Exotic projective structures and quasi-fuchsian spaces II

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Abstract

Let $P(S)$ denote the space of projective structures on a closed surface S, and let $Q(S)$ be the subset of $P(S)$ consisting of projective structures with quasi-fuchsian holonomy. It is known that $Q(S)$ has infinitely many connected components. In this paper, we show that the closure of any "exotic" component of $Q(S)$ is not a topological manifold with boundary, and that any two components of $Q(S)$ have intersecting closures.

Contents

1 Introduction

Let S be an oriented closed surface of genus $g > 1$. A projective structure on S is a maximal system of local coordinates modeled on the Riemann sphere $\widehat{\mathbf{C}}$, whose transition functions are Möbious transformations. For a given projective structure on S, we have a pair (f, ρ) of a local homeomorphism f from the universal cover $\tilde{S}$ of S to $\hat{\mathbf{C}}$, called a developing map, and a group homomorphism ρ of $\pi_1(S)$ into $\mathrm{PSL}_2(\mathbf{C})$, called a holonomy representation. Let $P(S)$ denote the space of all marked projective structures on S, and let $R(S)$ denote the space of all conjugacy classes of representations of $\pi_1(S)$ into $\mathrm{PSL}_2(\mathbf{C})$. Holonomy representations give a mapping $hol : P(S) \to R(S)$, which is called the holonomy map. It is known by Hejhal [He] that the map hol is a local homeomorphism. The quasi-fuchsian space $QF(S)$ is the subspace of $R(S)$ consisting of faithful representations whose holonomy images are quasi-fuchsian groups.

We are interested in the subset $Q(S) = hol^{-1}(QF(S))$ of $P(S)$. We say an element of $Q(S)$ is *standard* if its developing map is injective; otherwise it is *exotic*. For each connected component $\mathcal{Q}$ of $Q(S)$, the map $hol|\mathcal{Q}$ is a biholomorphic isomorphism from $\mathcal{Q}$ onto $QF(S)$. As a consequence of the result of Goldman [Go], it is known that the set of connected components of $Q(S)$ are in one-to-one correspondence with the set $\mathcal{ML}_{\mathbf{Z}}(S)$ of integral measured laminations (see 2.3 for a precise definition). We denote by $\mathcal{Q}_\lambda$ the component of $Q(S)$ corresponding to $\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)$, where $\mathcal{Q}_0$ is the component consisting of all standard projective structures. In this paper, we investigate the distribution of connected components of $Q(S)$.

The first important improvement on the distribution of exotic projective structures is the following theorem due to McMullen.

Theorem 1.1 (McMullen [Mc, Apendix A]). *There exists a sequence of exotic projective structures which converges to a point in the relative boundary $\partial\mathcal{Q}_0 = \overline{\mathcal{Q}_0} - \mathcal{Q}_0$ of the standard component $\mathcal{Q}_0$ in $P(S)$. As a consequence, we know that the closure of $QF(S)$ in $R(S)$ is not a topological manifold with boundary.*

In our previous paper [It], we investigated the phenomena in Theorem 1.1 more closely and obtained the following theorem.

Theorem 1.2 ([It]). *For any finite set* $\{\lambda_i\}_{i=1}^m$ *of* $\mathcal{ML}_{\mathbf{Z}}(S)-\{0\}$ *satisfying* $i(\lambda_j,\lambda_k)=0$ *for all* $j,k\in\{1,\ldots,m\}$, *we have*

$$\overline{\mathcal{Q}_0}\cap\left(\bigcap_{i=1}^m\overline{\mathcal{Q}_{\lambda_i}}\right)\neq\emptyset,$$

where $i(\cdot,\cdot)$ *denotes the geometric intersection number. Especially, we have* $\overline{\mathcal{Q}_0}\cap\overline{\mathcal{Q}_\lambda}\neq\emptyset$ *for any* $\lambda\in\mathcal{ML}_{\mathbf{Z}}(S)-\{0\}$, *and the closure of* $Q(S)$ *in* $P(S)$ *is connected.*

In this paper, we continue the above investigations and obtain the following theorems.

Theorem A. *For any* $\lambda\in\mathcal{ML}_{\mathbf{Z}}(S)-\{0\}$, *there exists a point* $Y\in\overline{\mathcal{Q}_0}\cap\overline{\mathcal{Q}_\lambda}$ *such that* $U\cap\mathcal{Q}_\lambda$ *is disconnected for any sufficiently small neighborhood* U *of* Y. *Especially,* $\overline{\mathcal{Q}_\lambda}$ *is not a topological manifold with boundary.*

Theorem B. *For any two elements* $\lambda,\mu\in\mathcal{ML}_{\mathbf{Z}}(S)$, *we have* $\overline{\mathcal{Q}_\lambda}\cap\overline{\mathcal{Q}_\mu}\neq\emptyset$.

Theorem C. *For any* $\lambda\in\mathcal{ML}_{\mathbf{Z}}(S)-\{0\}$, *the holonomy map* $hol:P(S)\to R(S)$ *is not injective on* $\overline{\mathcal{Q}_\lambda}$, *although the map hol is invective on* $\overline{\mathcal{Q}_0}$.

The following Theorem D is one of the essential observations in this paper. Theorem A follows immediately from this theorem. Theorem B is a consequence of Theorem 1.2 and Theorem D. Theorems C is also a corollary of Theorem D.

In 2.4, we define new elements $(\lambda,\mu)_\sharp$, $(\lambda,\mu)_\flat\in\mathcal{ML}_{\mathbf{Z}}(S)$ for $\lambda,\mu\in\mathcal{ML}_{\mathbf{Z}}(S)$. We remark that $(\lambda,\mu)_\sharp=(\lambda,\mu)_\flat$ if and only if $i(\lambda,\mu)=0$.

Theorem D. *Let* λ, $\mu\in\mathcal{ML}_{\mathbf{Z}}(S)-\{0\}$. *Assume that* λ *and* μ *have no parallel component in common and* $i(\lambda,\mu)\neq0$. *Then, there exist elements* $(\lambda,\mu)_\sharp$, $(\lambda,\mu)_\flat\in\mathcal{ML}_{\mathbf{Z}}(S)$, *a positive integer* N, *and sequences* $\{Y_n\}_{|n|>N}$, $\{Z_n\}_{|n|>N}$ *in* $Q(S)$ *which satisfy the following (see Figure 1):*

(1) $\{Y_n\}_{|n|>N}\subset\mathcal{Q}_\lambda$ *and* $Y_n\to Y_\infty\in\partial\mathcal{Q}_0\cap\partial\mathcal{Q}_\lambda$ *as* $|n|\to\infty$,

(2) $\{Z_n\}_{n>N}\subset\mathcal{Q}_{(\lambda,\mu)_\sharp}$, $\{Z_n\}_{n<-N}\subset\mathcal{Q}_{(\lambda,\mu)_\flat}$ *and* $Z_n\to Z_\infty\in\partial\mathcal{Q}_\mu$ *as* $|n|\to\infty$,

(3) $hol(Y_n)=hol(Z_n)$ *for any* $|n|>N$ *and* $hol(Y_\infty)=hol(Z_\infty)$.

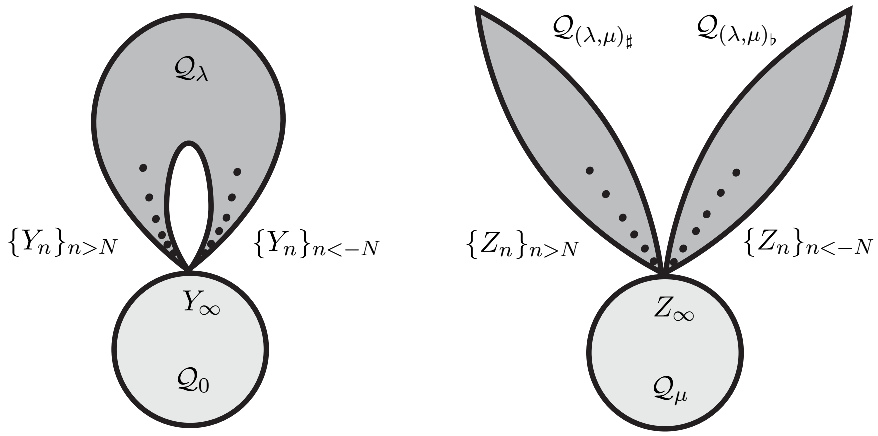

Figure 1: Schematic figure explaining Theorem D for the case of $i(\lambda, \mu) \neq 0$.

Figure 2 is a computer graphic created by Komori, Sugawa, Wada, and Yamashita (cf. [KS]). This shows a part of projective structures on a once-punctured torus with constant underlying complex structure. The white part corresponds to projective structures with quasi-fuchsian holonomy. The inner disk is a Bers slice (a slice of the standard component $\mathcal{Q}_0$) and the outer part is a slice of an exotic component. This graphic “seems to” show the claim of Theorem A.

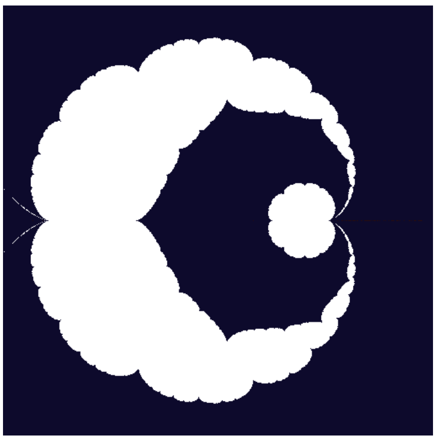

Figure 2: A part of a slice of $Q(S)$ in $P(S)$ (white part).

In [It] and this paper, we always make use of the sequence of exotic projective structures converging to a point in $\partial\mathcal{Q}_0$, which is constructed by McMullen in [Mc]. The original idea to construct such a sequence can be found in the theory of Kleinian groups: There is a sequence of Kleinian groups such that the algebraic limit is contained in the geometric limit "properly" and "twisted," where the "properly" part is due to Jørgensen [Jo] and Kerckhoff and Thurston [KT], and the "twisted" part is developed by Anderson and Canary [AC]. In Anderson-Canary [AC] and Anderson-Canary-McCullough [ACM], it is obtained a necessary and sufficient condition such that components of the set $AH(\Gamma)$ of discrete faithful representations of a finitely generated group Γ into $\mathrm{PSL}_2(\mathbf{C})$ have intersecting closures. On the other hand, Bromberg and Holt [BH] obtained a sufficient condition such that a component of $AH(\Gamma)$ is "self-bumping." Related topics can be found in Holt [Ho1], [Ho2]. Our results can be viewed as the projective structure analogues of those results: Theorem 1.2 and Theorem B correspond to the works in [AC] and [ACM]; on the other hand, Theorem A corresponds to the work in [BH].

Here, we explain more closely the phenomena which we discuss in this paper. We take a non-zero element $\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)$ and first review the proof of $\overline{\mathcal{Q}_0} \cap \overline{\mathcal{Q}_\lambda} \neq \emptyset$. Let $\mathrm{qf} : T(S) \times T(\bar{S}) \to QF(S)$ be the canonical homeomorphism, where $T(S)$ is the Teichmüller space of S, and $\bar{S}$ denotes S with its orientation reversed. We fix an element $(X, \bar{X}') \in T(S) \times T(\bar{S})$ and consider the sequence of representations

$$[\rho_n] = \mathrm{qf}(\tau^n X, \tau^{2n} \bar{X}') \in QF(S),$$

where τ is the Dehn twist corresponding to λ. Then, the sequence $[\rho_n]$ converges algebraically to a boundary group $[\rho_\infty] \in hol(\partial\mathcal{Q}_0)$ (see [Mc]). Take $Y_\infty \in \partial\mathcal{Q}_0$ such that $hol(Y_\infty) = [\rho_\infty]$, and take the sequence $\{Y_n\} \subset Q(S)$ such that $Y_n \to Y_\infty$ as $|n| \to \infty$ and that $hol(Y_n) = [\rho_n]$ for all sufficiently large $|n|$. In McMullen [Mc], it was shown that Y_n are exotic if $|n|$ are sufficiently large. Moreover, we showed in [It] that Y_n are eventually contained in the exotic component $\mathcal{Q}_\lambda$ as $|n| \to \infty$, which implies that $\overline{\mathcal{Q}_0} \cap \overline{\mathcal{Q}_\lambda} \neq \emptyset$.

For any $\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)$, there is a biholomorphic map $\mathrm{Gr}_\lambda : \mathcal{Q}_0 \to \mathcal{Q}_\lambda$ satisfying $hol \circ \mathrm{Gr}_\lambda = hol$, which is called the grafting map (see 2.3 for the definition). Let μ be a non-zero element of $\mathcal{ML}_{\mathbf{Z}}(S)$ such that the supports of μ and λ have no parallel component in common. Then, we can construct a grafting $Z_\infty = \mathrm{Gr}_\mu(Y_\infty)$ of $Y_\infty \in \partial\mathcal{Q}_0 \cap \partial\mathcal{Q}_\lambda$, which satisfies $hol(Z_\infty) = hol(Y_\infty)$. Moreover, we can show that $Z_\infty \in \partial\mathcal{Q}_\mu$ (see Proposition 2.12).

Since the map *hol* is a local homeomorphism, there exist open neighborhoods U of Y_∞, V of Z_∞ and a homeomorphism $\Phi : U \to V$ which satisfies $hol \circ \Phi = hol$ on U. We now obtain a new convergence sequence

$$Z_n = \Phi(Y_n) \to Z_\infty = \Phi(Y_\infty) \in \partial\mathcal{Q}_\mu$$

as $|n| \to \infty$. Note that $hol(Z_n) = [\rho_n]$ and $hol(Z_\infty) = [\rho_\infty]$ are satisfied. Then, the following theorem is the key theorem in this paper, which is a specific version of Theorem D.

Theorem D. *There exists a positive integer N such that $\{Z_n\}_{n>N} \subset \mathcal{Q}_{(\lambda,\mu)_\sharp}$ and $\{Z_n\}_{n<-N} \subset \mathcal{Q}_{(\lambda,\mu)_\flat}$.*

Let μ be an element of $\mathcal{ML}_{\mathbf{Z}}(S)$ as above. In addition, we assume that $i(\lambda, \mu) \neq 0$, which implies that $(\lambda, \mu)_\sharp \neq (\lambda, \mu)_\flat$. Then, $\{Z_n\}_{n>N}$ and $\{Z_n\}_{n<-N}$ are contained in distinct components of $Q(S)$. Since the holonomy map $hol : P(S) \to R(S)$ is a local homeomorphism, we obtain the following theorem.

Theorem 1.3. *For any sufficiently small neighborhood W of $[\rho_\infty]$, there exists a positive integer N such that $\{[\rho_n]\}_{n>N}$ and $\{[\rho_n]\}_{n<-N}$ are contained in distinct components of $W \cap QF(S)$.*

Now we obtain a proof of Theorem A.

Proof of Theorem A. Since the holonomy map is a local homeomorphism, for any sufficiently small neighborhood U of Y_∞, there exists a positive integer N such that $\{Y_n\}_{n>N}$ and $\{Y_n\}_{n<-N}$ are contained in distinct components of $U \cap \mathcal{Q}_\lambda$. This implies Theorem A. □

2 Preliminaries

2.1 Kleinian groups

A *Kleinian group* Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$, which acts on the hyperbolic space $\mathbf{H}^3$ as isometries, and on the sphere at infinity $S^2_\infty = \hat{\mathbf{C}}$ as conformal automorphisms. In this paper, we always assume that Kleinian groups are finitely generated. The *region of discontinuity* $\Omega(\Gamma)$ is the largest open subset of $\hat{\mathbf{C}}$ on which Γ acts properly discontinuously, and the *limit set* $\Lambda(\Gamma)$ of Γ is its complement $\hat{\mathbf{C}} - \Omega(\Gamma)$. The quotient manifold $N_\Gamma = \mathbf{H}^3 \cup \Omega(\Gamma)/\Gamma$ is called the *Kleinian manifold* of Γ. A Kleinian group Γ is said to be *geometrically finite* if some neighborhood of $CH(\Lambda(\Gamma))/\Gamma$ in

$\overline{\mathbf{H}^3}/\Gamma$ is a finite volume, where $CH(\Lambda(\Gamma))$ is the convex hull of $\Lambda(\Gamma)$ in $\overline{\mathbf{H}^3}$. A Kleinian group Γ is said to be a *quasi-fuchsian group* whose limit set $\Lambda(\Gamma)$ is a Jordan curve and which contains no element interchanging the two components of $\Omega(\Gamma)$. A Kleinian group Γ is said to be a *b-group* if there exists exactly one simply connected invariant component of $\Omega(\Gamma)$, which is denoted by $\Omega_0(\Gamma)$.

