

On continuous extension of grafting maps

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Abstract

The definition of the grafting operation for quasifuchsian groups is extended by Bromberg [Br] to all b -groups. Although the grafting maps are not necessarily continuous at boundary groups, in this paper, we show that the grafting maps take every “standard” convergent sequence to a convergent sequence. As a consequence of this result, we extend Goldman’s grafting theorem for quasifuchsian groups to all boundary b -groups.

1 Introduction

In this paper we are interested in the behavior of the holonomy map from projective structures to representations. Especially, we study the continuity and the non-continuity of the local inverse of the holonomy map at the boundary of the space of discrete faithful representations.

Let S be an oriented closed surface of genus $g > 1$. A projective structure on S is a (G, X) -structure where X is a Riemann sphere $\widehat{\mathbb{C}}$ and $G = \mathrm{PSL}_2(\mathbb{C})$ is the group of projective automorphism of $\widehat{\mathbb{C}}$. Let $P(S)$ denote the space of projective structures on S and $R(S)$ the space of conjugacy classes of representations $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{C})$. The holonomy map $hol : P(S) \rightarrow R(S)$ takes a projective structure to its holonomy representation.

We are interested in the quasifuchsian space $\mathcal{QF}(S) \subset R(S)$ of faithful representations with quasifuchsian images. Although $\mathcal{QF} = \mathcal{QF}(S)$ is a connected, contractible, open submanifold of $R(S)$, the topology of its closure $\overline{\mathcal{QF}}$ in $R(S)$ is very complicated; for instance McMullen [Mc] showed that \mathcal{QF} self-bumps, i.e., there exists $\rho \in \partial\mathcal{QF}$ such that $U \cap \mathcal{QF}$ is disconnected for any sufficiently small neighborhood U of ρ , by using projective structures and ideas of Anderson and Canary [AC]. In fact, since the holonomy map $hol : P(S) \rightarrow R(S)$ is a local homeomorphism, some kind of complexity of the topology of $\mathcal{QF} \subset R(S)$ can be observed by studying the inverse image $Q(S) = hol^{-1}(\mathcal{QF})$ of \mathcal{QF} in $P(S)$. By Goldman’s grafting theorem [Go], the set of components of $Q(S)$ is in one-to-one correspondence with the set $\mathcal{ML}_{\mathbb{N}}$ of integral points of measured laminations. For every $\lambda \in \mathcal{ML}_{\mathbb{N}}$, the associated component \mathcal{Q}_{λ} of $Q(S)$ is taken by the holonomy map biholomorphically onto \mathcal{QF} . We let $\Psi_{\lambda} : \mathcal{Q}_{\lambda} \rightarrow \mathcal{Q}_{\lambda}$ denote the univalent local branch of hol^{-1} and call

it the *grafting map* for $\lambda \in \mathcal{ML}_{\mathbb{N}}$. The *standard component* \mathcal{Q}_0 is the component of $Q(S)$ of projective structures with injective developing map. Then McMullen [Mc] actually showed that there exists a sequence in $Q(S) - \mathcal{Q}_0$ converging to $\partial\mathcal{Q}_0$, which implies that \mathcal{QF} self-bumps. Moreover, we showed in [It1, It3] that for any $n \in \mathbb{N}$, there exists n -components of $Q(S)$ which bump simultaneously (i.e., having intersecting closures), that any component of $Q(S)$ except for \mathcal{Q}_0 self-bumps, and that any two components of $Q(S)$ bump. All these phenomena are obtained by using exotic convergent sequence in \mathcal{QF} (see definition below) constructed by Anderson and Canary [AC].

Let $B : T(S) \times T(S) \rightarrow \mathcal{QF}$ be the parameterization of \mathcal{QF} of Bers' simultaneous uniformization and let $\rho_n = B(X_n, Y_n) \in \mathcal{QF}$ be a sequence converging to $\rho \in \partial\mathcal{QF}$. Then we say that the convergence $\rho_n \rightarrow \rho_\infty$ is *standard* if there exists a compact subset K of $T(S)$ which contains all X_n or all Y_n ; otherwise it is *exotic*. We let $\partial^\pm \mathcal{QF} \subset \partial\mathcal{QF}$ denote the set of limits of standard convergence sequences, or the set of all boundary b -groups (see §2.1). Then by using Bromberg's observation in [Br], the grafting map Ψ_λ is naturally extended to $\Psi_\lambda : \mathcal{QF} \sqcup \partial^\pm \mathcal{QF} \rightarrow \widehat{P}(S)$ for every $\lambda \in \mathcal{ML}_{\mathbb{N}}$. Here $\widehat{P}(S) = P(S) \cup \{\infty\}$ denotes the one-point compactification of $P(S)$. As described above, Anderson and Canary [AC] showed that there exist sequences $\rho_n \in \mathcal{QF}$ converging exotically to $\rho_\infty \in \partial^\pm \mathcal{QF}$, which course the non-continuity of the map $\Psi_\lambda : \mathcal{QF} \rightarrow P(S)$ at $\rho_\infty \in \partial^\pm \mathcal{QF}$ for all $\lambda \in \mathcal{ML}_{\mathbb{N}}$.

On the contrary, we study the continuity of the grafting maps in this paper. The following theorem (Theorem 5.1) is the main result of the paper, which states that each grafting map Ψ_λ behaves as a continuous map for every standard convergent sequence.

Theorem 1.1. *Let $\rho_n \in \mathcal{QF}$ be a sequence converging standardly to $\rho_\infty \in \partial^\pm \mathcal{QF}$. Then the sequence $\Psi_\lambda(\rho_n)$ converges to $\Psi_\lambda(\rho_\infty)$ in $\widehat{P}(S)$ for every $\lambda \in \mathcal{ML}_{\mathbb{N}}$.*

Let $\overline{\mathcal{Q}_0}$ denote the closure of \mathcal{Q}_0 in $P(S)$, *not* in $\widehat{P}(S)$. As we see in §3.3, a sequence $\Sigma_n \in \mathcal{Q}_0$ converges to $\Sigma_\infty \in \partial\mathcal{Q}_0$ if and only if the sequence $hol(\Sigma_n) \in \mathcal{QF}$ converges standardly to $hol(\Sigma_\infty) \in \partial^+ \mathcal{QF}$. Hence we have the next corollary (Theorem 6.10) of Theorem 1.1.

Corollary 1.2. *The map $Gr_\lambda := \Psi_\lambda \circ \Psi_0^{-1} : \mathcal{Q}_0 \rightarrow \mathcal{Q}_\lambda$ is extended continuously to $Gr_\lambda : \overline{\mathcal{Q}_0} \rightarrow \widehat{P}(S)$ for each $\lambda \in \mathcal{ML}_{\mathbb{N}}$, where $\overline{\mathcal{Q}_0}$ is the closure of \mathcal{Q}_0 in $P(S)$.*

An important remark is that, since we do not know whether \mathcal{Q}_0 self-bumps or not, the statement of Theorem 1.1 (or Corollary 1.2) is not trivial and worth considering.

The following theorem plays an important roll in the proof of Theorem 1.1, which have some interest in its own light (see §4). A Bers slice \mathcal{B} is a slice of \mathcal{QF} of the type $\{B(X, Y)\}_{Y \in T(S)}$ or $\{B(X, Y)\}_{X \in T(S)}$. In addition, we let $\pi : P(S) \rightarrow T(S)$ denote the projection which takes a projective structure to its underling conformal structure.

Theorem 1.3. *Let \mathcal{B} be a Bers slice and $\{\lambda_n\}$ a sequence of distinct elements of $\mathcal{ML}_{\mathbb{N}}$. Then the sequence $\{\pi \circ \Psi_{\lambda_n}(\mathcal{B})\}$ eventually escapes any compact subset K of $T(S)$; that is, $\pi \circ \Psi_{\lambda_n}(\mathcal{B}) \cap K = \emptyset$ for all large enough n .*

As a consequence of Theorem 1.1, we extend Goldman's grafting theorem as follows (see Theorem 6.1):

Theorem 1.4. *For every boundary b-groups $\rho \in \partial^{\pm} \mathcal{QF}$, all projective structures with holonomy ρ are obtained by grafting of ρ ; that is, $\text{hol}^{-1}(\rho) = \{\Psi_{\lambda}(\rho) : \lambda \in \mathcal{ML}_{\mathbb{N}}, \Psi_{\lambda}(\rho) \neq \infty\}$.*

As mentioned above, any two components of $Q(S) = \text{hol}^{-1}(\mathcal{QF})$ bump. On the contrary, we obtain the following theorem (Theorem 6.11), which reveals that only exotic convergent sequences cause the bumping of distinct components of $Q(S)$.

Theorem 1.5. *Set $\mathcal{QF}_K = \{B(X, Y) \in \mathcal{QF} : X \in K \text{ or } Y \in K\}$ for a compact subset $K \subset T(S)$. Then the inverse image $\text{hol}^{-1}(\mathcal{QF}_K)$ of \mathcal{QF}_K in $P(S)$ is discrete; that is, every connected component $\Psi_{\lambda}(\mathcal{QF}_K)$ of $\text{hol}^{-1}(\mathcal{QF}_K)$ has an open neighborhood in $P(S)$ which is disjoint from any other one.*

Remark. It is conjectured by Bers, Sullivan and Thurston that the closure of \mathcal{QF} is equal to the space $AH(S) \subset R(S)$ of discrete, faithful representations. This conjecture is closely related to Thurston's ending lamination conjecture, which was recently announced by Minsky to be solved affirmatively. But we do not make use of these deep results in this paper.

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2 Preliminaries

2.1 Quasifuchsian space

A Kleinian group Γ is a discrete subgroup of $\text{PSL}_2(\mathbb{C})$, which acts on hyperbolic 3-space \mathbb{H}^3 as isometries and on the sphere at infinity $\widehat{\mathbb{C}}$ as conformal automorphisms. The union $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \widehat{\mathbb{C}}$ is naturally topologized as a closed 3-ball so that $\text{PSL}_2(\mathbb{C})$ acts continuously on it. For a Kleinian group Γ , we let Ω_{Γ} denote the region of discontinuity and Λ_{Γ} the limit set. We associate to a Kleinian group Γ the following orbit spaces:

$$M_{\Gamma} = \mathbb{H}^3/\Gamma, \quad \overline{M}_{\Gamma} = (\mathbb{H}^3 \cup \Omega_{\Gamma})/\Gamma, \quad \partial M_{\Gamma} = \Omega_{\Gamma}/\Gamma,$$

where ∂M_{Γ} is called the *conformal boundary* of M_{Γ} . In general if \overline{M} is an oriented manifold with boundary ∂M , we orient ∂M by requiring that the frame (f, n) has positive orientation whenever f is a positively oriented frame on ∂M and n is an inward-pointing vector.

Let S be an oriented closed surface of genus $g > 1$. Let $R(S)$ be the space of conjugacy classes $[\rho]$ of representations $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{C})$ whose images $\rho(\pi_1(S))$ are non-abelian. The space $R(S)$ is equipped with the algebraic topology, the topology of convergence on generators up to conjugation. (By abuse of notation, we also denote $[\rho]$ by ρ if there is no confusion.) It is known that $R(S)$ is a $(6g - 6)$ -dimensional complex manifold (see Theorem 4.21 in [MT]).

Let $AH(S)$ be the subset of $R(S)$ of discrete, faithful representations, which is known to be a closed subset of $R(S)$ by Jørgensen [Jo]. Let $\rho_n \rightarrow \rho_\infty$ be an algebraic convergence sequence in $AH(S)$. Then it is known by Jørgensen and Marden [JM] that, by passing to a subsequence if necessary, the sequence $\Gamma_n = \rho_n(\pi_1(S))$ of Kleinian groups converges geometrically to some Kleinian group $\widehat{\Gamma}$, which contains the algebraic limit $\Gamma_\infty = \rho_\infty(\pi_1(S))$. Here the convergence $\Gamma_n \rightarrow \widehat{\Gamma}$ is *geometric* if and only if for any $\hat{\gamma} \in \widehat{\Gamma}$, there exist $\gamma_n \in \Gamma_n$ such that $\gamma_n \rightarrow \hat{\gamma}$ and if for every convergent sequence $\gamma_{n_j} \in \Gamma_{n_j}$ ($n_j \rightarrow \infty$), the limit is contained in $\widehat{\Gamma}$. A convergence $\rho_n \rightarrow \rho_\infty$ in $AH(S)$ is said to be *strong* if $\Gamma_n = \rho_n(\pi_1(S))$ converges geometrically to the algebraic limit $\Gamma_\infty = \rho_\infty(\pi_1(S))$.

