HAUSDORFF DIMENSION OF THE LIMIT SETS OF INFINITELY GENERATED KLEINIAN GROUPS

KATSUHIKO MATSUZAKI

Department of Mathematics, Ochanomizu University
Ohtsuka 2-1-1, Bunkyo-ku, Tokyo 112, Japan
matsuzak@math.ocha.ac.jp

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ABSTRACT. In this paper, we investigate the Hausdorff dimension of the accumulated point set by the orbit of a discrete subgroup of hyperbolic isometries, and obtain certain conditions for infinitely generated groups when the Hausdorff dimension is strictly less than the dimension of the sphere at infinity of the hyperbolic space.

§1. INTRODUCTION

There are several geometric interpretations of the Hausdorff dimension of the conical limit set $\Lambda_c(\Gamma)$, a set of points where the orbit by a Kleinian group Γ accumulates non-tangentially, via the critical exponent of convergence of the Poincaré series, such as the base eigenvalue in the L^2-spectra for the hyperbolic Laplacian and the hyperbolic isoperimetric constant on the hyperbolic manifold. However we do not know well how the Hausdorff dimension of the entire limit set $\Lambda(\Gamma)$ reflects the geometric structure of the manifold. For a geometrically finite Kleinian group Γ, the Hausdorff dimensions of $\Lambda_c(\Gamma)$ and $\Lambda(\Gamma)$ are coincident, but in general they are not. In this paper, we shall estimate the Hausdorff dimension of the entire $\Lambda(\Gamma)$ by using certain geometric values for infinitely generated Kleinian groups. (In the case $D = 1$, we especially call Γ a Fuchsian group.)

First we consider a necessary condition under which the limit sets of Kleinian groups Γ acting on the $(D+1)$-dimensional hyperbolic space ($D \geq 1$) have the Hausdorff dimension D. We shall obtain the following result, generalizing an argument due to Tukia [13] for geometrically finite groups of the second kind.

Proposition 1. *Let Γ be a Kleinian group acting on the upper half plane $\mathbb{H}^{D+1}$. If there is a positive constant L such that any point of the convex core C_Γ of the orbifold $N_\Gamma = \mathbb{H}^{D+1}/\Gamma$ is within the distance L of the boundary ∂C_Γ, then there is a constant $\alpha \in (0, D)$ depending only on L and D such that* $\dim \Lambda(\Gamma) \leq \alpha < D$, *where* $\dim \Lambda(\Gamma)$ *is the Hausdorff dimension of the limit set of* Γ.

Furthermore we loosen the assumption of this proposition to a certain extent; specifying a removable set in N_Γ, which is a disjoint union of certain regions of

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simple structure, we can derive the same conclusion if the convex core except the removable set is within the distance L of the boundary of the convex core. For the sake of simplicity, we state this result for Fuchsian groups as Theorem 1 in §2 and give a proof in §3. Higher dimensional generalization is obtained by the same proof, and thus we just state it as Theorem 3 in §5.

We show corollaries to these theorems in §5 and §6. They are concerning the Hausdorff dimension and the D-dimensional measure of the limit sets of Schottky groups generated by infinite circle packings.

In the opposite direction, we obtain a sufficient condition for the limit sets of Fuchsian groups to be of dimension 1, as an application of the recent work by Bishop and Jones [1]. Roughly speaking, if there is a sequence of points going away from the boundary of the convex core and the boundary (possibly with infinitely many components) is not so large, we see that the limit set is of the dimension 1. A proof is in §4.

In §7, we show concrete examples of Fuchsian groups and Kleinian groups which satisfy that $\dim \Lambda_c(\Gamma) < D$ but $\dim \Lambda(\Gamma) = D$ whereas the D-dimensional measure is zero. These examples indicate that our method of proving $\dim \Lambda(\Gamma) = D$ is not sharp yet. We illustrate our theorems with Fuchsian models of certain planar Riemann surfaces in §8. In the last section, we give a brief summary of results and a problem concerning the Hausdorff dimension of infinitely generated Kleinian groups.

§2. Results for Fuchsian groups

We consider the Hausdorff dimension of the limit set $\Lambda(\Gamma)$ ($\subset \hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$) of a Fuchsian group Γ acting on the upper half plane $\mathbb{H}^2$. We do not assume that Γ is finitely generated. The upper half plane is equipped with the hyperbolic metric, and then Γ is a properly discontinuous isometry group. The quotient hyperbolic orbifold is denoted by N_Γ. The hyperbolic distance between x and y is denoted by $\rho(x, y)$. For the sake of simplicity, we may assume that Γ has no elliptic elements, in other words, N_Γ is a hyperbolic Riemann surface, but this is not essential and can be removed. The Hausdorff dimension of the limit set is denoted by $\dim \Lambda(\Gamma)$. It is a kind of index to measure how large the ideal boundary of N_Γ is relatively to the border $(\hat{\mathbf{R}} - \Lambda(\Gamma))/\Gamma$. But this border is at infinity in the sense of hyperbolic geometry; useful is to consider the relative boundary of the convex core C_Γ of N_Γ instead of the border. Here C_Γ is the smallest convex subregion of N_Γ whose inclusion map induces homotopy equivalence. Concretely it is the quotient by Γ of the convex hull $C(\Lambda(\Gamma))$ of the limit set.

First we show a condition under which $\dim \Lambda(\Gamma)$ is strictly less than 1. For a Fuchsian group Γ of a compact bordered surface, the convex core C_Γ is compact and $\dim \Lambda(\Gamma) < 1$ is satisfied. Even if C_Γ is not compact but if its boundary ∂C_Γ is so large that any point of C_Γ is within a bounded distance of it, we can deduce the same conclusion. This is our proposition mentioned in the introduction. However this assumption is too restricted; cusps of N_Γ (parabolic elements in Γ) always violate the assumption while they do not seem to have much influence to the Hausdorff dimension. In fact, for a finitely generated Fuchsian group of the second kind, $\dim \Lambda(\Gamma)$ is strictly less than 1. We may allow a region with simple structure as well as a cusped region even if it is far from ∂C_Γ. We precisely define it as a removable set and state Theorem 1 below.

Definition. For a Fuchsian group Γ, we say a disjoint union of regions $A = \bigcup_{n\in\mathbf{N}} A_n$ in $\mathbb{H}^2$ is *removable* if it satisfies the following conditions:

(a) Each A_n is a simply connected open set in $\mathbb{H}^2$ which is either a hyperbolic disk, a horodisk tangential to $\hat{\mathbf{R}}$ or a neighborhood of a complete geodesic within a constant distance.

(b) The set A is invariant under Γ.

