# STRONG STARLIKENESS FOR A CLASS OF CONVEX FUNCTIONS

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Abstract. By means of the Briot-Bouquet differential subordination, we estimate the order of strong starlikeness of strongly convex functions of a prescribed order. We also make numerical experiments to examine our estimates.

## 1. Introduction

We denote by $\mathscr{A}$ the class of functions $f$ analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0$ and $f'(0) = 1$. Let $\mathscr{S}$ denote the class of normalized univalent analytic functions and, for each $0 \le k < 1$, let $\mathscr{S}(k)$ denote the subclass of $\mathscr{S}$ consisting of those functions which extend to $k$-quasiconformal mappings of the extended plane Let $g$ and $h$ be meromorphic functions in $\mathbb{D}$. We say that $g$ is *subordinate* to $h$ and express it by $g \prec h$ or conventionally by $g(z) \prec h(z)$ if $g = h \circ \omega$ for some analytic map $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$. When $h$ is univalent, the condition $g \prec h$ is equivalent to $g(\mathbb{D}) \subset h(\mathbb{D})$ and $g(0) = h(0)$.

An analytic function $f$ in the unit disk $\mathbb{D}$ is called *starlike* if $f$ is univalent and $f(\mathbb{D})$ is starlike with respect to $f(0)$. Also, $f$ is called *convex* if $f$ is univalent and $f(\mathbb{D})$ is convex.

It is well known that $f \in \mathscr{A}$ is starlike if and only if $\mathrm{Re}\,(zf'(z)/f(z)) > 0$ in $\mathbb{D}$ and $f \in \mathscr{A}$ is convex if and only if $\mathrm{Re}\,(1 + zf''(z)/f'(z)) > 0$ in $\mathbb{D}$ (see, for instance, [3]). The sets of starlike functions and convex functions in $\mathscr{A}$ are denoted by $\mathscr{S}^*$ and $\mathscr{K}$, respectively. Let $\alpha$ be a positive real number. A function $f$ in $\mathscr{A}$ is said to be *strongly starlike* of order $\alpha$ if $|\arg(zf'(z)/f(z))| < \pi\alpha/2$ for $z \in \mathbb{D}$. Similarly, $f \in \mathscr{A}$ is said to be *strongly convex* of order $\alpha$ if $|\arg(1+zf''(z)/f'(z))| < \pi\alpha/2$ for $z \in \mathbb{D}$. The sets of strongly starlike functions of order $\alpha$ and strongly convex functions of order $\alpha$ are denoted by $\mathscr{S}^*_\alpha$ and $\mathscr{K}_\alpha$, respectively. Many geometric characterizations of the class $\mathscr{S}^*_\alpha$, $0 < \alpha < 1$, are known (for a short survey, see [13]).

Throughout the paper, we will use the symbol $T$ to stand for the mapping of the unit disk onto the right-half plane which is defined by

$$T(z) = \frac{1+z}{1-z}.$$

1991 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C80.

*Key words and phrases.* strongly starlike, strongly convex, differential subordination.

The second author was partially supported by the Ministry of Education, Grant-in-Aid for Encouragement of Young Scientists (B), 14740100 and Grant-in-Aid for Scientific Research (B), 17340039.

For $0<\kappa\le 1$, we consider also the subclass $\mathscr{S}^*[\kappa]$ of $\mathscr{S}^*$ consisting of functions $f$ with $zf'(z)/f(z)\prec T(\kappa z)=(1+\kappa z)/(1-\kappa z)$. Hence, for $0<\kappa<1$, a function $f\in\mathscr{A}$ belongs to the class $\mathscr{S}^*[\kappa]$ if and only if the inequality

$$\left|\frac{zf'(z)}{f(z)}-\frac{1+\kappa^2}{1-\kappa^2}\right|<\frac{2\kappa}{1-\kappa^2}$$

holds in $\mathbb{D}$. Here are useful criteria for quasiconformal extensions.