2.2 Projective structures

Let S be an oriented closed surface of genus $g > 1$. A projective structure on S is a $(\mathrm{PSL}_2(\mathbf{C}), \hat{\mathbf{C}})$-structure. Let $P(S)$ denote the set of equivalence classes of pairs (g, Y); where Y is a closed surface with a projective structure and $g : S \to Y$ is an orientation preserving homeomorphism. Two pairs (g_1, Y_1) and (g_2, Y_2) are said to be equivalent if there is a projective isomorphism $h : Y_1 \to Y_2$ such that $h \circ g_1$ is isotopic to g_2. The equivalence class of (g, Y) is simply denoted by Y. The set $P(S)$ can be identified with the holomorphic cotangent bundle over the Teichmüller space $T(S)$ of S. By this identification, $P(S)$ is a complex manifold of dimension $6g - 6$.

For a projective structure $Y \in P(S)$, a *developing map*

$$f_Y : \tilde{Y} \to \hat{\mathbf{C}}$$

is a local homeomorphism, which is obtained by lifting the projective structure to the universal cover $\tilde{Y}$ of Y, and by continuing the coordinates analytically. For a developing map f_Y of Y, there is a group homeomorphism

$$\rho_Y : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$$

satisfying the condition $f_Y \circ \gamma = \rho_Y(\gamma) \circ f_Y$ for any $\gamma \in \pi_1(S)$. The map ρ_Y is called a *holonomy representation* for Y. Note that a projective structure Y determines the pair (f_Y, ρ_Y) uniquely up to the action of $\mathrm{PSL}_2(\mathbf{C})$; where the action of $\eta \in \mathrm{PSL}_2(\mathbf{C})$ is defined by $(f_Y, \rho_Y) \mapsto (\eta \circ f_Y, \eta \circ \rho_Y \circ \eta^{-1})$.

Let

$$R(S) = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbf{C}))/\mathrm{PSL}_2(\mathbf{C}).$$

denote the space of all conjugacy classes $[\rho]$ of representations of $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$ such that $\rho(\pi_1(S))$ is non-abelian. It is known that $R(S)$ is a $6g-6$ dimensional complex manifold (see [Gu] and [Fa]). The *quasi-fuchsian space* $QF(S)$ is the subset of $R(S)$ consisting of conjugacy classes of faithful representations whose images are quasi-fuchsian groups. It is known by Bers [Be] that the quasi-fuchsian space $QF(S) \subset R(S)$ is a complex submanifold of dimension $6g - 6$.

The *holonomy map*

$$hol : P(S) \to R(S)$$

is defined by $Y \mapsto [\rho_Y]$. The basic fact is that the holonomy map is a holomorphic local homeomorphism (see [He], [Ea] and [Hu]).

In this paper, we are mainly concerned with the pre-image of the quasi-fuchsian space: $Q(S) = hol^{-1}(QF(S))$. We say an element of $Q(S)$ is *standard* if its developing map is injective; otherwise it is *exotic*. By using the technique of quasi-conformal deformations of projective structures, developed by Shiga and Tanigawa [ST], we obtained the following:

Lemma 2.1 ([It, Proposition 2.3]). *For any connected component $\mathcal{Q}$ of $Q(S)$, the map $hol|\mathcal{Q} : \mathcal{Q} \to QF(S)$ is biholomorphic.*

2.3 Grafting

Let $\mathcal{S}$ denote the set of homotopy classes of non-trivial simple closed curves on S. By abuse of the notation, we also denote a representative of $C \in \mathcal{S}$ by C. Let $\mathcal{ML}_{\mathbf{Z}}(S)$ denote the set of integral measured laminations on S. Namely, each element $\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)$ is written as a formal summation $\sum k_i C_i$, where $\{k_i\}$ are positive integers and $\{C_i\}$ are elements of $\mathcal{S}$ such that $C_i \cap C_j = \emptyset$ for $i \neq j$. We regard $\mathcal{S}$ as a subset of $\mathcal{ML}_{\mathbf{Z}}(S)$. We shall contain the "zero" measured lamination in $\mathcal{ML}_{\mathbf{Z}}(S)$.

For $C_1, C_2 \in \mathcal{S}$, the geometric intersection number $i(C_1, C_2)$ is the minimum number of points in which the representations of C_1 and C_2 must intersect. Note that $i(C, C) = 0$ for any $C \in \mathcal{S}$. We can naturally extend the definition of the geometric intersection number for elements of $\mathcal{ML}_{\mathbf{Z}}(S)$. We can also define $m\lambda + n\mu \in \mathcal{ML}_{\mathbf{Z}}(S)$ for λ, $\mu \in \mathcal{ML}_{\mathbf{Z}}(S)$ with $i(\lambda, \mu) = 0$ and positive integers m and n.

Let $Y \in P(S)$ and let $\pi_Y : \tilde{Y} \to Y$ be the universal covering map. Fix an simple closed curve $C \in \mathcal{S}$. Let $\tilde{C} \subset \tilde{Y}$ be a connected component of ${\pi_Y}^{-1}(C)$, and let $c \in \pi_1(S)$ be a generator of the cyclic subgroup $\langle c \rangle$ of the covering transformation group ($= \pi_1(S)$) such that $\tilde{C}$ is $\langle c \rangle$-invariant.

Definition 2.2. A simple closed curve $C \in \mathcal{S}$ is called *admissible* on Y if $f_Y(\tilde{C})$ is a simple arc in $\hat{\mathbf{C}}$ and $\rho_Y(c)$ is a loxodromic element. Moreover, $\lambda = \sum_{i=1}^{l} k_i C_i \in \mathcal{ML}_{\mathbf{Z}}(S)$ is admissible if C_i $(1 \leq i \leq l)$ are admissible on Y.

Let $C \in \mathcal{S}$ be admissible on Y. Then $f_Y(\tilde{C})$ is invariant under the action of the loxodromic cyclic group $\langle \rho_Y(c) \rangle$. Let $\{p, q\}$ be the fixed points of $\rho_Y(c)$. Then, $\{p,q\}$ are the end points of the arc $f_Y(\tilde{C})$ and the closure of $f_Y(\tilde{C})$ in $\hat{\mathbf{C}}$ is $\overline{f_Y(\tilde{C})} = f_Y(\tilde{C}) \cup \{p,q\}$. Let

$$A_C = \left(\hat{\mathbf{C}} - \overline{f_Y(\tilde{C})}\right) / \langle \rho_Y(c) \rangle$$

be the quotient annulus equipped with the projective structure induced from that of $\hat{\mathbf{C}}$. Then we obtain a new projective structure $\mathrm{Gr}_C(Y)$ by cutting Y along C and inserting the annulus A_C at the cut locus without twisting. The new projective structure $\mathrm{Gr}_C(Y)$ is said to be obtained from Y by grafting along C, or simply called a grafting of Y. This definition does not depend on the choice of a representative C in its homotopy class (see [Go]). The basic fact is that the grafting operation does not change the holonomy representation; that is, $hol(\mathrm{Gr}_C(Y)) = hol(Y)$ is satisfied. Similarly, we can define the new projective structure $\mathrm{Gr}_{kC}(Y)$ for $kC \in \mathcal{ML}_{\mathbf{Z}}(S)$ by cutting Y along C and inserting k-copies of the annulus A_C at the cut locus. In general, we can naturally construct $\mathrm{Gr}_\lambda(Y)$ from Y by grafting along λ, if λ is admissible on Y. Note that $hol(\mathrm{Gr}_\lambda(Y)) = hol(Y)$ is satisfied again.

Now we describe the relationship between components of $Q(S)$ and the set $\mathcal{ML}_{\mathbf{Z}}(S)$. To this end, we recall the following:

Theorem 2.3 (Goldman [Go]). *If $Y \in Q(S)$ is a standard projective structure, then*

$$hol^{-1}(hol(Y)) = \{\mathrm{Gr}_\lambda(Y)\}_{\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)}.$$

Let $Y \in Q(S)$ be a standard projective structure and $\mathcal{Q}$ be a connected component of $Q(S)$. Then, from Lemma 2.1, $\mathcal{Q}$ contains a unique projective structure whose holonomy representation coincides with $hol(Y)$. On the other hand, this projective structure is contained in the set $\{\mathrm{Gr}_\lambda(Y)\}_{\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)}$ by Theorem 2.3. Therefore, we obtain the decomposition of $Q(S)$ into its connected components;

$$Q(S) = \coprod_{\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)} \mathcal{Q}_\lambda,$$

where $\mathcal{Q}_\lambda$ is the component containing $\mathrm{Gr}_\lambda(Y)$. Then $\mathcal{Q}_0$ is the component consisting of standard projective structures and any element in $\mathcal{Q}_\lambda$ $(\lambda \neq 0)$ is exotic. Moreover, the grafting map $\mathrm{Gr}_\lambda : \mathcal{Q}_0 \to P(S)$ defined on $\mathcal{Q}_0$ is a biholomorphic map onto $\mathcal{Q}_\lambda$ and satisfies $hol \circ \mathrm{Gr}_\lambda = hol$ for each $\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)$.

2.4 New operation on $\mathcal{ML}_{\mathbf{Z}}(S)$

Definition 2.4 (Realization, Support). Let $\lambda = \sum_{i=1}^{l} k_i C_i \in \mathcal{ML}_{\mathbf{Z}}(S)$. A disjoint union of simple closed curves on S is said to be a *realization* of λ if it consists of k_i simple closed curves $C_i^{(1)}, \ldots, C_i^{(k_i)}$ each of which is homotopic to C_i for each $1 \leq i \leq l$. A realization of λ is denoted by $\widehat{\lambda}$, that is, $\widehat{\lambda} = \bigcup_{i=1}^{l} \bigcup_{1 \leq q \leq k_i} C_i^{(q)}$. On the other hand, the disjoint union $\bigcup_{i=1}^{l} C_i$ is called the *support* of λ.

Let us take $\lambda, \mu \in \mathcal{ML}_{\mathbf{Z}}(S)$. We shall define new elements $(\lambda, \mu)_\sharp$ and $(\lambda, \mu)_\flat$ in $\mathcal{ML}_{\mathbf{Z}}(S)$. Let $\widehat{\lambda}$ and $\widehat{\mu}$ be realizations of λ and μ such that the geometric intersection number of $\widehat{\lambda}$ and $\widehat{\mu}$ is minimal. we now construct new disjoint unions of simple closed curves $\widehat{(\lambda, \mu)}_\sharp$ and $\widehat{(\lambda, \mu)}_\flat$ by drawing "zigzag" paths in $\widehat{\lambda} \cup \widehat{\mu}$. More precisely, we construct $\widehat{(\lambda, \mu)}_\sharp$ and $\widehat{(\lambda, \mu)}_\flat$ in the following way (see Figure 3):

- Construction of $\widehat{(\lambda, \mu)}_\sharp$: Take a point in $\widehat{\lambda}$ as a starting point. Go along $\widehat{\lambda}$ until one meets $\widehat{\mu}$, then turn to *right* and go along $\widehat{\mu}$ until one meets $\widehat{\lambda}$, then turn to *left*. We continue this process until one comes back to the starting point. Change the starting point, if necessary, and repeat the above process until $\widehat{(\lambda, \mu)}_\sharp$ covers $\widehat{\lambda} \cup \widehat{\mu}$.

- Construction of $\widehat{(\lambda, \mu)}_\flat$: Take a point in $\widehat{\lambda}$ as a starting point. Go along $\widehat{\lambda}$ until one meets $\widehat{\mu}$, then turn to *left* and go along $\widehat{\mu}$ until one meets $\widehat{\lambda}$, then turn to *right*. We continue this process until one comes back to the starting point. Change the starting point, if necessary, and repeat the above process until $\widehat{(\lambda, \mu)}_\flat$ covers $\widehat{\lambda} \cup \widehat{\mu}$.

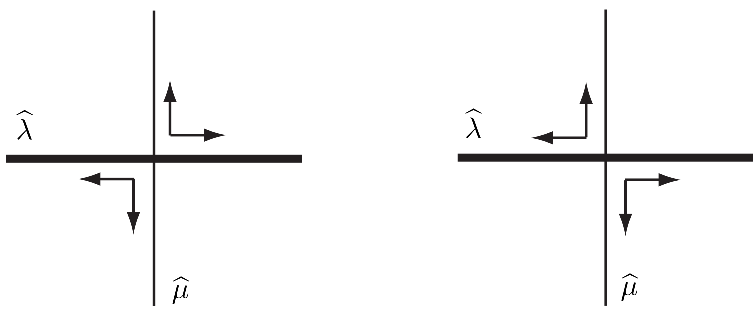

Figure 3: The rule to construct $\widehat{(\lambda, \mu)}_\sharp$ (left) and $\widehat{(\lambda, \mu)}_\flat$ (right).

Finally, we smooth the corners of $\widehat{(\lambda, \mu)}_\sharp$ and $\widehat{(\lambda, \mu)}_\flat$ so that they become unions of mutually disjoint, simple closed curves. Then we obtain new elements $(\lambda, \mu)_\sharp$ and $(\lambda, \mu)_\flat$ in $\mathcal{ML}_{\mathbf{Z}}(S)$, whose realizations are equal to $\widehat{(\lambda, \mu)}_\sharp$ and $\widehat{(\lambda, \mu)}_\flat$, respectively (see Figure 4). Note that this construction depends only on the orientation of S.

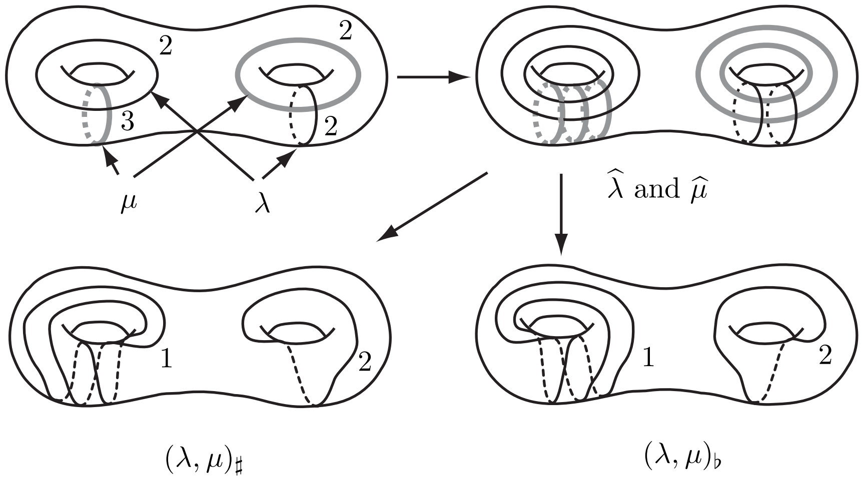

Figure 4: An example of $(\lambda, \mu)_\sharp$ and $(\lambda, \mu)_\flat$.

Note that, by definition, $(\lambda, \mu)_\sharp = (\mu, \lambda)_\flat$ is satisfied for any $\lambda, \mu \in \mathcal{ML}_{\mathbf{Z}}(S)$. We now collect some basic properties which we use in this paper. We leave these lemmas as exercises for the reader.

Lemma 2.5. *Let* $\lambda, \mu \in \mathcal{ML}_{\mathbf{Z}}(S)$. *Then* $i((\lambda, \mu)_\sharp, \mu) = i((\lambda, \mu)_\flat, \mu) = i(\lambda, \mu)$.

Lemma 2.6. *Let* $\lambda, \mu \in \mathcal{ML}_{\mathbf{Z}}(S)$. *If* $i(\lambda, \mu) = 0$, *then* $(\lambda, \mu)_\sharp = (\lambda, \mu)_\flat = \lambda + \mu$. *If* $i(\lambda, \mu) \neq 0$, *then* $(\lambda, \mu)_\sharp \neq (\lambda, \mu)_\flat$.

Lemma 2.7. *Let* $\lambda, \mu \in \mathcal{ML}_{\mathbf{Z}}(S) - \{0\}$. *Suppose that any component of the support of* μ *intersects* λ. *Then*

$$\begin{aligned}
((\lambda, \mu)_\sharp, \mu)_\flat &= \lambda = ((\lambda, \mu)_\flat, \mu)_\sharp, \\
((\lambda, \mu)_\sharp, \mu)_\sharp &= (\lambda, 2\mu)_\sharp, \\
((\lambda, \mu)_\flat, \mu)_\flat &= (\lambda, 2\mu)_\flat.
\end{aligned}$$

2.5 Algebraic and geometric limits of Kleinian groups

In this subsection, we describe the relationship between sequences of exotic projective structures and algebraic and geometric limits of their holonomy representations. We begin with the definition of geometric convergence of Kleinian groups.

