QUASICONFORMAL IMAGES OF SPHERES

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1 Introduction

The goal of this survey is to describe some topics in quasiconformal analysis of current interest. We try to emphasize ideas and leave proofs and technicalities aside, as far as possible. Some easily stated open problems are given. Most of this material is adopted from [AVV4] and [Vu4].

2 Quasiconformal maps and spheres

2.1. Categories of homeomorphisms. Below we shall discuss homeomorphisms of a domain of $\mathbb{R}^n$ onto another domain in $\mathbb{R}^n$, $n \geq 2$. Conformal maps provide a well-known subclass of such homeomorphisms. By Riemann's mapping theorem this class is very flexible and rich for $n = 2$ whereas Liouville's theorem shows that, for $n \geq 3$, conformal maps are the same as Möbius transformations, i.e., their class is very narrow. Thus the unit ball $B^n = \{x \in \mathbb{R}^n : \ |x| < 1\}$ can be mapped conformally only onto a half-space or a ball if the dimension is $n \geq 3$. Quasiconformal maps constitute a convenient interpolating category of maps, much wider than conformal maps, and less general than locally Hölder-continuous homeomorphisms. We also note that bilipschitz maps are a subclass of quasiconformal maps. Deferring the definition of a quasisymmetric map to 2.37, we note that bilipschitz maps are a subclass of quasisymmetric maps, which in turn are a subclass of quasiconformal maps.

2.2. Modulus of a curve family. Now follows perhaps the most technical part of this paper, the definition of the modulus of a curve family. This notion will be used later mainly in the definition of quasiconformal mappings. Note that an alternative definition of quasiconformal mappings can be given in terms of the geometric notion of linear dilatation (see 2.16). Let G be a domain in $\mathbb{R}^n$ and let Γ be a curve family in G. For $p>1$ the *p-modulus* $M_p(\Gamma)$ is defined by

$$M_p(\Gamma)=\inf_{\rho\in F(\Gamma)}\int_G \rho^p dm\,, \tag{2.3}$$

where $F(\Gamma)=\{\rho: G\to Y,\ \rho \text{ Borel: } \int_\gamma \rho ds\ge 1 \text{ for all locally rectifiable } \gamma\in\Gamma\}$, $Y=\{x\in R: x\ge 0\}\cup\{\infty\}$. The most important case is $p=n$ and we set $M(\Gamma)=M_n(\Gamma)$—in this case we just call $M(\Gamma)$ the modulus of Γ. The *extremal length* of Γ is $M(\Gamma)^{1/(1-n)}$. The modulus is *a conformal invariant*, i.e. $M(\Gamma)=M(h\Gamma)$ if h is a conformal map and $h\Gamma=\{h\circ\gamma:\ \gamma\in\Gamma\}$. For the basic properties of the modulus we refer the reader to [V1], [Car], [Oh], [Vu2].

2.4. Canonical rings. In view of this definition it is perhaps not surprising that $M(\Gamma)$ can be explicitly expressed in terms of special functions only in a few special cases. We now consider three cases where Γ joins the boundary components of a ring domain. For a domain $G\subset\mathbb{R}^n$ and $E,F\subset G$ let

$$\Delta(E,F;G)=\{\text{all curves joining } E \text{ and } F \text{ in } G\}.$$

If $E=\overline{B}^n, F=S^{n-1}(t), t>1$, then

$$M(\Delta(E,F;\mathbb{R}^n))=\omega_{n-1}(\log t)^{1-n}$$

where $\omega_{n-1}=n\pi^{n/2}/\Gamma(1+\frac{n}{2})$ is the $(n-1)$-dimensional surface area of the unit sphere $S^{n-1}=\partial B^n$. The so called *Grötzsch ring domain* has complementary components $E=\overline{B}^n, F=[te_1,\infty), t>1$, and we put

$$\gamma_n(t)\equiv M(\Delta(E,F;\mathbb{R}^n))\,.$$

The bounded Grötzsch ring is obtained if one reflects the Grötzsch ring in ∂B^n. Another important ring domain is *Teichmüller's ring* with complementary components $E=[-e_1,0], F=[se_1,\infty), s>0$, and we set

$$\tau_n(s)\equiv M(\Delta(E,F;\mathbb{R}^n))\,.$$

The function $\tau_n:(0,\infty)\to(0,\infty)$ defines a decreasing homeorphism with $\tau_n(t)=2^{1-n}\gamma_n(\sqrt{1+t})$. The functions $\gamma_n(t),\tau_n(t)$ are sometimes called *the capacities* of the Grötzsch and Teichmüller ring domains, resp.

2.5. Hypergeometric functions. For $a,b,c\in\mathbb{R},\ c\ne 0,-1,-2,..$ the (Gaussian) *hypergeometric function* is defined by the series

$$F(a,b;c;r)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)n!}r^n$$

for $|r|<1$, where $(a,0)=1,\ (a,n+1)=(a,n)(a+n),\ n=0,1,2,\dots$. Its importance is, in part, connected with its numerous particular cases: there are lists in [PBM] with hundreds of special cases of $F(a,b;c;r)$ for rational triples (a,b,c) expressed in terms of elementary functions. For our purposes, the main particular case of the hypergeometric function is the *complete elliptic integral* $\mathcal{K}(r)$

$$\mathcal{K}(r) = \frac{\pi}{2} F(\frac{1}{2}, \frac{1}{2}; 1; r^2),\ 0 \le r < 1. \tag{2.6}$$

2.7. Conformal map onto a disk minus a radial slit. A conformal mapping of a concentric annulus onto a disk minus a radial segment starting from the origin is provided by an elliptic function. Thus we see that for the dimension $n = 2$ there is a conformal mapping transforming the bounded Grötzsch ring onto an annulus. The length of such a segment depends on the ratio of the radii in a nonelementary fashion. In fact, if the inner and outer radius of the annulus are $t \in (0, 1)$ and 1, then the length $r \in (0, 1)$ of the radial segment satisfies the following transcendental equation, obtained by equating the capacities of these two ring domains;

$$\frac{2\pi}{\log \frac{1}{t}} = \frac{2\pi}{\mu(r)};\ \mu(r) = \frac{\pi}{2} \frac{\mathcal{K}(r')}{\mathcal{K}(r)}, \tag{2.8}$$

where $r' = \sqrt{1 - r^2}$ and we set $\mu(1) = 0$. For $n = 2$ the Grötzsch capacity can be expressed as

$$\gamma_2(s) = 2\pi/\mu(1/s),\ \ s > 1. \tag{2.9}$$

The properties of $\gamma_2(s)$ are explored in [AVV4].