For every $\rho \in AH(S)$ with image $\Gamma = \rho(\pi_1(S))$, Bonahon's theorem [Bo] guarantees that there exists an orientation preserving homeomorphism $\psi : S \times (-1, 1) \rightarrow M_\rho = M_\Gamma$, where $(-1, 1)$ is an open interval. The conformal boundary ∂M_ρ (possibly empty) of M_ρ decomposes into two parts $\partial^+ M_\rho \sqcup \partial^- M_\rho$, where $\partial^+ M_\rho$ (resp. $\partial^- M_\rho$) is the limit of $\psi(S \times \{t\})$ in $\overline{M_\rho}$ as $t \rightarrow -1$ (resp. $t \rightarrow 1$). Associated to this decomposition, Ω_Γ decomposes into $\Omega_\Gamma^+ \sqcup \Omega_\Gamma^-$ for which $\Omega_\Gamma^+/\Gamma = \partial^+ M_\rho$ and $\Omega_\Gamma^-/\Gamma = \partial^- M_\rho$.

Let $\Gamma = \rho(\pi_1(S))$ be the image of some $\rho \in AH(S)$. Then Γ is called a *b-group* if exactly one of Ω_Γ^+ or Ω_Γ^- is simply connected and Γ -invariant. For a *b-group* ρ , we denote by $\mathrm{para}(\rho)$ its *parabolic locus*, a collection of homotopy classes of disjoint simple closed curves c on S such that $\rho(c) \in \Gamma$ is parabolic. Similarly Γ is called a *quasifuchsian group* if both Ω_Γ^+ and Ω_Γ^- are simply connected and Γ -invariant. Quasifuchsian space $\mathcal{QF} = \mathcal{QF}(S)$ is the subset of $R(S)$ of faithful representations with quasifuchsian images. It is known by Marden [Mar] and Sullivan [Su] that \mathcal{QF} equals the interior of $AH(S)$. Hence \mathcal{QF} is a $(6g - 6)$ -dimensional complex manifold in $R(S)$. On the other hand, it is trivial that $\overline{\mathcal{QF}} \subset AH(S)$ and is conjectured that $\overline{\mathcal{QF}} = AH(S)$, which is so called the Bers-Sullivan-Thurston conjecture. We let $\partial\mathcal{QF}$ denote the relative boundary of \mathcal{QF} in $R(S)$, whose element is called a *boundary group*.

Now let $\rho \in \mathcal{QF}$. Then there exists an orientation preserving homeomorphism $\psi : S \times [-1, 1] \rightarrow \overline{M_\rho}$ which induces the representation $\psi_* = \rho : \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{C})$. Moreover, it provides orientation preserving homeomorphisms $\psi|_{S \times \{-1\}} : S \rightarrow \partial^+ M_\rho$ and $\psi|_{S \times \{1\}} : \bar{S} \rightarrow \partial^- M_\rho$, where \bar{S} denotes S with its orientation reversed. Hence $\rho \in \mathcal{QF}$ determines a pair of marked Riemann surfaces $(\partial^+ M_\rho, \partial^- M_\rho) \in T(S) \times T(\bar{S})$ in the product of the Teichmüller spaces. On the other hand, Bers [Be] showed that each pair $(X, \bar{Y}) \in T(S) \times T(\bar{S})$ has the unique simultaneous uniformization

$\rho = B(X, \bar{Y}) \in \mathcal{QF}$. Therefore the map

$$B : T(S) \times T(\bar{S}) \rightarrow \mathcal{QF}$$

gives us a global parameterization of \mathcal{QF} . We define *vertical* and *horizontal Bers slices* in \mathcal{QF} by $B_X = \{B(X, \bar{Y}) : \bar{Y} \in T(\bar{S})\}$ and $B_{\bar{Y}} = \{B(X, \bar{Y}) : X \in T(S)\}$, respectively. It is known by Bers that both B_X and $B_{\bar{Y}}$ are precompact in $R(S)$, whose frontiers are denoted by ∂B_X and $\partial B_{\bar{Y}}$. A representation $\rho \in AH(S)$ is called a *Bers boundary group* if it is contained in the frontier of some Bers slice.

2.2 Sequences in quasifuchsian space

We introduce the notion of “standard” and “exotic” convergence for a sequence $\rho_n \in \mathcal{QF}$ tending to a limit $\rho_\infty \in \partial\mathcal{QF}$. One of our main purpose of this paper is to show that every grafting map behaves as a continuous map for every standard convergent sequence (see Theorem 5.1). For a given subset K of $T(S)$, we set $\bar{K} = \{\bar{X} \in T(\bar{S}) : X \in K\} \subset T(\bar{S})$, where $\bar{X} \in T(\bar{S})$ denote the complex conjugation of $X \in T(S)$.

Definition 2.1 (Standard and exotic convergence). Suppose that a sequence $\rho_n = B(X_n, \bar{Y}_n)$ in \mathcal{QF} converges to $\rho_\infty \in \partial\mathcal{QF}$. Then the sequence ρ_n is said to converge *standardly* to ρ_∞ if there exists a compact subset $K \subset T(S)$ such that (i) $X_n \in K$ for all n , or (ii) $\bar{Y}_n \in \bar{K}$ for all n . Otherwise, we say that ρ_n converges *exotically* to ρ_∞ .

We let $\partial^+\mathcal{QF}$ and $\partial^-\mathcal{QF}$ denote the subsets of $\partial\mathcal{QF}$ consisting of limits of standard convergent sequences of type (i) and (ii), respectively, and set $\partial^\pm\mathcal{QF} = \partial^+\mathcal{QF} \sqcup \partial^-\mathcal{QF}$. It is easily seen that every Bers boundary group is an element of $\partial^\pm\mathcal{QF}$ and that every element in $\partial^\pm\mathcal{QF}$ is a *boundary b-group*, a *b-group* in $\partial\mathcal{QF}$. On the contrary, it follows from the arguments in the paper of Brock, Bromberg, Evans and Souto [BBES] that every boundary *b-group* is a Bers boundary group (see Theorem 2.3 below). Hence we see that the set $\partial^\pm\mathcal{QF}$ equals the set of all boundary *b-groups* and that the following hold:

$$\partial^+\mathcal{QF} = \bigsqcup_{X \in T(S)} \partial B_X, \quad \partial^-\mathcal{QF} = \bigsqcup_{\bar{Y} \in T(\bar{S})} \partial B_{\bar{Y}}.$$

Moreover, Bromberg [Br] obtained the following result by developing the grafting operation for elements in $\partial^-\mathcal{QF}$ (see §3) and by combining with Minsky’s result [Mi].

Theorem 2.2 (Bromberg). *Every b-group $\rho \in AH(S)$ without parabolics is a Bers boundary group.*

We now outline the proof of Theorem 2.3 by following the arguments in [BBES]. In the argument, we also obtain Corollary 2.4 below, which is required in §5.

Theorem 2.3 (Brock-Bromberg-Evans-Souto). *Every boundary b-group is a Bers boundary group.*

Corollary 2.4. *Let $\rho \in \partial^\pm \mathcal{QF}$ with parabolic locus $\text{para}(\rho)$. If $\rho \in \partial^+ \mathcal{QF}$ (resp. $\rho \in \partial^- \mathcal{QF}$), there exists a sequence $\rho_n = B(X_n, \bar{Y}_n) \in \mathcal{QF}$ which converges standardly to ρ and which satisfies $l_{\bar{Y}_n}(\text{para}(\rho)) \rightarrow 0$ (resp. $l_{X_n}(\text{para}(\rho)) \rightarrow 0$) as $n \rightarrow \infty$. Here $l_{\bar{Y}_n}(\text{para}(\rho))$ denotes the total sum of hyperbolic lengths of components of $\text{para}(\rho)$ on \bar{Y}_n .*

Outline of the proof of Theorem 2.3. Let $\rho \in \partial \mathcal{QF}$ with $\text{para}(\rho)$, which is possibly empty. We may assume that the positive part $\partial_c^+ M_\rho$ of the conformal boundary of M_ρ is homeomorphic to S , and hence determine a point $X \in T(S)$. By Theorem 3.1 in [BBES], the drilling theorem plays an important role in its proof, there exists a strong convergent sequence $\rho_n \rightarrow \rho$ in $\overline{\mathcal{QF}}$ for which each ρ_n is a geometrically finite representation whose parabolic locus $\text{para}(\rho_n)$ equals $\text{para}(\rho)$. Since the convergence $\rho_n \rightarrow \rho$ is strong, $X_n = \partial_c^+ M_{\rho_n}$ converges to X in $T(S)$. Moreover, for each n , there exists a strong convergent sequence $\rho_{n,k} \rightarrow \rho_n$ ($k \rightarrow \infty$) of quasifuchsian representations all of whose positive parts of the conformal boundaries equal $X_n \in T(S)$ (see Theorem 3.4 in [BBES]). Therefore $\rho_{n,k} = B(X_n, \bar{Y}_{n,k})$ is a sequence in the Bers slice B_{X_n} for each n . Since the convergence $\rho_{n,k} \rightarrow \rho_n$ is strong and since ρ_n is geometrically finite, the hyperbolic lengths $l_{\bar{Y}_{n,k}}(\text{para}(\rho))$ of $\text{para}(\rho)$ on $\bar{Y}_{n,k}$ tend to 0 as $k \rightarrow \infty$. By a diagonal argument, we can choose a sequence $\rho'_n = B(X_n, \bar{Y}_n)$ from $\{\rho_{n,k}\}_{n,k \in \mathbb{N}}$ so that $\rho'_n \rightarrow \rho$ and that $l_{\bar{Y}_n}(\text{para}(\rho)) \rightarrow 0$ as $n \rightarrow \infty$, which satisfies the desired property in Corollary 2.4. Now let us consider new sequence $\rho''_n = B(X, \bar{Y}_n)$ in B_X . Then this sequence $\rho''_n \in B_X$ also converges to ρ because maximal dilatations of quasiconformal automorphism of $\widehat{\mathbb{C}}$ conjugating ρ'_n to ρ''_n tend to 1 as $n \rightarrow \infty$. This implies that $\rho \in \partial B_X$. \square

We remark that the set $\partial \mathcal{QF} - \partial^\pm \mathcal{QF}$ is not empty; for instance, it contains limits of sequences which appear in Thurston's double limit theorem. On the other hand, Anderson and Canary [AC] showed that there exists a sequence in \mathcal{QF} which converges exotically to some point in $\partial^\pm \mathcal{QF}$. All the known such sequences are basically obtained by their technique and here is a typical example: let c be a simple closed curve on S and let $\tau = \tau_c$ be the Dehn twist along c . Then for a fixed pair $(X, \bar{Y}) \in T(S) \times T(\bar{S})$, the sequence

$$\rho_n = B(\tau^n X, \tau^{2n} \bar{Y}) \quad (n \in \mathbb{Z})$$

in \mathcal{QF} converges exotically to some $\rho_\infty \in \partial^+ \mathcal{QF}$ as $|n| \rightarrow \infty$. We will make use of this sequence in §3.3 to explain how the grafting maps fail to be extended to a continuous map.

2.3 Space of projective structures

We only give a brief summary of projective structures and refer to [It1] and elsewhere for more details.

A projective structure on S is a (G, X) -structure where X is a Riemann sphere $\widehat{\mathbb{C}}$ and $G = \mathrm{PSL}_2(\mathbb{C})$ is the group of projective automorphism of $\widehat{\mathbb{C}}$. Let $P(S)$ be the space of marked projective structures on S . A projective structure $\Sigma \in P(S)$ determine the underlying conformal structure $\pi(\Sigma) \in T(S)$. It is known that $P(S)$ is the holomorphic affine bundle over $T(S)$ with the projection $\pi : P(S) \rightarrow T(S)$, and that $P(S)$ is a $(6g - 6)$ -dimensional complex manifold.

A projective structure Σ on S can be lifted to that $\widetilde{\Sigma}$ on \widetilde{S} , where $\widetilde{S} \rightarrow S$ is the universal cover on which $\pi_1(S)$ acts as a covering group. Since $\widetilde{\Sigma}$ is simply connected, we obtain a developing map $f_\Sigma : \widetilde{S} \rightarrow \widehat{\mathbb{C}}$ by continuing charts of $\widetilde{\Sigma}$ analytically, which induce a holonomy representation $\rho_\Sigma : \pi_1(S) \cong \pi_1(\Sigma) \rightarrow \mathrm{PSL}_2(\mathbb{C})$ satisfying $f_\Sigma \circ \gamma = \rho_\Sigma(\gamma) \circ f_\Sigma$ for every $\gamma \in \pi_1(S)$. We remark that the pair (f_Σ, ρ_Σ) is determined uniquely up to $\mathrm{PSL}_2(\mathbb{C})$. We now define the *holonomy map*

$$hol : P(S) \rightarrow R(S)$$

by $\Sigma \mapsto [\rho_\Sigma]$. Hejhal [He] showed that the map hol is a local homeomorphism and Earle [Ea] and Hubbard [Hu] independently showed that the map is holomorphic.

