline{S})$: a multiconstant on $\overline{S}$

We write

$$f \equiv c(\overline{S}) \bmod \mathfrak{m}$$

if $\mathrm{ord}_P(f - c_Q) \geq \mathfrak{m}\,(P)$ for any $P \in S$ with $\rho(P) = Q$ at any $Q \in \overline{S}$.

Our formulation of a generalized Abel's theorem is the following.

Theorem 4. $D \in \mathrm{Div}(X_{\mathfrak{m}})$ *with* $\deg D = 0$

$\exists f \in \mathrm{Mer}(X)$ *with* $f \equiv c(\overline{S}) \bmod \mathfrak{m}$ *for some* $c(\overline{S})$ *such that* $D = (f)$

$\Longleftrightarrow$

$\exists$ 1-*chain* $c \in C_1(X \setminus S)$ *with* $\partial c = D$ *such that*

$$\int_c \rho^* \omega = 0, \quad \forall \omega \in H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$$

6 Proof of Theorem 4

$D \in \mathrm{Div}(X_{\mathfrak{m}})$, $X_D := \{P \in X; D(P) \geqq 0\}$

Definition 6. A C^∞ function f on X_D is called a weak solution of D if it satisfies the following condition:

$\forall P \in X$

$\exists (U, z)$: a coordinate nbd. of P with $z(P) = 0$

$\exists \psi$: C^∞ function on U with $\psi(P) \neq 0$ such that

$$f = \psi z^{D(P)} \quad \text{on} \quad U \cap X_D$$

Sheaf $\mathcal{E}_{\mathfrak{m}}^{(1)}$

$U \subset X_{\mathfrak{m}}$: an open set We define

$\mathcal{E}_{\mathfrak{m}}^{(1)}(U) := \{$a C^∞ 1-form ω on $U \setminus (U \cap \overline{S})$ satisfying the condition $(\star\star)\}$.

The condition $(\star\star)$:

Let $Q \in U \cap \overline{S}$. We set $\rho^{-1}(Q) = \{P_1, \ldots, P_k\}$. Let $V \subset U$ be an open neighbourhood of Q such that

$$\rho^{-1}(V) = \bigsqcup_{i=1}^{k} V_i \quad (P_i \in V_i),$$

(V_i, z_i) is a coordinate neighbourhood of P_i with $z_i(P_i) = 0$ and there exist C^∞ functions φ_i and ψ_i on $V_i \setminus \{P_i\}$ with

$$\rho^*\omega = \varphi_i dz_i + \psi_i d\overline{z}_i \quad \text{on} \quad V_i \setminus \{P_i\}.$$

Then limits

$$\lim_{P \to P_i} \varphi_i(P) z_i(P)^{\mathfrak{m}(P_i)} \quad \text{and} \quad \lim_{P \to P_i} \psi_i(P) \overline{z_i(P)}^{\mathfrak{m}(P_i)}$$

exist.

Then a presheaf $\{\mathcal{E}_{\mathfrak{m}}^{(1)}(U), r_V^U\}$ defines a sheaf $\mathcal{E}_{\mathfrak{m}}^{(1)}$ on $X_{\mathfrak{m}}$.

Lemma 1. *Suppose that* $c : [0,1] \longrightarrow X \setminus S$ *is a curve and* U *is a relatively compact open neighbourhood of* $c([0,1])$ *in* $X \setminus S$*. Then there exists a weak solution* f *of* ∂c *with* $f|(X \setminus U) = 1$ *such that for every* 1*-form* $\omega \in H^0(X_{\mathfrak{m}}, \mathcal{E}_{\mathfrak{m}}^{(1)})$ *with* $d\omega = 0$ *we have*

$$\frac{1}{2\pi\sqrt{-1}} \iint_X \frac{df}{f} \wedge \rho^*\omega = \int_c \rho^*\omega.$$

Lemma 2. *For any* $D \in \mathrm{Div}(X_{\mathfrak{m}})$ *the following two conditions are equivalent.*