2.10. Modulus and relative size. We define the *relative size* of the pair E, F by

$$r(E, F) = \min\{d(E), d(F)\}/d(E, F)\,,$$

where $d(E) = \sup\{|x - y| : x, y \in E\}$ and

$$d(E, F) = \inf\{|x - y| : x \in E \,\text{and}\, y \in F\}.$$

If E and F are disjoint continua then $M(\Delta(E, F; \mathbb{R}^n))$ and $r(E, F)$ are simultaneously small or large. In fact, there are increasing homeomorphisms $h_j : [0, \infty) \to [0, \infty)$ with $h_j(0) = 0,\ j = 1, 2,$ such that

$$h_1(r(E, F)) \le M(\Delta(E, F; \mathbb{R}^n)) \le h_2(r(E, F)) \tag{2.11}$$

(see [V1], [Vu2]). The explicit expressions for h_j in [Vu2, 7.41-7.42] and [H] involve special functions.

2.12. Quasiconformal maps. Let $K \ge 1$. A homeomorphism $f : G \to G'$ is termed K-*quasiconformal* if for all curve families Γ in G

$$M(f\Gamma)/K \le M(\Gamma) \le KM(f\Gamma). \tag{2.13}$$

The least constant K in (2.13) is called *the maximal dilatation* of f.

Note that conformal invariance is embedded in this definition: for $K = 1$ equality holds throughout in (2.13). This definition resembles the bilipschitz condition, but we will see below that quasiconformal maps can transform distances in a highly nonlinear and totally unlipschitz manner.

2.14. Schwarz lemma for quasiconformal maps. The Schwarz lemma for analytic functions is one of the basic results of complex analysis. A counterpart of this result also holds for quasiconformal maps in the following form.

2.15. Theorem. *Let $f : B^n \to fB^n \subset B^n$ be K-quasiconformal and $f(0) = 0$. Then, for $x \in B^n$,*

$$(1)\ |f(x)| \le \varphi_{K,n}(|x|) \le \lambda_n^{1-\alpha}|x|^\alpha,\ \alpha = K^{1/(1-n)},$$

$$(2)\ |f(x)| \le \psi_{K,n}(|x|) \equiv \sqrt{1 - \varphi_{1/K,n}(\sqrt{1-|x|^2})^2},$$

where $\varphi_{K,n}(r) \equiv 1/\gamma_n^{-1}(K\gamma_n(1/r))$ and $\varphi_{K,2}(r) = \mu^{-1}(\mu(r)/K)$. If, moreover, $fB^n = B^n$ and $\beta = 1/\alpha$, then

$$(3)\ |f(x)| \ge \varphi_{1/K,n}(|x|) \ge \lambda_n^{1-\beta}|x|^\beta,$$

$$(4)\ |f(x)| \ge \psi_{1/K,n}(|x|).$$

Note that in Theorem 2.15 both (1) and (2) are asymptotically sharp when$K \to 1$. Here $\lambda_2 = 4$, $\lambda_n \in [4, 2e^{n-1})$ is a constant [AVV4] depending only on n. It can be shown that, in (1) and (2), $\varphi_{K,n}(r)$ and $\psi_{K,n}(r)$ are different for $n > 2$ and identically equal for $n = 2$. The Schwarz lemma also shows that quasiconformal maps are locally Hölder continuous.

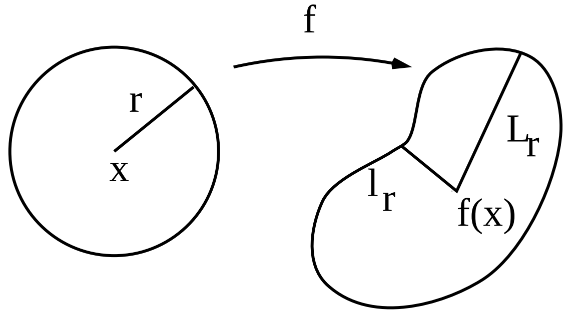
There are several equivalent ways of characterizing K-quasiconformal maps, which have the common feature that for $K = 1$ the class of conformal maps is obtained [Car]. When comparing two such definitions, it often happens that a mapping K_1-quasiconformal in the sense of one definition is K_2-quasiconformal in the sense of another definition, where K_2 depends from K_1 in an explicit way and, what is most important, $K_2 \to 1$ if $K_1 \to 1$. We shall next consider in 2.16 a definition equivalent to (2.13), based on the linear dilatation. However, finding a sharp bound for K_2 in terms of K_1 and the dimension is sometimes difficult. We shall see that in the case of this definition, finding such a constant K_2 explicitly has required a time span as long as the history of higher-dimensional quasiconformal maps.

2.16. Linear dilatation. For a homeomorphism $f : G \to G'$, $x_0 \in G$, $r \in (0, d(x_0, \partial G))$, let

$$H(x_0, f, r) = \sup\left[\frac{|f(x) - f(x_0)|}{|f(y) - f(x_0)|} :\ |x - x_0| = |y - x_0| = r\right],$$

$$H(x_0, f) = \limsup_{r\to 0} H(x_0, f, r).$$

Then $H(x_0, f)$ is called the *linear dilatation* of f at x_0.



There is an alternative characterization of quasiconformal maps, to the effect that a homeomorphism with bounded linear dilatation

$$\sup\{H(x,f) : x \in G\} \le L < \infty$$

is quasiconformal [V1]. We shall next review the known estimates for the constant L in terms of the maximal dilatation.