Definition 2.8 (Hausdorff topology, Geometric convergence). Let X be a locally compact Hausdorff space. We denote by $\mathcal{C}(X)$ the set of all closed subsets of X. A sequence $\{A_n\}$ of closed subsets of X converges to a closed subset $A \subset X$ in the *Hausdorff topology* on $\mathcal{C}(X)$ if every element $x \in A$ is the limit of a sequence $\{x_n \in A_n\}$ and if every accumulation point of every sequence $\{x_n \in A_n\}$ lies in A. A sequence of Kleinian groups $\{\Gamma_n\}$ is said to converge *geometrically* to a group $\hat{\Gamma}$ if $\{\Gamma_n\}$ converges to $\hat{\Gamma}$ in the Hausdorff topology on $\mathcal{C}(\mathrm{PSL}_2(\mathbf{C}))$.

We recall some basic facts on the convergence of representations. Let $\{\rho_n : \pi_1(S) \to \Gamma_n\}$ be a sequence of discrete faithful representations of $\pi_1(S)$ into $\mathrm{PSL}_2(\mathbf{C})$ converging algebraically to $\rho_\infty : \pi_1(S) \to \Gamma_\infty$. Then ρ_∞ is also a discrete faithful representation (see [Jo, Theorem 1]). Moreover, there is a subsequence of $\{\Gamma_n\}$ converging geometrically to a Kleinian group $\hat{\Gamma}$ which contains the algebraic limit Γ_∞ (see [JM, proposition 3.8]). It is said that a sequence $\{\rho_n : \pi_1(S) \to \Gamma_n\}$ converges *strongly* to $\rho_\infty : \pi_1(S) \to \Gamma_\infty$ if $\{\rho_n\}$ converges algebraically to ρ_∞ and $\{\Gamma_n\}$ converges geometrically to Γ_∞. The following theorem is due to Kerckhoff and Thurston [KT, Corollary 2.2].

Theorem 2.9 (Kerckhoff-Thurston). *Let $\{\rho_n : \pi_1(S) \to \Gamma_n\}$ be an algebraically convergent sequence of faithful representations onto quasi-fuchsian groups Γ_n. Assume that $\{\Gamma_n\}$ converges geometrically to $\hat{\Gamma}$. Then $\{\Lambda(\Gamma_n)\}$ converges to $\Lambda(\hat{\Gamma})$ in the Hausdorff topology on $\mathcal{C}(\hat{\mathbf{C}})$.*

2.6 Pullback of limit sets of Kleinian groups

Let Y be an element of $P(S)$ whose holonomy representation $\rho_Y : \pi_1(S) \to \Gamma$ is faithful and discrete. Let $\pi_Y : \tilde{Y} \to Y$ be the universal covering map and $f_Y : \tilde{Y} \to \hat{\mathbf{C}}$ be the developing map. Then, one can easily see that the preimage $f_Y^{-1}(\Lambda(\Gamma))$ of the limit set $\Lambda(\Gamma)$ in $\tilde{Y}$ is invariant under the action of the covering transformation group $\pi_1(Y)$. Then, the subset $f_Y^{-1}(\Lambda(\Gamma)) \subset \tilde{Y}$ descends to the subset

$$\Lambda_Y = \pi_Y \circ f_Y^{-1}(\Lambda(\Gamma))$$

in Y, which is called the pullback of the limit set $\Lambda(\Gamma)$ in Y. We note that we can also consider the pullback $\pi_Y \circ f_Y^{-1}(\Lambda(\hat{\Gamma})) \subset Y$ of the limit set $\Lambda(\hat{\Gamma})$ of a Kleinian group $\hat{\Gamma}$ containing Γ.

Theorem 2.10 (Goldman [Go]). *Let Y be an element of $Q(S)$. Then $Y \in \mathcal{Q}_\lambda$ if and only if $\Lambda_Y \subset Y$ is a realization of 2λ.*

Let $\{Y_n\}$ be a sequence in $P(S)$ converging to an element $Y_\infty \in P(S)$ as $n \to \infty$. A projective structure on S induces a complex structure on S and hence a hyperbolic structure. Therefore, with these canonical hyperbolic structures on Y_n and Y_∞, there exist K_n-quasi-isometric maps $\omega_n : Y_\infty \to Y_n$ with $K_n \to 1$ as $n \to \infty$.

Lemma 2.11 ([It, Lemma 3.3]). *Let $\{Y_n\}$ be a sequence in $Q(S)$ converging to an element Y_∞ in $\overline{Q(S)}$ as $n \to \infty$. Then the sequence of holonomy representations $\{\rho_{Y_n} : \pi_1(S) \to \Gamma_n\}$ converges algebraically to $\rho_{Y_\infty} : \pi_1(S) \to \Gamma_\infty$. Moreover, we assume that $\{\Gamma_n\}$ converges geometrically to a Kleinian group $\hat{\Gamma}$. Let $\omega_n : Y_\infty \to Y_n$ be a K_n-quasi-isometric map with $K_n \to 1$ as $n \to \infty$. In this situation, the sequence $\{\omega_n^{-1}(\Lambda_{Y_n})\}$ converges to $\hat{\Lambda}_{Y_\infty}$ in the Hausdorff topology on $\mathcal{C}(Y_\infty)$, where $\hat{\Lambda}_{Y_\infty} = \pi_{Y_\infty} \circ f_{Y_\infty}^{-1}(\Lambda(\hat{\Gamma}))$ is the pullback in Y_∞ of the limit set of the geometric limit $\hat{\Gamma}$.*

If there is no confusion, we simply say that $\{\Lambda_{Y_n}\}$ converges to $\hat{\Lambda}_{Y_\infty}$ and denote by $\Lambda_{Y_n} \to \hat{\Lambda}_{Y_\infty}$.

2.7 Grafting operation on $\partial\mathcal{Q}_0$

Let $\lambda, \mu \in \mathcal{ML}_{\mathbf{Z}}(S) - \{0\}$. Suppose that the supports of λ and μ have no parallel component in common. Let Y be a projective structure in $\partial\mathcal{Q}_0$ such that its holonomy image $\Gamma = \rho_Y(\pi_1(S))$ is a geometrically finite b-group whose parabolic locus is equal to the support of λ. Since f_Y is injective and since supports of λ and μ have no parallel component in common, μ is admissible on Y. Therefore, we obtain the grafting $Z = \mathrm{Gr}_\mu(Y)$ as in 2.3. On the other hand, since $hol(Z) = hol(Y)$ and since the holonomy map is a holomorphic local homeomorphism, there are neighborhoods U of Y and V of Z such that the map $\Phi = (hol|V)^{-1} \circ (hol|U) : U \to V$ is biholomorphic. Note that $Z = \mathrm{Gr}_\mu(Y) = \Phi(Y)$. In the following proposition, we will show that $Z = \mathrm{Gr}_\mu(Y)$ is contained in $\partial\mathcal{Q}_\mu$ by using the map $\Phi : U \to V$. (We do not know whether $\Phi \equiv \mathrm{Gr}_\mu$ on $U \cap \mathcal{Q}_0$ or not.)

Proposition 2.12. *Let λ and μ be non-zero elements in $\mathcal{ML}_{\mathbf{Z}}(S)$ whose supports have no parallel component in common. Let Y be a projective structure in $\partial\mathcal{Q}_0$ such that its holonomy image $\Gamma = \rho_Y(\pi_1(S))$ is a geometrically* *finite b-group whose parabolic locus is equal to the support of λ. Then the grafted projective structure $Z = \mathrm{Gr}_\mu(Y)$ is contained in $\partial \mathcal{Q}_\mu$.*

Proof. Since $\Gamma = \rho_Y(\pi_1(S))$ is a geometrically finite b-group, $[\rho_Y]$ is contained in the boundary of some Bers slice $B_X = \mathrm{qf}(\{X\} \times T(\bar{S}))$, and there exists a sequence $\{[\rho_n] = \mathrm{qf}(X, \bar{X}_n)\}$ in B_X which converges strongly to $[\rho_Y]$ (see [Ab]). Let us take a sequence $\{Y_n\} \subset \mathcal{Q}_0$ such that $hol(Y_n) = [\rho_n]$. Then $Y_n \to Y$ and $Z_n = \Phi(Y_n) \to Z = \mathrm{Gr}_\mu(Y) = \Phi(Y)$ as $n \to \infty$.

We will show that $Z_n \in \mathcal{Q}_\mu$, which implies that $Z \in \partial \mathcal{Q}_\mu$. Since $\Gamma_n = \rho_n(\pi_1(S))$ converge geometrically to Γ, $\Lambda_{Z_n} \subset Z_n$ converge to $\Lambda_Z \subset Z$ in the sense of Hausdorff by Lemma 2.11. For simplicity, we assume that μ is a simple closed curve C' of weight 1. Then, one can see that $\Lambda_Z \subset Z$ is contained in an annulus A whose core curve is homotopic to C'. Since $\Lambda_{Z_n} \to \Lambda_Z$, we can see that Λ_{Z_n} is also contained in A for $n \gg 0$, which implies that $Z_n \in \bigcup_{k>0} \mathcal{Q}_{kC'}$ (see [It, Lemma 4.2]).

We first consider the case of $i(\lambda, C') = 0$. Then Λ_Z consists of exactly two components. Since $[\rho_n] = \mathrm{qf}(X, \bar{X}_n)$ converge strongly to $[\rho]$, we can take annuli $A \subset X$ and $A_n \subset \bar{X}_n$ whose core curves are homotopic to C' and whose moduli are uniformly bounded below. Then we can see that the hyperbolic distance of any two component of Λ_{Z_n} in Z_n is bounded below by the same argument in the proof of [It, Theorem A]. Therefore, we obtain $Z_n \in \mathcal{Q}_{C'} = \mathcal{Q}_\mu$ for $n \gg 0$.

Secondly, we consider the case of $i(\lambda, C') \neq 0$. Then Λ_Z consists of exactly one component. Then, by the same argument as above, we can see that the components of Λ_{Z_n} contained in one grafted cylinder (which consists of two simple closed curves homotopic to C') are in a uniformly bounded distance from components contained in another grafted cylinder. Therefore, we also obtain $Z_n \in \mathcal{Q}_{C'} = \mathcal{Q}_\mu$ for $n \gg 0$. □

2.8 Hyperbolic Dehn filling theorem

We will make use of a variation of Thurston's hyperbolic Dehn filling theorem, which is stated in the thesis of Comar [Co]. Let $\hat{M}$ be a compact 3-manifold with l torus boundary components $T_1, \ldots, T_l$. Choose a meridian m_i and longitude l_i for the torus T_i and regard m_i and l_i as a basis for $\pi_1(T_i)$. Let (p_i, q_i) be relatively prime integers and let $(\mathbf{p}, \mathbf{q}) = (p_1, q_1; \ldots; p_l, q_l)$. Then $M(\mathbf{p}, \mathbf{q})$ is a manifold obtained by attaching a solid torus V_i to $\hat{M}$ by an orientation reversing homeomorphism which identifies the meridian of V_i with a simple closed curve in the homotopy class of $m_i^{p_i} l_i^{q_i}$ for each i.

Theorem 2.13 (Hyperbolic Dehn filling theorem [Co]). *Let $\hat{M}$ be a* *compact 3-manifold with l torus boundary components. Assume that $\hat{M}$ is uniformized by a geometrically finite Kleinian group $\hat{\Gamma}$ without a rank-one parabolic subgroup. There is a neighborhood U of $(\infty, \ldots, \infty)$ in $(\mathbf{R}^2 \cup \{\infty\})^l$ such that, if a collection of relative prime pairs $(\mathbf{p}, \mathbf{q}) = (p_1, q_1; \ldots; p_l, q_l)$ is contained in U, then there exists a group homomorphism $\chi_{\mathbf{p},\mathbf{q}} : \hat{\Gamma} \to$ $\mathrm{PSL}_2(\mathbf{C})$ which satisfies the following:*

(1) The image $\Gamma(\mathbf{p}, \mathbf{q}) = \chi_{\mathbf{p},\mathbf{q}}(\hat{\Gamma})$ is a convex co-compact Kleinian group which uniformizing $M(\mathbf{p}, \mathbf{q})$.

(2) The kernel of $\chi_{\mathbf{p},\mathbf{q}}$ is normally generated by $\{m_i^{p_i} l_i^{q_i}\}_{i=1}^l$.

(3) $\chi_{\mathbf{p},\mathbf{q}}$ converges algebraically to the identity representation of $\hat{\Gamma}$ as $(\mathbf{p}, \mathbf{q}) \to (\infty, \ldots, \infty)$.

(4) $\Gamma(\mathbf{p}, \mathbf{q})$ converges geometrically to $\hat{\Gamma}$ as $(\mathbf{p}, \mathbf{q}) \to (\infty, \ldots, \infty)$.

3 Proofs of main theorems

3.1 Outline of proofs

Most of this section (3.2–3.7) is devoted to the proof of Theorem D. Figure 5 should be helpful to understand the outline of the proof. Let us fix $\lambda = \sum k_i C_i \in \mathcal{ML}_{\mathbf{Z}}(S) - \{0\}$. In 3.2, we take and fix a geometrically finite Kleinian group $\hat{\Gamma}$ whose Kleinian manifold $N_{\hat{\Gamma}}$ is homeomorphic to $(S \times [0, 1]) - \bigcup_{i=1}^l C_i \times \{1/2\}$. In 3.3, we investigate the shape of the limit set $\Lambda(\hat{\Gamma})$ and prepare some notations. In 3.4, we define an immersion $w_\lambda : S \to N_{\hat{\Gamma}}$, which is called a wrapping map associated to λ. The wrapping map w_λ is constructed so that it is not homotopic into a boundary component of $N_{\hat{\Gamma}}$, and that the subgroup $\Gamma_\infty \subset \hat{\Gamma}$ corresponding to $w_\lambda(S)$ is a geometrically finite b-group. Let $\chi_n : \hat{\Gamma} \to \Gamma_n$ be representations from $\hat{\Gamma}$ onto quasi-fuchsian groups Γ_n which are obtained by performing $(-1, n)$ Dehn filling on each cusp of $N_{\hat{\Gamma}}$. Put $\rho_n = \chi_n \circ (w_\lambda)_* : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$ and $\rho_\infty = (w_\lambda)_* : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$, where $(w_\lambda)_* : \pi_1(S) \to \hat{\Gamma}$ is the group isomorphism induced by w_λ. Then, the sequence $\{\rho_n\}$ converges algebraically to ρ_∞, and the sequence $\{\rho_n(\pi_1(S)) = \Gamma_n\}$ converges geometrically to $\hat{\Gamma}$ as $|n| \to \infty$.

We will show in 3.5 that there is a projective structure $Y_\infty \in \partial\mathcal{Q}_0$ which satisfies $hol(Y_\infty) = [\rho_\infty]$. Since the holonomy map $hol : P(S) \to R(S)$ is a local homeomorphism, there exist $N > 0$ and sequences $\{Y_n\}_{|n|>N}$ in $Q(S)$ such that $Y_n \to Y_\infty$ as $|n| \to \infty$ and $hol(Y_n) = [\rho_n]$ for all $|n| > N$. As observed in [It], we can see that $\hat{\Lambda}_{Y_\infty}$ is a decoration of a realization of 2λ, which implies Λ_{Y_n} is a realization of 2λ for all $|n| > N$, if N is sufficiently large. Therefore, we obtain $Y_n \in \mathcal{Q}_\lambda$ for all $|n| > N$.

Now let us take an element $\mu \in \mathcal{ML}_{\mathbf{Z}}(S) - \{0\}$ such that the supports of λ and μ have no parallel component in common. In 3.6, we take the grafting $Z_\infty = \mathrm{Gr}_\mu(Y_\infty)$ of Y_∞ and sequences $\{Z_n\}_{|n|>N}$ such that $Z_n \to Z_\infty$ as $|n| \to \infty$ and $hol(Z_n) = hol(Y_n)$ for all $|n| > N$. The statement of Theorem D is that there exists $N > 0$ such that $\{Z_n\}_{n>N} \subset \mathcal{Q}_{(\lambda,\mu)_\sharp}$ and $\{Z_n\}_{n<-N} \subset \mathcal{Q}_{(\lambda,\mu)_\flat}$. Since $\Lambda_{Z_n} \to \hat{\Lambda}_{Z_\infty}$ as $|n| \to \infty$, we first observe the shape of $\hat{\Lambda}_{Z_\infty}$ in 3.7. In 3.8, we show that Λ_{Z_n} is a realization of $2(\lambda,\mu)_\sharp$ if $n \gg 0$, and $2(\lambda,\mu)_\flat$ if $n \ll 0$, which implies Theorem D. An important observation here is that, both limit sets $\Lambda(\Gamma_n)$ $(n \gg 0)$ and $\Lambda(\Gamma_n)$ $(n \ll 0)$ are close to $\Lambda(\hat{\Gamma})$, but spiral in opposite directions at each rank-two parabolic fixed point in $\Lambda(\hat{\Gamma})$.