(1) There exists a meromorphic function g *on* X *such that* $D = (g)$ *and we have a branch* f *of* $\log g$ *defined in a neighbourhood of* S *with the property*

$$\sum_{P \in \rho^{-1}(Q)} \mathrm{Res}_P(f\omega) = 0$$

for any point $Q \in \overline{S}$ *and for any* $\omega \in H^0(X, \rho^*\Omega_{\mathfrak{m}})$*.*

(2) There exist a meromorphic function g *on* X *and a multiconstant* $c(\overline{S})$ *such that*

$$D = (g) \quad \textit{and} \quad g \equiv c(\overline{S}) \bmod \mathfrak{m}.$$

Proof of Theorem 4 (Necessity)

Assumption

$\exists 1-\textit{chain}\ \ c \in C_1(X \setminus S)$ with $\partial c = D$ s.t.

$$\int_c \rho^*\omega = 0, \quad \forall \omega \in H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$$

By Lemma 1

$\exists f$: a weak solution of $D = \partial c$ s.t. $f|(X \setminus U) = 1$ and

$$\frac{1}{2\pi\sqrt{-1}} \iint_X \frac{df}{f} \wedge \rho^*\omega = \int_c \rho^*\omega$$

for every $\omega \in H^0(X_{\mathfrak{m}}, \mathcal{E}_{\mathfrak{m}}^{(1)})$ with $d\omega = 0$, where U is an open neighbourhood of the support of c with $U \subset\subset X \setminus S$.

Since $H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}}) \subset H^0(X_{\mathfrak{m}}, \mathcal{E}_{\mathfrak{m}}^{(1)})$, we obtain for every $\omega \in H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$

$$0 = \int_c \rho^*\omega = \frac{1}{2\pi\sqrt{-1}} \iint_X \frac{df}{f} \wedge \rho^*\omega = \frac{1}{2\pi\sqrt{-1}} \iint_X \frac{\overline{\partial} f}{f} \wedge \rho^*\omega$$

by the assumption.

$\sigma := \dfrac{\overline{\partial} f}{f}$: C^∞ (0,1)-form on X

Since $H^0(X, \Omega) \subset \rho^* H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$,

$$\frac{1}{2\pi\sqrt{-1}} \iint_X \sigma \wedge \eta = 0, \quad \forall \eta \in H^0(X, \Omega)$$

$\exists g$: C^∞ function on X s.t. $\overline{\partial} g = \sigma = \dfrac{\overline{\partial} f}{f}$

$F := e^{-g} f$ is also a weak solution of D, and meromorphic on X. Since $f = 1$ on a neighborhood of S, $F = e^{-g}$ there. Hence, $-g$ is a branch of $\log F$ on a neighborhood of S.

For any $\omega \in H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$ we have

$$\frac{1}{2\pi\sqrt{-1}} \iint_X \overline{\partial} g \wedge \rho^*\omega = \frac{1}{2\pi\sqrt{-1}} \iint_X \frac{\overline{\partial} f}{f} \wedge \rho^*\omega = \int_c \rho^*\omega = 0.$$

$Q \in \overline{S}$, $\rho^{-1}(Q) = \{P_1, \ldots, P_N\}$

$B_j(\varepsilon)$: a small disc centered at P_j with radius $\varepsilon > 0$

Since

$$\begin{aligned}\frac{1}{2\pi\sqrt{-1}} \iint_X \overline{\partial} g \wedge \rho^* \omega &= \lim_{\varepsilon \to 0} \frac{1}{2\pi\sqrt{-1}} \iint_{X \setminus \left(\cup_{j=1}^N B_j(\varepsilon)\right)} \overline{\partial} g \wedge \rho^* \omega \\ &= \lim_{\varepsilon \to 0} \left(\sum_{j=1}^N \frac{1}{2\pi\sqrt{-1}} \int_{\partial B_j(\varepsilon)} (-g) \rho^* \omega \right) \\ &= \sum_{P \in \rho^{-1}(Q)} \mathrm{Res}_P \left((-g) \rho^* \omega \right),\end{aligned}$$

we obtain

$$\sum_{P \in \rho^{-1}(Q)} \mathrm{Res}_P \left((-g) \rho^* \omega \right) = 0.$$

This is the condition (1) in Lemma 2. Then the condition (2) in Lemma 2 is satisfied: i.e.