Consider first the case $n = 2$. A. Mori proved in [Mor2] that if $f : G \to G'$, with $G, G' \subset \mathbb{R}^2$, is K-quasiconformal, then for all $x_0 \in G$

$$H(x_0, f) \le e^{\pi K}. \tag{2.17}$$

This bound is not sharp when $K \to 1$. The sharp bound

$$H(x_0, f) \le \lambda(K) = \frac{u^2}{1-u^2}, u = \varphi_{K,2}(1/\sqrt{2}), \tag{2.18}$$

is due to Lehto, Virtanen, and Väisälä [LVV] in the particular case $G = \mathbb{R}^2$ and due to Shah Dao-Shing and Fan Le-Le [SF] in the general case of a proper subdomain $G \subset \mathbb{R}^2$. Here for $n \ge 2, K > 0, r \in (0,1)$

$$\varphi_{K,n}(r) = 1/\gamma_n^{-1}(K\gamma_n(1/r))$$

defines a homeomorphism $\varphi_{K,n} : [0,1] \to [0,1]$.

Next we consider the case $n \ge 2$. If $f : G \to G'$, with $G, G' \subset \mathbb{R}^n$, is K-quasiconformal then, by a 1962 result of F.W. Gehring [G1, Lemma 8, pp. 371-372],

$$H(x_0, f) \le d(n,K) \equiv \exp\left[\left(\frac{K\omega_{n-1}}{\tau_n(1)}\right)^{1/(n-1)}\right] \tag{2.19}$$

for all $x_0 \in G$, where τ_n is the capacity of the Teichmüller condenser (see 2.4). For $n = 2$, the earlier result of A. Mori (2.17) is recovered as a particular case of (2.19), that is, $d(2,K) = e^{\pi K}$. Unfortunately $d(n,K) \nrightarrow 1$ as $K \to 1$. In 1986 M. Vuorinen sharpened the bound (2.19) to

$$\begin{aligned} H(x_0, f) \le c(n,K) &\equiv 1 + \tau_n^{-1}(\tau_n(1)/K) \\ &< \frac{1}{10} d(n,K). \end{aligned} \tag{2.20}$$

Note that $c(n,K) \to 2$ as $K \to 1$ [Vu2, 10.22, 10.32]. In 1990 Vuorinen proved for a K-quasiconformal map $f : \mathbb{R}^n \to \mathbb{R}^n$ of the whole space $\mathbb{R}^n$ [Vu3]

$$H(0,f) \le \exp(6(K+1)^2\sqrt{K-1}) \equiv s(K) \tag{2.21}$$

with the desirable property $s(K) \to 1$ as $K \to 1$. In 1996 P. Seittenranta [Se2] proved a similar result for maps of proper subdomains G of $\mathbb{R}^n$: a K-quasiconformal mapping $f : G \to G'$ satisfies

$$H(x_0, f) \le s(K) \tag{2.22}$$

for all $x_0 \in G$ with the same $s(K)$ as in (2.21). Note that (2.22) would easily follow from (2.21) if we could solve a local structure problem stated below. In fact, slightly better bounds than (2.21) and (2.22), involving the special function τ_n are known.

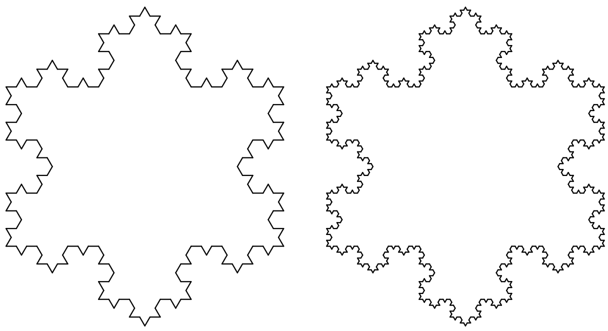
2.23. Open problem. Can the upper bound (2.22) be replaced by $s(n, K)$ with $\lim_{n\to\infty} s(n, K) = 1$ for each fixed $K > 1$?

2.24. Quasispheres and quasicircles. If $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$, is K-quasiconformal, then the set fS^{n-1} is called a K-*quasisphere* or, if $n = 2$, a K-*quasicircle*. Here, as usual, $S^{n-1} = \partial B^n$ and $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$.

Plane domains that are bounded by quasicircles, called quasidisks, have been studied extensively. See the surveys of Gehring [G5], [G7]. Compared to what is known for the dimension $n = 2$, very little is known in higher dimensions $n \geq 3$. We shall formulate below some open problems, both for the plane and the higher-dimensional case.

Part of the interest in quasispheres derives from the fact that these sets can have interesting geometric structure of fractal type. In fact, some of the differences between the categories of bilipschitz and quasiconformal maps can be understood if one studies the geometric structure of the images of spheres under these maps.

2.25. Examples of quasicircles. (1) Perhaps the most widely known example of a nonrectifiable quasicircle is the *snowflake curve* (also called *von Koch curve*), which is constructed in the following way. Take an equilateral triangle. To each side adjoin an equilateral triangle whose base agrees with the middle-third segment of the side; then remove this middle-third segment. Iterating this procedure recursively ad infinitum we get as a result a nonrectifiable Jordan curve of Hausdorff dimension > 1. Other similar examples are given in [GV2], [G5, p. 25], and [LV, p. 110].



(2) The Julia set J_f of an iteration $z \mapsto f(z)$ is the set of all those points that remain bounded under the repeated iterations. As a rule, J_f has an interesting fractal type structure, and for suitable f, J_f is a quasicircle. For the case of quadratic f, see [GM] and for rational f see [St].

(3) Images of circles under bilipschitz maps are always rectifiable (and hence of Hausdorff dimension 1) but they may fail to have tangents at some points. In fact, bilipschitz maps are differentiable only almost everywhere and if this "bad set" of zero measure is nonempty peculiar things may happen. See [VVW] for a construction of a bilipschitz circle which is $(q, 2)$- thick in the sense of definition **??** below.

(4) There are examples of Jordan domains with rectifiable boundaries which are not bounded by quasispheres. For instance, the "rooms and corridors"-type domains violating the Ahlfors condition in (2.30) can be used.