In 3.9, we prove Theorem B as a consequence of Theorem D and Theorem 1.2. In 3.10, we prove Theorem C as a corollary of Theorem D.

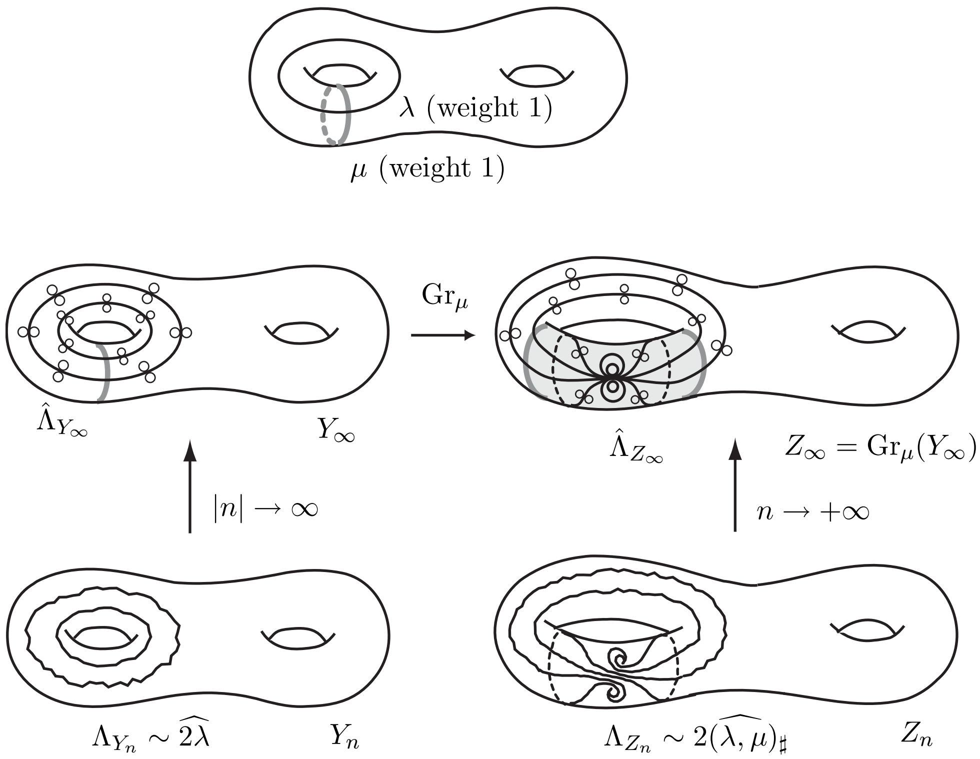

Figure 5: Schematic figure explaining the proof of Theorem D.

3.2 A Kleinian group $\hat{\Gamma}$

Take and fix an element $\lambda = \sum_{i=1}^{l} k_i C_i \in \mathcal{ML}_{\mathbf{Z}}(S) - \{0\}$, whose support is $\bigcup_{i=1}^{l} C_i \subset S$. Let

$$M_\lambda = (S \times [0,1]) - \bigcup_{i=1}^{l} C_i \times \{1/2\}$$

be a 3-manifold $S \times [0,1]$ from which simple closed curves $C_i \times \{1/2\}$ $(1 \le i \le l)$ are removed. Let $\hat{\Gamma}$ be a geometrically finite Kleinian group whose Kleinian manifold $N_{\hat{\Gamma}} = (\mathbf{H}^3 \cup \Omega(\hat{\Gamma}))/\hat{\Gamma}$ is homeomorphic to M_λ. The existence of such a Kleinian group $\hat{\Gamma}$ is guaranteed by Thurston's geometrization theorem (see [Mo]). From now on, we identify $N_{\hat{\Gamma}}$ with M_λ and $\hat{\Gamma} = \pi_1(N_{\hat{\Gamma}})$ with $\pi_1(M_\lambda)$. Then each tubular neighborhood of $C_i \times \{1/2\}$ in M_λ corresponds to a rank-two cusp end in $N_{\hat{\Gamma}}$. We fix a basis of the corresponding rank-two parabolic subgroup $\langle \gamma_i, \delta_i \rangle \subset \hat{\Gamma}$ so that $\gamma_i \in \pi_1(M_\lambda)$ is freely homotopic to $C_i \times \{0\}$ in M_λ, and that $\delta_i \in \pi_1(M_\lambda)$ is trivial in $S \times [0,1]$. Moreover, we fix the orientations of γ_i and δ_i so that the group $\langle \gamma_i, \delta_i \rangle$ is conjugate by some element of $\mathrm{PSL}_2(\mathbf{C})$ to $\langle \gamma_i'(z) = z+1, \delta_i'(z) = z + \tau_i \rangle$, and that the imaginary part $\Im \tau_i$ of τ_i is positive.

Note that $\Lambda(\hat{\Gamma})$ is connected, since each component of $\partial N_{\hat{\Gamma}}$ is incompressible. Therefore, each connected component of $\Omega(\hat{\Gamma})$ is simply connected. The region of discontinuity $\Omega(\hat{\Gamma})$ decomposes into two parts $\Omega^0(\hat{\Gamma}) = \pi_{\hat{\Gamma}}^{-1}(S \times \{0\})$ and $\Omega^1(\hat{\Gamma}) = \pi_{\hat{\Gamma}}^{-1}(S \times \{1\})$, where $\pi_{\hat{\Gamma}} : \mathbf{H}^3 \cup \Omega(\hat{\Gamma}) \to N_{\hat{\Gamma}}$ is the covering map. A subgroup $\Gamma \subset \hat{\Gamma}$ corresponding to $\pi_1(S \times \{0\})$ or $\pi_1(S \times \{1\})$ is a geometrically finite b-group whose invariant component $\Omega_0(\Gamma)$ is a component of $\Omega(\hat{\Gamma})$. Moreover, for a connected component Σ of $S - \bigcup_{i=1}^{l} C_i$, a subgroup $\Gamma' \subset \hat{\Gamma}$ corresponding to $\pi_1(\Sigma \times \{1/2\})$ is a quasi-fuchsian subgroup.

3.3 Structure of the limit set $\Lambda(\hat{\Gamma})$

To treat limit sets of Kleinian groups, we now prepare some terminology.

Definition 3.1 (Crescent-like domain). A domain $A \subset \hat{\mathbf{C}}$ is called a *crescent-like domain* if A is the interior of $B_2 - B_1$, where B_1 and B_2 are topological closed disks in $\hat{\mathbf{C}}$ such that $B_1 \subset B_2$ and $\partial B_1 \cap \partial B_2 = \{p\}$. We say that A is touching at $\{p\}$.

Definition 3.2 (Graph, Thickened graph, Skeleton, Decoration). Let Σ be a closed surface. A closed subset $\mathcal{G}$ of Σ is called a *graph* if there is a finite set of points $V = V(\mathcal{G})$ in $\mathcal{G}$ such that $\mathcal{G} - V$ consists of finite number of arcs or simple closed curves. We denote the set $\mathcal{G} - V$ by $E = E(\mathcal{G})$. Each element $v \in V(\mathcal{G})$ is called a *vertex* of $\mathcal{G}$, and each element $e \in E(\mathcal{G})$ is called an *edge* of $\mathcal{G}$.

A subset $\mathcal{N}$ of Σ is called a *thickened graph* of a graph $\mathcal{G}$ if $\mathcal{N}$ contains $\mathcal{G}$ and there exists a continuous map $r : \Sigma \to \Sigma$ homotopic to the identity map such that the restriction $r|\mathcal{N}$ is a deformation retraction of $\mathcal{N}$ onto $\mathcal{G}$.

Let $\mathcal{X}$ be a subset of Σ. A graph $\mathcal{G}$ contained in $\mathcal{X}$ is said to be a *skeleton* of $\mathcal{X}$ if there is a thickened graph $\mathcal{N}$ of $\mathcal{G}$ such that $\mathcal{X} \subset \mathcal{N}$ and that any connected component of $\mathcal{N}$ contains exactly one connected component of $\mathcal{X}$. We also say that $\mathcal{X}$ is a *decoration* of $\mathcal{G}$. A thickened graph $\mathcal{N}$ of $\mathcal{G}$ which satisfies the above condition is denoted by $\mathcal{N}(\mathcal{X}, \mathcal{G})$.

Let $R_i \subset S$ be mutually disjoint tubular neighborhoods of C_i. We parameterize R_i by a homeomorphism $\varphi_i : S^1 \times [-1, 1] \to R_i$ such that $\varphi_i(S^1 \times \{0\}) = C_i$. We denote $C_i^+ = \varphi_i(S^1 \times \{-1\})$ and $C_i^- = \varphi_i(S^1 \times \{1\})$. Now we take the following cylinders in $M_\lambda \subset S \times [0, 1]$ (see Figure 6):

$$W_i^- = C_i^- \times [0, 1], \quad W_i^+ = C_i^+ \times [0, 1], \quad R_i^0 = R_i \times \{0\}, \quad R_i^1 = R_i \times \{1\}.$$

The union of cylinders

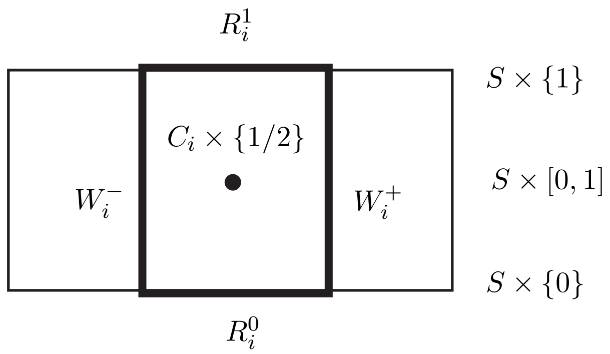

Figure 6: Cylinders in M_λ.

$$T_i = R_i^0 \cup W_i^+ \cup R_i^1 \cup W_i^-$$

is a torus bounding a tubular neighborhood of $C_i \times \{1/2\}$ in $M_\lambda = N_{\hat{\Gamma}}$.

Now we are going to consider a connected component of the pre-image of T_i in $\mathbf{H}^3 \cup \Omega(\hat{\Gamma})$ via the covering map $\pi_{\hat{\Gamma}} : \mathbf{H}^3 \cup \Omega(\hat{\Gamma}) \to N_{\hat{\Gamma}} = M_\lambda$. Let $\{p\}$

be a fixed point of a parabolic element in $\hat{\Gamma}$. Then, the stabilizer subgroup of $\{p\}$ in $\hat{\Gamma}$ is a rank-two parabolic subgroup $\langle \gamma, \delta \rangle$, which is conjugate to $\langle \gamma_i, \delta_i \rangle$ for some $i \in \{1, \dots, l\}$. More precisely, there exists $\eta \in \mathrm{PSL}_2(\mathbf{C})$ such that $\eta^{-1}\gamma\eta = \gamma_i$ and $\eta^{-1}\delta\eta = \delta_i$. Now we fix such i and abbreviate the suffix i for simplicity, that is, we denote C_i by C, R_i by R, and so on.

Let A be an annulus with boundary ∂A obtained by cutting the torus $T(= T_i)$ along $C^-(= C_i^-)$. We choose a connected component $\tilde{A}(p)$ of the pre-image $\pi_{\hat{\Gamma}}^{-1}(A) \subset \mathbf{H}^3 \cup \Omega(\hat{\Gamma})$ of $A \subset N_{\hat{\Gamma}}$ which is invariant under the action of the parabolic cyclic subgroup $\langle \gamma \rangle$. From now on, we simply say that $\tilde{A}(p)$ is a $\langle \gamma \rangle$-invariant "lift" of A. Let $\tilde{R}^0(p)$, $\tilde{W}^+(p)$, $\tilde{R}^1(p)$ and $\tilde{W}^-(p)$ be lifts of R^0, W^+, R^1 and W^- contained in $\tilde{A}(p)$, which satisfy

$$\tilde{A}(p) = \tilde{R}^0(p) \cup \tilde{W}^+(p) \cup \tilde{R}^1(p) \cup \tilde{W}^-(p).$$

We also take the lifts $\tilde{C}^0(p)$ and $\tilde{C}^1(p)$ of $C \times \{0\}$, $C \times \{1\}$ in $\tilde{R}^0(p)$ and $\tilde{R}^1(p)$, respectively. The parabolic element δ maps one connected component of $\partial\tilde{A}(p)$ to the other, where $\partial\tilde{A}(p)$ is a pre-image of ∂A in $\tilde{A}(p)$. (Note that $\partial\tilde{A}(p)$ does not contain $\{p\}$.) We may assume that δ maps the component of $\partial\tilde{A}(p)$ neighboring $\tilde{R}^0$ to the the component neighboring $\tilde{W}^-$.

Let $\tilde{T}(p)$ be the lift of T in $\mathbf{H}^3 \cup \Omega(\hat{\Gamma})$ which is invariant under the action of $\langle \gamma, \delta \rangle$. Then, $\tilde{T}(p) \cap \Omega^0(\hat{\Gamma}) = \bigcup_{m \in \mathbf{Z}} \delta^m(\tilde{R}^0(p))$ and $\tilde{T}(p) \cap \Omega^1(\hat{\Gamma}) = \bigcup_{m \in \mathbf{Z}} \delta^m(\tilde{R}^1(p))$ are satisfied. For each $m \in \mathbf{Z}$, we use the following notations

$$\begin{aligned} \tilde{C}^{2m}(p) &= \delta^m(\tilde{C}^0(p)), \quad \tilde{C}^{2m+1}(p) = \delta^m(\tilde{C}^1(p)), \\ \tilde{R}^{2m}(p) &= \delta^m(\tilde{R}^0(p)), \quad \tilde{R}^{2m+1}(p) = \delta^m(\tilde{R}^1(p)). \end{aligned}$$

Moreover, let $A_m(p)$ be the crescent-like domain touching at $\{p\}$, which is the one of the connected components of $\hat{\mathbf{C}} - \overline{\tilde{R}^m(p) \cup \tilde{R}^{m+1}(p)}$. Now we have obtained arrays of components

$$\{\tilde{C}^m(p)\}_{m \in \mathbf{Z}}, \quad \{\tilde{R}^m(p)\}_{m \in \mathbf{Z}}, \quad \{A_m(p)\}_{m \in \mathbf{Z}}.$$

With the above preparations, we are now in the position to consider connected components of $\Lambda(\hat{\Gamma}) - \{p\}$. Since $\bigcup_{m \in \mathbf{Z}} \tilde{R}^m(p)$ is contained in $\Omega(\hat{\Gamma})$, any connected component Ξ of $\Lambda(\hat{\Gamma}) - \{p\}$ is contained in some $A_m(p)$. Moreover, we have the following lemma (see Figure 7).

Lemma 3.3. *For each $m \in \mathbf{Z}$, there exists exactly one connected component $\Xi_m(p)$ of $\Lambda(\hat{\Gamma}) - \{p\}$ in $A_m(p)$. In addition, there exists a quasi-fuchsian subgroup $\Theta_m(p) \subset \hat{\Gamma}$ such that the set $\mathcal{X} = \Xi_m(p) \cup \{p\}$ is a decoration of a Jordan curve $\mathcal{G} = \Lambda(\Theta_m(p))$ with $\mathcal{N}(\mathcal{X}, \mathcal{G}) = A_m(p) \cup \{p\}$.*

From the above lemma, we obtain the following decomposition:

$$\Lambda(\hat{\Gamma}) - \{p\} = \bigcup_{m \in \mathbf{Z}} \Xi_m(p).$$

Proof of Lemma 3.3. We put $\Xi_m(p) = \Lambda(\hat{\Gamma}) \cap A_m(p)$ and are going to show that $\Xi_m(p)$ is connected. For simplicity, we only consider the case of $m = 0$. Let Ω^0 and Ω^1 be the connected component of $\Omega(\hat{\Gamma})$ containing $\tilde{R}^0$ and $\tilde{R}^1$, respectively. Moreover, we let $\Gamma^0 = \mathrm{Stab}_{\hat{\Gamma}}(\Omega^0)$ and $\Gamma^1 = \mathrm{Stab}_{\hat{\Gamma}}(\Omega^1)$ be the stabilizer subgroup of $\hat{\Gamma}$ associated to Ω^0 and Ω^1, respectively. Then, Γ^0 (resp. Γ^1) is a b-group corresponding to $\pi_1(S \times \{0\})$ (resp. $\pi_1(S \times \{1\})$), whose unique invariant component is exactly Ω^0 (resp. Ω^1). Note that $\Omega^0 \cap \Omega^1 = \emptyset$. Let Σ be the component of $S - \bigcup_i C_i$ which contains $C^+ \subset \partial R$. This implies that $W^+ = C^+ \times [0,1]$ is contained in $\Sigma \times [0,1]$. Let $\Theta_0(p) \subset \hat{\Gamma}$ be a quasi-fuchsian subgroup corresponding to $\pi_1(\Sigma \times [0,1])$ which is contained in both Γ^0 and Γ^1 as a component subgroup. Since $\tilde{R}^0(p) \subset \Omega^0$ and $\tilde{R}^1(p) \subset \Omega^1$, and since $\Lambda(\Theta_0(p))$ is contained in both $\partial\Omega^0 = \Lambda(\Gamma^0)$ and $\partial\Omega^1 = \Lambda(\Gamma^1)$, the Jordan curve $\Lambda(\Theta_0(p)) - \{p\}$ is contained in $A_0(p)$. Therefore, we have $\Lambda(\Theta_0(p)) - \{p\} \subset \Xi_0(p)$.