$\exists h \in \mathrm{Mer}(X)$, $\exists c(\overline{S})$: multiconstant s.t.

$$D = (h) \quad \text{and} \quad h \equiv c(\overline{S}) \bmod \mathfrak{m}$$

(Sufficiency)

Assumption

$\exists f \in \mathrm{Mer}(X)$ s.t.

$$D = (f) \quad \text{and} \quad f \equiv c(\overline{S}) \bmod \mathfrak{m} \text{ for some multiconstant } c(\overline{S})$$

$F : X \longrightarrow \mathbb{P}^1$ holomorphic map defined by f

$\forall \omega \in H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$

Trace($\rho^*\omega$): the trace of $\rho^*\omega$ by F

Trace($\rho^*\omega$) is a meromorphic 1-form on $\mathbb{P}^1$.

$F(S) := \{c_Q; Q \in \overline{S}\}$

It is obvious that Trace($\rho^*\omega$) is holomorphic on $\mathbb{P}^1 \setminus F(S)$.

By a careful investigation at a point in $F(S)$, we see it is holomorphic on the whole of $\mathbb{P}^1$.

Then Trace($\rho^*\omega$) $= 0$.

Therefore we can apply the usual argument.

7 Albanese varieties

$X_{\mathfrak{m}}$: a singular curve of genus $\pi = g + \delta$

$\{\omega_1, \ldots, \omega_\pi\}$: a basis of $H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$ s.t.

$\{\rho^*\omega_1, \ldots, \rho^*\omega_g\}$: a basis of $H^0(X, \Omega)$

$\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$: a canonical homology basis of X.

$S = \{P_1, \ldots, P_s\}$

γ_j : a small circle centered at P_j with anticlockwise direction

$\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_s\}$: a basis of $H_1(X \setminus S, \mathbb{Z}) = H_1(X_{\mathfrak{m}} \setminus \overline{S}, \mathbb{Z})$

$$A := H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}})^* / H_1(X_{\mathfrak{m}} \setminus \overline{S}, \mathbb{Z}).$$

Γ: a discrete subgroup generated by the following $2g+s$ vectors over $\mathbb{Z}$

$$\left(\int_{\alpha_i} \rho^*\omega_1, \ldots, \int_{\alpha_i} \rho^*\omega_\pi\right), \quad i=1,\ldots,g,$$
$$\left(\int_{\beta_i} \rho^*\omega_1, \ldots, \int_{\beta_i} \rho^*\omega_\pi\right), \quad i=1,\ldots,g,$$
$$\left(\int_{\gamma_j} \rho^*\omega_1, \ldots, \int_{\gamma_j} \rho^*\omega_\pi\right), \quad j=1,\ldots,s$$

$$A = H^0(X_\mathfrak{m}, \Omega_\mathfrak{m})^*/H_1(X\setminus S, \mathbb{Z}) \cong \mathbb{C}^\pi/\Gamma \quad \text{as a complex Lie group}$$

We write it $\mathrm{Alb}^{an}(X_\mathfrak{m})$ emphasizing its analytic structure.