(5) We next give a construction of a bilipschitz map $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(0) = 0$ which carries rays passing through 0 to "logarithmic spirals" through 0. We first fix

an integer $p \geq 5$ and note that there exists $L \geq 1$ and an $L-$bilipschitz mapping of the annulus $\overline{B}^2(p) \setminus B^2$ which is identity on $S^1(p)$ and a restriction of the rotation $z \mapsto e^{i\theta} z, \theta \in (0, \pi/(2p))$, on $S^1(1)$. The boundary values of this map guarantee that this mapping can be extended to an $L-$bilipschitz map of the whole plane, which in the annuli $B(p^{k+1}) \setminus \overline{B}(p^k)\,, k \in Z$, agrees with our original map up to conjugations by suitable rotations and dilations. For a similar construction, see Luukkainen and Väisälä [LuV, 3.10 (4), 4.11].

(6) The univalent function

$$f(z) = \int_0^z \exp\{ib \sum_{k=0}^{\infty} \zeta^{2^k}\} d\zeta, \quad b < \frac{1}{4}\,,$$

defined in the unit disk B^2, provides an analytic representation of a quasicircle $\Gamma = f(\partial D)$ that fails to have a tangent at each of its points. For details see Ch. Pommerenke [Po, pp.304-305].

2.26. Particular classes of domains. The unit ball in $\mathbb{R}^n$ is the standard domain for most applications in quasiconformal analysis. Since the early 1960's several classes of domains have been introduced in studies on quasiconformal maps. It is not our goal to review such studies, but we note that at least the following two types of domain classes have been studied:

(1) domains satisfying a geometric condition;

(2) domains characterized by conditions involving moduli of curve families, capacities, or other analytic conditions.

Domains of type (1) include so-called uniform domains and their various generalizations. Domains of type (2) include, e.g., so-called QED-domains. A domain $G \subset \mathbb{R}^n$ is called $c-$QED, $c \in (0, 1]$ if, for each pair of disjoint continua $F_1, F_2 \subset G$, it is true that $M(\Delta(F_1, F_2; G)) \geq cM(\Delta(F_1, F_2; \mathbb{R}^n))$. There is a useful survey of some of these classes by J. Väisälä [V6].

Let us look at a property of the unit ball. For nondegerate continua $E, F \subset B^n$ we have

$$M(\Delta(E, F; \mathbb{R}^n)) \geq M(\Delta(E, F; B^n)) \geq$$

$$M(\Delta(E, F; \mathbb{R}^n))/2 \geq \frac{1}{2} h_1(r(E, F))$$

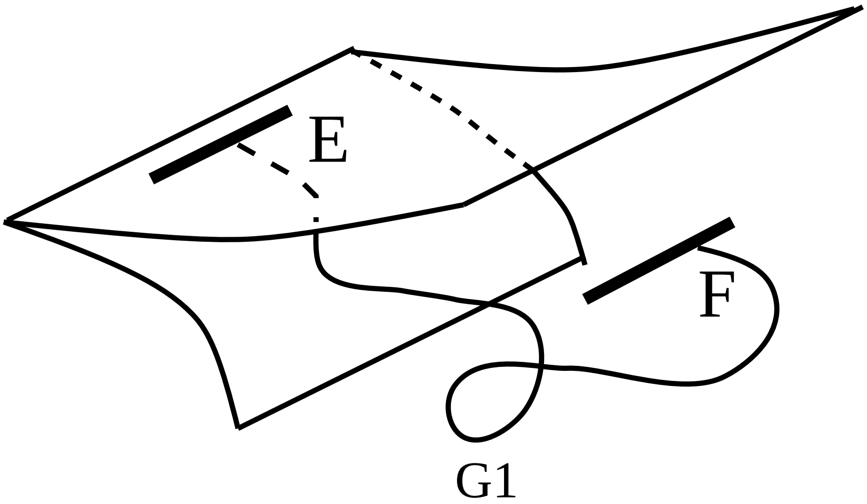
by [G4] and (2.11). (In particular, the unit ball is 1/2-QED.) For a domain $D \subset \mathbb{R}^n$ and $r_0 > 0$ we set

(2.27) $$L(D, r_0) = \inf_{r(E,F) \geq r_0} M(\Delta(E, F; D))\,,$$

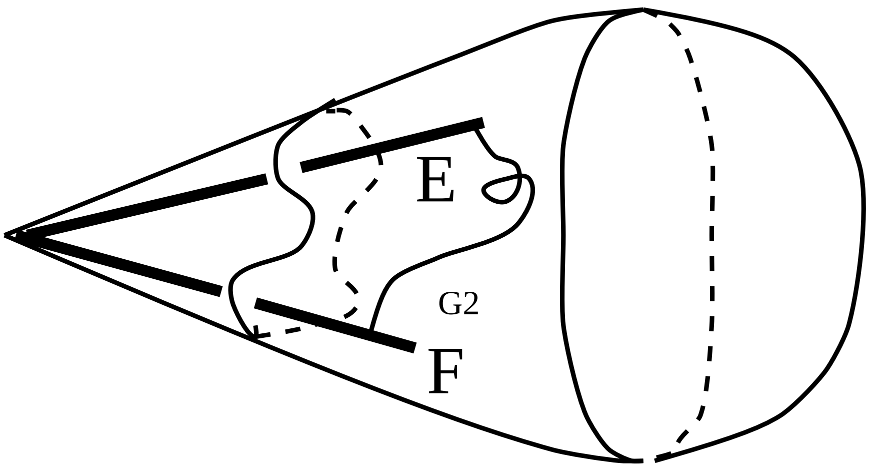
where E and F are continua. For all dimensions $n \geq 2$ it is easy to construct "rooms and corridors" type Jordan domains with $L(D, r_0) = 0$ (only simplest estimates of moduli are needed from [V1, pp. 20-24]). For dimensions $n \geq 3$ one can construct such domains also in the form

$$D_g = \{(x, y, z) \in \mathbb{R}^3 : \ x > 0, \ |y| < g(x)\}$$

for a suitable homeomorphism $g : [0, \infty) \to [0, \infty), \ g(0) = 0, \ g'(0) = 0$; now the access to the "ridge" $A \equiv \{(0, y, 0) : \ y \in \mathbb{R}\}$ of the domain gets narrower and narrower as we approach A from within D_g.