Suppose that $\Xi_0(p)$ is not connected. Then there is a connected component K of $\Xi_0(p)$ such that $K \cap \Lambda(\Theta_0(p)) = \emptyset$. If the closure $\overline{K}$ of K in $\hat{\mathbf{C}}$ does not contain $\{p\}$, this contradicts the fact that $\Lambda(\hat{\Gamma})$ is connected. If $\{p\}$ is contained in $\overline{K}$, one can easily see that $\overline{K}$ separates $\Lambda(\Theta_0(p)) - \{p\}$ from $\tilde{C}^0$ or $\tilde{C}^1$, which contradicts the fact that both $\partial\Omega_0 = \Lambda(\Gamma^0)$ and $\partial\Omega_1 = \Lambda(\Gamma^1)$ contain $\Lambda(\Theta_0(p))$. Therefore, we have proved that $\Xi_0(p)$ is connected and that $\mathcal{X} = \Xi_0(p) \cup \{p\}$ is a decoration of a Jordan curve $\mathcal{G} = \Lambda(\Theta_0(p))$ with $\mathcal{N}(\mathcal{X}, \mathcal{G}) = A_0(p) \cup \{p\}$. □

3.4 Wrapping map and associated representations

We now construct an immersion $w_\lambda : S \to M_\lambda \cong N_{\hat{\Gamma}}$, which is called a *wrapping map* associated with $\lambda = \sum_{i=1}^{l} k_i C_i \in \mathcal{ML}_{\mathbf{Z}}(S)$.

Recall that $R_i \subset S$ are mutually disjoint tubular neighborhoods of C_i. We make use of the cylinders R_i^0, W_i^+, R_i^1 and W_i^- in $M_\lambda \cong N_{\hat{\Gamma}}$, which are defined in 3.3, again. Let $\Psi_i^0 : R_i \to R_i^0 \subset M_\lambda$ be the inclusion map. We now define a continuous map $\Psi_i^{k_i} : R_i \to M_\lambda$ which is wrapping a neighborhood of $C_i \times \{1/2\}$ for k_i times, that is, let $\Psi_i^{k_i} : R_i \to M_\lambda$ be a continuous map which satisfies

(1) $\Psi_i^{k_i} = \Psi_i^0$ on ∂R_i,

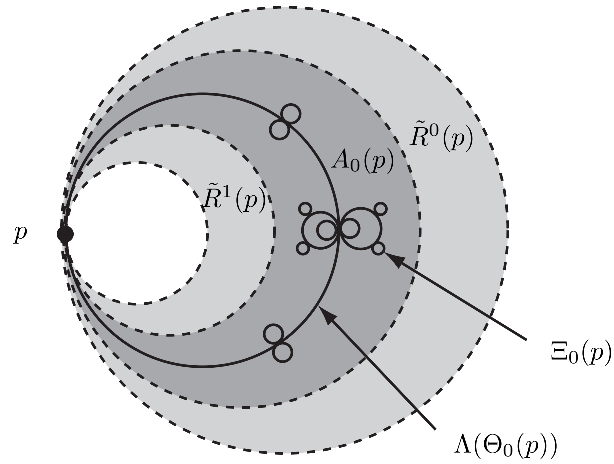

Figure 7: The component $\Xi_0(p)$ of $\Lambda(\hat{\Gamma}) - \{p\}$ in the crescent-like domain $A_0(p)$.

(2) $\Psi_i^{k_i}$ is homotopic to Ψ_i^0 relative boundary in $S \times [0, 1]$,

(3) The image of R_i by $\Psi_i^{k_i}$ can be considered as the self-overlapping cylinder in M_λ obtained by joining the cylinders R_i^0, W_i^+, R_i^1, W_i^- in this order for k_i times and by finally joining the cylinder R_i^0.

Let $w_0 : S \to S \times \{0\} \subset M_\lambda$ be the inclusion map. The wrapping map $w_\lambda : S \to M_\lambda$ is defined by $w_\lambda \equiv w_0$ on $S - \bigcup_{i=1}^l R_i$, and $w_\lambda \equiv \Psi_i^{k_i}$ on R_i for each $1 \leq i \leq l$.

By performing $(-1, n)$ Dehn filling ($n \in \mathbf{Z}$) on each cusp end of $N_{\hat{\Gamma}}$, we obtain from Theorem 2.13 a sequence of representations $\{\chi_n : \hat{\Gamma} \to \mathrm{PSL}_2(\mathbf{C})\}$ which satisfies the following:

- $\Gamma_n = \chi_n(\hat{\Gamma})$ is a quasi-fuchsian group.
- The kernel of χ_n is normally generated by $\gamma_1^n \delta_1^{-1}, \ldots, \gamma_l^n \delta_l^{-1}$.
- $\{\chi_n\}$ converges algebraically to the identity representation of $\hat{\Gamma}$.
- $\{\Gamma_n\}$ converges geometrically to $\hat{\Gamma}$.

Then, for each $i \in \{1, \ldots, l\}$, $\chi_n(\gamma_i) \to \gamma_i$ and $\chi_n(\gamma_i)^n = \chi_n(\gamma_i^n) = \chi_n(\delta_i) \to \delta_i$ as $|n| \to \infty$.

We put

$$\rho_n = \chi_n \circ (w_\lambda)_* : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$$

and

$$\rho_\infty = (w_\lambda)_* : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C}),$$

where $(w_\lambda)_* : \pi_1(S) \to \pi_1(N_{\hat{\Gamma}}) = \hat{\Gamma}$ is the group isomorphism induced by w_λ. Then, ρ_n are faithful representations onto the quasi-fuchsian groups $\Gamma_n = \chi_n(\hat{\Gamma})$ and the sequence $\{\rho_n\}$ converges algebraically to ρ_∞. We denote the algebraic limit $\rho_\infty(\pi_1(S))$ by Γ_∞, which is a proper subgroup of the geometric limit $\hat{\Gamma}$. Note that Γ_∞ is a geometrically finite b-group.

3.5 Sequences $\{Y_n\}_{|n|>N}$ and their limit Y_∞

In this subsection, we obtain a projective structure $Y_\infty \in \partial\mathcal{Q}_0$ and sequences of projective structures $\{Y_n\}_{|n|>N} \subset \mathcal{Q}_\lambda$ which satisfy $Y_n \to Y_\infty\,(|n| \to \infty)$, $hol(Y_n) = [\rho_n]\,(|n| > N)$ and $hol(Y_\infty) = [\rho_\infty]$. Here, we denote two sequences $\{Y_n\}_{n>N}$ and $\{Y_n\}_{n<-N}$ simply by $\{Y_n\}_{|n|>N}$.

Let $\pi : \tilde{S} \to S$ be the universal covering map whose covering transformation group is identified with $\pi_1(S)$. Let us denote by $\widetilde{w_\lambda} : \tilde{S} \to \mathbf{H}^3 \cup \Omega(\hat{\Gamma})$ the lift of $w_\lambda : S \to N_{\hat{\Gamma}}$ which satisfies

$$\widetilde{w_\lambda} \circ \gamma = \rho_\infty(\gamma) \circ \widetilde{w_\lambda}$$

for all $\gamma \in \pi_1(S)$.

Proposition 3.4. *There exists a projective structure $Y_\infty \in \partial\mathcal{Q}_0$ such that $hol(Y_\infty) = [\rho_\infty]$.*

Proof. We claim that there exists an equivariant isotopy between $\widetilde{w_\lambda}$ and an embedding $F_1 : \tilde{S} \to \hat{\mathbf{C}}$. More precisely, there is a continuous map

$$F_t(x) = F(x,t) : \tilde{S} \times [0,1] \to \mathbf{H}^3 \cup \hat{\mathbf{C}}$$

such that $F_t : \tilde{S} \to \mathbf{H}^3 \cup \hat{\mathbf{C}}$ $(t \in [0,1])$ are embedding satisfying

$$F_t \circ \gamma = \rho_\infty(\gamma) \circ F_t$$

for all $\gamma \in \pi_1(S)$, $F_0 = \widetilde{w_\lambda}$ and the image of F_1 is contained in $\hat{\mathbf{C}}$. Then, the embedding F_1 induces a projective structure Y_∞ on S such that $f_{Y_\infty} = F_1$ and $\rho_{Y_\infty} = \rho_\infty$. Since $f_{Y_\infty} : \tilde{S} \to \hat{\mathbf{C}}$ is injective and the holonomy image $\Gamma_\infty = \rho_{Y_\infty}(\pi_1(S))$ is a b-group, we know that Y_∞ is contained in $\partial\mathcal{Q}_0$.

We now show the above claim. We use the same notations in 3.3 and 3.4. Recall that $\bigcup_{i=1}^{l} R_i$ is a regular neighborhood of $\bigcup_{i=1}^{l} C_i$ in S. Note that each connected component of $\tilde{S} - \bigcup_{i=1}^{l} \pi^{-1}(R_i)$ is mapped by $\widetilde{w_\lambda}$ into distinct components of $\Omega^0(\hat{\Gamma}) = \pi_{\hat{\Gamma}}^{-1}(S \times \{0\}) \subset \Omega(\hat{\Gamma})$. We now take and fix $i \in \{1, \ldots, l\}$. Let $\tilde{R}$ be a component of $\pi^{-1}(R_i) \subset \tilde{S}$, and let c be an element of $\pi_1(S)$ such that $\tilde{R}$ is $\langle c \rangle$-invariant. Then $\gamma = (w_\lambda)_*(c) \in \Gamma_\infty$ is a parabolic element with fixed point $\{p\}$. We can see that $\widetilde{w_\lambda}(\tilde{R}) \cap \hat{\mathbf{C}}$ is a $2k+1$ succeeding sequence of $\{\tilde{R}^m(p)\}_{m \in \mathbf{Z}}$, say $\tilde{R}^1(p), \ldots, \tilde{R}^{2k+1}(p)$, where $k = k_i$ is the weight of C_i in λ. Moreover, $\widetilde{w_\lambda}(\tilde{R})$ cuts off $2k$ component $D_1, \ldots, D_{2k}$ from $\mathbf{H}^3$ such that $\overline{D_j} \cap \hat{\mathbf{C}} = \overline{A_j(p)}$ for $j \in \{1, \ldots, 2k\}$ (see Figure 8). This observation shows that the map $\widetilde{w_\lambda}|\tilde{R}$ is injective. Then, it easily follows that the map $\widetilde{w_\lambda}$ is injective on the whole $\tilde{S}$. Now, we can construct an isotopy from $\widetilde{w_\lambda}|\tilde{R} : \tilde{R} \to \mathbf{H}^3 \cup \hat{\mathbf{C}}$ to an embedding into $\hat{\mathbf{C}}$ which is constant on $\widetilde{w_\lambda}^{-1}(\bigcup_{m=1}^{2k+1} \tilde{R}^m(p))$ and equivariant under the action of $\langle \gamma \rangle$. By applying the isotopy constructed as above at each component of $\bigcup_{i=1}^{l} \pi^{-1}(R_i)$, we obtain a desired isotopy $F_t(x) = F(x,t) : \tilde{S} \times [0,1] \to \mathbf{H}^3 \cup \hat{\mathbf{C}}$ on the whole $\tilde{S}$. □

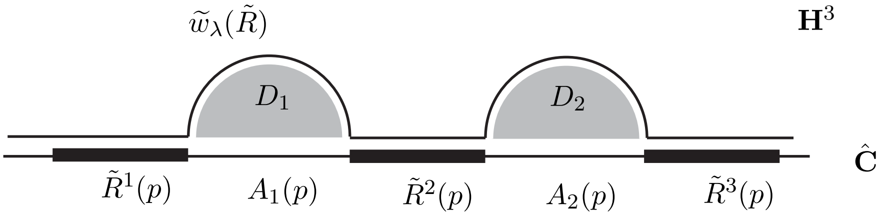

Figure 8: The image of $\tilde{R}$ by the wrapping map w_λ, where $k = 1$.

Here, we summarize our knowledge and make some observations. In the above proposition, we obtain $Y_\infty \in \partial \mathcal{Q}_0$ such that $hol(Y_\infty) = [\rho_\infty]$. Moreover, the holonomy image $\Gamma_\infty = \rho_\infty(\pi_1(S))$ is a geometrically finite b-group, whose invariant component $\Omega_0(\Gamma_\infty)$ is equal to the image $f_{Y_\infty}(\tilde{Y}_\infty)$ of the injective developing map $f_{Y_\infty} : \tilde{Y}_\infty \to \hat{\mathbf{C}}$. Recall that $\bigcup_{i=1}^{l} R_i$ is a regular neighborhood of $\bigcup_{i=1}^{l} C_i$ in Y_∞. Let $\tilde{R}$ be a component of $\pi_{Y_\infty}^{-1}(R_i) \subset \tilde{Y}_\infty$ for some $i \in \{1, \ldots, l\}$, and let c be an element of $\pi_1(Y_\infty)$ such that $\tilde{R}$ is $\langle c \rangle$-invariant. Then $\gamma = \rho_\infty(c) \in \Gamma_\infty$ is a parabolic element with fixed point $\{p\}$. From Lemma 3.3 and Proposition 3.4, we have

$$f_{Y_\infty}(\tilde{R}) \cap \Lambda(\hat{\Gamma}) = \bigcup_{m=1}^{2k_i} \Xi_m(p).$$

Moreover, since $f_{Y_\infty}(\tilde{Y}_\infty - \bigcup_{i=1}^{l} \pi^{-1}(R_i))$ is contained in $\Omega^0(\hat{\Gamma}) \subset \Omega(\hat{\Gamma})$, we have

$$\Omega_0(\Gamma_\infty) \cap \Lambda(\hat{\Gamma}) = f_{Y_\infty}(\tilde{Y}_\infty) \cap \Lambda(\hat{\Gamma}) = \bigcup_{\tilde{R}} f_{Y_\infty}(\tilde{R}) \cap \Lambda(\hat{\Gamma}),$$

where $\tilde{R}$ varies over all components of $\bigcup_{i=1}^{l} \pi_{Y_\infty}^{-1}(R_i) \subset \tilde{Y}_\infty$.

With the above preparations, we can now obtain the sufficient information on the shape of $\hat{\Lambda}_{Y_\infty} = \pi_{Y_\infty} \circ f_{Y_\infty}^{-1}(\Lambda(\hat{\Gamma}))$ in Y_∞.