Theorem 2.5 (Hejhal, Earle and Hubbard). *The holonomy map $hol : P(S) \rightarrow R(S)$ is a holomorphic local homeomorphism.*

We denote by $Q(S) = hol^{-1}(\mathcal{QF})$ the set of projective structures with quasifuchsian holonomy. An element of $Q(S)$ is said to be *standard* if its developing map is injective; otherwise it is *exotic*. We denote by $\mathcal{Q}_0 \subset Q(S)$ the subset of standard projective structures. Let $\rho = B(X, Y) \in \mathcal{QF}$ with image $\Gamma = \rho(\pi_1(S))$. Then the quotient surface $\Sigma = \Omega_\Gamma^+ / \Gamma$ can be regarded as a standard projective structure on S with bijective developing map $f_\Sigma : \widetilde{S} \rightarrow \Omega_\Gamma^+$, with holonomy representation $\rho_\Sigma = \rho$, and with underlying conformal structure $X \in T(S)$. Let

$$\Psi_0 : \mathcal{QF} \rightarrow \mathcal{Q}_0$$

be the map defined by the correspondence $\rho \mapsto \Omega_\Gamma^+ / \Gamma$ as described above. Then the map Ψ_0 turns out to be a univalent local branch of hol^{-1} onto the connected component \mathcal{Q}_0 of $Q(S)$, which is called the *standard component*. It is known by Bers that every Bers slice $B_X \subset \mathcal{QF}$ is embedded by the map Ψ_0 into a bounded domain $\Psi_0(B_X)$ of a fiber $\pi^{-1}(X) \subset P(S)$. Observe that $hol|_{\mathcal{Q}_0} : \overline{\mathcal{Q}_0} \rightarrow \mathcal{QF} \sqcup \partial^+ \mathcal{QF}$ is bijective, where $\overline{\mathcal{Q}_0}$ is the closure of \mathcal{Q}_0 in $P(S)$.

3 Grafting

3.1 Grafting maps on quasifuchsian space

We let $\mathcal{ML}_\mathbb{N} = \mathcal{ML}_\mathbb{N}(S)$ denote the set of integral points of measured laminations on S . In other words, each element of $\lambda \in \mathcal{ML}_\mathbb{N}$ is a isotopy class of disjoint union $\sqcup_{i=1}^l k_i c_i$ of homotopically distinct simple closed curves c_i on S with positive integer

k_i weights. We do not distinguish the isotopy class λ and its representative if there is no confusion. The “zero-lamination” 0 is also contained in $\mathcal{ML}_{\mathbb{N}}$. In what follows, the parabolic locus $\text{para}(\rho)$ of a b -group ρ is also regarded as an element of $\mathcal{ML}_{\mathbb{N}}$.

For each non-zero $\lambda \in \mathcal{ML}_{\mathbb{N}}$, we will explain how to obtain the grafting map

$$\text{Gr}_{\lambda} : \mathcal{Q}_0 \rightarrow P(S),$$

which satisfies $\text{hol} \circ \text{Gr}_{\lambda} \equiv \text{hol}$ on \mathcal{Q}_0 . We give here two equivalent definitions of grafting operation; the first one is as usual, and the second one is introduced by Bromberg in [Br] so that it also makes sense for elements of $\partial^- \mathcal{QF}$. For a while, we assume that λ is a simple closed curve c of weight one for simplicity. In addition, we fix our notation as follows: for a given $\rho \in \mathcal{QF}$, let Γ be the quasifuchsian image of ρ and $\Omega_{\Gamma} = \Omega_{\Gamma}^+ \sqcup \Omega_{\Gamma}^-$ the region of discontinuity of Γ . We regard quotient surfaces $\Sigma = \Omega_{\Gamma}^+/\Gamma = \Psi_0(\rho_0)$ and $\Sigma^- = \Omega_{\Gamma}^-/\Gamma$ as projective structures on S and \bar{S} , respectively. Let $c^+ \subset \Sigma$ and $c^- \subset \Sigma^-$ be simple closed curves corresponding to $c \subset S$ and suppose that $\gamma \in \Gamma \cong \pi_1(S)$ is a representative of the homotopy class of c . Let $\tilde{c}^+ \subset \Omega_{\Gamma}^+$ and $\tilde{c}^- \subset \Omega_{\Gamma}^-$ be the $\langle \gamma \rangle$ -invariant lifts of $c^+ \subset \Sigma$ and $c^- \subset \Sigma^-$, respectively.

Notational remark. Let A, B be open subsets of a topological space \mathcal{X} such that $A \cap B = \emptyset$ and that $\overline{A} \cap \overline{B} \neq \emptyset$. Then $A \amalg B$ denotes the interior of $\overline{A \sqcup B}$.

Definition 3.1 (Grafting). In the situation as described above, the *grafting* $\text{Gr}_{\lambda}(\Sigma)$ of the standard projective structure $\Sigma = \Psi_0(\rho)$ along c is a projective structure obtained by the following (equivalent) procedures:

I let A_c be a cylinder $(\widehat{\mathbb{C}} - \tilde{c}^+)/\langle \gamma \rangle$ equipped with a projective structure induced from that of $\widehat{\mathbb{C}}$. We obtain $\text{Gr}_c(\Sigma)$ by cutting Σ along c and inserting A_c at the cut locus without twisting; that is,

$$\text{Gr}_c(\Sigma) = (\Sigma - c) \amalg A_c.$$

II Here we further assume that c separates S into two surfaces S_1 and S_2 with boundaries. (The non-separating case is described precisely in [Br].) Accordingly, Σ and Σ^- decompose into $\Sigma - c^+ = \Sigma_1 \sqcup \Sigma_2$ and $\Sigma^- - c^- = \Sigma_1^- \sqcup \Sigma_2^-$, respectively. Let i denotes 1 or 2. Let Δ_i be the connected component of the inverse image of $\Sigma_i^- \subset \Sigma^-$ in Ω_{Γ}^- which contains \tilde{c}^- in its closure $\overline{\Delta}_i \subset \widehat{\mathbb{C}}$. Then the stabilizer subgroup $\Gamma_i = \text{Stab}_{\Gamma}(\Delta_i)$ of $\Gamma \cong \pi_1(S)$ is identified with $\pi_1(S_i)$. Since Γ_i is a purely loxodromic free group with non-empty region of discontinuity, Maskit’s result [Mas] implies that Γ_i is a Schottky group. Note that the conformal boundary $\partial M_{\Gamma_i} = \Omega_{\Gamma_i}/\Gamma_i$ of M_{Γ_i} with natural projective structure containing both projective surfaces Σ_i and Σ_i^- . Then $\text{Gr}_c(\Sigma)$ is obtained from projective surfaces $\Omega_{\Gamma_1}/\Gamma_1 - \Sigma_1^-$ and $\Omega_{\Gamma_2}/\Gamma_2 - \Sigma_2^-$ by gluing their boundaries without twisting (see Figure 1); that is,

$$\text{Gr}_c(\Sigma) = (\Omega_{\Gamma_1}/\Gamma_1 - \Sigma_1^-) \amalg (\Omega_{\Gamma_2}/\Gamma_2 - \Sigma_2^-).$$

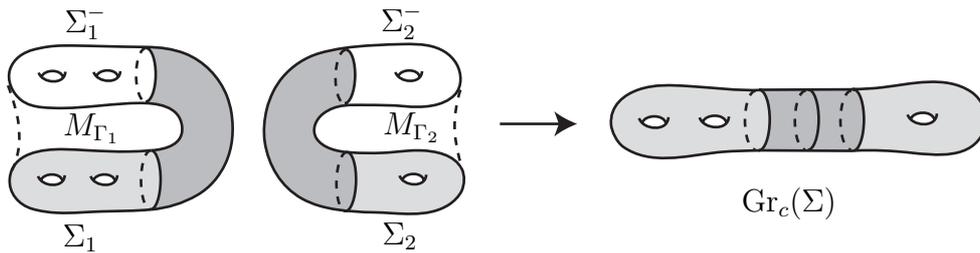


Figure 1: The grafting $\text{Gr}_c(\Sigma)$ of Σ along c .

Observe that the Definitions I and II are equivalent. The grafting $\text{Gr}_\lambda(\Sigma)$ of Σ along general $\lambda = \sqcup k_i c_i \in \mathcal{ML}_{\mathbb{N}}$ is similarly obtained; for instance, $\text{Gr}_{kc}(\Sigma)$ is obtained by inserting k -copies of A_c in Definition I. An important fact is that the grafting operation does not change the holonomy representation, that is, $\text{hol}(\text{Gr}_\lambda(\Sigma)) = \text{hol}(\Sigma)$ is always satisfied.

Since the grafting map $\text{Gr}_\lambda : \mathcal{Q}_0 \rightarrow P(S)$ satisfies $\text{hol} \circ \text{Gr}_\lambda \equiv \text{hol}$ on \mathcal{Q}_0 and since $\text{hol}|_{\mathcal{Q}_0} : \mathcal{Q}_0 \rightarrow \mathcal{QF}$ is a biholomorphic map with its inverse Ψ_0 , the map Gr_λ takes \mathcal{Q}_0 biholomorphically onto the image $\text{Gr}_\lambda(\mathcal{Q}_0)$, which is denoted by \mathcal{Q}_λ . Hence we obtain a univalent local branch

$$\Psi_\lambda : \mathcal{QF} \rightarrow \mathcal{Q}_\lambda$$

of hol^{-1} which is defined by $\Psi_\lambda = \text{Gr}_\lambda \circ \Psi_0$ and is also called the *grafting map* for λ (see the commutative diagram below). By Goldman's grafting theorem [Go] below,

$$\begin{array}{ccc}
 P(S) \supset \mathcal{Q}_0 & \xrightarrow{\text{Gr}_\lambda} & \mathcal{Q}_\lambda \\
 \text{hol} \downarrow & \Psi_0 \uparrow & \nearrow \Psi_\lambda \\
 R(S) \supset \mathcal{QF} & &
 \end{array}$$

we obtain the decomposition $\bigsqcup_{\lambda \in \mathcal{ML}_{\mathbb{N}}} \mathcal{Q}_\lambda$ of $\mathcal{Q}(S)$ into its connected components.

Theorem 3.2 (Goldmann [Go]). *For every $\rho \in \mathcal{QF}$, all projective structures with holonomy ρ are obtained by grafting of ρ ; that is, we have $\text{hol}^{-1}(\rho) = \{\Psi_\lambda(\rho) : \lambda \in \mathcal{ML}_{\mathbb{N}}\}$.*

3.2 Extension of grafting maps

Let $\widehat{P}(S)$ denotes the one-point compactification $P(S) \cup \{\infty\}$ of $P(S)$. We now extend the grafting map $\Psi_\lambda : \mathcal{QF} \rightarrow \mathcal{Q}_\lambda$ to $\Psi : \mathcal{QF} \sqcup \partial^\pm \mathcal{QF} \rightarrow \widehat{P}(S)$. (In fact $\Psi_\lambda(\rho)$ is defined for all b -groups $\rho \in \text{AH}(S)$ by the same manner.) Observe that Definition I works well even for $\rho \in \partial^+ \mathcal{QF}$ whenever γ is loxodromic, because there

still exists a $\langle \gamma \rangle$ -invariant simple arc \tilde{c}^+ in non-degenerate component Ω_{Γ}^+ , for which $(\widehat{\mathbb{C}} - \tilde{c}^+)/\langle \gamma \rangle$ is still an annulus. On the other hand, Definition II works well for $\rho \in \partial^- \mathcal{QF}$ whenever every connected component of $\text{para}(\rho)$ intersects c essentially. In fact, on this assumption, Γ_1 and Γ_2 in Definition II are still Schottky groups, γ is still loxodromic, and there still exists a $\langle \gamma \rangle$ -invariant simple arc \tilde{c}^- in non-degenerate component Ω_{Γ}^- . For general $\lambda \in \mathcal{ML}_{\mathbb{N}}$, we can also define the grafting $\Psi_{\lambda}(\rho) \in P(S)$ of ρ along λ if the pair (λ, ρ) satisfies the following condition:

Definition 3.3 (Admissible). The pair (λ, ρ) of $\lambda \in \mathcal{ML}_{\mathbb{N}}$ and $\rho \in \partial^{\pm} \mathcal{QF}$ is said to be *admissible* if

- $\rho \in \partial^+ \mathcal{QF}$ and $\text{para}(\rho)$ and λ have no parallel component in common, or
- $\rho \in \partial^- \mathcal{QF}$ and every component of $\text{para}(\rho)$ intersects λ essentially.

If the pair (λ, ρ) is not admissible, we set $\Psi_{\lambda}(\rho) = \infty \in \widehat{P}(S)$. The extended grafting map $\Psi_{\lambda} : \mathcal{QF} \sqcup \partial^{\pm} \mathcal{QF} \rightarrow \widehat{P}(S)$ is also denoted by the same symbol Ψ_{λ} .