**Theorem A.**

(i) $\mathscr{S}^*_\alpha\subset\mathscr{S}(\sin(\pi\alpha/2))$ *for* $0<\alpha<1$.

(ii) $\mathscr{S}^*[\kappa]\subset\mathscr{S}(\kappa)$ *for* $0<\kappa<1$.

Relation (i) is due to Fait, Krzyż and Zygmunt [4], and (ii) is due to Brown [1] (see also [12]). It is easy to see that $\mathscr{S}^*[\kappa]\subset\mathscr{S}^*_\alpha$ for $\alpha=(2/\pi)\arcsin(2\kappa/(1+\kappa^2))$ because of $T(\kappa z)\prec T^\alpha(z)$.

Obviously, a convex function is starlike, in other words, $\mathscr{K}\subset\mathscr{S}^*$. Moreover, Mocanu showed the relation $\mathscr{K}_\alpha\subset\mathscr{S}^*_\alpha$ for $0<\alpha\le 2$ in [6]. Therefore, it is natural to consider the problem of finding the number

$$\beta^*(\alpha)=\inf\{\beta:\mathscr{K}_\alpha\subset\mathscr{S}^*_\beta\}$$

for each $\alpha>0$. By the maximum principle, we have $\mathscr{K}_\alpha\subset\mathscr{S}^*_{\beta^*(\alpha)}$. Hence, $\beta^*(\alpha)$ is the minimal number $\beta$ so that $\mathscr{K}_\alpha\subset\mathscr{S}^*_\beta$.

Later, Mocanu proved the following in [7]. For $0<\beta<1$, set

$$\begin{aligned}\gamma(\beta)&=\frac{2}{\pi}\arctan\left[\tan\frac{\pi\beta}{2}+\frac{\beta}{(1+\beta)^{\frac{1+\beta}{2}}(1-\beta)^{\frac{1-\beta}{2}}\cos(\pi\beta/2)}\right]\\&=\beta+\frac{2}{\pi}\arctan\left[\frac{\beta\cos(\pi\beta/2)}{(1+\beta)^{\frac{1+\beta}{2}}(1-\beta)^{\frac{1-\beta}{2}}+\beta\sin(\beta\pi/2)}\right].\end{aligned}\tag{1.1}$$

**Theorem B** (Mocanu)**.** *A strongly convex function of order* $\gamma(\beta)$ *is strongly starlike of order* $\beta$ *for* $0<\beta<1$.

The function $\gamma(\beta)$ is continuous and strictly increases from 0 to 1 when $\beta$ moves from 0 to 1. We denote by $\gamma^{-1}:(0,1)\to(0,1)$ the inverse function of $\gamma$. The theorem then implies the relation $\mathscr{K}_\alpha\subset\mathscr{S}^*_{\gamma^{-1}(\alpha)}$, namely, $\beta^*(\alpha)\le\gamma^{-1}(\alpha)$ for $0<\alpha<1$. The same result was re-proved later by Nunokawa [8] and by Nunokawa and Thomas [9]. It is further claimed in [9] that the result is best possible, namely, $\beta^*(\alpha)=\gamma^{-1}(\alpha)$ for $0<\alpha<1$. This is, however, wrong as we see in the following result.

**Theorem 1.1.** *The function* $\beta^*(\alpha)$ *is continuous and strictly increasing in* $0<\alpha<1$. *Moreover,* $\beta^*(\alpha)<\gamma^{-1}(\alpha)$ *holds for each* $0<\alpha<1$.

In the same way as above, we denote by $\kappa^*(\alpha)$ the minimal number $\kappa$ so that $\mathscr{K}_\alpha \subset \mathscr{S}^*[\kappa]$. It is clear that $\kappa^*(\alpha) \le 1$. It seems, however, that no bounds of $\kappa^*(\alpha)$ were given in the literature. The next theorem implies that for each $0 < \alpha < 1$, there exists a $\kappa \in (0,1)$ such that $\mathscr{K}_\alpha \subset \mathscr{S}^*[\kappa]$.