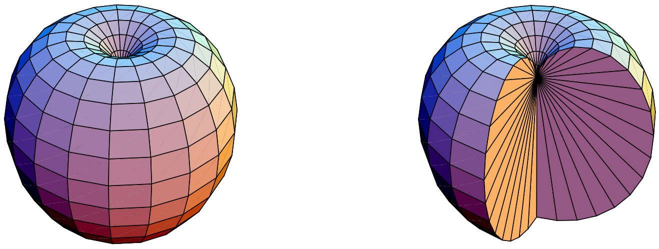
It is not difficult to show with the help of (2.11) that the class of domains with $L(D, r_0) > 0$ is invariant under quasiconformal maps of $\mathbb{R}^n$. Hence we see that boundaries of domains with $L(D, r_0) = 0$ cannot be quasispheres.



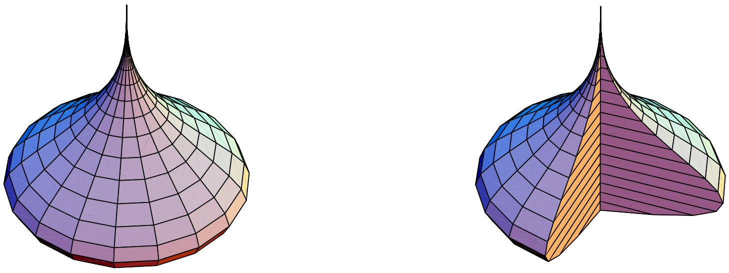
One can also construct domains $D \subset \mathbb{R}^n$ such that for a pair of disjoint continua $E, F \subset D$ with $r(E, F) = \infty$ we have $M(\Delta(E, F; D)) < \infty$.

2.28. Quasiconformal images of B^3. By Liouville's theorem, the unit ball B^n, $n \geq 3$, can be mapped conformally only onto another ball or a half-space. Gehring and Väisälä [GV1] created an extensive theory which gives necessary (and, in certain cases, sufficient) conditions for a domain to be of the form fB^n where $f : B^n \to \mathbb{R}^n$ is quasiconformal. They also exhibited several interesting domains illuminating their results which we shall now discuss.

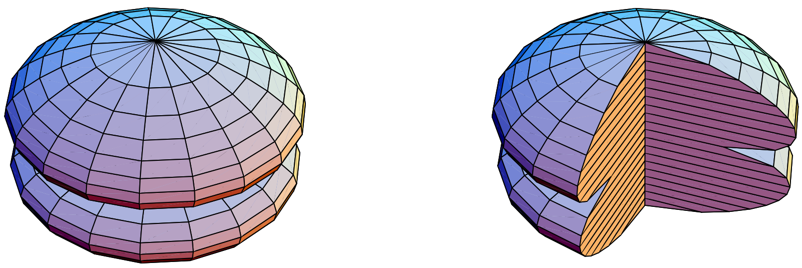
(1) The first example is an apple-shaped domain (cf. picture). By [GV1] such a domain cannot, in general, be mapped quasiconformally onto B^3.



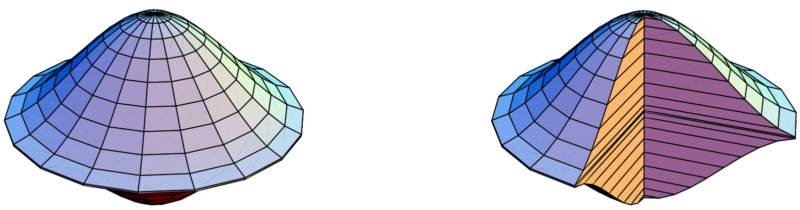
(2) On the other hand, there are onion-shaped domains that can be so mapped.



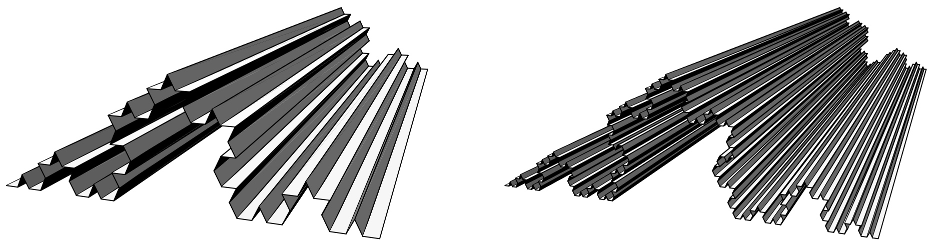
(3) In examples (1) and (2) above, the critical behavior takes place near one boundary point at the tip of a spire. In the case of an onion-shaped domain the spire is outwards-directed and for apple-shaped domains it is inwards-directed. In this and the following example the critical set consists of the edge of a boundary "ridge". An example of a domain with inward-directed ridge is shown ("yoyo-domain") in the picture below. The shape of the yoyo can be so chosen that the domain is a quasiconformal image of B^3.



(4) Consider now a "ufo-shaped" domain where the ridge is outward-directed (cf. the picture below). In this case the shape can be so chosen that the domain is not quasiconformally equivalent to B^3.



(5) P. Tukia [Tu2] used an example of S. Rickman to construct a domain whose boundary is the Cartesian product $K \times \mathbb{R}$ where K is a snowflake-style curve with a periodic structure. The domain underneath the surface fails to be quasiconformally equivalent to B^3.



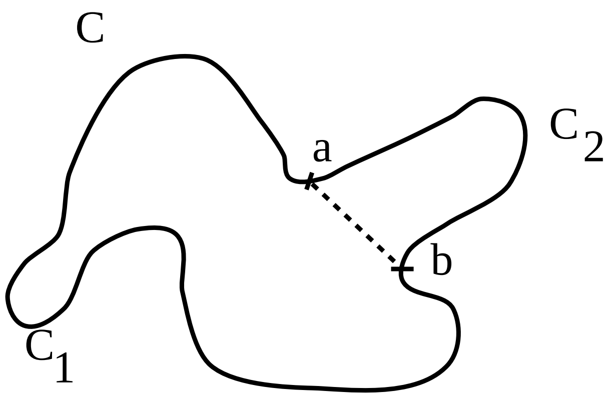
(6) Note that for dimensions $n \geq 3$ it is possible that a Jordan domain can be quasiconformally mapped onto B^n but that its complement fails to have this property.