3.3 Non-continuity of grafting maps

We collect in Table 1 below the equivalent conditions with standard/exotic convergence of quasifuchsian representations (see [Mc, Appendix A] and [It1, §3]). The situation in which we consider is as follows: let $\rho_n \in \mathcal{QF}$ be a sequence converging to $\rho_{\infty} \in \partial^+ \mathcal{QF}$. We may assume that the sequence $\Gamma_n = \rho_n(\pi_1(S))$ converges geometrically to a Kleinian group $\widehat{\Gamma}$, which contains the algebraic limit $\Gamma_{\infty} = \rho_{\infty}(\pi_1(S))$. Let Σ_{∞} be a unique projective structure in $\partial \mathcal{Q}_0$ such that $\text{hol}(\Sigma_{\infty}) = \rho_{\infty}$ and let $\Phi : U \rightarrow P(S)$ be a univalent local branch of hol^{-1} which is defined on a neighborhood U of ρ_{∞} and takes ρ_{∞} to Σ_{∞} . Then $\Sigma_n = \Phi(\rho_n)$ converges to $\Sigma_{\infty} = \Phi(\rho_{\infty})$. In this situation, all conditions in the same line in Table 1 are equivalent.

Table 1: Equivalent conditions with standard/exotic convergence.

$\rho_n \rightarrow \rho_{\infty} : \text{standard}$	$\rho_n \rightarrow \rho_{\infty} : \text{exotic}$
Γ_{∞} is a component subgroup of $\widehat{\Gamma}$	Γ_{∞} is not a component subgroup of $\widehat{\Gamma}$
$\Omega_{\Gamma_{\infty}}^+ \cap \Lambda(\widehat{\Gamma}) = \emptyset$	$\Omega_{\Gamma_{\infty}}^+ \cap \Lambda(\widehat{\Gamma}) \neq \emptyset$
Σ_n are standard ($n \gg 0$)	Σ_n are exotic ($n \gg 0$)

In the same setting as above, let $\rho_n = B(\tau^n X, \tau^{2n} \bar{Y})$ be the sequence introduced at the end of §2.1 which converges exotically to ρ_{∞} . Then Σ_n are exotic projective structure for all large enough $|n|$. Since $\Sigma_n \rightarrow \Sigma_{\infty} \in \partial \mathcal{Q}_0$, the sequence $\Psi_0(\rho_n) \in \mathcal{Q}_0$ of standard projective structure can not accumulates on a point in $\partial \mathcal{Q}_0$ and hence diverges. This implies that the map $\Psi_0 : \mathcal{QF} \sqcup \partial^{\pm} \mathcal{QF} \rightarrow \widehat{P}(S)$ is not continuous

at $\rho_\infty \in \partial^+ \mathcal{QF}$. In addition, we have shown in [It1] that $\Sigma_n = \Psi_c(\rho_n) \in \mathcal{Q}_c$ and that $\lim_{n \rightarrow +\infty} \Psi_c(\rho_n) = \lim_{n \rightarrow -\infty} \Psi_c(\rho_n)$. On the other hand, we have shown in [It3] that $\lim_{n \rightarrow +\infty} \Psi_{c'}(\rho_n) \neq \lim_{n \rightarrow -\infty} \Psi_{c'}(\rho_n)$ for every simple closed curve c' on S which intersects c essentially. This implies that the grafting map $\Psi_{c'} : \mathcal{QF} \sqcup \partial^\pm \mathcal{QF} \rightarrow \widehat{P}(S)$ is not continuous at $\rho_\infty \in \partial^+ \mathcal{QF}$. More general, we have the following

Theorem 3.4 ([It3]). *For any $\lambda \in \mathcal{ML}_\mathbb{N}$, the grafting map $\Psi_\lambda : \mathcal{QF} \sqcup \partial^\pm \mathcal{QF} \rightarrow \widehat{P}(S)$ is not continuous.*

3.4 Pull-backs of region of discontinuities

Let $\Sigma = (f_\Sigma, \rho_\Sigma) \in P(S)$ with $\Gamma = \rho_\Sigma(\pi_1(S))$ and let X be a subset of $\widehat{\mathbb{C}}$ which is Γ -invariant. Then we naturally obtain a subset $f_\Sigma^{-1}(X)/\pi_1(S)$ of Σ since the inverse image $f_\Sigma^{-1}(X) \subset \widetilde{\Sigma}$ of X is also invariant under the action of the covering group $\pi_1(S)$ of the covering map $\widetilde{\Sigma} \rightarrow \Sigma$. We call $f_\Sigma^{-1}(X)/\pi_1(S)$ the *pull-back* of X in Σ .

For $\lambda = \sqcup_i k_i c_i \in \mathcal{ML}_\mathbb{N}$, a *realization* $\widehat{\lambda}$ of λ is a disjoint union of simple closed curves on S which realize each weighted simple closed curve $k_i c_i$ by k_i parallel disjoint simple closed curves which are homotopic to c_i . Let $\mathcal{N}(\widehat{\lambda})$ denotes a regular neighborhood of $\widehat{\lambda}$ in S .

Let $\rho \in \mathcal{QF}$ with $\Gamma = \rho(\pi_1(S))$, let $\Sigma = \Psi_\lambda(\rho)$ be the grafting of ρ along $\lambda \in \mathcal{ML}_\mathbb{N}$, and let s denotes either $+$ or $-$. Then we denote by Ω_Σ^s the pull-back $f_\Sigma^{-1}(\Omega_\Gamma^s)/\pi_1(S) \subset \Sigma$ of an invariant component Ω_Γ^s of Ω_Γ . From the definition of grafting, we have the following lemma (see also the proof of Theorem 4.2).

Lemma 3.5. *Let $\Sigma \in \mathcal{Q}_\lambda$. Then there is a automorphism of Σ which is homotopic to the identity and which takes $\Sigma - \mathcal{N}(\widehat{\lambda})$ onto Ω_Σ^+ and $\mathcal{N}(\widehat{\lambda})$ onto Ω_Σ^- .*

Therefore the topological type of the pull-backs Ω_Σ^+ and Ω_Σ^- are constant for every elements $\Sigma \in \mathcal{Q}_\lambda$. The following lemma states that the similar result holds for a grafting $\Sigma = \Psi_\lambda(\rho)$ of $\rho \in \partial^\pm \mathcal{QF}$ and for the pull-back of the unique invariant component of Ω_Γ (see Figure 2). In fact, we use this fact in §5 to show that $\Sigma = \Psi_\lambda(\rho)$ for $\rho \in \partial^\pm \mathcal{QF}$ is actually a limit of a sequence in \mathcal{Q}_λ . Similar arguments can be found in [It1] and [It3].

Lemma 3.6. *Let (λ, ρ) be an admissible pair of $\lambda \in \mathcal{ML}_\mathbb{N}$ and $\rho \in \partial^\pm \mathcal{QF}$, and set $\Sigma = \Psi_\lambda(\rho)$.*

- *If $\rho \in \partial^+ \mathcal{QF}$, there is a one-to-one correspondence between the set of connected components of $\Omega_\Sigma^+ \subset \Sigma$ and that of $\Sigma - \mathcal{N}(\widehat{\lambda}) \subset \Sigma$. Moreover, for each connected component of $\omega \subset \Omega_\Sigma^+$, there is a homeomorphism from ω onto the corresponding component of $\Sigma - \mathcal{N}(\widehat{\lambda})$ which is isotopic to the identity.*
- *If $\rho \in \partial^- \mathcal{QF}$, there is a one-to-one correspondence between the set of connected components of $\Omega_\Sigma^- \subset \Sigma$ and that of $\mathcal{N}(\widehat{\lambda}) \subset \Sigma$. Moreover, for each connected component of $\omega \subset \Omega_\Sigma^-$, there is a homeomorphism from ω onto the corresponding component of $\mathcal{N}(\widehat{\lambda})$ which is isotopic to the identity.*

Proof. We discuss in the same setting of Definition 3.1 and §3.2, except for one change; here Σ denotes the grafting $\Psi_\lambda(\rho)$ but not a standard projective structure. We first suppose that $\rho \in \partial^+ \mathcal{QF}$. Let us consider the quotient torus $T_c = \widehat{\mathbb{C}}/\langle\gamma\rangle$, in which the arc $\tilde{c}^+ \subset \Omega_\Gamma^+$ descends to an essential curve c^+ in T_c . Then the grafting annulus $A_c = (\widehat{\mathbb{C}} - \tilde{c}^+)/\langle\gamma\rangle$ is identified with the subset $T_c - c^+$ of T_c . On the other hand, since the unique invariant component Ω_Γ^+ of Ω_Γ is simply connected, $\Omega_\Gamma^+/\langle\gamma\rangle \subset T_c$ is also an annulus with core curve c^+ . Hence the intersection $A_c \cap (\Omega_\Gamma^+/\langle\gamma\rangle)$ equals $\Omega_\Gamma^+/\langle\gamma\rangle - c^+$ in T_c , which is a disjoint union of two essential annuli A_1 and A_2 . We now regard that $A_1 \sqcup A_2 \subset A_c \subset \Sigma$. Then the pull-back $\Omega_\Sigma^+ \subset \Sigma$ of Ω_Γ^+ coincides with the union $(\Sigma - A_c) \amalg A_1 \amalg A_2 \subset \Sigma$, which is obtained by gluing those pieces along copies of c^+ . Thus the result follows in this case.

Next we suppose that $\rho_\infty \in \partial^- \mathcal{QF}$. By the same argument as above, we see that the quotient $\Omega_\Gamma^-/\langle\gamma\rangle$ of the unique invariant component Ω_Γ^- of Ω_Γ is an annulus with core curve $c^- = \tilde{c}^-/\langle\gamma\rangle$, and that the disjoint union $\Omega_\Gamma^- - \tilde{c}^- = \Omega_1^- \sqcup \Omega_2^-$ descends to that of essential annuli $A_1^- \sqcup A_2^-$ in $\Omega_\Gamma^-/\langle\gamma\rangle$, where $A_i^- = \Omega_i^-/\langle\gamma\rangle$ for $i = 1, 2$. We may assume that $\Delta_i \subset \Omega_i^-$ for $i = 1, 2$. Recall that $\Gamma_i = \text{Stab}_\Gamma(\Delta_i)$ is a Schottky group. Since $\text{Stab}_{\Gamma_1}(\Omega_2^-) = \langle\gamma\rangle$, the image of $\Omega_\Gamma^- = \Omega_1^- \amalg \Omega_2^-$ via the covering map $\Omega_{\Gamma_1} \rightarrow \Omega_{\Gamma_1}/\Gamma_1$ is $\Sigma_1^- \amalg A_2^- \subset \Omega_{\Gamma_1}/\Gamma_1$, which is obtained from $\Sigma_1^- = \Delta_1/\Gamma_1$ and $A_2^- = \Omega_2^-/\langle\gamma\rangle$ by gluing along c^- . Similarly, the image of $\Omega_\Gamma^- = \Omega_1^- \amalg \Omega_2^-$ via the covering map $\Omega_{\Gamma_2} \rightarrow \Omega_{\Gamma_2}/\Gamma_2$ is $\Sigma_2^- \amalg A_1^- \subset \Omega_{\Gamma_2}/\Gamma_2$. Since Σ is obtained from $\Omega_{\Gamma_1}/\Gamma_1 - \Sigma_1^-$ and $\Omega_{\Gamma_2}/\Gamma_2 - \Sigma_2^-$ by gluing along c^- , we see that the pull-back $\Omega_\Sigma^- \subset \Sigma$ of Ω_Γ^- equals the annulus $A_1^- \amalg A_2^- = \Omega_1^-/\langle\gamma\rangle \amalg \Omega_2^-/\langle\gamma\rangle = \Omega_\Gamma^-/\langle\gamma\rangle \subset \Sigma$ with core curve c^- . Thus we have obtained the result also in this case. \square

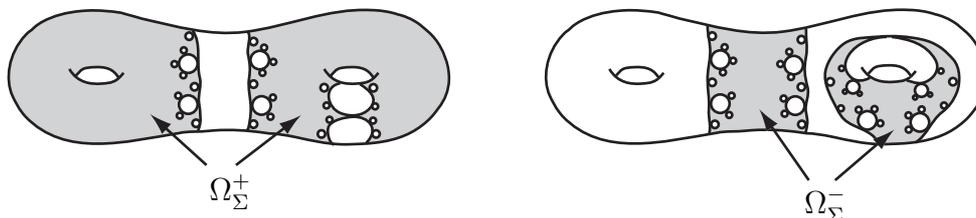


Figure 2: Pull-backs $\Omega_\Sigma^+ \subset \Sigma$ for $\rho \in \partial^+ \mathcal{QF}$ and $\Omega_\Sigma^- \subset \Sigma$ for $\rho \in \partial^- \mathcal{QF}$.