Proposition 3.5 ([It, Lemma 4.1]). *The set $\hat{\Lambda}_{Y_\infty} \subset Y_\infty$ is a decoration of a realization $\widehat{2\lambda}$ of $2\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)$.*

Proof. From the above observations, to investigate the shape of

$$\hat{\Lambda}_{Y_\infty} = \pi_{Y_\infty} \circ f_{Y_\infty}^{-1}(\Lambda(\hat{\Gamma})) \subset Y_\infty,$$

we only have to consider the shape of

$$\pi_{Y_\infty} \circ f_{Y_\infty}^{-1}(f_{Y_\infty}(\tilde{R}) \cap \Lambda(\hat{\Gamma})) = \pi_{Y_\infty} \circ f_{Y_\infty}^{-1}\left(\bigcup_{m=1}^{2k_i} \Xi_m(p)\right) \subset Y_\infty,$$

where $\tilde{R}$ is a component of $\pi_{Y_\infty}^{-1}(R_i) \subset \tilde{Y}_\infty$ for some i. From Lemma 3.3, we know that, for each m, $\Xi_m(p) \cup \{p\}$ is a decoration of $\Lambda(\Theta_m(p))$ with thickened graph $A_m(p) \cup \{p\}$. Therefore, the set

$$\mathcal{X}_i = R_i \cap \hat{\Lambda}_{Y_\infty} = \bigcup_{m=1}^{2k_i} \pi_{Y_\infty} \circ f_{Y_\infty}^{-1}(\Xi_m(p))$$

is a decoration of the graph

$$\mathcal{G}_i = \bigcup_{m=1}^{2k_i} \pi_{Y_\infty} \circ f_{Y_\infty}^{-1}(\Lambda(\Theta_m(p)) - \{p\})$$

with a thickened graph

$$\mathcal{N}(\mathcal{X}_i, \mathcal{G}_i) = \bigcup_{m=1}^{2k_i} \pi_{Y_\infty} \circ f_{Y_\infty}^{-1}(A_m(p)),$$

where $\pi_{Y_\infty} \circ f_{Y_\infty}^{-1}(A_m(p))$ is an annulus whose core curve $\pi_{Y_\infty} \circ f_{Y_\infty}^{-1}(\Lambda(\Theta_m(p)) - \{p\})$ is a simple closed curve homotopic to C_i for each m. Therefore, we can see that the set $\hat{\Lambda}_{Y_\infty} = \bigcup_i \mathcal{X}_i \subset Y_\infty$ is a decoration of a realization $\widehat{2\lambda} = \bigcup_i \mathcal{G}_i$ of 2λ with the thickened graph $\bigcup_i \mathcal{N}(\mathcal{X}_i, \mathcal{G}_i)$. □

For the later use, we prepare the following notations: $\mathcal{X}_{Y_\infty} = \bigcup_i \mathcal{X}_i (= \hat{\Lambda}_{Y_\infty})$, $\mathcal{G}_{Y_\infty} = \bigcup_i \mathcal{G}_i (= \widehat{2\lambda})$ and $\mathcal{N}_{Y_\infty} = \bigcup_i \mathcal{N}(\mathcal{X}_i, \mathcal{G}_i)$.

Recall that the sequence $\{[\rho_n]\}$ converges algebraically to $[\rho_\infty]$ and that $hol(Y_\infty) = [\rho_\infty]$. Since the holonomy map is a local homeomorphism, we can take a neighborhood U of Y_∞ such that $hol|U$ is a homeomorphism. There exists a positive integer N such that $[\rho_n] \in hol(U)$ for all $|n| > N$. Now let

$$Y_n = (hol|U)^{-1}([\rho_n])$$

for $|n| > N$. Then the set $\{Y_n\}_{|n|>N}$ decomposes into two sequences, $\{Y_n\}_{n>N}$ and $\{Y_n\}_{n<-N}$, both of which converge to Y_∞ as $|n| \to \infty$.

Since $\hat{\Lambda}_{Y_\infty} \subset Y_\infty$ is a decoration of a realization of 2λ from Proposition 3.5, and since $\Lambda_{Y_n} \to \hat{\Lambda}_{Y_\infty}$ $(|n| \to \infty)$ from Lemma 2.11, we have $\Lambda_{Y_n} \neq \emptyset$ for all $|n| \gg 0$, which implies that $\{Y_n\}_{|n|>N}$ are exotic projective structures for sufficiently large N (see [Mc]). Moreover, one can see that $\Lambda_{Y_n} \subset Y_n$ are realizations of 2λ for all $|n| \gg 0$. Then, from Theorem 2.10, we can see that $\{Y_n\}_{|n|>N} \subset \mathcal{Q}_\lambda$ for sufficiently large N. Therefore, we obtain the following theorem, which is the one of the main theorems in [It].

Theorem 3.6 ([It, Theorem A]). *For any $\lambda \in \mathcal{ML}_{\mathbf{Z}}(S) - \{0\}$, we have $\overline{\mathcal{Q}_0} \cap \overline{\mathcal{Q}_\lambda} \neq \emptyset$.*

3.6 Sequences $\{Z_n\}_{|n|>N}$ and their limit $Z_\infty = \mathrm{Gr}_\mu(Y_\infty)$

Let μ be an element of $\mathcal{ML}_{\mathbf{Z}}(S) - \{0\}$ such that the supports of μ and λ have no parallel component in common. Then, since μ is admissible on Y_∞, we can obtain the grafting $Z_\infty = \mathrm{Gr}_\mu(Y_\infty)$ of Y_∞ along μ.

Remark. On the other hand, μ is not admissible on Y_n for any $|n| > N$, if $i(\lambda, \mu) \neq 0$.

As explained in 2.7, we have a homeomorphism $\Phi = (hol|V)^{-1} \circ (hol|U) : U \to V$ from a neighborhood U of Y_∞ onto a neighborhood V of Z_∞. Now let $Z_n = \Phi(Y_n)$ for $|n| > N$. Then, both sequences, $\{Z_n\}_{n>N}$ and $\{Z_n\}_{n<-N}$, converge to $Z_\infty = \mathrm{Gr}_\mu(Y_\infty)$ as $|n| \to \infty$. Note that $hol(Z_n) = hol(Y_n) = [\rho_n]$ is satisfied for any $|n| > N$.

In the following two subsection, we shall prove Theorem D, which state that $\{Z_n\}_{n>N} \subset \mathcal{Q}_{(\lambda,\mu)_\sharp}$ and $\{Z_n\}_{n<-N} \subset \mathcal{Q}_{(\lambda,\mu)_\flat}$ for sufficiently large N. This statement is equivalent to the following statement: $\Lambda_{Z_n} \subset Z_n$ are realizations of $2(\lambda, \mu)_\sharp$ for all $n > N$, and of $2(\lambda, \mu)_\flat$ for all $n < -N$. To show the latter statement, we first examine the shape of $\hat{\Lambda}_{Z_\infty} = \pi_{Z_\infty} \circ f_{Z_\infty}^{-1}(\Lambda(\hat{\Gamma}))$ in Z_∞, since both sequences, $\{\Lambda_{Z_n}\}_{n>N}$ and $\{\Lambda_{Z_n}\}_{n<-N}$, converge to $\hat{\Lambda}_{Z_\infty}$ as $|n| \to \infty$ by Lemma 2.11.

3.7 The shape of $\hat{\Lambda}_{Z_\infty}$

We first define a graph $\mathcal{G} = \mathcal{G}(\widehat{2\lambda}, \widehat{2\mu})$ in S for $\lambda, \mu \in \mathcal{ML}_{\mathbf{Z}}(S)$, which will turn out to be a skeleton of $\hat{\Lambda}_{Z_\infty}$ in Z_∞. Take realizations $\widehat{\lambda}$ and $\widehat{\mu}$ so that the number of intersection points $\sharp(\widehat{\lambda} \cap \widehat{\mu})$ is minimal. Let $\mathcal{N}(\widehat{\lambda})$ and $\mathcal{N}(\widehat{\mu})$ be regular neighborhoods of $\widehat{\lambda}$ and $\widehat{\mu}$, respectively. Then, the boundaries $\partial\mathcal{N}(\widehat{\lambda})$ and $\partial\mathcal{N}(\widehat{\mu})$ can be seen as realizations $\widehat{2\lambda}$, $\widehat{2\mu}$ of 2λ and 2μ, respectively. By taking $\mathcal{N}(\widehat{\lambda})$ and $\mathcal{N}(\widehat{\mu})$ properly, we may assume that the number of intersection points $\sharp(\widehat{2\lambda} \cap \widehat{2\mu})$ is also minimal. Then, for each intersecting point $x \in \widehat{\lambda} \cap \widehat{\mu}$, there is a square component $B(x)$ of $S - (\widehat{2\lambda} \cup \widehat{2\mu})$ containing x. Let $\psi : S \to S$ be a continuous map homotopic to the identity, such that $\psi(B(x)) = \{x\}$ for any $x \in \widehat{\lambda} \cap \widehat{\mu}$, and that the restriction of ψ to $S - \bigcup_{x \in \widehat{\lambda} \cap \widehat{\mu}} B(x)$ is a homeomorphism. We now define a graph $\mathcal{G}(\widehat{2\lambda}, \widehat{2\mu})$ in S by

$$\mathcal{G}(\widehat{2\lambda}, \widehat{2\mu}) = \psi(\widehat{2\lambda} \cup \widehat{2\mu})$$

(see Figure 9).

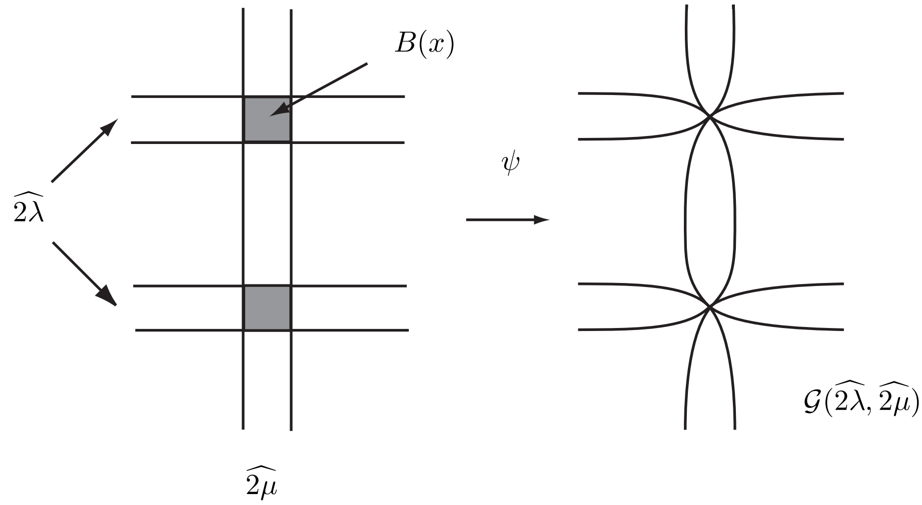

Figure 9: Construction of a graph $\mathcal{G}(\widehat{2\lambda}, \widehat{2\mu})$.

Throughout this subsection, we prove the following proposition.

Proposition 3.7. *The graph $\mathcal{G} = \mathcal{G}(\widehat{2\lambda}, \widehat{2\mu})$ is a skeleton of $\hat{\Lambda}_{Z_\infty}$ in Z_∞.*

Recall that $\lambda = \sum_{i=1}^{l} k_i C_i \in \mathcal{ML}_{\mathbf{Z}}(S)$. For simplicity, we only consider the case that $\mu \in \mathcal{ML}_{\mathbf{Z}}(S)$ is a simple closed curve $C' \in \mathcal{S}$ of weight 1.

(The general case can be proved by parallel argument.) Let $c' \in \pi_1(S)$ be a representative of the homotopy class C'. Recall that Z_∞ is obtained from Y_∞ by cutting Y_∞ along C' and inserting a cylinder

$$A' = \left(\hat{\mathbf{C}} - \overline{f_{Y_\infty}(\tilde{C'})}\right) / \langle \rho_{Y_\infty}(c') \rangle,$$

where $\tilde{C'}$ is the $\langle c' \rangle$-invariant lift of C' in $\tilde{Y}_\infty$. We also consider the quotient torus

$$T' = (\hat{\mathbf{C}} - \{p, q\}) / \langle \rho_{Y_\infty}(c') \rangle,$$

where $\{p, q\}$ are fixed points of loxodromic element $\rho_{Y_\infty}(c')$. We denote the covering map by

$$\pi' : \hat{\mathbf{C}} - \{p, q\} \to T'.$$

By abuse of the notation, we also denote the simple closed curve $\pi'(f_{Y_\infty}(\tilde{C'}))$ in T' by C'. Then, $A' = T' - C'$. Let

$$\iota : Y_\infty - C' \hookrightarrow Z_\infty$$

and

$$\iota' : A' \hookrightarrow Z_\infty$$

be natural inclusion maps. We denote this situation simply by

$$Z_\infty = \iota(Y_\infty - C') \sqcup \iota'(A').$$

We now decompose $\hat{\Lambda}_{Z_\infty} \subset Z_\infty$ into two parts: $\hat{\Lambda}_{Z_\infty} \cap \iota(Y_\infty - C')$ and $\hat{\Lambda}_{Z_\infty} \cap \iota'(A')$. Here, one can see that the set $\hat{\Lambda}_{Z_\infty} \cap \iota(Y_\infty - C')$ is equal to the image $\iota(\hat{\Lambda}_{Y_\infty} \setminus C')$ of $\hat{\Lambda}_{Y_\infty} \setminus C'$, where the shape of $\hat{\Lambda}_{Y_\infty}$ is already known in Proposition 3.5. Therefore, we only have to consider the set

$$\hat{\Lambda}_{Z_\infty} \cap \iota'(A') = \iota' \left(\pi'(\Lambda(\hat{\Gamma}) - \{p, q\}) \setminus C' \right)$$

in $\iota'(A')$, which is equivalent to consider how the sets $\pi'(\Lambda(\hat{\Gamma}) - \{p, q\})$ and C' are sitting in T'. From now on, we denote $\pi'(\Lambda(\hat{\Gamma}) - \{p, q\})$ simply by $\pi'(\Lambda(\hat{\Gamma}))$, if there is no confusion.

We first consider the case of $i(\lambda, C') = 0$. Then, $\rho_{Y_\infty}(c') \in \Gamma_\infty$ is contained in some component subgroup of Γ_∞. In this case, one can easily see that $\pi'(\Lambda(\hat{\Gamma})) \subset T'$ is a decoration of a realization of $2C'$ and $\pi'(\Lambda(\hat{\Gamma})) \cap C' = \emptyset$. This implies that $\hat{\Lambda}_{Z_\infty}$ is a realization of $2(\lambda + \mu)$, which is equal to $\mathcal{G}(\widehat{2\lambda}, \widehat{2\mu})$. Moreover, since Λ_{Z_n} converge to $\hat{\Lambda}_{Z_\infty}$ as $|n| \to \infty$ in the sense of Hausdorff, we can see that Λ_{Z_n} is a realization of $2(\lambda + \mu)$ for any $|n| > N$, if N is sufficiently large. Then, we have $\{Z_n\}_{|n|>N} \subset \mathcal{Q}_{\lambda+\mu}$ from Theorem 2.10. Therefore, we obtain a proof of Theorem D in the case of $i(\lambda, \mu) = 0$.

In the following, we only consider the case of $i(\lambda, C') \neq 0$. Recall that we denote by $\pi_{Y_\infty} : \tilde{Y}_\infty \to Y_\infty$ the universal covering map, and identify $\pi_1(Y_\infty)$ with the covering transformation group. Let $\tilde{\mathcal{C}}$ be the set of connected components of $\pi_{Y_\infty}^{-1}(\bigcup_{i=1}^l C_i) \subset \tilde{Y}_\infty$ which intersect $\tilde{C}'$, where $\bigcup_{i=1}^l C_i$ is the support of $\lambda = \sum_{i=1}^l k_i C_i$. Then the cyclic group $\langle c' \rangle$, which fixes $\tilde{C}'$, naturally acts on the set $\tilde{\mathcal{C}}$. We number the set $\tilde{\mathcal{C}} = \{\tilde{C}_{i(n)}\}_{n \in \mathbf{Z}}$ so that $\tilde{C}_{i(n)}$ separates $\tilde{C}_{i(n-1)}$ from $\tilde{C}_{i(n+1)}$ in $\tilde{Y}_\infty$ for all $n \in \mathbf{Z}$. Moreover, we assume that $\pi_{Y_\infty}(\tilde{C}_{i(n)}) = C_{i(n)}$ for all $n \in \mathbf{Z}$. Let $\{\tilde{C}_{i(1)}, \ldots, \tilde{C}_{i(s)}\}$ be the set of succeeding representatives of the quotient set $\tilde{\mathcal{C}}/\langle c' \rangle$, where $s = i(\bigcup_i C_i, C')$. (Note that the map $i : \{1, \ldots, s\} \to \{1, \ldots, l\}$ defined by $j \mapsto i(j)$ is not necessarily onto nor injective.) Let $\tilde{R}_{i(j)} \subset \tilde{Y}_\infty$ be the lift of $R_{i(j)} \subset Y_\infty$ which contains $\tilde{C}_{i(j)}$ for each $0 \leq j \leq s$. Recall that $R_{i(j)}$ is a regular neighborhood $C_{i(j)}$ in Y_∞.