2.29. Ahlfors' condition for quasicircles. Quasicircles have been studied extensively and many characterizations for them given by many authors. For interesting surveys, see [G5], [G7]. Chronologically, one of the first characterizations was given by L. V. Ahlfors in [Ah1] and this result still continues to be the most popular one and it reads as follows: A Jordan curve $C \subset \overline{\mathbb{R}}^2$ is a quasicircle if and only if there exists a constant $m \geq 1$ such that for all finite points $a, b \in C$

$$\min\{d(C_1), d(C_2)\} \leq m|a - b|\,, \tag{2.30}$$

where C_1 and C_2 are the components of $C \setminus \{a, b\}$ and where d stands for the Euclidean diameter.

Note that this formulation shows that (2.30) guarantees the existence of a K-quasiconformal mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $C = fS^1$. However, the least upper bound for K in terms of m, is not known.



2.31. Open problem. Generalize Ahlfors' condition to quasispheres.

2.32. Bilipschitz circles and spheres. In harmony with our hierarchy of the categories of maps in 1.1, it is natural to ask if a criterion similar to (2.30) exists also for bilipschitz circles or surfaces. The case $n = 2$ was settled by P. Tukia [Tu1] in 1980 and also by D. Jerison- C. Kenig [JK] in 1982. The case $n \geq 3$ is open. Some results of this type were obtained by S. Semmes [S1], [S2] and T. Toro [To1], [To2].

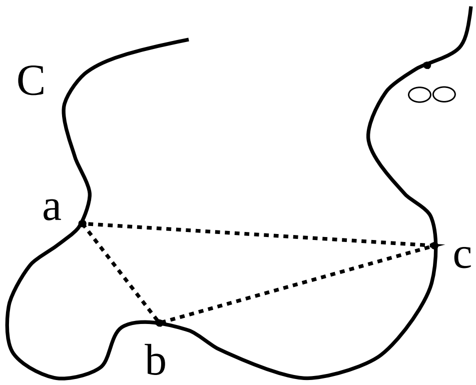
2.33. Open problem. Find the least K for which a quadrilateral with given dimensions is a K-quasicircle. A particular case is the rectangle. R. Kühnau

[Küh2, p. 104] has proved that a triangle with the least angle $\alpha\pi(<\pi/3)$ is a K-quasicircle with $K^2 \geq (1+d)/(1-d), d = |1-\alpha|$, with equality for the equilateral triangle ($\alpha = 1/3$). (In fact, equality holds for all $\alpha \in (0, 1/3)$ by S. Werner [We].)

2.34. Open problem - triangle condition. We say that a Jordan curve $C \subset \overline{\mathbb{R}}^2$ with $\infty \in C$ satisfies a triangle condition if there exists a constant $M \geq 1$ such that for all successive finite points $a, b, c \in C$ we have

$$|a-b| + |b-c| \leq M|a-c| \tag{2.35}$$

Show that there exists a constant $K \geq 1$ such that $C = f\mathbb{R}$ where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is K-quasiconformal. Give $K = K(M)$ explicitly in terms of M with $K(M) \to 1$ as $M \to 1$.



2.36. Remarks. (1) From a result of S. Agard - F.W. Gehring [AG] it follows that $K(M) \geq 1 + 0.25(M-1)$ for $M \in (1, 2)$.

(2) D. Trotsenko has informed the author (1996) about an idea to settle the open problem 2.34 with $K(M) \leq 1 + c_1\sqrt{M-1},\ c_1 = 10^5$, for $M < 1 + 10^{-13}$. See also [Tr].

2.37. Quasisymmetric maps. Let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism with $\eta(0) = 0$ and let $f : G \to G'$ be a homeomorphism, where $G, G' \subset \mathbb{R}^n$. We say [TV1] that f is *η-quasisymmetric* if, for all $a, b, c \in G$ with $a \neq c$,

$$\frac{|f(a) - f(b)|}{|f(a) - f(c)|} \leq \eta\left(\frac{|a-b|}{|a-c|}\right) \tag{2.38}$$

2.39. Beurling - Ahlfors extension result. A. Beurling and L. Ahlfors [BAh] introduced the class of homeomorphisms $h : \mathbb{R} \to \mathbb{R}$ satisfying

$$\frac{1}{M} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq M \tag{2.40}$$

for all $x \in \mathbb{R}, t > 0$, and for some $M > 1$. Such homeomorphisms were later termed quasisymmetric. Note that, for maps of the real axis, condition (2.40) agrees with (2.38) under the additional constraint $|a-b| = |a-c|$. Beurling and Ahlfors also proved that a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ of the real axis can be extended to a K-quasiconformal map $f^* : \mathbb{R}^2 \to \mathbb{R}^2$ iff f satisfies (2.40). We remark that again there is a problem of finding the optimal constant K if $M > 1$ is given. It is known by [L, p. 34] that one can choose $K \leq \min\{M^{3/2}, 2M-1\}$.

2.41. Quasisymmetry - quasiconformality. If $f : G \to G'$ satisfies (2.38) it follows easily that $H(x_0, f) \le \eta(1)$ for all $x_0 \in G$. By the alternative characterization of quasiconformality in terms of the linear dilatation 2.16, we thus see that quasisymmetric maps constitute a subclass of quasiconformal maps. As a rule, these two classes of maps are different. However, if $G = \mathbb{R}^n$ then quasiconformal maps are η-quasisymmetric, by a result of P. Tukia and J. Väisälä [TV1]. Much more delicate is the question of finding for a given $K > 1$ an explicit η_K which is "asymptotically sharp" when $K \to 1$. In [Vu3] it was shown, for the first time, that an explicit $\eta_{K,n}(t)$ exists which tends to t as $K \to 1$: If $f : \mathbb{R}^n \to \mathbb{R}^n, n \ge 2$, is K-quasiconformal, then f is $\eta_{K,n}$-quasisymmetric with

$$\begin{cases} \eta_{K,n}(1) \le \exp(6(K+1)^2\sqrt{K-1}), & \\ \eta_{K,n}(t) \le \eta_{K,n}(1)\varphi_{K,n}(t), & 0 < t < 1, \\ \eta_{K,n}(t) \le \eta_{K,n}(1)/\varphi_{1/K,n}(1/t), & t > 1. \end{cases} \tag{2.42}$$