**Theorem 1.2.** *The function $\kappa^*(\alpha)$ is continuous and strictly increasing in $0 < \alpha < 1$. Moreover, $\kappa^*(\alpha) < 1$ holds for each $0 < \alpha < 1$.*

Explicit expressions of $\beta^*(\alpha)$ and $\kappa^*(\alpha)$ will be given in Section 3 in terms of a solution to a Briot-Bouquet differential equation (Proposition 3.1). The proof of our theorems depends on geometric properties of the solution. Section 3 will also be devoted to investigation of the solution.

The above two theorems are, however, not quantitative. In order to obtain better and concrete upper bounds for $\beta^*(\alpha)$ and $\kappa^*(\alpha)$, we need more efforts. We propose a method of giving a better estimate for them. Due to some technicality, the presentation of the method will be postponed to Section 4. The next section is used to the preparation of necessary materials for the proof of the theorems and for development of our methods.

We end this introduction with the remark that, using Theorem A, we obtain quasiconformal extension criteria for the class $\mathscr{K}_\alpha$, though we do not state them separately.

## 2. Preliminaries

Our arguments will be largely based on results proved by Miller and Mocanu. We state it in convenient forms for the present aim. The first result is the following.

**Theorem C** (Miller and Mocanu [5, Theorems 3.2a and 3.2j]). *Let $h$ be a convex function in the unit disk with $h(0) = 1$ and $\operatorname{Re} h(z) > 0$ in $|z| < 1$ and let $q$ be the analytic solution to the differential equation*

$$q(z) + \frac{zq'(z)}{q(z)} = h(z), \quad z \in \mathbb{D}, \quad \textit{and} \quad q(0) = 1.$$

*Then $q$ is univalent and subordinate to $h$. Moreover, if an analytic function $p$ in the unit disk with $p(0) = 1$ satisfies the subordination*

$$p(z) + \frac{zp'(z)}{p(z)} \prec h(z),$$

*then $p(z) \prec q(z)$.*

The second result that we will need is contained in [5, Theorem 3.4h] with the choice of $\theta(w) = w$ and $\phi(w) = 1/w$.

**Theorem D** (Miller and Mocanu). *Let $q$ be a non-vanishing univalent function in $\mathbb{D}$ with $q(0) = 1$ and set $Q(z) = zq'(z)/q(z)$ and $h = q + Q$. Suppose that $Q$ is starlike and* *that* $\mathrm{Re}\,(q(z) + zQ'(z)/Q(z)) > 0$ *in* $\mathbb{D}$. *Then* $h$ *is univalent. Furthermore, if an analytic function* $p$ *in* $\mathbb{D}$ *with* $p(0) = 1$ *satisfies the relation*

$$p(z) + \frac{zp'(z)}{p(z)} \prec q(z) + \frac{zq'(z)}{q(z)} = h(z),$$

*then* $p(z) \prec q(z)$.

Note that, when $\mathrm{Re}\,q(z) > 0$, the condition $\mathrm{Re}\,(q(z) + zQ'(z)/Q(z)) > 0$ is fulfilled automatically because $\mathrm{Re}\,(zQ'(z)/Q(z)) > 0$.

We also use the following version of the Julia-Wolff lemma, which is a combination of known facts.

**Lemma 2.1.** *Let* $z_0 \in \partial\mathbb{D}$. *Suppose that an analytic function* $\omega$ *in* $\mathbb{D} \cup \{z_0\}$ *satisfies* $|\omega(z)| < 1$ *in* $|z| < 1, \omega(0) = 0$ *and* $|\omega(z_0)| = 1$. *Then*

$$m = \frac{z_0\omega'(z_0)}{\omega(z_0)}$$

*is a positive real number with* $m \geq 1$. *Furthermore,* $m = 1$ *only if* $\omega(z) \equiv \omega(z_0)z/z_0$.