Theorem 1. *Let Γ be a Fuchsian group acting on the upper half plane $\mathbb{H}^2$. If there is a positive constant L and a removable set A for Γ such that any point of the convex hull $C(\Lambda(\Gamma))$ is in A or within the distance L of the boundary $\partial C(\Lambda(\Gamma))$, then there is a constant $\alpha \in (0,1)$ depending only on L such that the Hausdorff dimension of the limit set of Γ satisfies $\dim \Lambda(\Gamma) \leq \alpha < 1$.*

Our next problem is to find a sufficient condition for $\dim \Lambda(\Gamma)$ to be 1. In general, the estimate of the Hausdorff dimension from below is more difficult than from above. In a recent work [1], Bishop and Jones have deduced that $\Lambda(\Gamma)$ is in the full dimension by showing that the Lebesgue measure of $\Lambda(\Gamma)$ is positive. Their proof is originally for Kleinian groups but of course it is also applicable to Fuchsian. They deal with analytically finite groups but here we modify their argument to analytically infinite cases.

Theorem 2. *Let N_Γ be a Riemann surface of infinite topological type and let $\{c_n\}_{n=1,2,\ldots}$ be the components of the boundary of the convex core $\partial C_\Gamma \subset N_\Gamma$. If the hyperbolic lengths $l(c_n)$ satisfy*

$$\sum_n l(c_n)^{\frac{1}{2}} < \infty \ ,$$

then the Hausdorff dimension of the limit set of Γ is equal to 1.

§3. Proof of Theorem 1

This section is entirely devoted to a proof of Theorem 1. The argument is based on Tukia's paper [13] but several improvements are added.

To begin with, we fix notation used in the proof. We denote by $\mathcal{K}$ the set of all the closed intervals Q in the real axis $\mathbf{R}$. For $Q \in \mathcal{K}$, let a_Q and b_Q be the end points of Q and $d(Q)$ the length $b_Q - a_Q$. For $Q \in \mathcal{K}$ and an integer $q \in \mathbf{N}$, we define $\mathcal{K}(Q,q)\,(\subset \mathcal{K})$ as the set of q number of subintervals of Q which are obtained by dividing it equally.

The following are defined for an interval Q and for a given positive constant L, however we omit the index L, for it does not change in the proof. We take two points α_Q and β_Q in $\mathbb{H}^2$ so that their real coordinates satisfy

$$\operatorname{Re}\alpha_Q = (3a_Q + b_Q)/4 \quad \text{and} \quad \operatorname{Re}\beta_Q = (a_Q + 3b_Q)/4$$

and the hyperbolic distance from α_Q (β_Q) to the geodesic $\{\operatorname{Re} z = a_Q\}$ ($\{\operatorname{Re} z = b_Q\}$ respectively) is $2L$. Then we denote the closed rectangle with four vertices at $z =$

$(a_Q, 0)$, $(b_Q, 0)$, $(b_Q, \operatorname{Im}\beta_Q)$ and $(a_Q, \operatorname{Im}\alpha_Q)$ by Q_+. Note that for any Q and Q' in $\mathcal{K}$, Q_+ and Q'_+ are equivalent under an euclidean similarity. We define a positive number k independent on Q as the ratio of the lengths of the vertical side to the horizontal side of Q_+. A smaller closed rectangle in Q_+ with vertices at $z = \alpha_Q$, β_Q, $(\operatorname{Re}\beta_Q, \frac{1}{2}kd(Q))$ and $(\operatorname{Re}\alpha_Q, \frac{1}{2}kd(Q))$ is denoted by Q_*, where $kd(Q) = \operatorname{Im}\alpha_Q = \operatorname{Im}\beta_Q$ (Figure 1).

We call this Q_* the *face* of Q. For $0 \le s \le t \le 1$, $Q_{[s,t]}$ is a part of Q_+ between the horizontal lines $\operatorname{Im} z = skd(Q)$ and $\operatorname{Im} z = tkd(Q)$ including the lines. Set $\overline{Q} = Q_{[1,1]}$. Let $l = l(L)$ be the hyperbolic diameter of $\overline{Q}$.

FIGURE 1. rectangle Q_+

Here we show a simple but crucial lemma in our argument. Let $A = \bigcup_{n \in \mathbf{N}} A_n$ be the removable set. Due to a technical reason, we thin out each A_n by L; we consider the region

$$B_n = \{ z \in A_n \mid \rho(z, \partial A_n) > L \}$$

for each n. This may be empty. Set $B = \bigcup_{n \in \mathbf{N}} B_n$. Then we have the following:

Lemma 1. *Under the same assumptions as in Theorem 1, we have an integer $q = q(L)$ satisfying the condition that, for any interval $Q \in \mathcal{K}$ with $Q_* \not\subset B$, there is $Q' \in \mathcal{K}(Q, q)$ such that $Q' \cap \Lambda(\Gamma) = \emptyset$.*

Proof. We may assume that $Q_* \subset C(\Lambda(\Gamma))$. Let q_Q be the minimal integer for which there is $Q' \in \mathcal{K}(Q, q_Q)$ with $Q' \cap \Lambda(\Gamma) = \emptyset$. Put

$$t_Q = \inf\{ t > 0 \mid \partial C(\Lambda(\Gamma)) \cap Q_+ \subset Q_{[0,t]} \}.$$

It is clear that if q_Q tends to ∞ then t_Q tends to 0. Suppose that t_Q is so small that

$$\rho(z, \partial C(\Lambda(\Gamma)) \cap Q_+) > 2L$$

for every $z \in Q_*$. On the other hand, we see

$$\rho(z, \partial C(\Lambda(\Gamma)) - Q_+) > 2L$$

because the shortest geodesic from z to $\partial C(\Lambda(\Gamma)) - Q_+$ must go across one of the geodesics $\{\operatorname{Re} z = a_Q\}$ and $\{\operatorname{Re} z = b_Q\}$. Therefore $\rho(z, \partial C(\Lambda(\Gamma))) > 2L$ for any $z \in Q_*$ (Figure 2). But this is a contradiction because $\rho(z, \partial C(\Lambda(\Gamma))) \le 2L$ for $z \in Q_* - B$ by the assumption. Thus q_Q is uniformly bounded. $\square$

FIGURE 2. distance from Q_*

Hereafter we fix this integer $q = q(L)$, and continue several definitions. For $Q \in \mathcal{K}$, we define a subset of the indices for the removable regions as

$$I_Q = \{\, n \in \mathbf{N} \mid B_n \cap Q_{[\frac{1}{q},\frac{1}{q}]} \neq \emptyset \,\} ,$$

and set $B(Q) = \bigcup_{n \in I_Q} B_n$. Since $\{A_n\}$ are mutually disjoint and since B_n disappears for a thin A_n, we see that there is an integer $N = N(L)$ such that $\#I_Q \leq N$ for any $Q \in \mathcal{K}$.