Here $\varphi_{K,n}(t)$ is the distortion function in the quasiconformal Schwarz lemma with

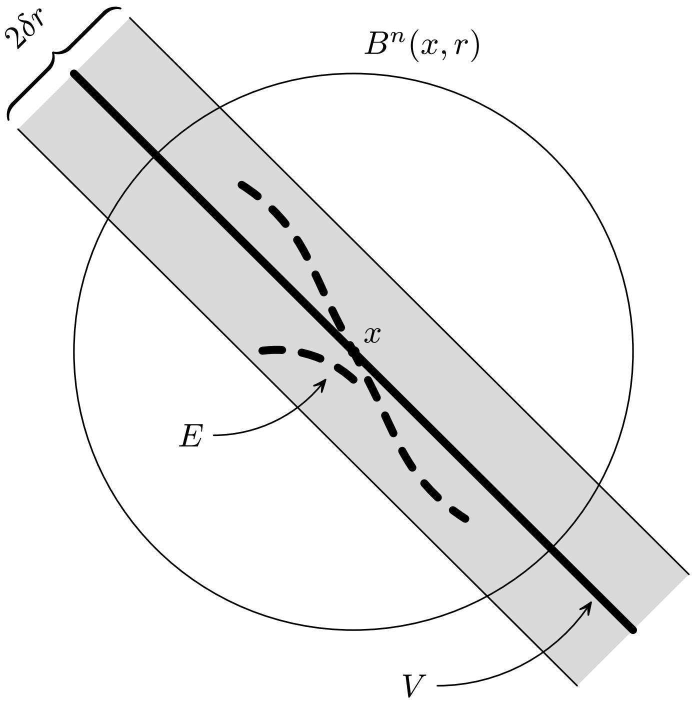
$$\lambda_n^{1-\beta} r^\beta \le \varphi_{1/K,n}(r) \le \varphi_{K,n}(r) \le \lambda_n^{1-\alpha} r^\alpha \,, \tag{2.43}$$

$\alpha = K^{1/(1-n)} = 1/\beta,\ \lambda_n \in [4, 2e^{n-1})$. A K-quasiconformal map of B^n need not be quasisymmetric, but its restriction to $\overline{B}^n(s),\ s \in (0,1)$, is quasisymmetric. In fact, P. Seittenranta [Se2] proved that for prescribed $K > 1$ and $n \ge 2$, there exists an explicit $s \in (0,1)$ such that $f|\overline{B}^n(s)$ is $\overline{\eta}_{K,n}$-quasisymmetric where $\overline{\eta}_{K,n}$ is of the same type as in (2.42).

2.44. Linear approximation property. Our examples of quasicircles in 2.25 show that quasicircles need not have tangents at any point. On the other hand, when $K \to 1$, we expect that K-quasicircles become more like usual circles. We next introduce a definition which enables us to quantify such a passage to the limit:

Given integers $n \ge 2,\ p \in \{1, ..., n-1\}$, and positive numbers $r_0 > 0,\ \delta \in (0,1)$, we say that a compact set $E \subset \mathbb{R}^n$ satisfies the *linear approximation property* with parameters (p, δ, r_0) if for every $x \in E$ and every $r \in (0, r_0)$ there exists a p-dimensional hyperplane $V_r \ni x$ such that

$$E \cap B^n(x,r) \subset \{w \in \mathbb{R}^n :\ d(w, V_r) \le \delta r\}.$$



P. Mattila and M. Vuorinen proved in 1990 [MatV] that quasispheres satisfy this property.

2.45. Theorem. *Let $K_2 > 1$ be such that*

$$c = \eta_{K,n}(1)^{-2}/2 > 15/32$$

for all $K \in (1, K_2]$. Then a K-quasisphere $E = fS^{n-1}$ satisfies the linear approximation property with parameters

$$(n-1, 4g(K), d(E)g(K)),\ g(K) = \sqrt{1-2c}\,. \tag{2.46}$$

Observe that here $\delta = 4g(K) \to 0$ as $K \to 1$.

This limit behavior shows that, the closer $K-1$ is to 0, the better K-quasispheres can be locally approximated by $(n-1)$-dimensional hyperplanes. Note that at a point $x \in E$ the approximating hyperplanes V_r may depend on r: they will very strongly depend on r if x is a "bad" point. An example of such bad behavior is a quasicircle which logarithmically spirals in a neighborhood of a point x.

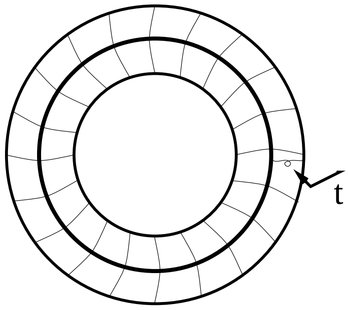
2.47. Jones' β-parameters. In the same year as [MatV] appeared, P. Jones [Jo] introduced "β-parameters" for the analysis of geometric properties of plane sets. In fact, the particular case $n = 2$, $p = 1$, of the linear approximation property is very close to the condition used by Jones in his investigations. Later on, Jones' β-parameters were used extensively by C. Bishop - P. Jones [BJ1], G. David - S. Semmes [DS], K. Okikiolu [Ok], and H. Pajot [Paj].

2.48. Open problem. For $n = 2$ the parameter δ of the linear approximation property in (2.46) is roughly $\sqrt{K-1}$. Can this be reduced, say to $K-1$, when K is close to 1?

2.49. Open problem. The Hausdorff dimension of a K-quasicircle has a majorant of the form $1 + 10(K-1)^2$ (see [BP2], [MatV, 1.8]). Is there a similar bound for the Hausdorff dimension of a K-quasisphere in $\mathbb{R}^n$, e.g. in the form $n-1+c(K-1)^2$ where c is a constant?