*Proof.* First, by the Julia-Wolff lemma [11, Proposition 4.13], we see that $m = z_0\omega'(z_0)/\omega(z_0)$ is a positive real number. Here we recall the boundary Schwarz lemma due to Osserman [10, Lemma 1]: $|\omega'(z_0)| \geq 2/(1 + |\omega'(0)|)$. The remaining assertion now follows from the Schwarz lemma: $|\omega'(0)| \leq 1$ and equality holds only if $\omega$ is a rotation about the origin. □

The next strange-looking result generates a new family of starlike functions from a single starlike function.

**Lemma 2.2.** *Let* $a$ *and* $b$ *be complex numbers with* $a \neq 0$ *and* $|a| + |b| \leq 1$. *For a starlike function* $f : \mathbb{D} \to \mathbb{C}$ *with* $f(0) = 0$, *the function* $g : \mathbb{D} \to \mathbb{C}$ *defined by*

$$g(z) = \frac{f\left(\frac{az}{1-bz}\right)}{1 - bz}$$

*is also starlike.*

Note that $az/(1 - bz) \in \mathbb{D}$ for $z \in \mathbb{D}$ whenever $|a| + |b| \leq 1$. In this lemma, $f$ and $g$ satisfy only the condition $f(0) = g(0) = 0$, thus, $f$ and $g$ might not be normalized so that $f'(0) = g'(0) = 1$. In order to obtain a transformation of $\mathscr{S}^*$ into itself, we may consider the operator $I_{a,b}$ defined by $I_{a,b}[f](z) = f(az/(1 - bz))/(a(1 - bz))$. It might be interesting to observe that $I_{a,b}[f](z) = z/(1 - bz)^2 \to z/(1 - z)^2$ (the Koebe function) as $(a, b) \to (0, 1)$ for each $f \in \mathscr{S}^*$.

*Proof.* Let $\varphi(z) = zf'(z)/f(z)$. Then the starlikeness implies $\mathrm{Re}\,\varphi > 0$ in $\mathbb{D}$. We need to see that

$$\frac{zg'(z)}{g(z)} = \frac{\varphi\left(\frac{az}{1-bz}\right) + bz}{1 - bz}$$

has positive real part. First we consider the special case when $\varphi = \varphi_\zeta$ for some $\zeta \in \partial\mathbb{D}$, where $\varphi_\zeta(z) = (1+\zeta z)/(1-\zeta z)$. Then, a straightforward computation gives

$$\frac{zg'(z)}{g(z)} = \frac{1+(a\zeta+b)z}{1-(a\zeta+b)z}.$$

By assumption, we have $|a\zeta+b| \le |a|+|b| \le 1$ and thus $\mathrm{Re}\,(zg'(z)/g(z)) > 0$.

To show the general case, we use the Herglotz representation of a function with positive real part (cf. [3]). For the general $\varphi$, there exists a Borel probability measure $\mu$ on the unit circle $\partial\mathbb{D}$ such that

$$\varphi(z) = \int_{\partial\mathbb{D}} \frac{1+\zeta z}{1-\zeta z} d\mu(\zeta) = \int_{\partial\mathbb{D}} \varphi_\zeta(z) d\mu(\zeta).$$

Therefore,

$$\begin{aligned}\frac{\varphi\left(\frac{az}{1-bz}\right)+bz}{1-bz} &= \int_{\partial\mathbb{D}} \frac{\varphi_\zeta\left(\frac{az}{1-bz}\right)+bz}{1-bz} d\mu(\zeta)\\ &= \int_{\partial\mathbb{D}} \frac{1+(a\zeta+b)z}{1-(a\zeta+b)z} d\mu(\zeta).\end{aligned}$$

This shows that $zg'(z)/g(z)$ has positive real part. $\square$

We recall also the following simple fact (cf. [2]).

**Lemma 2.3.** *Let $f : \mathbb{D} \to \mathbb{C}$ be a convex univalent function and $\Delta$ be an open disk contained in $\mathbb{D}$. Then $f(\Delta)$ is also convex.*

## 3. Proof of Theorems 1.1 and 1.2

In this section, we prove Theorems 1.1 and 1.2. To this end, we introduce a mapping associated with the function $T^\alpha$.