We define a period map φ with base point $P_0 \in X\setminus S$ by

$$\varphi : X\setminus S \longrightarrow \mathrm{Alb}^{an}(X_\mathfrak{m}), \quad P \longmapsto \left[\left(\int_{P_0}^P \rho^*\omega_1, \ldots, \int_{P_0}^P \rho^*\omega_\pi\right)\right].$$

G : a commutative complex Lie group

$\psi : X\setminus S \longrightarrow G$: a holomorphic map, $\forall D \in \mathrm{Div}(X_\mathfrak{m})$

$$\psi(D) := \sum_{P\in X\setminus S} D(P)\psi(P)$$

$g \in \mathrm{Mer}(X)$ with $g \equiv c(\overline{S}) \bmod \mathfrak{m}$ for some $c(\overline{S})$

$$\psi((g)) := \sum_{P\in X\setminus S} \mathrm{ord}_P(g)\psi(P) \quad \text{well-defined}$$

Definition 7. A holomorphic map $\psi : X \setminus S \longrightarrow G$ admits $\mathfrak{m}$ for a modulus

$$\underset{\text{def}}{\Longleftrightarrow} \quad \psi((f)) = 0, \forall f \in \mathrm{Mer}(X) \text{ with } f \equiv c(\overline{S}) \bmod \mathfrak{m} \text{ for some } c(\overline{S})$$

Remark. *In [R] and [Ser],* $\boxed{f \equiv 1 \bmod \mathfrak{m}}$ *is considered.*

[R] M. Rosenlicht, Generalized jacobian varieties, Ann. of Math., **59** (1954), 505–530.

[Ser] J.-P. Serre, Groupes algébriques et corps de classes, Hermann, Paris, 1959.

Proposition 1. *The period map* $\varphi : X \setminus S \longrightarrow \mathrm{Alb}^{an}(X_{\mathfrak{m}})$ *defined above admits* $\mathfrak{m}$ *for a modulus. Furthermore, it is a holomorphic embedding if* $g \geq 1$.

Theorem 5. *The map* $\varphi : (X \setminus S)^{(\pi)} \longrightarrow \mathrm{Alb}^{an}(X_{\mathfrak{m}})$ *is surjective.*

$(X \setminus S)^{(\pi)}$: the π-symmetric product of $X \setminus S$

Corollary 1. $\overline{\mathrm{Div}^0(X_{\mathfrak{m}})} \cong \mathrm{Alb}^{an}(X_{\mathfrak{m}})$ *as groups*

Theorem 6. *The map* $\varphi : (X \setminus S)^{(\pi)} \longrightarrow \mathrm{Alb}^{an}(X_{\mathfrak{m}})$ *is bimeromorphic.*

$$\mathrm{Alb}^{an}(X_{\mathfrak{m}}) = \mathbb{C}^p \times (\mathbb{C}^*)^q \times \mathfrak{Q}$$

$\mathfrak{Q}$: an r-dimensional quasi-abelian variety of kind 0, $p + q + r = \pi$

$\mathfrak{Q} = \mathbb{C}^r/\Gamma_0$, rank $\Gamma_0 = r + s$

$\mathfrak{Q} \longrightarrow A_0$: principal $(\mathbb{C}^*)^{r-s}$-bundle over an abelian variety A_0

$\overline{\mathfrak{Q}}$: the standard compactification of $\mathfrak{Q}$

$\overline{\mathrm{Alb}^{an}(X_{\mathfrak{m}})} := (\mathbb{P}^1)^{p+q} \times \overline{\mathfrak{Q}}$: the standard compactification of $\mathrm{Alb}^{an}(X_{\mathfrak{m}})$

Remark. *The map* $\varphi : X \setminus S \longrightarrow \mathrm{Alb}^{an}(X_{\mathfrak{m}})$ *does not extend to a holomorphic map* $\overline{\varphi} : X \longrightarrow \overline{\mathrm{Alb}^{an}(X_{\mathfrak{m}})}$.

Theorem 7 (Universality Property)**.** *Let* G *be a commutative complex Lie group, and let* P_0 *be the base point of the map* $\varphi : X \setminus S \longrightarrow \mathrm{Alb}^{an}(X_{\mathfrak{m}})$. *Then, for any holomorphic map* $\psi : X \setminus S \longrightarrow G$ *which admits* $\mathfrak{m}$ *for a modulus there exists uniquely a homomorphism* $\Psi : \mathrm{Alb}^{an}(X_{\mathfrak{m}}) \longrightarrow G$ *between complex Lie groups such that* $\psi = \Psi \circ \varphi + g_0$, *where* $g_0 = \psi(P_0)$.