We define the following set of subintervals of Q:

$$\mathcal{L}(Q) = \{\, Q' \in \bigcup_{i=0}^{\infty} \mathcal{K}(Q, 2^i q) \mid Q' \text{ satisfies } (d) \text{ and } (m) \,\} ,$$

where

(d) $Q' \cap \Lambda(\Gamma) \neq \emptyset$ and $Q'_* \not\subset B(Q)$;

(m) there is no $Q'' \in \bigcup_{i=0}^{\infty} \mathcal{K}(Q, 2^i q)$ such that Q'' contains Q' properly and satisfies the condition (d).

Moreover, we set

$$\mathcal{L}'(Q) := \mathcal{L}(Q) - \mathcal{K}(Q, q).$$

For each B_n, we define its *base* point(s) $v_n \in \hat{\mathbf{R}}$ as follows: For a horodisk, v_n is the tangential point; for a hyperbolic disk, v_n is the vertical projection point of its center; for a neighborhood of a complete geodesic, there are two base points v_n^+ and v_n^- which are the end points of the geodesic. We denote by V the set of all the base points for B and by V_Q the set of those for $B(Q)$. Note that $\#V_Q \leq 2\#I_Q \leq 2N$.

Now we prepare two lemmas.

Lemma 2. *The intervals $\mathcal{L}(Q) = \{Q'\}$ cover $Q \cap (\Lambda(\Gamma) - V_Q)$.*

Proof. Take an arbitrary point x in $Q \cap \Lambda(\Gamma)$. If no intervals in $\mathcal{L}(Q)$ cover x, we have an infinite chain of the faces $\{Q_*^0, Q_*^1, \ldots, Q_*^i, \ldots\}$ which are all contained in $B(Q)$ and converge to x, where $Q^i \in \mathcal{K}(Q, 2^i q)$ and $Q^i \supset Q^{i+1}$ (Figure 3). But since the chain is connected, this is possible only if the chain is contained in one of the elements of $B(Q)$ and hence x must be its base point. This proves the lemma. □

FIGURE 3. chain of faces

Lemma 3. *For any $Q' \in \mathcal{L}(Q)$, we have $Q'_* \not\subset B$.*

Proof. If $Q' \in \mathcal{K}(Q, q)$, then Q'_* has the intersection with $Q_{[\frac{1}{q},\frac{1}{q}]}$. Thus the condition $Q'_* \subset B$ implies $Q'_* \subset B(Q)$. If $Q' \in \mathcal{L}'(Q)$, then $Q' \in \mathcal{K}(Q, 2^j q)$ for some $j \geq 1$, and thus there is $Q'' \in \mathcal{K}(Q, 2^{j-1} q)$ such that $Q'' \supset Q'$. Since Q' satisfies the condition (m), Q'' does not satisfy (d). From $Q'' \cap \Lambda(\Gamma) \neq \emptyset$, we see $Q''_* \subset B(Q)$. Namely, there is B_n for some $n \in I_Q$ such that $Q''_* \subset B_n$. Since $Q''_* \cap Q'_* \neq \emptyset$, the condition $Q'_* \subset B$ actually implies $Q'_* \subset B_n \subset B(Q)$. □

Regarding $\mathcal{L}$ as an operator for the intervals, we construct the hierarchy of coverings of $\Lambda(\Gamma) - V$ inductively. Without loss of generality, we may assume $\infty \notin \Lambda(\Gamma)$. First, we choose $\mathcal{L}_0$ as a set of just one interval that contains the whole $\Lambda(\Gamma)$. For $i \geq 0$, we define

$$\mathcal{L}_{i+1} = \bigcup_{Q \in \mathcal{L}_i} \mathcal{L}(Q).$$

By Lemma 2, we know $\mathcal{L}_i$ covers $\Lambda(\Gamma) - V$ for every $i \geq 1$.

At this stage, we have the following result as a consequence of Lemmas 1, 2 and 3.

Corollary 1. *Let Γ be a Fuchsian group satisfying the assumption of Theorem 1. Then the 1-dimensional measure of $\Lambda(\Gamma)$ is zero.*

Proof. By Lemma 1 and the definition of $\mathcal{L}(Q)$, if $Q \in \mathcal{K}$ satisfies the condition $Q_* \not\subset B$, then

$$\sum_{Q' \in \mathcal{L}(Q)} d(Q') \leq (1 - \frac{1}{q}) d(Q).$$

Any $Q \in \mathcal{L}_i$ $(i \geq 1)$ holds the condition $Q_* \not\subset B$ by Lemma 3, and thus

$$\sum_{Q' \in \mathcal{L}_{i+1}} d(Q') \leq (1 - \frac{1}{q}) \sum_{Q \in \mathcal{L}_i} d(Q).$$

Since $\mathcal{L}_i$ covers $\Lambda(\Gamma) - V$ for every $i \geq 1$ by Lemma 2 and since V is countable, we see the 1-dimensional measure of $\Lambda(\Gamma)$ is zero. $\square$

Remark. We have not yet utilized the condition (a) of the removable regions about the shape (disk, horodisk and geodesic neighborhood). Hence we may choose each A_n to be a domain in $\mathbb{H}^2$ whose euclidean closure intersects $\mathbf{R}$ with zero measure, only for the purpose of Corollary 1. We shall discuss this later in §6.

However we need more in order to prove Theorem 1.

Lemma 4. *There are positive constants α and c strictly less than 1 depending only on L such that*

$$\sum_{Q' \in \mathcal{L}(Q)} d(Q)^{\alpha} \leq c d(Q)^{\alpha}$$

for any $Q \in \mathcal{K}$ with $Q_* \not\subset B$.

In the proof below, we need the following observation.

Lemma 5. *If $Q' \in \mathcal{L}'(Q)$, then there is an index $n = n(Q') \in I_Q$ such that $\overline{Q'}$ is within the distance l of B_n.*

Proof. If $Q' \in \mathcal{K}(Q, 2^j q)$ for some $j \geq 1$, then there is $Q'' \in \mathcal{K}(Q, 2^{j-1} q)$ such that $\overline{Q'} \cap Q''_* \neq \emptyset$ and $Q''_* \subset B(Q)$, as is seen in the proof of Lemma 3. Since the diameter of $\overline{Q'}$ is l, we have the conclusion. $\square$

Proof of Lemma 4. By change of scaling, we may assume that $d(Q) = 1$.

Lemma 5 says that we can define a map $n : \mathcal{L}'(Q) \to I_Q$ by the correspondence $Q' \mapsto n(Q')$ such that $\overline{Q'}$ is in the l-neighborhood of $B_{n(Q')}$. We will find an area where this neighborhood can be.