For each $j \in \{1, \ldots, s\}$, let $c_{i(j)} \in \pi_1(Y_\infty)$ be a generator of a cyclic subgroup $\langle c_{i(j)} \rangle \subset \pi_1(Y_\infty)$ which fixes $\tilde{C}_{i(j)} \subset \tilde{Y}_\infty$. Then $\rho_{Y_\infty}(c_{i(j)})$ is a parabolic element, whose fixed point is denoted by $p_{i(j)}$. Then $\alpha_{i(j)} := f_{Y_\infty}(\tilde{C}_{i(j)}) \cup \{p_{i(j)}\}$ is a simple closed curve which is $\langle \rho_{Y_\infty}(c_{i(j)}) \rangle$-invariant. Note that $\{\pi'(\alpha_{i(j)})\}_{j=1}^s$ is the set of mutually homotopic, mutually disjoint simple closed curves in T', which satisfies $i(\pi'(\alpha_{i(j)}), C') = 1$ for all j. Moreover, $\pi'(f_{Y_\infty}(\tilde{R}_{i(j)}) \cup \{p_{i(j)}\}) \subset T'$ $(0 \leq j \leq s)$ are mutually disjoint and each of which contains $\pi'(\alpha_{i(j)})$.

To understand $\pi'(\Lambda(\hat{\Gamma})) \subset T'$, we first consider the set $\pi'(\Lambda(\hat{\Gamma}) \cap \Omega_0(\Gamma_\infty))$ in T'. As observed in 3.5, for each $1 \leq j \leq s$, the components of $\Lambda(\hat{\Gamma}) - \{p_{i(j)}\}$ which is contained in $\Omega_0(\Gamma_\infty) = f_{Y_\infty}(\tilde{Y}_\infty)$ are contained in $f_{Y_\infty}(\tilde{R}_{i(j)})$. Hence, $\pi'(\Lambda(\hat{\Gamma}) \cap \Omega_0(\Gamma_\infty))$ in T' is contained in $\bigcup_{j=1}^s \pi'(f_{Y_\infty}(\tilde{R}_{i(j)}))$. Recall that $f_{Y_\infty}(\tilde{R}_{i(j)}) \cap \Lambda(\hat{\Gamma})$ can be written as $\bigcup_{m=1}^{2k_{i(j)}} \Xi_m(p_{i(j)})$. Hence, for each $1 \leq j \leq s$ and $1 \leq m \leq 2k_{i(j)}$, we can see that $\pi'(\Xi_m(p_{i(j)}) \cup \{p_{i(j)}\})$ is a decoration of a simple closed curve $\pi'(\Theta_m(p_{i(j)}))$ with thickened graph $\pi'(A_{i(j)} \cup \{p_{i(j)}\})$, where $\pi'(\Theta_m(p_{i(j)}))$ is homotopic to $\pi'(\alpha_{i(j)})$ relative $\pi'(\{p_{i(j)}\})$.

Secondly, we consider the set $\pi'(\Lambda(\hat{\Gamma}) \setminus \Omega_0(\Gamma_\infty))$ in T'. A point in $\pi'(\Lambda(\hat{\Gamma}) \setminus \Omega_0(\Gamma_\infty))$ is contained in $\pi'(\Lambda(\Gamma_\infty)) = \pi'(\partial\Omega_0(\Gamma_\infty))$ or separated by $\pi'(\Lambda(\Gamma_\infty))$ from $\pi'(\Omega_0(\Gamma_\infty))$. The proof of the following lemma is left for the reader.

Lemma 3.8. *There exists a quasi-fuchsian component subgroup $\Theta_j \subset \Gamma_\infty$ such that $p_{i(j)}$, $p_{i(j+1)} \in \Lambda(\Theta_j)$ for each $j \in \{1, \ldots, s\}$, where $s+1$ is regarded as* 1.

Moreover, since $\Lambda(\Theta_j) \subset \Lambda(\Gamma_\infty) = \partial\Omega_0(\Gamma_\infty)$, one can easily see that

$$\Theta_j = \Theta_{2k_{i(j)}+1}(p_{i(j)}) = \Theta_0(p_{i(j+1)})$$

for each $j \in \{1, \ldots, s\}$. We now obtain a string of beads $B = \bigcup_{j=1}^{s} \pi'(\Lambda(\Theta_j))$ in T', whose complement $T' - B$ consists s quasi-disks and one annular domain which contains C' as a core curve.

We now claim that the string of beads $B = \bigcup_{j=1}^{s} \pi'(\Lambda(\Theta_j))$ is a skeleton of the set $\pi'(\Lambda(\hat{\Gamma}) \setminus \Omega_0(\Gamma_\infty))$ in T'. To show this claim, we prepare some notations (see Figure 10). For a while, we simply denote $p = p_{i(j)}$ and $k = k_{i(j)}$. Let $U(p)$ be a topological open disk containing p such that $A_m(p) \cap U(p)$ consists of exactly two components for $-1 \leq m \leq 2k+2$. Moreover, we assume that both $B^-_{-2}(p) \setminus U(p)$ and $B^+_{2k+3}(p) \setminus U(p)$ are non-empty and connected, where $B^-_{m-}(p)$ and $B^+_m(p)$ are the connected components of $\hat{\mathbf{C}} - \overline{\tilde{R}^m(p)}$ such that $A_{m-1}(p) \subset B^-_{m-1}(p)$ and $A_m(p) \subset B^+_m(p)$, respectively. In addition, we also take a topological open disk $V(p)$ which satisfies the same property with $U(p)$ and $\overline{V(p)} \subset U(p)$. We fix the following notations for the later use:

$$\begin{aligned} \widehat{V}(p) &= B^-_{-1}(p) \cup V(p) \cup B^+_{2k+2}(p), \\ \widehat{\widehat{V}}(p) &= B^-_{-2}(p) \cup A_{-1}(p) \cup V(p) \cup A_{2k+2}(p) \cup B^+_{2k+3}(p). \end{aligned}$$

Note that $\widehat{V}(p)$ is topological disk containing p and that $\widehat{\widehat{V}}(p) \subset \widehat{V}(p)$. Now let us return to the above claim. Let us take a regular neighborhood A_j of the set $\Lambda(\Theta_j) - \{p_{i(j)}, p_{i(j+1)}\}$ which satisfies

$$A_j \cap V(p_{i(j)}) = A_{2k_{i(j)+1}}(p_{i(j)}) \cap V(p_{i(j)}),$$

and

$$A_j \cap V(p_{i(j+1)}) = A_0(p_{i(j+1)}) \cap V(p_{i(j+1)}).$$

Moreover, we assume that each of two connected components of $\Lambda(\hat{\Gamma}) - \{p_{i(j)}, p_{i(j+1)}\}$ which contains a component of $\Lambda(\Theta_j) - \{p_{i(j)}, p_{i(j+1)}\}$ is contained in A_j. Then, we can see that the set $\pi'(\Lambda(\hat{\Gamma}) \setminus \Omega_0(\Gamma_\infty))$ is a decoration of the graph $B = \bigcup_{j=1}^{s} \pi'(\Lambda(\Theta_j))$ with a thickened graph $\bigcup_{j=1}^{s} \pi'(A_j) \cup \bigcup_{j=1}^{s} \pi'(\widehat{V}(p_{i(j)}))$.

From the above observations, we can see that the set $\mathcal{X}_{T'} = \pi'(\Lambda(\hat{\Gamma}))$ is a decoration of the graph

$$\mathcal{G}_{T'} = \bigcup_{j=1}^{s} \bigcup_{m=1}^{2k_{i(j)}} \pi'\left(\Lambda(\Theta_m(p_{i(j)}))\right) \cup \bigcup_{j=1}^{s} \pi'(\Lambda(\Theta_j))$$

in T', whose associated thickened graph $\mathcal{N}_{T'} = \mathcal{N}(\mathcal{X}_{T'}, \mathcal{G}_{T'})$ can be taken as

$$\mathcal{N}_{T'} = \bigcup_{j=1}^{s} \bigcup_{m=1}^{2k_{i(j)}} \pi'\left(A_m(p_{i(j)})\right)\right) \cup \bigcup_{j=1}^{s} \pi'(A_j) \cup \bigcup_{j=1}^{s} \pi'\left(\widehat{V}(p_{i(j)})\right).$$

From $\mathcal{G}_{Y_\infty}, \mathcal{X}_{Y_\infty}$ and $\mathcal{N}_{Y_\infty}$ in 3.5, and $\mathcal{G}_{T'}, \mathcal{X}_{T'}$ and $\mathcal{N}_{T'}$ above, we obtain the graph

$$\mathcal{G}_{Z_\infty} = (\mathcal{G}_{Y_\infty} \setminus C') \sqcup (\mathcal{G}_{T'} \setminus C')$$

and the thickened graph

$$\mathcal{N}_{Z_\infty} = (\mathcal{N}_{Y_\infty} \setminus C') \sqcup (\mathcal{N}_{T'} \setminus C')$$

such that $\mathcal{G}_{Z_\infty}$ is a skeleton of $\mathcal{X}_{Z_\infty} = \hat{\Lambda}_{Z_\infty}$ with a thickened graph $\mathcal{N}_{Z_\infty}$.

Now we can easily see that the graph $\mathcal{G}_{Z_\infty}$ is equivalent to the graph $\mathcal{G} = \mathcal{G}(\widehat{2\lambda}, \widehat{2\mu})$, where the string of beads $B = \bigcup_{j=1}^{s} \pi'(\Lambda(\Theta_j)) \subset \mathcal{G}_{T'} \setminus C'$ corresponds to the subgraph $\psi(\widehat{2\mu})$ of the graph $\mathcal{G}(\widehat{2\lambda}, \widehat{2\mu}) = \psi(\widehat{2\lambda} \cup \widehat{2\mu})$. Now we have completed the proof of Proposition 3.7.

3.8 The shape of Λ_{Z_n}

We complete the proof of the following Theorem D in this subsection. Figure 10 should be helpful to understand the arguments throughout this subsection.

Theorem D. *There exists a positive integer N such that $\{Z_n\}_{n>N} \subset \mathcal{Q}_{(\lambda,\mu)_\sharp}$ and $\{Z_n\}_{n<-N} \subset \mathcal{Q}_{(\lambda,\mu)_\flat}$.*

Since $Z_n \to Z_\infty$ ($|n| \to \infty$), we have $\Lambda_{Z_n} \to \hat{\Lambda}_{Z_\infty}$ ($|n| \to \infty$) in the sense of Hausdorff by Lemma 2.11. We will show that there exist $N > 0$ and subsets $\mathcal{N}_\sharp$, $\mathcal{N}_\flat$ of $\mathcal{N}_{Z_\infty}$ which satisfy the following:

- $\Lambda_{Z_n} \subset \mathcal{N}_\sharp$ for all $n > N$ and $\Lambda_{Z_n} \subset \mathcal{N}_\flat$ for all $n < -N$,
- $\mathcal{N}_\sharp$ and $\mathcal{N}_\flat$ are some regular neighborhoods of realizations of $2(\lambda, \mu)_\sharp$ and $2(\lambda, \mu)_\flat$, respectively.

Now let $\mathcal{N}'_{Z_\infty}$ be a subset of $\mathcal{N}_{Z_\infty}$, which is obtained from $\mathcal{N}_{Z_\infty}$ by replacing $\pi'\left(\widehat{V}(p_{i(j)})\right)$ in the definition of $\mathcal{N}_{T'} \subset \mathcal{N}_{Z_\infty}$ with $\pi'\left(\widehat{\widehat{V}}(p_{i(j)})\right)$ for each $1 \le j \le s$. Let p stand for some $p_{i(j)}$ and put $k = k_{i(j)}$. we let D^+_m, D^-_m be the two components of $(U(p) - V(p)) \cap \tilde{R}^m(p)$ for each $-1 \le m \le 2k+3$, where the sign $\pm$ of $D^{\pm}_m$ is determined so that $\gamma(D^+_m) \subset U(p)$ and $\gamma(D^-_m) \not\subset U(p)$. Then, we have

$$\pi'\left(\bigcup_{m=-1}^{2k+3}(D^+_m \cup D^-_m)\right) = \pi'(U(p)) \setminus \mathcal{N}'_{Z_\infty}.$$

Lemma 3.9. *There exist $N > 0$ and two families of mutually disjoint arcs, $\{\alpha^+_j\}_{j=0}^{2k+2}$ and $\{\alpha^-_j\}_{j=0}^{2k+2}$ in $U(p)$, which satisfy the following conditions for each $j \in \{0, \dots, 2k+2\}$:*

- *$\alpha^+_j \subset V(p) \cup D^+_{j-1} \cup D^-_{j+1}$, and one of the end points of α^+_j is in D^+_{j-1} and the other is in D^-_{j+1}. Moreover, $\alpha^+_j \subset \Omega(\Gamma_n)$ for any $n > N$,*
- *$\alpha^-_j \subset V(p) \cup D^-_{j-1} \cup D^+_{j+1}$, and one of the end points of α^-_j is in D^-_{j-1} and the other is in D^+_{j+1}. Moreover, $\alpha^-_j \subset \Omega(\Gamma_n)$ for any $n < -N$.*

Proof. We only consider the case that n is positive and sufficiently large. Recall that $\{\Gamma_n\}$ converges geometrically to $\hat{\Gamma}$ and hence $\{\Lambda(\Gamma_n)\}$ converges to $\Lambda(\hat{\Gamma})$ in the Hausdorff topology on $\mathcal{C}(\hat{\mathbf{C}})$ by Theorem 2.9. Hence, $\{\Omega(\Gamma_n)\}$ converges to $\Omega(\hat{\Gamma})$ in the sense of Carathéodory (see [KT]). Since $\{p\}$ is a parabolic fixed point in $\Lambda(\hat{\Gamma})$, there is a rank-two parabolic subgroup $\langle\gamma, \delta\rangle$ in $\hat{\Gamma}$ which fixes $\{p\}$. Moreover, $\langle\gamma, \delta\rangle$ is conjugate to some $\langle\gamma_i, \delta_i\rangle$ in $\hat{\Gamma}$. Let $\chi_n : \hat{\Gamma} \to \Gamma_n$ be group homomorphisms as in 3.4 and let $\gamma_n = \chi_n(\gamma) \in \Gamma_n$. Then, the cyclic groups $\langle\gamma_n\rangle$ converge geometrically to the rank-two parabolic subgroup $\langle\gamma, \delta\rangle$ as $n \to +\infty$. More precisely, γ_n converge to γ and γ_n^n converge to δ as $n \to +\infty$.

Choose $m \in \{-1, \dots, 2k+1\}$. For this m, we take and fix a point $x_0 \in \tilde{C}^m(p) \setminus U(p)$. Note that $\gamma^k(x_0) \in \tilde{C}^m(p)$ and $\gamma^k\delta(x_0) \in \tilde{C}^{m+2}(p)$ for any $k \in \mathbf{Z}$. Let a^- and a^+ be the connected components of $\tilde{C}^m(p) - \{x_0\}$ such that a^+ containing $\{\gamma^k(x_0)\}_{k=1}^\infty$. Then $\delta(a^-)$ is contained in $\tilde{C}^{m+2}(p)$ and contains $\{\gamma^{-k}\delta(x_0)\}_{k=1}^\infty$.

For any $\epsilon > 0$, there exists $N_0 > 0$ such that $d(\gamma_n(x_0), \gamma(x_0)) < \epsilon$ for any $n > N_0$, where $d(\cdot,\cdot)$ denotes the spherical metric on $\hat{\mathbf{C}}$. Let b_n be an arc connecting x_0 and $\gamma_n(x_0)$ such that $b_n \subset \mathcal{N}_\epsilon(a^+) \cap \Omega(\Gamma_n)$, where $\mathcal{N}_\epsilon(a^+)$ denotes the ϵ-neighborhood of a^+. Let $a_n = \bigcup_{k=1}^n \gamma_n^k(b_n)$. Then, a_n be an arc in $\Omega(\Gamma_n)$ connecting x_0 and $\gamma_n^n(x_0)$.

We will show that, for any $\epsilon > 0$, there exists $N > 0$ such that $a_n \subset \mathcal{N}_\epsilon(a^+ \cup \delta(a^-))$ for any $n > N$. Let $B_\epsilon(p)$ be the ϵ-neighborhood of $\{p\}$. Note that $B_\epsilon(p) \subset \mathcal{N}_\epsilon(a^+ \cup \delta(a^-))$. Let $N_1(> N_0)$ be the smallest integer such that both $\gamma^{N_1}(x_0)$ and $\gamma^{-N_1}\delta(x_0)$ are contained in $B_{\epsilon/2}(p)$. Then there exists $N_2(> N_1)$ such that $\gamma_n^{N_1}(x_0)$, $\gamma_n^{-N_1}\delta(x_0) \in B_\epsilon(p)$ for all $n > N_2$.