8 The reason why $\mathrm{Div}(X_{\mathfrak{m}})$ is sufficient

$D \in \mathrm{Div}(X_{\mathfrak{m}}) \iff D :$ a divisor prime to S

We should consider divisors on the whole $X_{\mathfrak{m}}$.

$\mathcal{M}_{\mathfrak{m}}$: the quotient sheaf of $\mathcal{O}_{\mathfrak{m}}$

The divisor sheaf $\mathcal{D}_{\mathfrak{m}}$ on $X_{\mathfrak{m}}$ is

$$\mathcal{D}_{\mathfrak{m}} = \mathcal{M}_{\mathfrak{m}}^{*}/\mathcal{O}_{\mathfrak{m}}^{*}.$$

An element in $H^0(X_{\mathfrak{m}}, \mathcal{D}_{\mathfrak{m}})$ is identified with a divisor

$$D = \sum_{Q \in X_{\mathfrak{m}}} D(Q)Q$$

$D(Q) = \sum_{P \in \rho^{-1}(Q)} n_P$, $n_P \in \mathbb{Z}$ with $|n_P| \geqq \mathfrak{m}(P)$ and $n_P n_{P'} > 0$,

$\forall P, P' \in \rho^{-1}(Q)$ if $Q \in \overline{S}$ and $D(Q) \neq 0$,

$D(Q) \in \mathbb{Z}$ if $Q \notin X_{\mathfrak{m}} \setminus \overline{S}$.

The number of points with $D(Q) \neq 0$ is finite.

$\widetilde{\mathrm{Div}_{\mathfrak{m}}}(X_{\mathfrak{m}})$: the group of all such divisors

$\forall f \in \mathrm{Mer}(X_{\mathfrak{m}})$, $f \neq 0$

$$(f) := \sum_{Q \in X_{\mathfrak{m}}} \mathrm{ord}_Q(f)Q \in \widetilde{\mathrm{Div}_{\mathfrak{m}}}(X_{\mathfrak{m}})$$

Definition 8. $D_1, D_2 \in \widetilde{\mathrm{Div}_{\mathfrak{m}}}(X_{\mathfrak{m}})$

$$D_1 \sim_{\mathfrak{m}} D_2 \quad \underset{\text{def}}{\iff} \quad \exists f \in \mathrm{Mer}(X_{\mathfrak{m}}) \; s.t. \; D_1 - D_2 = (f)$$

Lemma 3. $\forall \widetilde{D} \in \widetilde{\mathrm{Div}}_{\mathfrak{m}}(X_{\mathfrak{m}})$

$\exists f \in \mathrm{Mer}(X_{\mathfrak{m}})$ *s.t.* $\widetilde{D} - (f) \in \mathrm{Div}(X_{\mathfrak{m}})$

Proof. Assume: $Q \in \overline{S}$, $M := \widetilde{D}(Q) \neq 0$

It suffices to consider the case $M > 0$.

$\rho^{-1}(Q) = \{P_1, \ldots, P_N\}$

$\forall P_i$, $\exists n_i \in \mathbb{N}$ with $n_i \geqq \mathfrak{m}(P_i)$ s.t. $M = \sum_{i=1}^{N} n_i$

z_i: a local coordinate at P_i

$r_i(z_i) := z_i^{n_i}$

$\forall P \in S \setminus \{P_1, \ldots, P_N\}$, $r_P(z_P) := 1 + z_P^{\mathfrak{m}(P)}$

z_P: a local coordinate at P

$\exists f \in \mathrm{Mer}(X)$ s.t.

$$\begin{cases} \mathrm{ord}_P(f - r_i) > n_i & \text{if } P = P_i \text{ for some } i = 1, \ldots, N, \\ \mathrm{ord}_P(f - r_P) > \mathfrak{m}(P) & \text{if } P \in S \setminus \{P_1, \ldots, P_N\} \end{cases}$$