Let $q_\alpha$ be the analytic function in the unit disk determined by

$$q_\alpha(z) + \frac{zq_\alpha'(z)}{q_\alpha(z)} = \left(\frac{1+z}{1-z}\right)^\alpha = T^\alpha(z), \quad \text{and} \quad q_\alpha(0) = 1. \tag{3.1}$$

Since $T^\alpha$ is analytic in $\overline{\mathbb{D}} \setminus \{\pm 1\}$, $q_\alpha$ is also analytically continued in a neighborhood of $\overline{\mathbb{D}} \setminus \{\pm 1\}$. Hence, we can argue the value of $q_\alpha(e^{i\theta})$ for $0 < |\theta| < \pi$.

As an immediate consequence of Theorem C, we obtain explicit expressions of the quantities $\beta^*(\alpha)$ and $\kappa^*(\alpha)$ in terms of the function $q_\alpha$ for $0 < \alpha < 1$.

**Proposition 3.1.** *Let $0 < \alpha < 1$ and let $\beta^*(\alpha)$ and $\kappa^*(\alpha)$ be the minimal numbers $\beta$ and $\kappa$, respectively, such that $\mathscr{K}_\alpha \subset \mathscr{S}^*_\beta, \mathscr{K}_\alpha \subset \mathscr{S}^*[\kappa]$. Then they are expressed by*

$$\beta^*(\alpha) = \sup_{0<\theta<\pi} \arg q_\alpha(e^{i\theta}) \quad \text{and} \quad \kappa^*(\alpha) = \sup_{0<\theta<\pi} \left|\frac{q_\alpha(e^{i\theta})-1}{q_\alpha(e^{i\theta})+1}\right|.$$

*Proof.* Let $f \in \mathscr{K}_\alpha$ and set $p(z) = zf'(z)/f(z)$. By the relation $p(z) + zp'(z)/p(z) = 1 + zf''(z)/f'(z) \prec T^\alpha(z)$, we conclude that $p \prec q_\alpha$ by Theorem C. In particular,

$$\sup_{z\in\mathbb{D}} |\arg p(z)| \le \sup_{z\in\mathbb{D}} |\arg q_\alpha(z)| = \sup_{0<\theta<\pi} \arg q_\alpha(e^{i\theta}).$$

Here, equality holds when we take $f$ so that $1+zf''(z)/f(z) = T^\alpha(z)$, namely, $zf'(z)/f(z) = q_\alpha(z)$. Thus, we have shown the first relation. The second one can be deduced in the same way. $\square$

*Remark.* A direct computation gives $q_1(z) = 1/(1-z)$. In the case when $\alpha = 1$, the above argument thus yields that for $f \in \mathscr{K}_1 = \mathscr{K}$, $p(z) = zf'(z)/f(z)$ is subordinate to $q_1(z) = 1/(1-z)$, that is, $p(\mathbb{D}) \subset q_1(\mathbb{D}) = \{\mathrm{Re}\, w > 1/2\}$. In other words, a convex function is starlike of order 1/2. This is known as Strohhäcker's theorem, see [3, p. 251].

In order to get information about $\beta^*(\alpha)$ and $\kappa^*(\alpha)$, it is thus important to know about the function $q_\alpha$. We summarize geometric properties of $q_\alpha$ in the following proposition.

**Proposition 3.2.** *For $0 < \alpha < 1$, the function $q_\alpha$ maps the unit disk univalently onto a symmetric bounded Jordan domain contained in $\{w \in \mathbb{C} : |\arg w| < \pi\alpha/2, \mathrm{Re}\, w > 1/2\}$ in such a way that $q_\alpha(x) \in \mathbb{R}$ for $x \in (-1, 1)$ and $q_\alpha'(0) = \alpha$.*

The proof is divided into several steps as follows.