Corollary 3.7. *Let $\lambda, \mu \in \mathcal{ML}_\mathbb{N}$, $\lambda \neq \mu$ and $\rho \in \partial^\pm \mathcal{QF}$. If both pairs (λ, ρ) and (μ, ρ) are admissible then $\Psi_\lambda(\rho) \neq \Psi_\mu(\rho)$.*

4 Finiteness

The aim of this section is to prove Theorem 4.3, which guarantees some finiteness: suppose that a sequence $\rho_n \in \mathcal{QF}$ converges standardly to $\rho \in \partial^\pm \mathcal{QF}$ and that $\Phi : U \rightarrow P(S)$ is a univalent local branch of hol^{-1} defined on a neighborhood U of ρ . Then the sequence $\Phi(\rho_n)$ is actually contained in some finite union of components

of $Q(S)$. This finiteness theorem plays an important role in the proof of our main theorem (Theorem 5.1), which guarantees some continuity of the grafting maps and which can be seen also as the uniqueness theorem.

4.1 Length-Modulus inequality

Recall that the *modulus* $\text{Mod}(A)$ of a conformal annulus A is defined uniquely as the ratio of the height and the circumference of an Euclidian annulus which is conformally equivalent to A . The following inequality is a direct consequence of the geometric and analytic definitions of the extremal length.

Lemma 4.1. *Let R be a complete hyperbolic surface of finite area $\text{Area}(R) < \infty$ and let A be an essential annular domain in R . Then we have*

$$\text{Mod}(A) l_R(c)^2 \leq \text{Area}(R),$$

where $l_R(c)$ is the hyperbolic length of the homotopy class c of a core curve of A .

Proof. Let $E_R(c)$ denotes the extremal length of the homotopy class c in R . From the analytical definition of $E_R(c)$, we have

$$E_R(c) := \sup_{\rho} \frac{(\inf_{c'} \int_{c'} \rho(z) |dz|)^2}{\iint \rho(z)^2 |dz|^2} \geq \frac{(l_R(c))^2}{\text{Area}(R)},$$

where the supremum is taken over all metrics ρ consistent with the conformal structure of R and the infimum is taken over all closed curves c' in the homotopy class c . On the other hand, from the geometrical definition of $E_R(c)$, we have

$$E_R(c) := \frac{1}{\sup_{A' \subset R} \text{Mod}(A')} \leq \frac{1}{\text{Mod}(A)},$$

where the supremum is taken over all annuli $A' \subset R$ whose core curve is in the homotopy class c . From the above two inequality, the desired inequality follows. \square

4.2 Quasiconformal deformations

We introduce the notion of a quasiconformal deformation of a projective structure with quasifuchsian holonomy, which was developed by Shiga and Tanigawa in [ST]. Let $\Sigma \in Q(S)$ and let $\rho_{\Sigma} \in \mathcal{QF}$ be its holonomy. Suppose that $\rho' \in \mathcal{QF}$ is a quasiconformal deformation of ρ_{Σ} induced by a quasiconformal automorphism $q : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$, whose Beltrami differential is denoted by μ . Then we obtain a new projective structures Σ' with holonomy ρ' which is characterized as follows:

1. There is a quasiconformal map $\tilde{\varphi} : \tilde{\Sigma} \rightarrow \tilde{\Sigma}'$ whose Beltrami differential is equal to the pull-back $f_{\Sigma}^*(\mu)$ of μ via the developing map $f_{\Sigma} : \tilde{\Sigma} \rightarrow \widehat{\mathbb{C}}$. Moreover, the map $\tilde{\varphi} : \tilde{\Sigma} \rightarrow \tilde{\Sigma}'$ descends to a quasiconformal map $\varphi : \Sigma \rightarrow \Sigma'$, which is consistent with their markings.

2. The developing map of Σ' is defined by $f_{\Sigma'} = q \circ f_{\Sigma} \circ \tilde{\varphi}^{-1} : \tilde{\Sigma}' \rightarrow \widehat{\mathbb{C}}$.

Here we say that a map between projective structures is *quasiconformal* if it is a quasiconformal map between their underlying conformal structures. We call Σ' as the *quasiconformal deformation* of Σ . We remark that every grafting map $\Psi_{\lambda} : \mathcal{QF} \rightarrow \mathcal{Q}_{\lambda}$ is obtained by quasiconformal deformations of some fixed $\Sigma \in \mathcal{Q}_{\lambda}$ and its holonomy $\rho_{\Sigma} \in \mathcal{QF}$.

4.3 Length estimates

Theorem 4.2. *Fix $X \in T(S)$ arbitrarily. Suppose that $\lambda \in \mathcal{ML}_{\mathbb{N}}$ contains a weighted simple closed curve kc of weight $k \in \mathbb{N}$. Let $\rho \in B_X \cup B_{\bar{X}}$ and let $X' \in T(S)$ denotes the underlying conformal structure of $\Psi_{\lambda}(\rho)$. Then we have $l_{X'}(c)^2 \leq 4(g-1)l_X(c)/k$, where g denotes the genus of S .*

Remark. We make use of the inequality above mostly in the following form:

$$\frac{l_{X'}(c)}{l_X(c)} \leq \sqrt{\frac{4(g-1)}{k \cdot l_X(c)}},$$

which ensures that if $k \cdot l_X(c)$ is large enough then the Teichmüller distance $d_{T(S)}(X, X')$ between X and X' is also large.

Proof of Theorem 4.2. We first treat the case where λ is a simple closed curve c of weight one. Let $\rho^0 = B(X, \bar{X}) \in \mathcal{QF}$ be the Fuchsian representation with image $\Gamma^0 = \rho^0(\pi_1(S))$. We normalize ρ^0 so that $\Omega_{\Gamma^0}^+$ equals the upper half plane $H = \{x + iy \in \mathbb{C} : y > 0\}$, $\Omega_{\Gamma^0}^-$ equals the lower half plane $L = \{x + iy \in \mathbb{C} : y < 0\}$, and that the hyperbolic element $\gamma = \rho^0(c) \in \Gamma^0$ fixes the positive imaginary axis $i\mathbb{R}_+$. We let Σ^0 denote the standard projective structure $\Psi_0(\rho^0) = \Omega_{\Gamma^0}^+/\Gamma^0$. Recall that the grafting $\Psi_c(\rho^0) = \text{Gr}_c(\Sigma^0)$ of Σ_0 , which is denoted by Σ_c^0 , is obtained from Σ^0 by cutting along c and inserting the annulus $A = (\mathbb{C} - i\mathbb{R}_+)/\langle \gamma \rangle$. We set $H - i\mathbb{R}_+ = H_L \sqcup H_R$. Then $\mathbb{C} - i\mathbb{R}_+ = H_R \sqcup L \sqcup H_L$ and then $A = A_1 \amalg A_2 \amalg A_3$, where $A_1 = H_R/\langle \gamma \rangle$, $A_2 = L/\langle \gamma \rangle$ and $H_L/\langle \gamma \rangle$. Here $\text{Mod}(A_1) = \text{Mod}(A_3) = \pi/2l_X(c)$ and $\text{Mod}(A_2) = \pi/l_X(c)$. From now on, we regard $A = A_1 \amalg A_2 \amalg A_3$ as a subset of Σ_c^0 . By definition of grafting, it holds that $A \cap \Omega_{\Sigma_c^0}^+ = A_1 \sqcup A_3$ and that $\Omega_{\Sigma_c^0}^- = A_2$ on Σ_c^0 , where $\Omega_{\Sigma_c^0}^+ \subset \Sigma_c^0$ (resp. $\Omega_{\Sigma_c^0}^- \subset \Sigma_c^0$) is the pull-back of the upper half plane $H = \Omega_{\Gamma^0}^+$ (resp. the lower half plane $L = \Omega_{\Gamma^0}^-$).

Let $\rho \in B_X \cup B_{\bar{X}}$ be a quasiconformal deformation of $\rho^0 = B(X, \bar{X})$ which is induced by a quasiconformal map $q : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$. If ρ is contained in B_X (resp. $B_{\bar{X}}$) we may assume that the support of the Beltrami differential of q is contained in the lower half plane $L = \Omega_{\Gamma^0}^-$ (resp. the upper half plane $H = \Omega_{\Gamma^0}^+$). Associated to the quasiconformal deformation ρ of ρ^0 , we have a quasiconformal deformation $\Sigma_c = \Psi_c(\rho)$ of $\Sigma_c^0 = \Psi_c(\rho^0)$ and a quasiconformal map $\varphi : \Sigma_c^0 \rightarrow \Sigma_c$. It is important to notice that the map φ takes $\Omega_{\Sigma_c^0}^+$ (resp. $\Omega_{\Sigma_c^0}^-$) conformally into Σ_c if $\rho \in B_X$ (resp. $\rho \in B_{\bar{X}}$). Here we claim that the annulus $\varphi(A) \subset \Sigma_c$ always satisfies $\text{Mod}(\varphi(A)) \geq$

$\pi/l_X(c)$. In fact, if $\rho \in B_X$ then the map φ takes $A_1 \sqcup A_3 \subset \Omega_{\Sigma_0}^+$ conformally into $\varphi(A) \subset \Sigma_c$ and thus $\text{Mod}(\varphi(A)) \geq \text{Mod}(\varphi(A_1)) + \text{Mod}(\varphi(A_3)) = \pi/l_X(c)$. Similarly, if $\rho \in B_{\bar{X}}$ then φ takes $A_2 \subset \Omega_{\Sigma_0}^-$ conformally into $\varphi(A) \subset \Sigma_c$ and thus $\text{Mod}(\varphi(A)) \geq \text{Mod}(\varphi(A_2)) = \pi/l_X(c)$. Recall that X' denote the underlying conformal structure of $\Sigma_c = \Psi_c(\rho)$. By Lemma 4.1, we have $\text{Mod}(\varphi(A)) l_{X'}(c)^2 \leq \text{Area}(X') = 4\pi(g-1)$. By combining this inequality with the inequality $\text{Mod}(\varphi(A)) \geq \pi/l_X(c)$, the result follows in the case where $\lambda = c$.

Next suppose that $\lambda \in \mathcal{ML}_{\mathbb{N}}$ contains a weighted simple closed curve kc with $k \in \mathbb{N}$. Then $\Sigma_\lambda = \Psi_\lambda(\rho)$ contains an annulus \tilde{A} which is a union of succeeding k parallel annuli each of whose modulus $\geq \pi/l_X(c)$. Thus $\text{Mod}(\tilde{A}) \geq k\pi/l_X(c)$ and hence we have obtained the desired inequality in general. \square

From Theorem 4.2, we see that the both $\pi \circ \Psi_\lambda(B_X)$ and $\pi \circ \Psi_\lambda(B_{\bar{X}})$ are proper subsets of $T(S)$ for every non-zero $\lambda \in \mathcal{ML}_{\mathbb{N}}$. In fact if $kc \subset \lambda$, they are contained in the proper subset

$$\{X' \in T(S) : l_{X'}(c) \leq \sqrt{4(g-1)l_X(c)/k}\}$$

of $T(S)$. Compare this fact with a result which was independently obtained by Gallo [Ga] and Tanigawa [Ta]; they showed that the map $\pi \circ \Psi_\lambda$ takes the Fuchsian space $\mathcal{F} = \{B(X, \bar{X}) \in \mathcal{QF} : X \in T(S)\} \subset \mathcal{QF}$ bijectively onto $T(S)$ for every $\lambda \in \mathcal{ML}_{\mathbb{N}}$. In addition, we remark that both $\pi \circ \Psi_\lambda(B_X)$ and $\pi \circ \Psi_\lambda(B_{\bar{X}})$ are non-precompact in $T(S)$ for every non-zero $\lambda \in \mathcal{ML}_{\mathbb{N}}$. In fact if $c \subset \lambda$, we can make the length $l_{X'}(c)$ arbitrarily small by varying $\rho = B(X, \bar{Y})$ in B_X so that $l_{\bar{Y}}(c) \rightarrow 0$ or $\rho = B(Y, \bar{X})$ in $B_{\bar{X}}$ so that $l_Y(c) \rightarrow 0$.

4.4 Finiteness theorem

As a consequence of Theorem 4.2, we obtain the following

Theorem 4.3 (Finiteness). *Suppose that a sequence $\Sigma_n \in Q(S)$ of quasifuchsian projective structures converges to $\Sigma_\infty \in P(S)$ and that the sequence $\rho_{\Sigma_n} \in \mathcal{QF}$ of their holonomies converges standardly to $\rho_{\Sigma_\infty} \in \partial^\pm \mathcal{QF}$. Then there is a finite union $Q_{\lambda_1} \sqcup \cdots \sqcup Q_{\lambda_l}$ of components of $Q(S)$ in which Σ_n are contained for all n .*

Instead proving this theorem directly, we prove the following theorem, which is a contraposition of Theorem 4.3.