$\exists g \in \mathrm{Mer}(X_{\mathfrak{m}})$ s.t. $f = \rho^* g$

$\widetilde{D} - (g) = 0$ at Q □

$$\left[\widetilde{\mathrm{Div}}_{\mathfrak{m}}(X_{\mathfrak{m}})\right] := \widetilde{\mathrm{Div}}_{\mathfrak{m}}(X_{\mathfrak{m}}) / \sim_{\mathfrak{m}}$$

$$\widetilde{\mathrm{Div}}^0_{\mathfrak{m}}(X_{\mathfrak{m}}) := \{\widetilde{D} \in \widetilde{\mathrm{Div}}_{\mathfrak{m}}(X_{\mathfrak{m}}); \deg \widetilde{D} = 0\}$$

$$\left[\widetilde{\mathrm{Div}}^0_{\mathfrak{m}}(X_{\mathfrak{m}})\right] := \widetilde{\mathrm{Div}}^0_{\mathfrak{m}}(X_{\mathfrak{m}}) / \sim_{\mathfrak{m}}$$

Proposition 2.

$$\left[\widetilde{\mathrm{Div}}_{\mathfrak{m}}(X_{\mathfrak{m}})\right] \cong \overline{\mathrm{Div}(X_{\mathfrak{m}})}$$

$$\left[\widetilde{\mathrm{Div}}^{0}_{\mathfrak{m}}(X_{\mathfrak{m}})\right] \cong \overline{\mathrm{Div}^{0}(X_{\mathfrak{m}})}$$

9 Appendix

p.14 lines 5–6

Let $Q \in \overline{S}$

$\forall P \in \rho^{-1}(Q)$, $m(\geqq \mathfrak{m}(P))$: the multiplicity of F at P

$\exists t$: a local coordinate at c_Q

$\exists w$: a local coordinate at P s.t.

$$F \text{ is represented as } t = w^m.$$

$\exists h(w)$: meromorphic function in a nbd. of P s.t.

$$\rho^*\omega = h(w)dw \quad \text{and} \quad h(w) = \sum_{n \geqq -\mathfrak{m}(P)} c_n w^n.$$

By $dt = mw^{m-1}dw$, $\rho^*\omega = \dfrac{h(w)}{mw^{m-1}}dt$

$\zeta^i w$ $(i = 0, 1, \ldots, m-1)$: the preimages of $t = w^m$

$(\zeta = \exp(\sqrt{-1}\frac{2\pi}{m}))$

Then

$$\begin{aligned} &\sum_{i=0}^{m-1} \frac{h(\zeta^i w)}{mw^{m-1}}dt \\ &= \frac{1}{m} \sum_{n \geqq -\mathfrak{m}(P)} c_n \left(\sum_{i=0}^{m-1} \zeta^{i(n-m+1)} \right) w^{n-m+1}dt \end{aligned} \qquad (*)$$

If $n-m+1 \neq km$, then $\sum_{i=0}^{m-1} \zeta^{i(n-m+1)} = 0$.

Since $n \geqq -\mathfrak{m}(P)$ and $m \geqq \mathfrak{m}(P)$, we have

$$(*) = \sum_{k \geqq 0} c_{km-1} t^{k-1} dt.$$

Noting $c_{-1} = \mathrm{Res}_P(\rho^* \omega)$, we obtain the expression of $\mathrm{Trace}(\rho^* \omega)$ at c_Q as follows:

$$\mathrm{Trace}(\rho^* \omega) = \left(\left(\sum_{P \in \rho^{-1}(Q)} \mathrm{Res}_P(\rho^* \omega) \right) \frac{1}{t} + \text{holomorphic part} \right) dt$$

$$\sum_{P \in \rho^{-1}(Q)} \mathrm{Res}_P(\rho^* \omega) = 0 \quad \text{for} \quad \omega \in H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$$

Then $\mathrm{Trace}(\rho^* \omega)$ is holomorphic at c_Q.