Since $T^\alpha$ is convex and has positive real part, the functions $h = T^\alpha$ and $q = q_\alpha$ satisfy the assumptions of Theorem C. Therefore, $q_\alpha$ is univalent and $|\arg q_\alpha(z)| < \pi\alpha/2$ in $\mathbb{D}$. By the symmetry of the equation (3.1), the solution $q_\alpha$ is symmetric, namely, $\overline{q_\alpha(z)} = q_\alpha(\bar{z})$. Therefore, $q_\alpha(\mathbb{D})$ is symmetric in the real axis and $q_\alpha$ maps real numbers to real numbers. Since $T^\alpha \prec T$, the theorem yields also the relation $q_\alpha \prec q_1$. Therefore, $q_\alpha(\mathbb{D})$ lies in the domain $q_1(\mathbb{D}) = \{\mathrm{Re}\, w > 1/2\}$. The relation $q_\alpha'(0) = \alpha > 0$ can be verified directly. In particular, we see that $\mathrm{Im}\, q_\alpha(z) > 0$ for $\mathrm{Im}\, z > 0$.

The following lemma gives an upper bound for $|q_\alpha(z)|$.

**Lemma 3.3.** *The function $q_\alpha$ is bounded in the unit disk for $0 < \alpha < 1$.*

*Proof.* For a fixed $\theta \in \mathbb{R}$, we consider the function $u(r) = q_\alpha(re^{i\theta})$ in $0 \le r \le 1$. We express $u$ also in the form $u(r) = R(r)e^{i\Theta(r)}$, where $R(r) > 0$ and $\Theta(r) \in \mathbb{R}$ with $\Theta(0) = 0$. Then

$$R(r)e^{i\Theta(r)} + \frac{rR'(r)}{R(r)} + ir\Theta'(r) = u(r) + \frac{ru'(r)}{u(r)} = T^\alpha(re^{i\theta}).$$

As observed above, $|\Theta(r)| \le \pi\alpha/2$ holds. Therefore, taking the real part of the above relation, we obtain

$$cR(r) + \frac{rR'(r)}{R(r)} \le \mathrm{Re}\, T^\alpha(re^{i\theta}) \le \left(\frac{1+r}{1-r}\right)^\alpha,$$

where $c = \cos(\pi\alpha/2)$. We now define the positive function $g(r),\ 0 < r \le 1$, by the relation

$$\log g(r) = \int_{1/2}^{r} \frac{cR(x)}{x} dx.$$

Then, $rg'(r)/g(r) = cR(r)$ and thus

$$\frac{g''(r)}{g'(r)} = \frac{1}{r}\left[cR(r) + \frac{rR'(r)}{R(r)} - 1\right] \le \frac{1}{r}\left[\left(\frac{1+r}{1-r}\right)^{\alpha} - 1\right].$$

An integration yields

$$\log \frac{g'(r)}{g'(1/2)} \le \Phi(r) = \int_{1/2}^{r} \frac{1}{x}\left[\left(\frac{1+x}{1-x}\right)^{\alpha} - 1\right] dx$$

for $1/2 < r < 1$. Since

$$\Phi(r) < 2^{1+\alpha} \int_{1/2}^{1} \frac{dx}{(1-x)^{\alpha}} = \frac{4^{\alpha}}{1-\alpha},$$

and $g'(1/2) = 2cR(1/2)$, the inequality

$$g'(r) < 2cR(\tfrac{1}{2}) \exp \frac{4^{\alpha}}{1-\alpha}$$

follows for $1/2 < r < 1$. Since $g' > 0$, we have $g(r) > g(1/2) = 1$ for $1/2 < r < 1$. Therefore,

$$R(r) = \frac{rg'(r)}{cg(r)} < 2R(\tfrac{1}{2}) \exp \frac{4^{\alpha}}{1-\alpha}$$

for $1/2 < r < 1$. We now recall the growth theorem for functions $f$ in $\mathscr{S}$ (cf. [3, p. 33]):

$$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2}, \quad |z| = r < 1.$$

Since $(q_\alpha - 1)/\alpha$ belongs to $\mathscr{S}$, we have $|q_\alpha(z)| \le 1 + \alpha r/(1-r)^2,\ |z| = r$. Letting $r = 1/2$, we obtain the estimate $R(1/2) \le 1 + 2\alpha$. Therefore,

$$|q_\alpha(z)| = R(r) \le 2(1+2\alpha) \exp \frac{4^{\alpha}}{1-\alpha}$$

for $z = re^{i\theta}, 1/2 < r < 1$. The last inequality is valid for all $z \in \mathbb{D}$ by the maximum modulus principle. □