First we assume that the component $B_n = B_{n(Q')}$ is a disk or a horodisk. Since $B_n \not\supset Q_*$ whereas $B_n \cap Q_{[\frac{1}{q},\frac{1}{q}]} \neq \emptyset$, we see that the euclidean radius of B_n cannot be so large. Hence there exists a constant $m = m(L) > 0$ independent on n such that the l-neighborhood of B_n is over a parabola $y = m(x - v_n)^2$ touching $\mathbf{R}$ at the base point of B_n. Thus any point $(x, y) \in \overline{Q'}$ holds $y \geq m(x - v_{n(Q')})^2$ in this case (Figure 4).

FIGURE 4. size of B_n

Next we assume that $B_n = B_{n(Q')}$ is r-neighborhood of a geodesic for some $r > 0$. We want to show that $\overline{Q'}$ is either in the region $y \geq m(x - v^+_{n(Q')})^2$ or in the region $y \geq m(x - v^-_{n(Q')})^2$, by choosing a smaller positive constant m instead of the previous m if necessary. If r is uniformly bounded from above, this is easy. Thus we have only to consider the case where r is greater than a sufficiently large constant. Then, as in the previous case where B_n is a horodisk, B_n cannot be so large in the euclidean sense. Hence we can find a desired constant m also in this case.

Summing up, we have shown that, for any $Q' \in \mathcal{L}'(Q)$, any point $x \in Q'$ satisfies

$$kd(Q') \geq m(x - v_{Q'})^2, \tag{1}$$

where $v_{Q'}$ is either $v^+_{n(Q')}$ or $v^-_{n(Q')}$ according to the parabola over which $\overline{Q'}$ lies if $B_{n(Q')}$ is a neighborhood of a geodesic, and otherwise $v_{Q'}$ is $v_{n(Q')}$.

Next for arbitrary $\epsilon \in (0, \frac{1}{q})$ and $v \in V_Q$, we define a subset of $\mathcal{L}'(Q)$ as

$$\mathcal{L}_v(Q;\epsilon) = \{\, Q' \in \mathcal{L}'(Q) \mid v_{Q'} = v,\ d(Q') \leq \epsilon \,\}$$

and the finite union over v as

$$\mathcal{L}(Q;\epsilon) = \bigcup_v \mathcal{L}_v(Q;\epsilon).$$

For $x \in Q' \in \mathcal{L}_v(Q;\epsilon)$, the inequality (1) yields

$$k\epsilon \geq kd(Q') \geq m(x - v)^2, \tag{2}$$

in other words,

$$Q' \subset \{\, x \mid |x - v| \leq (\frac{k\epsilon}{m})^{\frac{1}{2}} \,\}.$$

Now we consider the sum of $d(Q')^\alpha$ for $0 < \alpha < 1$. By the above (2), we have

$$\begin{aligned}\sum_{Q' \in \mathcal{L}_v(Q;\epsilon)} d(Q')^\alpha &= \sum_{Q'} \int_{Q'} d(Q')^{\alpha-1} dx \\ &\leq \int_{|x-v| \leq (k\epsilon/m)^{1/2}} \{\frac{m}{k}(x-v)^2\}^{\alpha-1} dx \\ &= \frac{2}{2\alpha - 1}(\frac{k}{m})^{\frac{1}{2}} \epsilon^{\alpha - \frac{1}{2}}.\end{aligned}$$

Hence by $\#V_Q \leq 2N$, we have

$$\sum_{Q' \in \mathcal{L}(Q;\epsilon)} d(Q')^\alpha \leq 2N \frac{2}{2\alpha - 1} (\frac{k}{m})^{\frac{1}{2}} \epsilon^{\alpha - \frac{1}{2}}. \tag{3}$$

We describe here how to choose α and ϵ. First choose $\epsilon \in (0, \frac{1}{q})$ so that

$$2N \frac{2}{2\alpha - 1} (\frac{k}{m})^{\frac{1}{2}} \epsilon^{\alpha - \frac{1}{2}} \leq \frac{1}{3q} \tag{4}$$

holds for any $\alpha > 2/3$. After this, choose $\alpha \in (\frac{2}{3}, 1)$ so that

$$\epsilon^\alpha \leq (1 + \frac{1}{3q})\epsilon. \tag{5}$$

We fix them in the remainder.

Finally we consider the sum of $d(Q')$ taken over all the Q' in $\mathcal{L}(Q)$. If $Q' \in \mathcal{L}(Q) - \mathcal{L}(Q;\epsilon)$, then $d(Q') > \epsilon$. Hence by (5),

$$\sum_{Q' \in \mathcal{L}(Q) - \mathcal{L}(Q;\epsilon)} d(Q')^\alpha \leq \sum_{Q'} (1 + \frac{1}{3q}) d(Q') \leq (1 + \frac{1}{3q})(1 - \frac{1}{q}). \tag{6}$$

Setting $c = 1 - \frac{1}{3q} < 1$, we obtain from (3), (4) and (6)

$$\begin{aligned} \sum_{Q' \in \mathcal{L}(Q)} d(Q')^\alpha &= \sum_{\mathcal{L}(Q;\epsilon)} + \sum_{\mathcal{L}(Q) - \mathcal{L}(Q;\epsilon)} \\ &\leq \frac{1}{3q} + (1 + \frac{1}{3q})(1 - \frac{1}{q}) \\ &\leq 1 - \frac{1}{3q} = c = c d(Q). \end{aligned}$$

This completes the proof. □

From this lemma and the fact that each $\mathcal{L}_i$ covers $\Lambda(\Gamma) - V$, we immediately see that the Hausdorff dimension of $\Lambda(\Gamma)$ is not greater than α in a similar way to Corollary 1. Thus we obtain Theorem 1.

§4. Proof of Theorem 2

Theorem 2 is actually proved in the following form.

Theorem 2′. *Let N_Γ be a hyperbolic surface of infinite topological type whose base eigenvalue $\lambda_0(\Gamma)$ for the hyperbolic Laplacian is positive. Let $\{c_n\}_{n=1,2,\ldots}$ be the components of the boundary of the convex core $\partial C_\Gamma \subset N_\Gamma$. If the hyperbolic lengths $l(c_n)$ satisfy*

$$\sum_n l(c_n)^{\frac{1}{2}} < \infty ,$$

then the 1*-dimensional measure of the limit set of* Γ *is positive.*

Indeed, for a complete hyperbolic manifold N_Γ, the relation between the base eigenvalue $\lambda_0(\Gamma)$ and the critical exponent $\delta(\Gamma)$ of the Kleinian group Γ is known by the Elstrodt-Patterson-Sullivan theorem ([12]) as

$$\lambda_0(\Gamma) = \begin{cases} D^2/4 & \delta(\Gamma) \le D/2 \\ \delta(\Gamma)(D - \delta(\Gamma)) & \delta(\Gamma) \ge D/2 \ . \end{cases}$$

And $\delta(\Gamma)$ is equal to the Hausdorff dimension of the conical limit set $\Lambda_c(\Gamma)$. (See [8] for Fuchsian groups and [1] for general Kleinian groups.) Hence if $\lambda_0(\Gamma) = 0$, then $\dim(\Lambda(\Gamma)) \ge \dim(\Lambda_c(\Gamma)) = 1$, and we have nothing to prove. Thus we may assume that $\lambda_0(\Gamma) > 0$.