Theorem 4.4. *Suppose that a sequence $\rho_n \in \mathcal{QF}$ converges standardly to $\rho_\infty \in \partial^\pm \mathcal{QF}$ and $\{\lambda_n\}$ is a sequence of distinct element of $\mathcal{ML}_{\mathbb{N}}$. Then the sequence $\Psi_{\lambda_n}(\rho_n)$ converges to ∞ in $\hat{P}(S)$ as $n \rightarrow \infty$, that is, the set $\{\Psi_{\lambda_n}(\rho_n) : n \in \mathbb{N}\}$ has no accumulation point in $P(S)$.*

Proof. Set $\rho_n = B(X_n, \bar{Y}_n)$. We first assume that $\rho_\infty \in \partial^+ \mathcal{QF}$. Then $\rho_\infty \in \partial B_X$ for some $X \in T(S)$ and $X_n \rightarrow X$ in $T(S)$. One can choose a sequence of weighted simple closed curves $k_n c_n \subset \lambda_n$ which satisfies $k_n \cdot l_X(c_n) \rightarrow \infty$ as $n \rightarrow \infty$. Let

$X'_n \in T(S)$ denote the underlying conformal structure of $\Psi_{\lambda_n}(\rho_n)$. Then it follows from Theorem 4.2 that

$$\frac{l_{X'_n}(c_n)}{l_{X_n}(c_n)} \leq \sqrt{\frac{4(g-1)}{k_n \cdot l_{X_n}(c_n)}}$$

and that the Teichmüller distance $d_{T(S)}(X_n, X'_n)$ between X_n and X'_n tends to ∞ as $n \rightarrow \infty$. Since $X_n \rightarrow X$ in $T(S)$, $d_{T(S)}(X, X'_n)$ also tends to ∞ , which implies that $\Psi_{\lambda_n}(\rho_n) \rightarrow \infty$ in $\widehat{P}(S)$. The proof for $\rho_\infty \in \partial^- \mathcal{QF}$ is completely parallel. \square

5 Continuity

The following is our main theorem in this paper.

Theorem 5.1 (Continuity). *Let $\rho_n \in \mathcal{QF}$ be a sequence which converges standardly to $\rho_\infty \in \partial^\pm \mathcal{QF}$. Then the sequence $\Psi_\lambda(\rho_n)$ converges to $\Psi_\lambda(\rho_\infty)$ in $\widehat{P}(S)$ for every $\lambda \in \mathcal{ML}_\mathbb{N}$.*

We devote this section to the proof of the above theorem, which is divided into two cases; the case where $\Psi_\lambda(\rho_\infty) \in P(S)$ and that where $\Psi_\lambda(\rho_\infty) = \infty$. Throughout this proof, we let s denote either $+$ or $-$ for which $\rho_\infty \in \partial^s \mathcal{QF}$.

5.1 Case where $\Psi_\lambda(\rho_\infty) \in P(S)$

Let $\Phi : U \rightarrow V$, $\rho_\infty \mapsto \Psi_\lambda(\rho_\infty)$ be a univalent local branch of hol^{-1} from a neighborhood U of ρ_∞ to that V of $\Psi_\lambda(\rho_\infty)$ and set $\Sigma_n := \Phi(\rho_n)$ and $\Sigma_\infty := \Phi(\rho_\infty) = \Psi_\lambda(\rho_\infty)$. Then the sequence Σ_n converges to Σ_∞ in $P(S)$. We claim in Lemma 5.2 below that $\Sigma_n \in \mathcal{Q}_\lambda$ for all large enough n . Assuming this lemma, we have $\Sigma_n = \Psi_\lambda(\rho_n)$ and thus obtain the desired convergence $\Psi_\lambda(\rho_n) = \Sigma_n \rightarrow \Sigma_\infty = \Psi_\lambda(\rho_\infty)$.

Lemma 5.2. $\Sigma_n \in \mathcal{Q}_\lambda$ for all large enough n .

The rest of this subsection is devoted to the proof of Lemma 5.2. The idea of the proof is as follows: by Lemma 3.6, we have already known the topological type of the set $\Omega_{\Sigma_\infty}^s \subset \Sigma_\infty = \Psi_\lambda(\rho_\infty)$ in relation to λ . As we will see in Lemma 5.5, the set $\Omega_{\Sigma_\infty}^s \subset \Sigma_\infty$ coincides with the essential part of the Carathéodory limit of the sequence $\Omega_{\Sigma_n}^s \subset \Sigma_n$ as $n \rightarrow \infty$. It is essential that the sequence Σ_n is contained in a finite union of components of $Q(S)$ by Theorem 4.3. Then we will show that for any subsequence $\{\Sigma_{n_j}\}_{j=1}^\infty$ of $\{\Sigma_n\}_{n=1}^\infty$ contained in a component of $Q(S)$, the topological types of $\Omega_{\Sigma_{n_j}}^s \subset \Sigma_{n_j}$ is the same to that of $\Omega_{\Sigma_\infty}^s \subset \Sigma_\infty$ for all j , which implies that $\Sigma_n \in \mathcal{Q}_\lambda$ for all n .

Now we fill in the details. We may assume that the sequence $\Gamma_n = \rho_n(\pi_1(S))$ of quasifuchsian groups converges geometrically to some Kleinian group $\widehat{\Gamma}$, which contains the algebraic limit $\Gamma_\infty = \rho_\infty(\pi_1(S))$. Then Kerckhoff and Thurston [KT, Corollary 2.2] showed that the sequence Ω_{Γ_n} of the regions of discontinuity of Γ_n converges to the region of discontinuity $\Omega_{\widehat{\Gamma}}$ of $\widehat{\Gamma}$ in the sense of Carathéodory. Here

$\Omega_{\Gamma_n} \rightarrow \Omega_{\widehat{\Gamma}}$ in this sense if (i) for every compact $K \subset \Omega_{\widehat{\Gamma}}$, we have $K \subset \Omega_{\Gamma_n}$ for all large enough n and if (ii) for any open set $U \subset \Omega_{\Gamma_{n_j}}$ ($n_j \rightarrow \infty$), we have $U \subset \Omega_{\widehat{\Gamma}}$. Moreover $\Omega_{\widehat{\Gamma}}$ decomposes into two $\widehat{\Gamma}$ -invariant regions $\Omega_{\widehat{\Gamma}}^+$, $\Omega_{\widehat{\Gamma}}^-$, each of which are the Carathéodory limits of $\Omega_{\Gamma_n}^+$, $\Omega_{\Gamma_n}^-$, respectively. Since the convergence $\rho_n \rightarrow \rho_\infty \in \partial^s \mathcal{QF}$ is standard, the argument in the proof of [KT, Proposition 2.3] reveals that $\Omega_{\Gamma_\infty}^s$ is a component of $\Omega_{\widehat{\Gamma}}^s$, that $\Omega_{\widehat{\Gamma}}^s = \bigcup_{\delta \in \widehat{\Gamma}} \delta(\Omega_{\Gamma_\infty}^s)$, and that $\delta \in \widehat{\Gamma}$ fixes $\Omega_{\Gamma_\infty}^s$ if and only if $\delta \in \Gamma_\infty$. We put them together in the following

Lemma 5.3 (Kerckhoff-Thurston). *We have $\Omega_{\widehat{\Gamma}}^s = \bigsqcup_{\delta \in \widehat{\Gamma}/\Gamma_\infty} \delta(\Omega_{\Gamma_\infty}^s)$.*

Recall that $\Omega_{\Sigma_n}^s$ denotes the pull-back of $\Omega_{\Gamma_n}^s$ in Σ_n . Similarly $\Omega_{\Sigma_\infty}^s$ is the pull-back of $\Omega_{\Gamma_\infty}^s$ in Σ_∞ . In addition, we let $\widehat{\Omega}_{\Sigma_\infty}^s$ denotes the pull-back of $\Omega_{\widehat{\Gamma}}^s$ in Σ_∞ . By the same argument of [It1, Lemma 3.3], we obtain the following

Lemma 5.4. *The sequence $\Omega_{\Sigma_n}^s \subset \Sigma_n$ converges to $\widehat{\Omega}_{\Sigma_\infty}^s \subset \Sigma_\infty$ in the sense of Carathéodory. More precisely, there exist K_n -bi-Lipshiz diffeomorphisms $\psi_n : \Sigma_\infty \rightarrow \Sigma_n$ between hyperbolic surfaces (consistent with their markings) such that $K_n \rightarrow 1$ as $n \rightarrow \infty$ and that the sequence $\psi_n^{-1}(\Omega_{\Sigma_n}^s) \subset \Sigma_\infty$ converges to $\widehat{\Omega}_{\Sigma_\infty}^s \subset \Sigma_\infty$ in the sense of Carathéodory.*

We claim in the following lemma that $\Omega_{\Sigma_\infty}^s \subset \Sigma_\infty$ carries the full fundamental group of $\widehat{\Omega}_{\Sigma_\infty}^s \subset \Sigma_\infty$.

Lemma 5.5. *Each connected component of $\widehat{\Omega}_{\Sigma_\infty}^s - \Omega_{\Sigma_\infty}^s$ is homotopically trivial in Σ_∞ .*

Proof. Suppose for contradiction that there exists a connected component ω of $\widehat{\Omega}_{\Sigma_\infty}^s - \Omega_{\Sigma_\infty}^s$ which contains an essential closed curve α . Let $\tilde{\omega} \subset \widetilde{\Sigma}_\infty$ be a lift (i.e., a connected component of the inverse image) of $\omega \subset \Sigma_\infty$ and $\tilde{\alpha} \subset \widetilde{\Sigma}_\infty$ a lift of the curve $\alpha \subset \omega$ which is contained in $\tilde{\omega}$. Let Ω be the connected component of $\Omega_{\widehat{\Gamma}}^s - \Omega_{\Gamma_\infty}^s$ which contains the developing image $f_{\Sigma_\infty}(\tilde{\omega}) \subset \widehat{\mathbb{C}}$. Let $\gamma = \rho_\infty(\alpha)$ be an element of Γ_∞ which fixes $f_{\Sigma_\infty}(\tilde{\alpha})$, and hence Ω .

Now we see that γ is loxodromic from Definition 3.3 and Lemma 3.5. In fact if $\rho_\infty \in \partial^+ \mathcal{QF}$, then each connected component of $\Sigma_\infty - \Omega_{\Sigma_\infty}^+$ is an annulus whose core curve is homotopic to a component of λ . Hence the curve α in $\omega \subset \Sigma_\infty - \Omega_{\Sigma_\infty}^+$ is homotopic into λ and hence is not homotopic into $\text{para}(\rho_\infty)$. On the other hand, if $\rho_\infty \in \partial^- \mathcal{QF}$ then each component of $\text{para}(\rho_\infty)$ must traverse λ , and thus $\Omega_{\Sigma_\infty}^-$, in Σ_∞ . Therefore the curve α in $\Sigma_\infty - \Omega_{\Sigma_\infty}^-$ is not homotopic into $\text{para}(\rho_\infty)$.

From Lemma 5.3, there exists $\delta \in \widehat{\Gamma} - \Gamma_\infty$ such that $\Omega = \delta(\Omega_{\Gamma_\infty}^s)$. Since γ fixes the component Ω , $\delta^{-1}\gamma\delta$ fixes the component $\Omega_{\Gamma_\infty}^s$, which implies that $\delta^{-1}\gamma\delta \in \Gamma_\infty$. But this contradicts to the following lemma (Lemma 2.4 in [ACCS]). \square

Lemma 5.6 (Anderson-Canary-Culler-Shalen). *Let G be a finitely generated group. Let $\rho_n : G \rightarrow \Gamma_n \subset \text{PSL}_2(\mathbb{C})$ be a sequence of discrete faithful representations converging algebraically to $\rho_\infty : G \rightarrow \Gamma_\infty$. Moreover, assume that Γ_n converges to $\widehat{\Gamma}$ geometrically. Then for any $\delta \in \widehat{\Gamma} - \Gamma_\infty$, $\Gamma_\infty \cap \delta^{-1}\Gamma_\infty\delta$ is $\{\text{id}\}$ or rank-one parabolic subgroup.*

Let us combine the above observations to complete the proof of Lemma 5.2, which states that $\Sigma_n \in \mathcal{Q}_\lambda$ for all large enough n . Note that the sequence Σ_n is contained in a finite union of components of $Q(S)$ by Theorem 4.3. Therefore to show Lemma 5.2, we only need to show that $\lambda' = \lambda$ by assuming that the sequence Σ_n is contained in a component $\mathcal{Q}_{\lambda'}$ of $Q(S)$.

Let $\rho_0 \in \mathcal{QF}$ be a fixed element and set $\Sigma_0 = \Psi_{\lambda'}(\rho_0)$. Then every Σ_n is a quasiconformal deformation of Σ_0 in $\mathcal{Q}_{\lambda'}$, which is induced by the quasiconformal deformation ρ_n of ρ_0 , and which induces the quasiconformal map $\varphi_n : \Sigma_0 \rightarrow \Sigma_n$. Note that the map φ_n is consistent with their markings and takes $\Omega_{\Sigma_0}^s$ onto $\Omega_{\Sigma_n}^s$. Since the convergence $\rho_n \rightarrow \rho_\infty \in \partial^s \mathcal{QF}$ is standard, the dilatation of $\varphi_n|_{\Omega_{\Sigma_0}^s}$ tends to 1, and hence we assume that $\varphi_n|_{\Omega_{\Sigma_0}^s} : \Omega_{\Sigma_0}^s \rightarrow \Omega_{\Sigma_n}^s$ is conformal for every n for simplicity.