We first show that there exists $N > 0$ such that the set $\{\gamma_n^k(x_0)\}_{k=1}^n$ is contained in $\mathcal{N}_\epsilon(a^+ \cup \delta(a^-))$ for any $n > N$. To this end, we consider the set of accumulation points of the set $\bigcup_{n>N}\{\gamma_n^k(x_0)\}_{k=1}^n$ in $\hat{\mathbf{C}} - B_{\epsilon/2}(p)$. Assume that a (sub)sequence $\{\gamma_n^{k_n}(x_0)\}$ $(0 \leq k_n \leq n)$ converges to a point $x \notin B_{\epsilon/2}(p)$ as $n \to +\infty$. Since there exists $N_3(> N_2)$ such that the fixed points of γ_n are contained in $B_{\epsilon/2}(p)$ for all $n > N_3$, $\{\gamma_n^{k_n}\}$ is not a divergence sequence in $\mathrm{PSL}_2(\mathbf{C})$. Therefore, by taking a subsequence (which is denoted by the same symbol), if necessary, $\{\gamma_n^{k_n}\}$ converges to $\gamma^r\delta^s \in \langle\gamma, \delta\rangle$. Here, we have $x = \gamma^r\delta^s(x_0)$ and hence $\gamma_n^{k_n-r+sn}(x_0) \to x_0$. Here, we remark that, since $\{\Omega(\Gamma_n)\}$ converges to $\hat{\Gamma}$ in the sense of Carathéodory, there exists $N_4(> N_3)$ and a neighborhood $U(x_0)$ of x_0 such that $U(x_0) \subset \Omega(\Gamma_n)$ for all $n > N_4$. Then, there exists $N_5(> N_4)$ such that $\gamma_n^{k_n-r+sn} \equiv \mathrm{id}$ for all $n > N_5$. Since $0 \leq k_n \leq n$ and $x \notin B_{\epsilon/2}(p)$, we have $x \in \{\gamma^k(x_0)\}_{k=1}^{N_1} \cup \{\gamma^{-k}\delta(x_0)\}_{k=1}^{N_1}$. Hence, there exists $N_6(> N_5)$ such that for any $n > N_6$, the set $\{\gamma_n^{-k}(x_0)\}_{k=1}^n$ is contained in the set

$$B_\epsilon(p) \cup \bigcup_{k=1}^{N_1} B_{\epsilon/2}(\gamma^k(x_0)) \cup \bigcup_{k=1}^{N_1} B_{\epsilon/2}(\gamma^{-k}\delta(x_0)),$$

which is contained in $\mathcal{N}_\epsilon(a^+ \cup \delta(a^-))$.

From the above observations, we can see that there exists $N(> N_6)$ such that the arc $a_n = \bigcup_{k=1}^n \gamma_n^k(b_n)$ is contained in $\mathcal{N}_\epsilon(a^+ \cup \delta(a^-))$ for all $n > N$. We now obtain a desired arc α_{m+1}^+ as a sub-arc of a_n with $n > N$. Moreover, by taking $m = -1$ and considering curves $a_{(2k+2)n} = \bigcup_{k=1}^{(2k+2)n} b_n$ in $\Omega(\Gamma_n)$, a similar argument reveals that one can cut off mutually disjoint desired arcs $\{\alpha_j^+\}_{j=0}^{2k+2}$ from $a_{(2k+2)n}$ if n is sufficiently large. □

We now define the subsets $\mathcal{N}_\sharp, \mathcal{N}_\flat \subset \mathcal{N}'_{Z_\infty}$ so that $\mathcal{N}_\sharp$ and $\mathcal{N}_\flat$ are equal to $\mathcal{N}'_{Z_\infty}$ in the complement $Z_\infty - \bigcup_{j=1}^s U(p_{i(j)})$ of $\bigcup_{j=1}^s U(p_{i(j)})$, and that

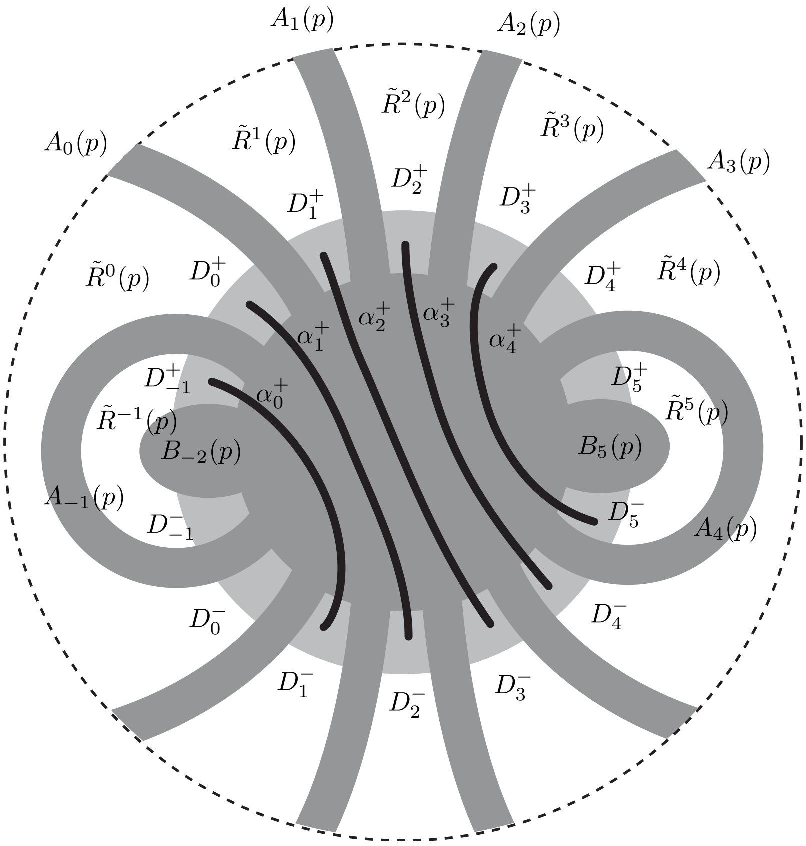

Figure 10: $\mathcal{N}'_\infty$ in a neighborhood of p and family of arcs $\{\alpha_j^+\}_{j=0}^{2k+2} \subset U(p)$. (The case of $k = 1$.)

the following conditions are satisfied for each $p = p_{i(j)}$ $(1 \leq j \leq s)$:

$$\mathcal{N}_\sharp \cap \pi'(U(p)) = \mathcal{N}'_{Z_\infty} \cap \pi' \left(U(p) - \bigcup_{0 \leq j \leq 2k+2} \alpha_j^+ \right),$$

$$\mathcal{N}_\flat \cap \pi'(U(p)) = \mathcal{N}'_{Z_\infty} \cap \pi' \left(U(p) - \bigcup_{0 \leq j \leq 2k+2} \alpha_j^- \right).$$

Observe that $\mathcal{N}_\sharp$ and $\mathcal{N}_\flat$ can be regarded as a regular neighborhood of a realization of $2(\lambda, \mu)_\sharp$ and $2(\lambda, \mu)_\flat$, respectively. Moreover, we have $\Lambda_{Z_n} \subset \mathcal{N}_\sharp$ for any $n > N$ and $\Lambda_{Z_n} \subset \mathcal{N}_\flat$ for any $n < -N$ by definition. Therefore, we now obtain the following lemma, and complete the proof of Theorem D.

Lemma 3.10. *There exists $N > 0$ such that*

- *Λ_{Z_n} is a decoration of a realization of $2(\lambda, \mu)_\sharp$ whose thickened graph is $\mathcal{N}_\sharp$ for any $n > N$,*
- *Λ_{Z_n} is a decoration of a realization of $2(\lambda, \mu)_\flat$ whose thickened graph is $\mathcal{N}_\flat$ for any $n < -N$.*

3.9 Proof of Theorem B

We will show that any two connected components of $Q(S) = \coprod_{\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)} \mathcal{Q}_\lambda = hol^{-1}(QF(S))$ have intersecting closures.

Theorem B. *For any two elements $\lambda, \mu \in \mathcal{ML}_{\mathbf{Z}}(S)$, we have $\overline{\mathcal{Q}_\lambda} \cap \overline{\mathcal{Q}_\mu} \neq \emptyset$.*

Proof. We combine Theorem 1.2 and Theorem D to prove Theorem B.

Take two elements λ and μ in $\mathcal{ML}_{\mathbf{Z}}(S)$, arbitrarily. If $i(\lambda, \mu) = 0$, than we have the conclusion from Theorem 1.2. Hence, we may assume that $i(\lambda, \mu) \neq 0$. Let us decompose μ into $\mu = \mu' + \mu''$ so that μ', $\mu'' \in \mathcal{ML}_{\mathbf{Z}}(S)$, and that $i(\lambda, \mu) = i(\lambda, \mu')$. We now divide the proof into two parts: the case of $\mu'' = 0$, and the case of $\mu'' \neq 0$.

We first consider the case of $\mu'' = 0$. As in 3.5, we can construct a sequence $\{Y_n\}$ in $\mathcal{Q}_{(\lambda,\mu)_\flat}$ conversing to $Y_\infty \in \overline{\mathcal{Q}_0} \cap \overline{\mathcal{Q}_{(\lambda,\mu)_\flat}}$ as $|n| \to \infty$. Since the supports of $(\lambda, \mu)_\flat$ and μ have no parallel component in common, we obtain a grafting $\mathrm{Gr}_\mu(Y_\infty)$ and a homeomorphism $\Phi : U \to V$ from a neighborhood U of Y_∞ onto a neighborhood V of $\mathrm{Gr}_\mu(Y_\infty)$. Now we have a sequence $\{Z_n = \Phi(Y_n)\}$ conversing to $\mathrm{Gr}_\mu(Y_\infty) \in \partial \mathcal{Q}_\mu$ as $|n| \to \infty$. Since $((\lambda, \mu)_\flat, \mu)_\sharp = \lambda$ by Lemma 2.7, we have $Z_n \in \mathcal{Q}_{((\lambda,\mu)_\flat,\mu)_\sharp} = \mathcal{Q}_\lambda$ for any $n \gg 0$ by Theorem D. Therefore, we have $\overline{\mathcal{Q}_\lambda} \cap \overline{\mathcal{Q}_\mu} \neq \emptyset$.

Secondly, we consider the case of $\mu'' \neq 0$. Since $i(\lambda, \mu'') = 0$ and $i(\mu', \mu'') = 0$, we have $i((\lambda, \mu')_\flat, \mu'') = 0$. Then, by Theorem 1.2, we have $\overline{\mathcal{Q}_0} \cap \overline{\mathcal{Q}_{(\lambda,\mu')_\flat}} \cap \overline{\mathcal{Q}_{\mu''}} \neq \emptyset$. More precisely, we can construct sequences $\{Y_n\} \subset \mathcal{Q}_{(\lambda,\mu')_\flat}$ and $\{Y'_n\} \subset \mathcal{Q}_{\mu''}$ both of which converge to an element $Y_\infty \in \partial\mathcal{Q}_0$. Since both the supports of $(\lambda, \mu')_\flat$ and μ'' have no parallel component in common with that of μ', we obtain a grafting $\mathrm{Gr}_{\mu'}(Y_\infty)$ and a homeomorphism $\Phi : U \to V$ from a neighborhood U of Y_∞ onto a neighborhood V of $\mathrm{Gr}_{\mu'}(Y_\infty)$. Now we have sequences, $\{Z_n = \Phi(Y_n)\}$ and $\{Z'_n = \Phi(Y'_n)\}$, both of which converge to $Z_\infty = \mathrm{Gr}_{\mu'}(Y_\infty) \in \partial\mathcal{Q}_{\mu'}$ as $|n| \to \infty$. From Theorem D, we can see that $Z_n \in \mathcal{Q}_{((\lambda,\mu')_\flat,\mu')_\sharp} = \mathcal{Q}_\lambda$ and $Z'_n \in \mathcal{Q}_{\mu'+\mu''} = \mathcal{Q}_\mu$ for any $n \gg 0$. Hence, we have $\overline{\mathcal{Q}_\lambda} \cap \overline{\mathcal{Q}_\mu} \neq \emptyset$. □

3.10 Proof of Theorem C

Theorem C. *For any $\lambda \in \mathcal{ML}_{\mathbf{Z}}(S) - \{0\}$, the holonomy map hol : $P(S) \to R(S)$ is not injective on $\overline{\mathcal{Q}_\lambda}$, although hol is invective on $\overline{\mathcal{Q}_0}$.*

Proof. It is well-known that the map *hol* is invective on $\overline{\mathcal{Q}_0}$. Let us take $\mu \in \mathcal{ML}_{\mathbf{Z}}(S) - \{0\}$ such that any component of the support of μ intersects with λ. Let us take a sequence $\{Y_n\}_{n\in\mathbf{Z}} \subset \mathcal{Q}_\mu$ which converges to $Y_\infty \in \partial\mathcal{Q}_0 \cap \partial\mathcal{Q}_\mu$ as $|n| \to \infty$ as before.

Since both the supports $(\mu, \lambda)_\flat$ and $(\mu, \lambda)_\sharp$ have no parallel components in common with that of μ, we can take $Z'_\infty = \mathrm{Gr}_{(\mu,\lambda)_\flat}(Y_\infty)$ and $Z''_\infty = \mathrm{Gr}_{(\mu,\lambda)_\sharp}(Y_\infty)$. Now we have the following equations from Lemma 2.7:

$$
\begin{aligned}
(\mu, (\mu, \lambda)_\flat)_\sharp &= ((\mu, \lambda)_\flat, \mu)_\flat = ((\lambda, \mu)_\sharp, \mu)_\flat = \lambda,\\
(\mu, (\mu, \lambda)_\flat)_\flat &= ((\mu, \lambda)_\flat, \mu)_\sharp = ((\lambda, \mu)_\sharp, \mu)_\sharp = (\lambda, 2\mu)_\sharp,\\
(\mu, (\mu, \lambda)_\sharp)_\flat &= ((\mu, \lambda)_\sharp, \mu)_\sharp = ((\lambda, \mu)_\flat, \mu)_\sharp = \lambda,\\
(\mu, (\mu, \lambda)_\sharp)_\sharp &= ((\mu, \lambda)_\sharp, \mu)_\flat = ((\lambda, \mu)_\flat, \mu)_\flat = (\lambda, 2\mu)_\flat.
\end{aligned}
$$

Let $\Phi' : U \to V'$ be a homeomorphism from a neighborhood U of Y_∞ onto a neighborhood V' of Z'_∞ such that $hol \circ \Phi' \equiv hol$. Then, $Z'_n = \Phi'(Y_n) \to Z'_\infty$ as $|n| \to \infty$. Moreover, there exists $N > 0$ such that $\{Z'_n\}_{n>N} \subset \mathcal{Q}_{(\mu,(\mu,\lambda)_\flat)_\sharp} = \mathcal{Q}_\lambda$ and $\{Z'_n\}_{n<-N} \subset \mathcal{Q}_{(\mu,(\mu,\lambda)_\flat)_\flat} = \mathcal{Q}_{(\lambda,2\mu)_\sharp}$ from Theorem D. Similarly, we have a homeomorphism $\Phi'' : U \to V''$ onto a neighborhood V'' of Z''_∞ such that $hol \circ \Phi'' \equiv hol$, and a sequence $Z''_n = \Phi''(Y_n) \to Z''_\infty$ as $|n| \to \infty$ such that $\{Z''_n\}_{n>N} \subset \mathcal{Q}_{(\lambda,2\mu)_\flat}$ and $\{Z''_n\}_{n<-N} \subset \mathcal{Q}_\lambda$.

Suppose that $Z'_\infty = Z''_\infty$. Then $\Phi' = \Phi''$ on U and $Z'_n = Z''_n$ for any $|n| \gg 0$. This implies $\lambda = (\lambda, 2\mu)_\flat$, which is a contradiction. Therefore, we have $Z'_\infty \neq Z''_\infty$. Now we have two sequences $\{Z'_n\}_{n>N}$, $\{Z''_n\}_{n<-N}$ in

$\mathcal{Q}_\lambda$ such that $\lim_{n\to+\infty} Z'_n = Z'_\infty \neq Z''_\infty = \lim_{n\to-\infty} Z''_n$. On the other hand, since $hol(Z'_n) = hol(Z''_n) = hol(Y_n)$ are satisfied for all $n \in \mathbf{Z}$, we have $\lim_{n\to+\infty} hol(Z'_n) = \lim_{n\to-\infty} hol(Z''_n) = hol(Y_\infty)$. Therefore, we have shown that hol is not injective on $\overline{\mathcal{Q}_\lambda}$. $\square$

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