Further we can reduce Theorem $2'$ to the following result. The proof is going along the same line as Bishop and Jones [1].

Theorem 3. *Let* N_Γ *be a hyperbolic surface such that* $\lambda_0(\Gamma) > 0$*. If there is an infinite sequence* $X = \{x_m\}_{m \in \mathbf{N}}$ *of non-accumulating points in* C_Γ *such that the injective radii at* x_m *are uniformly bounded away from zero and*

$$\sum_n l(c_n)^{-\frac{1}{2}} \int_{c_n} \exp(-\frac{\lambda_0}{2}\rho(X, y)) ds(y) < \infty \ ,$$

where ds *is the hyperbolic line element, then* $\Lambda(\Gamma)$ *has positive* 1*-dimensional measure.*

Proof. Let $p(x, y, t)$ be the heat kernel for N_Γ. An estimate of its upper bound is known as

$$0 \le p(x, y, t) \le C \,\mathrm{area}(B(x, 1))^{-\frac{1}{2}} \,\mathrm{area}(B(y, 1))^{-\frac{1}{2}} \exp(-\frac{\lambda_0}{2}t - \frac{\rho(x, y)^2}{5t})$$

for $\rho(x, y) \ge 1$, where C is some constant and $B(x, r)$ is a portion of N_Γ within r of x (cf. [1]). Let U be the edge of ∂C_Γ of width 1 in the exterior of the convex core. We define E_m as the expected time a Brownian motion started at x_m spends in U, namely,

$$E_m = \int_0^\infty \int_U p(x_m, y, t) dm(y) dt \ ,$$

where dm is the hyperbolic area element. By the above estimate of the kernel and the fact that $\mathrm{area}(B(x_m, 1))$ is bounded away from zero, we have

$$E_m \le C' \int_U \left[\mathrm{area}(B(y, 1))^{-\frac{1}{2}} \int_0^\infty \exp(-\frac{\lambda_0}{2}t - \frac{\rho(x_m, y)^2}{5t}) dt \right] dm(y).$$

Here

$$\begin{aligned} \int_0^\infty \exp(-\frac{\lambda_0}{2}t - \frac{\rho(x_m, y)^2}{5t}) dt &= \int_0^{\rho(x_m, y)} + \int_{\rho(x_m, y)}^\infty \\ &\le \frac{2}{\lambda_0}(\exp(-\frac{\rho(x_m, y)}{5}) + \exp(-\frac{\lambda_0 \rho(x_m, y)}{2})) \ , \end{aligned}$$

and since $\lambda_0 \leq 1/4$, this is bounded by

$$\frac{4}{\lambda_0}\exp(-\frac{\lambda_0}{2}\rho(x_m,y)) \leq \frac{4}{\lambda_0}\exp(-\frac{\lambda_0}{2}\rho(X,y)) \ .$$

We consider the integral majorant $\int_U \text{area}(B(y,1))^{-\frac{1}{2}}\frac{4}{\lambda_0}\exp(-\frac{\lambda_0}{2}\rho(X,y))dm(y)$. We can easily see that this integral over U is bounded by a constant multiple of the line integral over ∂C_Γ. Let c_n be a component of ∂C_Γ. There is a constant $\eta > 0$ not depending on n such that if $y \in c_n$, then $\text{area}(B(y,1)) \geq \eta l(c_n)$. Hence

$$\begin{aligned}&\int_U \text{area}(B(y,1))^{-\frac{1}{2}}\exp(-\frac{\lambda_0}{2}\rho(X,y))dm(y)\\ &\leq C'' \sum_n \int_{c_n} l(c_n)^{-\frac{1}{2}}\exp(-\frac{\lambda_0}{2}\rho(X,y))ds(y) \ ,\end{aligned}$$

which is finite by the assumption. Therefore by the dominated convergence theorem,

$$\begin{aligned}&\lim_{m\to\infty}\int_U \text{area}(B(y,1))^{-\frac{1}{2}}\exp(-\frac{\lambda_0}{2}\rho(x_m,y))dm(y)\\ &= \int_U \text{area}(B(y,1))^{-\frac{1}{2}}\left[\lim_{m\to\infty}\exp(-\frac{\lambda_0}{2}\rho(x_m,y))\right]dm(y) \ ,\end{aligned}$$

and since $\lim_{m\to\infty}\rho(x_m,y)\to\infty$ for each y, we see $E_m \to 0$ as $m \to \infty$.

Assume that the 1-dimensional measure of $\Lambda(\Gamma)$ is zero. This is equivalent to the condition that the harmonic measure of the ideal boundary of C_Γ is zero. Let β be the union of the core curves in the annular components of U. Then this assumption also implies that the hitting probability of a Brownian motion to β is equal to 1 not depending on the starting point x_m. But the expected time a Brownian motion starting at $y \in \beta$ spends in U is bounded away from zero independent on where y is and thus the Markov property yields E_m is also bounded away from zero. This contradiction proves that the 1-dimensional measure of $\Lambda(\Gamma)$ is positive. □

To prove Theorem 2′ from Theorem 3, we have only to remark the following lemma.

Lemma 6. *In the convex core C_Γ of a hyperbolic Riemann surface N_Γ of infinite topological type, there is an infinite non-accumulating sequence of points where the injective radii are uniformly bounded away from zero.*

Proof. There is a universal constant $r_0 > 0$ such that each component of the thin part of N_Γ where the injective radius is less than r_0 is either a cusp neighborhood or an annulus. Since C_Γ removed the thin part remains non-compact, we can choose the required sequence. □

§5. Higher dimensional case

Tukia [13] proved that for arbitrary $D \geq 1$, the Hausdorff dimension of the limit set of a $(D+1)$-dimensional geometrically finite Kleinian group of the second kind is strictly less than D. Our proof of Theorem 1 is obtained by generalizing his argument. For the sake of simplicity, we have proved it only for Fuchsian groups ($D = 1$), however we can find that the dimension is a problem of no matter. In this section, we just state the higher dimensional generalization of Theorem 1 without proof.

Definition. For a Kleinian group Γ acting on the $(D+1)$-dimensional hyperbolic space $\mathbb{H}^{D+1}$, we say a disjoint union of regions $A = \bigcup_{n \in \mathbf{N}} A_n$ in $\mathbb{H}^{D+1}$ is *removable* if it satisfies the following conditions:

(a) Each A_n is a simply connected open set in $\mathbb{H}^{D+1}$ which is either a hyperbolic ball, a horoball tangential to $\partial\mathbb{H}^{D+1}$ or a neighborhood of a complete geodesic within a constant distance.
(b) The set A is invariant under Γ.