2.50. Rectifiability of quasispheres. Snowflake-type quasicircles provide examples of locally nonrectifiable curves. We now briefly review conditions under which quasicircles will be rectifiable. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is K-quasiconformal and $t \in (0, 1/2)$, then for convenience of notation we set

$$K(t) = K(f|A(t)),\ A(t) = \bigcup_{x \in S^{n-1}} B^n(x,t)\,. \tag{2.51}$$



A natural question is this: Does $K(t) \to 1$ as $t \to 0$ imply that fS^{n-1} is rectifiable? For $n = 2$, J. Becker and Ch. Pommerenke [BP1] have shown that the answer is in the negative. Imposing a stronger condition for the convergence $K(t) \to 1$, we have a positive result [MatV]:

2.52. Theorem. *If*

$$\int_0^{1/2} \frac{1-\alpha(t)}{t} dt < \infty,\ \alpha(t) = K(t)^{1/(1-n)}\,, \tag{2.53}$$

then fS^{n-1} is rectifiable.

An alternative proof of Theorem 2.52 was given by Yu. G. Reshetnyak in [Re2, pp. 378-382]. For some related results see also [GuV]. For $n = 2$ one can replace condition (2.53) by a slightly weaker one, as shown in [ABL], [Carle].

2.54. Quasiconformal maps of S^{n-1}. Many of the peculiarities of quasiconformal maps exhibited above are connected with the interesting geometric structure of quasispheres. We will now briefly discuss the simplest case when $f : \mathbb{R}^n \to \mathbb{R}^n$ is a K-quasiconformal map with $fS^{n-1} = S^{n-1}$. Let $g = f|S^{n-1}$. Then $H(x,g) \le H(x,f)$ for every $x \in S^{n-1}$. By the alternative characterization mentioned in 2.16, we see that if $n-1 \ge 2$, then g is quasiconformal [note: we have not defined quasiconformality in dimension 1]. Thus for $n \ge 3$ the restriction g satisfies all the properties of a quasiconformal map. In particular, g is absolutely continuous with respect to the $(n-1)$-dimensional Hausdorff measure on S^{n-1}. For $n = 2$ the situation is drastically different, as the following result of Beurling and Ahlfors shows.

2.55. Beurling - Ahlfors' singular function. In [BAh] Beurling and Ahlfors constructed a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ satisfying the condition (2.40) for some $M > 1$ such that h is not absolutely continuous with respect to the 1-dimensional Lebesgue measure. By their extension result mentioned in 2.39, h is the restriction of a quasiconformal mapping h^* of $\mathbb{R}^2$. If g is a Möbius transformation with $g(S^1) = \mathbb{R}$, then the conjugation $g^{-1} \circ h^* \circ g$ is the required counterexample.

2.56. Tukia's quasisymmetric function. Answering a question of W.K. Hayman and A. Hinkkanen, P. Tukia constructed in [Tu3] an example showing that a quasisymmetric map f of $\mathbb{R}$ can map a set E, with H-dim $E < \varepsilon$ onto a set with H-dim$(\mathbb{R} \setminus fE) < \varepsilon$. See also [BS] and [Ro].

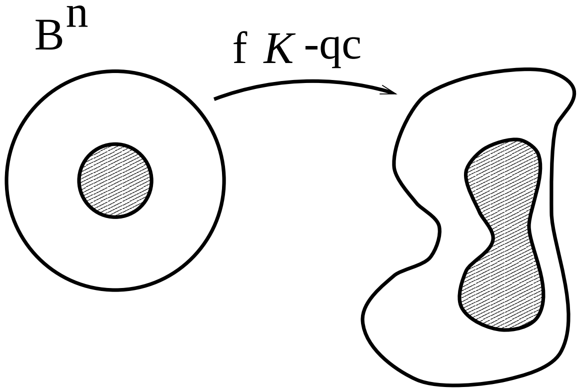
2.57. Thick sets. We conclude this section with a discussion of a property opposite to the linear approximation property. Let $c > 0$, $p \in \mathbb{N}$. We say that $A \subset \mathbb{R}^n$ is (c,p)-*thick* if, for every $x \in A$ and for all $r \in (0, d(A)/3)$, there exists a p-simplex Δ with vertices in $A \cap B^n(x,r)$ with $m_p(\Delta) \geq cr^p$ [VVW], [V5].

Snowflake-type curves are examples of $(c,2)$-thick curves. One can even show that for every $K > 1$ there are $(\frac{\sqrt{K}-1}{768}, 2)$-thick K-quasicircles. For this purpose one uses a snowflake-style construction, but replaces the angles $\frac{\pi}{3}$ by smaller ones that tend to 0 as $K \to 1$ [VVW].

A condition similar to thickness is the notion of wiggly sets [BJ2].

2.58. Open problem. Are there quasispheres in $\mathbb{R}^n$, $n \geq 3$, which are (c,n)-thick for some $c > 0$?

2.59. Books. The existing books on quasiconformal maps include [Car], [KK], [L], [LV], [V1]. Generalizations to the case of noninjective mappings, so-called *quasiregular mappings*, are studied in [HKM], [I1], [IM2], [Re2], [Ri], [V2], [Vu2].



2.60. Local structure problem (from [Vu2, p. 193]) . Prove or disprove the following assertion. For each $n \geq 2$, $r \in (0,1)$, and $K \geq 1$ there exists a number $d(n,K,r)$ with $d(n,K,r) \to d(n,K)$ as $r \to 0$ and $d(n,K) \to 1$ as $K \to 1$ such that whenever $f : B^n \to \mathbb{R}^n$ is K–qc, then $fB^n(r)$ is a $d(n,K,r)$–quasiball. More precisely, the representation $fB^n(r) = gB^n$ holds where $g : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ is a $d(n,K,r)$–qc mapping with $g(\infty) = \infty$. (Note: It was kindly pointed out by J. Becker that we can choose $d(2,1,r) = (1+r)/(1-r)$ either by [BC, pp. 39–40] or by a more general result of S. L. Krushkal' [KR].)

3 Concluding remarks

The change of Hausdorff dimension under quasiconformal maps was studied in [IM2] and [Ast]. A subclass of quasicircles, so-called *asymptotically conformal curves*, was studied, for instance, in [BP1], [ABL], [GuR].

During the past few years, there has been progress in the study of fractal objects in analysis. In this connection also highly irregular surfaces, which admit parametrizations in terms of quasisymmetric maps, have been studied in [DS]. It has turned out that the linear approximation property is similar to a condition of Reifenberg. See for instance a joint paper of G. David and T. Toro [DT] and also [DKT]. Various other results in this context area include [BJe].

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