Now let $\mathcal{C}(\Omega_{\Sigma_0}^s), \mathcal{C}(\Omega_{\Sigma_\infty}^s)$ denote the set of connected components of $\Omega_{\Sigma_0}^s, \Omega_{\Sigma_\infty}^s$, respectively. Then Lemma 3.6 tells us that $\lambda = \lambda'$ if and only if there exists a bijection

$$g : \mathcal{C}(\Omega_{\Sigma_0}^s) \rightarrow \mathcal{C}(\Omega_{\Sigma_\infty}^s)$$

which satisfies $\pi_1(\omega) = \pi_1(g(\omega))$ for every $\omega \in \mathcal{C}(\Omega_{\Sigma_0}^s)$. Here we identify $\pi_1(\Sigma_0)$ and $\pi_1(\Sigma_\infty)$ with $\pi_1(S)$ via their markings. Note that $\psi_n^{-1} \circ \varphi_n(\Omega_{\Sigma_0}^s) \subset \Sigma_\infty$ converge to $\widehat{\Omega}_{\Sigma_\infty}^s \subset \Sigma_\infty$ in the sense of Lemma 5.4 and that $\pi_1(\omega) = \pi_1(\psi_n^{-1} \circ \varphi_n(\omega))$ hold for all $\omega \in \mathcal{C}(\Omega_{\Sigma_0}^s)$ and n . We first define a map $h : \mathcal{C}(\Omega_{\Sigma_\infty}^s) \rightarrow \mathcal{C}(\Omega_{\Sigma_0}^s)$, which will turn out to be the inverse of g . Let $\omega' \in \mathcal{C}(\Omega_{\Sigma_\infty}^s)$ and let $K \subset \omega'$ be a compact subset such that $\pi_1(K) = \pi_1(\omega')$. Then there exists $\omega \in \mathcal{C}(\Omega_{\Sigma_0}^s)$ such that $K \subset \psi_n^{-1} \circ \varphi_n(\omega)$ for all large enough n , and hence that $\pi_1(K) = \pi_1(\omega') \subset \pi_1(\omega)$. Now we let $h(\omega') = \omega$. Next we define the map $g : \mathcal{C}(\Omega_{\Sigma_0}^s) \rightarrow \mathcal{C}(\Omega_{\Sigma_\infty}^s)$ as follows: let $\omega \in \mathcal{C}(\Omega_{\Sigma_0}^s)$. By passing to a subsequence if necessary, the sequence $\psi_n^{-1} \circ \varphi_n(\omega) \subset \Sigma_\infty$ converges to an open subset $\widehat{\omega}$ of $\widehat{\Omega}_{\Sigma_\infty}^s \subset \Sigma_\infty$. We claim that $\widehat{\omega}$ contains exactly one essential component ω' of $\widehat{\Omega}_{\Sigma_\infty}^s \subset \Sigma_\infty$, which satisfies $\pi_1(\omega) \subset \pi_1(\omega')$. We postpone the proof of this claim and continue the argument. We have $\omega' \in \mathcal{C}(\Omega_{\Sigma_\infty}^s)$ by Lemma 5.5 and let $g(\omega) = \omega'$. Then it follows immediately that g is a bijective map with $g^{-1} = h$ and hence that $\lambda = \lambda'$. We have completed the proof of Lemma 5.2, and hence that of Theorem 5.1 in the case where $\Psi_\lambda(\rho_\infty) \in P(S)$.

We only need to show the above claim. Suppose for contradiction that $\widehat{\omega} \subset \Sigma_\infty$ is trivial or contains more than two essential connected components. Since $\psi_n^{-1} \circ \varphi_n(\omega)$ converge to $\widehat{\omega}$, $\varphi_n(\omega) \subset \Sigma_n$ then become more and more constricted, that is, there exists an arc $\alpha_n \subset \varphi_n(\omega)$ joining two distinct components of $\varphi_n(\partial\omega)$ whose hyperbolic length tends to 0 as $n \rightarrow \infty$. But since the map $\varphi_n|_{\Omega_{\Sigma_0}^s} : \Omega_{\Sigma_0}^s \rightarrow \varphi_n(\Omega_{\Sigma_0}^s)$ is conformal, this contradicts to Lemma 5.7 below. Thus we have proved the claim.

Lemma 5.7 ([It1, Lemma 4.4]). *Let R be a hyperbolic surface, $c \subset R$ a simple closed curve and $A \subset R$ is an annulus whose core curve is homotopic to c . Then there is a positive constant $C > 0$, depending only on $l_R(c)$ and $\text{Mod}(A)$, such that $l_R(\alpha) \geq C$ holds for every arcs $\alpha \subset A$ joining points in distinct components of ∂A .*

5.2 Case where $\Psi_\lambda(\rho_\infty) = \infty$

We first prepare the following

Lemma 5.8. *Let $\lambda \in \mathcal{ML}_{\mathbb{N}}$ and let $\rho_n = B(X_n, \bar{Y}_n) \in \mathcal{QF}$ be a sequence which converges standardly to $\rho_\infty \in \partial^\pm \mathcal{QF}$ and which satisfies the property as in Corollary 2.4. Suppose that $\Psi_\lambda(\rho_\infty) = \infty$. Then the sequence $\Psi_\lambda(\rho_n) \rightarrow \infty$ in $\widehat{P}(S)$.*

Proof. We first suppose that $\rho_\infty \in \partial^+ \mathcal{QF}$. Since (λ, ρ_∞) is not admissible, λ and $\text{para}(\rho_\infty)$ have parallel components in common and let c be one of such components. Since $l_{\bar{Y}_n}(c) \rightarrow 0$, there are annular neighborhoods A_n of c in \bar{Y}_n such that $\text{Mod}(A_n) \rightarrow \infty$ as $n \rightarrow \infty$. We set $\Sigma_n = \Psi_\lambda(\rho_n)$. Then observe that the annulus A_n is conformally embedded in $\Omega_{\Sigma_n}^- \subset \Sigma_n$ for every n . Hence the underlying conformal structures $\pi(\Psi_\lambda(\rho_n))$ of $\Psi_\lambda(\rho_n)$ diverge in $T(S)$, and hence $\Psi_\lambda(\rho_n) \rightarrow \infty$ in $\widehat{P}(S)$.

Next we suppose that $\rho_\infty \in \partial^- \mathcal{QF}$. Then there is a component c of $\text{para}(\rho_\infty)$ which do not intersect λ essentially. Since $l_{X_n}(c) \rightarrow 0$, there are annular neighborhoods A_n of c in X_n such that $\text{Mod}(A_n) \rightarrow \infty$ as $n \rightarrow \infty$. Then observe that the annulus A_n is conformally embedded in $\Omega_{\Sigma_n}^+ \subset \Sigma_n = \Psi_\lambda(\rho_n)$ for every n . By the same argument as above, we see that $\Psi_\lambda(\rho_n) \rightarrow \infty$ in $\widehat{P}(S)$. \square

Now we back to the proof of Theorem 5.1 in the case where $\Psi_\lambda(\rho_\infty) = \infty$; we intend to show that $\Psi_\lambda(\rho_n) \rightarrow \Psi_\lambda(\rho_\infty) = \infty$ for every standard convergent sequence $\mathcal{QF} \ni \rho_n \rightarrow \rho_\infty \in \partial^\pm \mathcal{QF}$. Set $\Sigma_n = \Psi_\lambda(\rho_n)$. To obtain a contradiction, we suppose that the sequence Σ_n has a subsequence (which is denoted by the same symbols) converging to some $\Sigma_\infty \in P(S)$. In addition, let $\rho'_n \in \mathcal{QF}$ be a sequence converging to ρ_∞ as in Lemma 5.8 and let $\Phi : U \rightarrow P(S)$, $\rho_\infty \mapsto \Sigma_\infty$ be a univalent local branch of hol^{-1} . Then we have $\Sigma_n = \Phi(\rho_n)$ for all large enough n . Note that both $\Sigma_n = \Phi(\rho_n)$ and $\Sigma'_n := \Phi(\rho'_n)$ converge to Σ_∞ . From Theorem 4.3, the sequence Σ'_n has a subsequence (also denoted by the same symbols) which is contained in a component \mathcal{Q}_μ of $Q(S)$ for some $\mu \in \mathcal{ML}_{\mathbb{N}}$. Then we have $\Sigma'_n = \Psi_\mu(\rho'_n)$, and thus $\Psi_\mu(\rho'_n) = \Sigma'_n \rightarrow \Sigma_\infty \neq \infty$, which implies that $\Psi_\mu(\rho_\infty) \neq \infty$ by Lemma 5.8 and that $\lambda \neq \mu$. Since both ρ_n, ρ'_n converge standardly to ρ_∞ , Theorem 5.1 for admissible pairs guarantees that both $\Psi_\mu(\rho_n), \Psi_\mu(\rho'_n)$ converge to $\Psi_\mu(\rho_\infty)$. On the other hand, since $\Psi_\mu(\rho'_n) = \Sigma'_n \rightarrow \Sigma_\infty$, we have $\Psi_\mu(\rho_n) \rightarrow \Sigma_\infty$. Now we obtain two sequences $\Psi_\lambda(\rho_n), \Psi_\mu(\rho_n)$ both of which converge to Σ_∞ . But since $\lambda \neq \mu$, this contradicts to the fact that the map hol is a local homeomorphism. Now we have completed the proof of Theorem 5.1.

6 Applications

We collect in this section some results which are obtained as consequences of Theorem 5.1.

6.1 Grafting theorem for boundary b -groups

The following statement is conjectured by Bromberg in [Br].

Theorem 6.1. *For every boundary b -groups $\rho \in \partial^\pm \mathcal{QF}$, all projective structure with holonomy ρ are obtained by grafting of ρ , that is, we have*

$$\text{hol}^{-1}(\rho) = \{\Psi_\lambda(\rho) : \lambda \in \mathcal{ML}_\mathbb{N}, \Psi_\lambda(\rho) \in P(S)\}.$$

Proof. Let $\rho \in \partial^\pm \mathcal{QF}$ and $\Sigma \in \text{hol}^{-1}(\rho)$. We only need to show that $\Sigma = \Psi_\lambda(\rho)$ for some $\lambda \in \mathcal{ML}_\mathbb{N}$. Let $\rho_n \in \mathcal{QF}$ be a sequence converging standardly to ρ . Since the map hol is a local homeomorphism, there exists a convergent sequence $\Sigma_n \rightarrow \Sigma$ of projective structures with $\text{hol}(\Sigma_n) = \rho_n$ for all n . From Theorem 4.3, there exists a subsequence $\{\Sigma_{n_j}\}_{j=1}^\infty$ of the sequence $\{\Sigma_n\}_{n=1}^\infty$ which is contained in a component \mathcal{Q}_λ of $Q(S)$. Then we have $\Sigma_{n_j} = \Psi_\lambda(\rho_{n_j})$ for all j and thus $\Psi_\lambda(\rho_{n_j}) = \Sigma_{n_j} \rightarrow \Sigma$ as $j \rightarrow \infty$. On the other hand, since the convergence $\rho_n \rightarrow \rho$ is standard, Theorem 5.1 implies that $\Psi_\lambda(\rho_n) \rightarrow \Psi_\lambda(\rho)$ in $\widehat{P}(S)$. Hence $\Sigma = \Psi_\lambda(\rho)$ and the result follows. \square

From Bromberg's result (Theorem 2.2), we obtain the following

Corollary 6.2. *Suppose that $\rho \in AH(S)$ is a b -group with no parabolics. Then we have $\text{hol}^{-1}(\rho) = \{\Psi_\lambda(\rho) : \lambda \in \mathcal{ML}_\mathbb{N}\}$.*

6.2 Analytic continuations and lifts of paths

Theorem 6.3. *Suppose that $\Psi_\lambda(\rho) \in P(S)$ for $\lambda \in \mathcal{ML}_\mathbb{N}$ and $\rho \in \partial^\pm \mathcal{QF}$. Then there exists a path $\alpha : [0, 1] \rightarrow R(S)$ with $\alpha(0) \in \mathcal{QF}$ and $\alpha(1) = \Psi_\lambda(\rho)$ along which the grafting map $\Psi_\lambda : \mathcal{QF} \rightarrow P(S)$ is continued analytically to a univalent local branch $\Phi : U \rightarrow P(S)$, $\rho \mapsto \Psi_\lambda(\rho)$ of hol^{-1} which is defined on some neighborhood U of ρ .*