Theorem 4. *Let Γ be a Kleinian group acting on $\mathbb{H}^{D+1}$. If there is a positive constant L and a removable set A for Γ such that any point of the convex hull $C(\Lambda(\Gamma))$ is in A or within the distance L of the boundary $\partial C(\Lambda(\Gamma))$, then there is a constant $\alpha = \alpha(L, D)$ depending only on L and D such that the Hausdorff dimension of the limit set of Γ satisfies* $\dim \Lambda(\Gamma) \leq \alpha < D$.

As an application of this theorem, we can prove a result by Schwartz [11] concerning the Hausdorff dimension of the limit set of a certain Kleinian group generated by the reflections with respect to infinitely many circles. The result is true for any dimension $D+1$ greater than or equal to 3, however we use $(2+1)$-dimensional terminology in the statement and the proof; a ball is $B^{D+1} = \{x \in \mathbf{R}^{D+1} \mid |x-a| < r\}$, a sphere ∂B^{D+1}, a disk B^D, a circle ∂B^D and a segment B^{D-1}.

Corollary 2. *Let $\mathcal{S} = \{S_n\}_{n=1}^{\infty}$ be a family of an infinite number of circles with mutually disjoint interiors in the euclidean space $\mathbf{R}^D$ ($D \geq 2$), whose radii are greater than $k > 0$ and less than $K < \infty$. Let Γ be a discrete group generated by the reflections with respect to $\mathcal{S}$. Then there is a constant $\beta = \beta(k, K, D)$ depending only on k, K and D such that* $\dim \Lambda(\Gamma) \leq \beta < D$.

Proof. Let $\hat{S}_n$ be the hemisphere in $\mathbb{H}^{D+1} = \{(x,t) \mid x \in \mathbf{R}^D,\ t > 0\}$ spanning S_n and P the common exterior of all the hemispheres $\{\hat{S}_n\}$. Since P is a fundamental region of Γ in $\mathbb{H}^{D+1}$, we have only to consider the points in P for our problem claimed below. Let t_0 ($\leq K$) be the highest t-coordinate of the points on ∂P, and consider the horoball $A_1 = \{(x,t) \mid t > t_0\}$. We may regard the union $A = \Gamma(A_1)$ of the orbits as removable in our sense. Thus, by Theorem 4, if we show any point of $(P - A) \cap C(\Lambda(\Gamma))$ is within a distance L of the boundary $\partial C(\Lambda(\Gamma))$ depending only on k, K and D , we obtain the desired result (Figure 5).

FIGURE 5. hemispheres and $\partial C(\Lambda(\Gamma))$

First we consider an easier situation. Remark that there are no limit points of Γ in the common exterior of $\mathcal{S}$. We take an annular edge ω_n of width $k/100$ inside each circle S_n. If Γ has no limit points in $\bigcup \omega_n$, the above claim is easy to prove. Indeed, for x in the common exterior of $\mathcal{S}$, there is $t_1 \geq k/100$ such that $z_1 = (x, t_1) \in \partial C(\Lambda(\Gamma))$; hence if $z = (x, t) \in (P - A) \cap C(\Lambda(\Gamma))$ has such x coordinate, the hyperbolic distance from z to $\partial C(\Lambda(\Gamma))$ is not greater than

$$\rho(z, z_1) \leq \log \frac{t_0}{t_1} \leq \log \frac{100K}{k} \ ,$$

and otherwise $z' \in (P - A) \cap C(\Lambda(\Gamma))$ finds such z within a bounded distance depending only on k, K and D.

In case there is a limit point of Γ in $\bigcup \omega_n$, say in ω_1, another circle, say S_2, must has the intersection with the reflected image of ω_1 with respect to S_1. This means that the euclidean distance between S_1 and S_2 is less than about $k/100$, and between them there is a narrow interstice where the other circles are forbidden.

The extreme case is when S_1 and S_2 are tangential at a point x. In this case, even though x is a limit point of Γ, we can take doubly cusped disks R_1 and R_2 where there are no limit points, mutually tangentially at x and transversally to S_1 and S_2. Moreover we can choose them uniformly large because the radii of the circles are bounded from below by k. If a point $z \in (P - A) \cap C(\Lambda(\Gamma))$ goes away from $\partial C(\Lambda(\Gamma))$, it must tend to $\mathbf{R}^D$, and in the present case it must tend to x in a region bounded by four hemispheres $\hat{S}_1$, $\hat{S}_2$, $\hat{R}_1$ and $\hat{R}_2$. Since the region grows thinner near x in the hyperbolic sense, and $C(\Lambda(\Gamma))$ is contained in the convex hull $C(\mathbf{R}^D - (R_1 \cup R_2))$, we see that $\rho(z, \partial C(\Lambda(\Gamma))) \to 0$ as $z \to x$. Let ϵ be a fixed positive constant. Then we can take a uniformly large euclidean neighborhood U of x such that $\rho(z, \partial C(\Lambda(\Gamma))) \leq \epsilon$ for $z \in U$ because R and R' are uniformly large. Outside U, we can apply the argument in the previous paragraph.

Even if S_1 and S_2 are not tangential yet, a similar argument works. We replace the point x with a segment l joining the fixed points of the hyperbolic transformation that is the composition of the reflections with respect to S_1 and S_2. Instead of the cusped disks, we can take doubly truncated disks R_1' and R_2' which are amalgamated along l (Figure 6). As before, $C(\Lambda(\Gamma))$ is contained in $C(\mathbf{R}^D - (R_1' \cup R_2'))$, and there is an euclidean neighborhood of l where $\rho(z, \partial C(\Lambda(\Gamma))) \leq \epsilon$ for $z \in (P - A) \cap C(\Lambda(\Gamma))$. We can keep the radius of this neighborhood uniformly bounded away from zero in the process to the extreme case because at the cusp it is uniformly large as is seen above. Thus we can also apply the previous argument outside the neighborhood, and finish the proof. $\square$

FIGURE 6. between hemispheres

§6. The D-dimensional measure

We have remarked after Corollary 1 that we need not impose uniformity of the shape upon a removable region in order to prove only the result of the corollary. In this section, we state rigorously such a weaker assumption under which the nullity of the D-dimensional measure of the limit set follows. As in the previous section, our statement does not necessarily remain in the Fuchsian case any longer. Now our task is to define a generalized removable set for which the proofs of Lemmas 1, 2 and 3 work.