Let $0 < \alpha < \beta < 1$. Since $T^\alpha \prec T^\beta$, Theorem C implies that $q_\alpha \prec q_\beta$. Note also that $\omega = q_\beta^{-1} \circ q_\alpha$ is analytically continued across the border $\partial\mathbb{D} \setminus \{\pm 1\}$. We now show the following.

**Lemma 3.4.** *Let $\omega = q_\beta^{-1} \circ q_\alpha : \mathbb{D} \to \mathbb{D}$ for $0 < \alpha < \beta < 1$. Then $|\omega(e^{i\theta})| < 1$ for each $\theta \in (0, \pi)$.*

*Proof.* Suppose, to the contrary, that $|\omega(z_0)| = 1$ for some $z_0 = e^{i\theta_0},\ \theta_0 \in (0, \pi)$. Lemma 2.1 implies that $m = z_0\omega'(z_0)/w_0 \geq 1$, where we set $w_0 = \omega(z_0)$. Note that $|w_0| = 1$ and $\operatorname{Im} w_0 > 0$. On the other hand, the relation $q_\alpha = q_\beta \circ \omega$ yields

$$
\begin{aligned}
T^\alpha(z_0) &= q_\alpha(z_0) + \frac{z_0 q'_\alpha(z_0)}{q_\alpha(z_0)} \\
&= q_\beta(w_0) + \frac{z_0 q'_\beta(w_0)\omega'(z_0)}{q_\beta(w_0)} \\
&= q_\beta(w_0) + \frac{m w_0 q'_\beta(w_0)}{q_\beta(w_0)} \\
&= -(m-1)q_\beta(w_0) + mT^\beta(w_0).
\end{aligned}
$$

By Theorem B, $0 \leq \arg q_\beta(w_0) \leq \pi\gamma^{-1}(\beta)/2 < \pi\beta/2 = \arg T^\beta(w_0)$. Therefore, an elementary geometry tells us that the argument of $-(m-1)q_\beta(w_0) + mT^\beta(w_0)$ is in between $\pi\beta/2$ and $\pi\beta/2 + \pi$. This contradicts the fact that $\arg T^\alpha(z_0) = \pi\alpha/2 < \pi\beta/2$. Thus the inequality in question has been shown. □

**Lemma 3.5.** *Let $0 < \alpha < 1$. The curve $\gamma_\alpha : (0, \pi) \to \mathbb{C}$ defined by $\gamma_\alpha(\theta) = q_\alpha(e^{i\theta})$ is a Jordan arc of finite length.*

*Proof.* If $\gamma_\alpha$ is not injective, then $\gamma_\alpha$ bounds a domain $D$. If $\beta > \alpha$ is close enough to $\alpha$, the boundary of $q_\beta(\mathbb{D})$ must go through $D$, which implies that the curve $\gamma_\beta$ has a common point with the curve $\gamma_\alpha$. This is, however, impossible by Lemma 3.4. In order to see finiteness of the length of $\gamma_\alpha$, we use the Hardy spaces. Since $T \in H^p$ for all $p < 1$, we see that $T^\alpha \in H^p$ for all $p < 1/\alpha$. In particular, $T^\alpha \in H^1$. Here, by (3.1), $zq'_\alpha(z) = q_\alpha(z)(T^\alpha(z) - q_\alpha(z))$. Since $q_\alpha$ is bounded by Lemma 3.3, we have $q'_\alpha \in H^1$. The length of $\gamma_\alpha$ is now estimated by

$$
\int_0^\pi |\gamma'_\alpha(\theta)| d\theta = \frac{1}{2}\int_0^{2\pi} |q'_\alpha(e^{i\theta})| d\theta = \pi \|q'_\alpha\|_{H^1} < \infty.
$$

□

*Completion of the proof of Proposition 3.2.* The remaining part is to show that $q_\alpha(\mathbb{D})$ is a Jordan domain.