Proof. We may assume that $\rho \in \partial^+ \mathcal{QF}$, and hence that $\rho \in \partial B_X$ for some $X \in T(S)$. Let $D \subset T(S)$ be a bounded domain with $X \in D$ and set $B_D = \sqcup_{X' \in D} B_{X'}$. We claim that there exists a neighborhood U of ρ such that $\Phi \equiv \Psi_\lambda$ holds on $U \cap B_D$. For otherwise there is a sequence $\rho_n \in B_D$ converging to ρ for which $\Phi(\rho_n) \neq \Psi_\lambda(\rho_n)$ hold for all n . On the other hand, Theorem 5.1 implies that both sequences $\Phi(\rho_n), \Psi_\lambda(\rho_n)$ converge to $\Phi(\rho) = \Psi_\lambda(\rho)$, which contradicts to the fact that Φ is univalent. Now let $\alpha : [0, 1] \rightarrow R(S)$ be an arc such that $\alpha(0) \in \mathcal{QF}$, $\alpha(1) = \rho$ and that $\alpha([0, 1]) \subset B_D \cup U$. Then $\Psi_\lambda : \mathcal{QF} \rightarrow P(S)$ is continued analytically to $\Phi : U \rightarrow P(S)$ along α . \square

Let $\alpha : [0, 1] \rightarrow R(S)$ be a path as in Theorem 6.3. Then the above theorem implies that there exists the lift $\tilde{\alpha} : [0, 1] \rightarrow P(S)$ of α via the holonomy map with the starting point $\Psi_\lambda(\alpha(0))$. On the other hand, we have the following

Theorem 6.4. *Suppose that $\Psi_\lambda(\rho) = \infty$ for $\lambda \in \mathcal{ML}_\mathbb{N}$ and $\rho \in \partial^\pm \mathcal{QF}$. Then there exists a path $\alpha : [0, 1] \rightarrow R(S)$ of which there is no lift to $P(S)$ with the starting point $\Psi_\lambda(\alpha(0))$.*

Proof. We may assume that $\rho \in \partial^{\pm} \mathcal{QF}$, and hence that $\rho \in \partial B_X$ for some $X \in T(S)$. Let take a path $\alpha : [0, 1] \rightarrow R(S)$ with $\alpha(0) \in \mathcal{QF}$ and $\alpha(1) = \rho$ for which there exists a sequence $0 < t_n < 1$, $t_n \rightarrow 1$ such that $\alpha(t_n) \in B_D = \sqcup_{X' \in D} B_{X'}$ for some bounded domain $D \subset T(S)$ with $X \in D$. Then the sequence $\alpha(t_n) \in B_D$ converges standardly to ρ , and thus $\Psi_{\lambda}(\alpha(t_n)) \rightarrow \infty$ by Theorem 5.1. Hence there is no lift of α to $P(S)$ with the starting point $\Psi_{\lambda}(\alpha(0))$. \square

6.3 Obstructions for hol to be a covering map

We first fix our terminology.

Definition 6.5. Let \mathcal{X}, \mathcal{Y} be topological spaces. A continuous map $f : \mathcal{Y} \rightarrow \mathcal{X}$ is said to be a *covering map* or a *weak-covering map* if for any $x \in \mathcal{X}$ there is a neighborhood U of x such that $f|_V : V \rightarrow U$ is a homeomorphism

- for any connected component V of $f^{-1}(U)$, or
- for any $y \in f^{-1}(x)$ and some neighborhood V of y ,

respectively.

Although the holonomy map $hol : P(S) \rightarrow R(S)$ is a local homeomorphism and the map $hol|_{Q(S)} : Q(S) \rightarrow \mathcal{QF}$ is a covering map, Hejhal showed the following

Theorem 6.6 ([He, Theorem 8]). *The holonomy map $hol : P(S) \rightarrow R(S)$ is not a covering map onto its image.*

We explain this fact in our context. Let $\rho \in \partial^{\pm} \mathcal{QF}$ with $\text{para}(\rho) \neq \emptyset$. Then there are $\lambda, \mu \in \mathcal{ML}_{\mathbb{N}}$ such that $\Psi_{\lambda}(\rho) \in P(S)$ and $\Psi_{\mu}(\rho) = \infty$. One can find a path $\alpha : [0, 1] \rightarrow R(S)$ with $\alpha(0) \in \mathcal{QF}$ and $\alpha(1) = \rho$ which satisfies both conditions in the proofs of Theorems 6.3 and 6.4. Then there is the lift of the path α with the starting point $\Psi_{\lambda}(\alpha(0))$ but there is no lift with the starting point $\Psi_{\mu}(\alpha(0))$, which implies that the map hol is not a covering map. We remark that the idea used in the argument above is the same to that of Hejhal [He], but he made use of a path $\alpha : [0, 1] \rightarrow R(S)$ of Shottky representations.

Moreover we claim in Corollary 6.8 below that the covering map $hol|_{Q(S)} : Q(S) \rightarrow \mathcal{QF}$ is not extended any more even in the sense of weak-covering. The essential observation is the following

Theorem 6.7. *Let $\rho \in \partial^{\pm} \mathcal{QF}$. For every $\lambda \in \mathcal{ML}_{\mathbb{N}}$ with $\Psi_{\lambda}(\rho) \in P(S)$, suppose that $\Phi_{\lambda} : U_{\lambda} \rightarrow P(S)$, $\rho \mapsto \Psi_{\lambda}(\rho)$ is a univalent local branch of hol^{-1} defined on a neighborhood U_{λ} of ρ . Then ρ can not be an interior point of $\bigcap_{\lambda} U_{\lambda}$, where the intersection is taken over all $\lambda \in \mathcal{ML}_{\mathbb{N}}$ with $\Psi_{\lambda}(\rho) \in P(S)$.*

Proof. We may assume that $\rho \in \partial B_X$ for some $X \in T(S)$. Then there exists a sequence ρ_n of maximal cusps converging to ρ in ∂B_X for which the parabolic locus $\lambda_n = \text{para}(\rho_n) \in \mathcal{ML}_{\mathbb{N}}$ has no parallel component in common with $\text{para}(\rho)$ (which

is possibly empty) for every n (see [It2, Theorem 5.5]). Since $\Psi_{\lambda_n}(\rho) \neq \infty$, there exists a univalent local branch $\Phi_{\lambda_n} : U_{\lambda_n} \rightarrow P(S)$, $\rho \mapsto \Psi_{\lambda_n}(\rho)$ for every n . On the other hand, since $\Psi_{\lambda_n}(\rho_n) = \infty$, $\rho_n \notin U_{\lambda_n}$ for every n . Therefore we have obtained the result. \square

Corollary 6.8. *For any open neighborhood O of \mathcal{QF} in $R(S)$, the map $hol|_{hol^{-1}(O)} : hol^{-1}(O) \rightarrow O$ is not a weak-covering map.*

Proof. For any $\rho \in \partial\mathcal{QF}$ and its neighborhood U contained in O , one can find a boundary b -group $\rho' \in \partial^\pm\mathcal{QF}$ in U . Then the result follows from the above theorem. \square

6.4 Further continuity

We extend our definition of standard convergence to sequences in $\overline{\mathcal{QF}}$. Let K be a compact subset of $T(S)$ and $\bar{K} \subset T(S)$ its complex conjugation. We set $B_K = \bigsqcup_{X \in K} B_X$ and $B_{\bar{K}} = \bigsqcup_{\bar{X} \in \bar{K}} B_{\bar{X}}$. Recall that a sequence $\rho_n \in \mathcal{QF}$ converges standardly to $\rho_\infty \in \partial\mathcal{QF}$ if there is a compact subset $K \subset T(S)$ such that $\rho_n \in B_K \cup B_{\bar{K}}$ for all n . Similarly we say that a sequence $\rho_n \in \overline{\mathcal{QF}}$ converges *standardly* to $\rho_\infty \in \partial\mathcal{QF}$ if there is a compact subset $K \subset T(S)$ such that all ρ_n are contained in the closure of $B_K \cup B_{\bar{K}}$ in $R(S)$. Note that the limit of a standard convergent sequence in $\overline{\mathcal{QF}}$ is contained in $\partial^\pm\mathcal{QF}$ and that if a sequence $\rho_n \in \partial\mathcal{QF}$ converges standardly to $\partial^s\mathcal{QF}$ then $\rho_n \in \partial^s\mathcal{QF}$ for all large enough n , where s denotes $+$ or $-$.

Theorem 6.9. *Let $\rho_n \rightarrow \rho_\infty$ be a standard convergent sequence in $\overline{\mathcal{QF}}$. Then the sequence $\Psi_\lambda(\rho_n)$ converges to $\Psi_\lambda(\rho_\infty)$ in $\hat{P}(S)$ for every $\lambda \in \mathcal{ML}_{\mathbb{N}}$.*

Proof. We only have to consider the case where $\rho_n \in \partial\mathcal{QF}$ for all n . Then $\rho_n \rightarrow \rho_\infty$ in $\partial^\pm\mathcal{QF}$. If $\Psi_\lambda(\rho_\infty) \in P(S)$ then the result follows from Theorem 6.3. Hence we suppose that $\Psi_\lambda(\rho_\infty) = \infty$. (We remark that the argument below also works in the case where $\Psi_\lambda(\rho_\infty) \in P(S)$.) To obtain a contradiction, we assume that the sequence $\Psi_\lambda(\rho_n)$ converges to some $\Sigma' \in P(S)$. Then for each n there exists a sequence $\rho_{n,k} \in \mathcal{QF}$ converging standardly to $\rho_n \in \partial^\pm\mathcal{QF}$ as $k \rightarrow \infty$, which is taken by Ψ_λ to a convergent sequence $\Psi_\lambda(\rho_{n,k}) \rightarrow \Psi_\lambda(\rho_n)$ from Theorem 5.1. Then by a diagonal argument, we can choose a sequence ρ'_n in \mathcal{QF} which converges standardly to ρ_∞ and which satisfies $\Psi_\lambda(\rho'_n) \rightarrow \Sigma'$. On the other hand, $\Psi_\lambda(\rho'_n) \rightarrow \Psi_\lambda(\rho_\infty) = \infty$ by Theorem 5.1, which is a contradiction. Now we have obtained the result. \square

Let $\overline{\mathcal{Q}_0}$ denote the closure of \mathcal{Q}_0 in $P(S)$, *not* in $\hat{P}(S)$. From the observation in §3.3, we see that a sequence $\Sigma_n \in \overline{\mathcal{Q}_0}$ converges to $\Sigma_\infty \in \partial\mathcal{Q}_0$ in $P(S)$ if and only if the sequence $\rho_{\Sigma_n} \in \mathcal{QF} \sqcup \partial^+\mathcal{QF}$ converges standardly to $\rho_{\Sigma_\infty} \in \partial^+\mathcal{QF}$. Hence, as a direct consequence of the above theorem, we obtain the following

Theorem 6.10. *The map $\text{Gr}_\lambda : \mathcal{Q}_0 \rightarrow \mathcal{Q}_\lambda$ is extended continuously to $\text{Gr}_\lambda : \overline{\mathcal{Q}_0} \rightarrow \hat{P}(S)$ for each $\lambda \in \mathcal{ML}_{\mathbb{N}}$, where $\overline{\mathcal{Q}_0}$ is the closure of \mathcal{Q}_0 in $P(S)$.*

6.5 Discreteness of inverse images

Theorem 6.11. *Let K be a given compact subset of $T(S)$ and set $\mathcal{QF}_K = B_K \cup B_{\bar{K}}$. Then the inverse image*

$$\text{hol}^{-1}(\mathcal{QF}_K) = \bigsqcup_{\lambda \in \mathcal{ML}_{\mathbb{N}}} \Psi_{\lambda}(\mathcal{QF}_K)$$

of \mathcal{QF}_K in $P(S)$ is discrete; that is, every connected component of $\text{hol}^{-1}(\mathcal{QF}_K)$ has open neighborhood which is disjoint from any other one.

Proof. Suppose for contradiction that there exist $\rho \in \overline{\mathcal{QF}_K} \cap \partial^{\pm} \mathcal{QF}$ and $\lambda \in \mathcal{ML}_{\mathbb{N}}$ such that $\Psi_{\lambda}(\rho)$ is a limit of a sequence Σ_n in $\text{hol}^{-1}(\mathcal{QF}_K) - \Psi_{\lambda}(\mathcal{QF}_K)$. By Theorem 4.3, we may assume that the sequence Σ_n is contained in a component \mathcal{Q}_{μ} of $Q(S)$ with $\lambda \neq \mu$. Since the sequence $\rho_n = \text{hol}(\Sigma_n) \in \mathcal{QF}_K$ converges standardly to ρ , we have $\Sigma_n = \Psi_{\mu}(\rho_n) \rightarrow \Psi_{\mu}(\rho)$ by Theorem 5.1. But this contradicts the fact that $\Psi_{\lambda}(\rho) \neq \Psi_{\mu}(\rho)$ (Corollary 3.7). \square

Compare the above theorem with the following

Theorem 6.12 ([It3]). *Any pair of connected components of the inverse image*

$$\text{hol}^{-1}(\mathcal{QF}) = \bigsqcup_{\lambda \in \mathcal{ML}_{\mathbb{N}}} \Psi_{\lambda}(\mathcal{QF})$$

of \mathcal{QF} have intersecting closures in $P(S)$.

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