Definition. For a Kleinian group Γ acting on the $(D+1)$-dimensional hyperbolic space $\mathbb{H}^{D+1}$, we say a disjoint union of regions $A = \bigcup_{n\in\mathbf{N}} A_n$ in $\mathbb{H}^{D+1}$ is *weakly removable* if it satisfies the following conditions:

(a') Each A_n is an open set in $\mathbb{H}^{D+1}$ whose euclidean closure intersects $\mathbf{R}^D$ with the D-dimensional measure zero.

(b') The set A is invariant under Γ.

Theorem 5. *Let Γ be a Kleinian group acting on $\mathbb{H}^{D+1}$. If there is a positive constant L and a weakly removable set A for Γ such that any point of the convex hull $C(\Lambda(\Gamma))$ is in A or within the distance L of the boundary $\partial C(\Lambda(\Gamma))$, then the D-dimensional measure of the limit set of Γ is zero.*

As a corollary to this theorem, we see the 2-dimensional measure of the limit set vanishes for an infinitely generated reflection group with respect to a circle packing of the complex plane with bounded valency. This result was first proved by He [5] in the course of an argument about the order of the circle packing constant. We state our result a little generally where we need not restrict ourselves to a circle packing; a circle need not be tangential to another.

Corollary 3. *Let $\mathcal{S} = \{S_n\}_{n=1}^{\infty}$ be a family of an infinite number of circles with mutually disjoint interiors and without a point of accumulation in the euclidean space $\mathbf{R}^D$ $(D \geq 2)$ which satisfies the following condition:*

(u) *There is a constant $\kappa > 1$ such that, for any circle $S \in \mathcal{S}$, the ratio of the distance between the center of S and that of any other circle to the radius of S is greater than κ.*

Let Γ be a discrete group generated by the reflections with respect to $\mathcal{S}$. Then the D-dimensional measure of $\Lambda(\Gamma)$ is zero.

Proof. From the proof of Corollary 2, we maintain the notations; $\hat{S}_n$ is the hemisphere in $\mathbb{H}^{D+1}$ spanning S_n and P is the common exterior of all the hemispheres $\{\hat{S}_n\}$ which is a fundamental region of Γ in $\mathbb{H}^{D+1}$. However, in the proof here, we reset $A_1 = \mathrm{Int}(P \cap C(\Lambda(\Gamma)))$. Then we may regard $A = \Gamma(A_1)$ as weakly removable because the euclidean closure of A_1 intersects $\mathbf{R}^D$ only with tangent points of the circles $\mathcal{S}$, which are countable. Thus, by Theorem 5, if we show any point of $\partial P \cap C(\Lambda(\Gamma))$ is within a bounded distance of the boundary $\partial C(\Lambda(\Gamma))$, we obtain the desired result.

We pick up any component $\hat{S}$ of ∂P and we have only to prove the above claim for $\hat{S} \cap C(\Lambda(\Gamma))$. We may change the scale so that the radius of S is 1. We choose an exterior annular edge ω' of S with the width less than $(\kappa-1)/2$. Then the condition (u) implies that if another circle has the intersection with ω', its radius must be greater than $(\kappa-1)/2$. Thus radii of circles close to S are uniformly bounded away from zero and the arguments in the proof of Corollary 2 are applicable. We conclude that there is a constant L such that any point of $\hat{S} \cap C(\Lambda(\Gamma))$ is within L of $\partial C(\Lambda(\Gamma))$. $\square$

If a circle packing of $\mathbf{R}^2$ has bounded valency, namely, the number of tangential circles to any circle is uniformly bounded by a constant, the ring lemma [10] implies that the packing circles satisfy the condition (u). Thus the result of He follows.

§7. Examples for $\dim \Lambda_c(\Gamma) \neq \dim \Lambda(\Gamma)$

We think that the assumptions of Theorem $2'$ and Theorem 3 are far from the sharp condition for $\dim \Lambda(\Gamma) = 1$ because their conclusion is that $\Lambda(\Gamma)$ has positive 1-dimensional measure. In fact, we shall exhibit an example of a Fuchsian group Γ with $\dim \Lambda(\Gamma) = 1$, which is out of application of Theorem $2'$.

To construct such an example, we consider the hyperbolic isoperimetric inequality for a hyperbolic manifold N_Γ. The Cheeger isoperimetric constant $h(\Gamma)$ is defined as

$$h(\Gamma) = \inf \frac{\mathrm{vol}(\partial A)}{\mathrm{vol}(A)} ,$$

where the infimum is taken over all the relatively compact domains A in N_Γ with smooth boundary ∂A. When $h(\Gamma)$ is positive, we say N_Γ satisfies the hyperbolic isoperimetric inequality. It is known that $h(\Gamma) > 0$ if and only if $\lambda_0(\Gamma) > 0$ (cf. Buser [2], Fernández and J. Rodríguez [4]). Recall that these conditions are equivalent to $\dim \Lambda_c(\Gamma) < 1$.

Theorem 6. *There is a Fuchsian group* Γ *such that* $\dim \Lambda(\Gamma) = 1$*,* $\dim \Lambda_c(\Gamma) < 1$ *and the* 1*-dimensional measure of* $\Lambda(\Gamma)$ *is zero.*

Proof. Our example Γ is a subgroup of a Fuchsian group for a closed Riemann surface R_0. First we construct a Riemann surface $R_1 = \mathbb{H}^2/\Gamma_1$ such that the 1-dimensional measure of $\Lambda(\Gamma_1)$ is zero and $\dim \Lambda(\Gamma_1) = 1$ as follows (Figure 7). We take a $\mathbf{Z}$-cover of R_0 by cutting it along the meridian α_0 of a handle and connecting its copies infinitely. Then we divide the $\mathbf{Z}$-cover into the two parts by a copy α of α_0 and replace one of them with the annular cover. We denote the resulting Riemann surface by R_1. From the fact that the $\mathbf{Z}$-cover of a closed Riemann surface does not admit Green's function, we see that the harmonic measure of the ideal boundary of R_1 is zero, which implies that the 1-dimensional measure of $\Lambda(\Gamma_1)$ is zero. We can easily see that R_1 does not satisfy the hyperbolic isoperimetric inequality. Thus the base eigenvalue $\lambda_0(\Gamma_1)$ is zero, and in particular $\dim \Lambda(\Gamma_1) = 1$.

FIGURE 7. covering of R_0

As the next step, we construct a planar normal cover R of R_1. It is defined by the normal closure generated by the homotopy class of the curve α in the fundamental group of R_1. The Fuchsian model of this R is our desired Γ. Since Γ is normal in Γ_1, we have $\Lambda(\Gamma) = \Lambda(\Gamma_1)$. Thus $\Lambda(\Gamma)$ is also of the 1-dimensional measure zero and of the Hausdorff dimension 1. Since R is a planar cover of a closed Riemann surface, the injective radii of R are uniformly bounded away from zero, which implies that $h(\Gamma) > 0$ (cf. [4]). Thus $\lambda_0(\Gamma) > 0$, and $\dim \Lambda_c(\Gamma) < 1$. □

We can construct such an example as in Theorem 6 also for higher dimensional cases.

Theorem 7. *For any* $D \geq 2$*, there is a* $(D+1)$*-dimensional Kleinian group* Γ *such that* $\dim \Lambda(\Gamma) = D$*,* $\dim \Lambda_c(\Gamma) < D$ *and the* D*-dimensional measure of* $\Lambda(\Gamma)$ *is zero.*

Proof. First we choose a Kleinian group G of the second kind such that $\dim \Lambda(G) = D$ and the D-dimensional measure of $\Lambda(G)$ is zero. For example, the existence of such G for any dimension $D+1$ ($D \geq 2$) is seen by the following argument. As in the proof of Theorem 6, if we have a closed hyperbolic manifold R_0 with an embedded totally geodesic submanifold α_0 of codimension 1 such that $R_0 - \alpha_0$ is connected, then we can construct its covering manifold R_1 in the same way. The Kleinian model G of R_1 has the required property by the same reason as the previous proof. The existence of such R_0 is guaranteed by Milson [7]. If we are restricted to $D = 2$, we may also take a totally degenerate group as G because the 2-dimensional measure of $\Lambda(G)$ is zero, which is well-known as a partial answer to the Ahlfors conjecture, and $\dim \Lambda(G) = 2$, which is proved by Bishop and Jones [1].

In a fundamental domain of G in $\mathbf{R}^D$, we take two circles with disjoint interiors. Let γ be a Möbius transformation which maps the interior of one circle to the exterior of the other. Then by the classical Klein combination theorem, the group $\hat{G}$ generated by G and γ is also Kleinian and it is easy to see that $\hat{G}$ also satisfies that $\dim \Lambda(\hat{G}) = D$. Moreover, the D-dimensional measure of $\Lambda(\hat{G})$ is zero. To see this, we may apply a result in Maskit [6], for example. Our example Γ is a subgroup of $\hat{G}$ generated by $\{g\gamma g^{-1}\}_{g \in G}$.

We show that Γ fulfills the three requirements. The inclusion $\Lambda(G) \subset \Lambda(\Gamma)$ is easily seen. Thus $\dim \Lambda(\Gamma) = D$. Since Γ is a subgroup of $\hat{G}$, we know that the D-dimensional measure of $\Lambda(\Gamma)$ is zero. The remainder task is to show that $\dim \Lambda_c(\Gamma) < D$. Note that Γ is a Schottky group generated by an infinite number of Möbius transformations pairing circles. Hence the claim follows from the result by Phillips and Sarnak [9] for $D \geq 3$ and Doyle [3] for $D = 2$ that there is a positive constant depending only on D such that the base eigenvalue $\lambda_0(\Gamma)$ for any classical Schottky group Γ is greater than the constant. It is no matter that our Γ is infinitely generated. □

Remark that this construction is not applicable to Fuchsian groups. The reason why the proof of Theorem 7 does not work is essentially the same as why we restrict Corollaries 2 and 3 to $D \geq 2$. Namely, the uniform estimate about classical Schottky groups is not valid for $D = 1$.

§8. Planar Riemann surfaces

We apply our theorems to certain planar domains. In this section, we always assume that a Riemann surface R is $\mathbf{C} - E$, where E is a closed set of a countable number of connected components which are isolated from the others. Though this assumption on E can be weakened if necessary, it is adopted here for the sake of simplicity.

Corollary 4. *Let $R = \mathbf{C} - E$ be as above and $\{e_n\}_{n=1}^{\infty}$ the components of E. For each e_n, we consider all the doubly connected regions in R with e_n a boundary component and define $m(e_n)$ as the supremum of the moduli of them. If they satisfy*

$$\sum_n m(e_n)^{-\frac{1}{2}} < \infty \ ,$$

then the base eigenvalue of R is zero, or equivalently R does not hold the hyperbolic isoperimetric inequality. In particular, the Hausdorff dimension of the limit set of the Fuchsian model of R is 1.

Proof. If all the e_n are singletons, R does not admit Green's function and it follows that the base eigenvalue is zero. Thus we may assume that E has continuum and then we know that the 1-dimensional measure of the limit set of the Fuchsian model of R is zero. On the other hand, the assumption about the convergence of the sum of the moduli implies the condition in Theorem $2'$ about the lengths of boundary curves of the convex core. Therefore the result immediately follows from Theorem $2'$. □

If the injective radii of R are uniformly bounded away from zero, then R satisfies the hyperbolic isoperimetric inequality. The converse is not true. However, Corollary 4 shows that if R violates the uniformity so rapid, then R violates the hyperbolic isoperimetric inequality too. If R is a Denjoy domain $R = \mathbf{C} - E$, where E is in the real axis $\mathbf{R}$, we may have another formulation of Corollary 4 in terms of the euclidean lengths of the slits e_n and the intervals between the slits.

Conversely, we think a condition under which the Hausdorff dimension of the Fuchsian model of a Denjoy domain R is strictly less than 1. Though Corollary 2 is not true for Fuchsian groups, we have a similar result for certain Denjoy domains directly from Theorem 1, except for the estimate of the Hausdorff dimension.

Corollary 5. *Let $R = \mathbf{C} - E$ be a Denjoy domain such that all the slits e_n have the same length and all the intervals in $\mathbf{R} - E$ have the same length. Then the Hausdorff dimension of the limit set of the Fuchsian model is strictly less than 1.*

§9. Conclusion

We compare a result in this paper with one in Bishop and Jones [1]. They proved in particular the following.

Proposition 2. *Let $N_\Gamma = \mathbb{H}^{D+1}/\Gamma$ be a $(D+1)$-dimensional hyperbolic manifold and suppose that the boundary ∂C_Γ of the convex core is compact. If there is a sequence of points $\{x_n\}$ in the convex core C_Γ which satisfies the following condition $(*)$, then $\dim \Lambda(\Gamma) = D$.*

$(*)$ *$\rho(x_n, \partial C_\Gamma) \to \infty$ as $n \to \infty$; and there is a positive constant r_0 such that the injective radii at x_n are greater than r_0 for all n.*

On the other hand, we have obtained in particular the following result in this paper.

Proposition 3. *For an arbitrary $(D+1)$-dimensional hyperbolic manifold $N_\Gamma = \mathbb{H}^{D+1}/\Gamma$, if there is no sequence of points $\{x_n\}$ in the convex core C_Γ which satisfies the above condition $(*)$, then $\dim \Lambda(\Gamma) < D$.*

Thus problems remain when the boundary ∂C_Γ is not compact and the thick part of C_Γ is not bounded from ∂C_Γ. In this case, the growth order of C_Γ relative to ∂C_Γ should be important.

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