Since the curve $\gamma_\alpha$ has finite length, it extends to a continuous map on $[0, \pi]$, which will be denoted by the same symbol $\gamma_\alpha$. By the symmetry of $q_\alpha$, it is now obvious that the image $q_\alpha(\mathbb{D})$ is bounded by $\gamma_\alpha([0, \pi])$ and its reflection in $\mathbb{R}$. Thus the assertion has been proved. □

We are now ready to prove Theorems 1.1 and 1.2. It is a standard fact in the theory of ordinary differential equations that the solutions of initial value problems are continuous with respect to parameters. The continuity of $\beta^*(\alpha)$ and $\kappa^*(\alpha)$ immediately follows from this fact. Proposition 3.2 yields that $q_\alpha(\mathbb{D})$ is bounded and contained in the half-plane $\operatorname{Re} w > 1/2$, which implies $\kappa^*(\alpha) < 1$. (A crude estimate of $\kappa^*(\alpha)$ can be given by using the estimate in the proof of Lemma 3.3.)

It follows from the next lemma that $\beta^*(\alpha)$ and $\kappa^*(\alpha)$ are strictly increasing.

**Lemma 3.6.** *Let $0 < \alpha_1 < \alpha_2 < 1$. Then $q_{\alpha_1}(\mathbb{D})$ is a relatively compact subdomain of $q_{\alpha_2}(\mathbb{D})$.*

*Proof.* Since $q_\alpha(\mathbb{D})$ is a Jordan domain, Carathéodory's theorem ensures that $q_\alpha$ extends to a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{q_\alpha(\mathbb{D})}$, which will still be denoted by $q_\alpha$. We show now that $q_{\alpha_1}(1) < q_{\alpha_2}(1)$. Let $f_\alpha$ be the function in $\mathscr{A}$ determined by the relation $1+zf_\alpha''(z)/f_\alpha'(z) = T^\alpha(z)$. Then $q_\alpha$ is expressed by $q_\alpha(z) = zf_\alpha'(z)/f_\alpha(z)$. Letting $S_\alpha(z) = (T^\alpha(z)-1)/z$, we obtain $f_\alpha''/f_\alpha' = S_\alpha$. Integrating both sides of the last relation, we have $\log f_\alpha'(z) = \int_0^z S_\alpha(\zeta)d\zeta =: U_\alpha(z)$. Therefore,

$$\frac{1}{q_\alpha(1)} = \frac{f_\alpha(1)}{f_\alpha'(1)} = \int_0^1 e^{U_\alpha(x)-U_\alpha(1)}dx.$$

Since $U_\alpha(x)-U_\alpha(1) = -\int_x^1 S_\alpha(t)dt$ is strictly decreasing in $0<\alpha<1$ for a fixed $x \in (0,1)$, it is concluded that $q_\alpha(1)$ is strictly increasing. Thus the claim follows. In the same way, we can show that $q_{\alpha_2}(-1) < q_{\alpha_1}(-1)$.

Let $\omega = q_{\alpha_2}^{-1} \circ q_{\alpha_1} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$. Note that $\omega$ is continuous in $\overline{\mathbb{D}}$. What we have seen above means that $|\omega(\zeta)| < 1$ for $\zeta = \pm 1$. On the other hand, Lemma 3.4 asserts that this is valid for $\zeta \in \partial\mathbb{D} \setminus \{1,-1\}$. Therefore, we conclude that $\max_{|\zeta|=1} |\omega(\zeta)| < 1$, which implies the required assertion. □

Finally, we show that $\beta^*(\alpha) < \gamma^{-1}(\alpha)$ for $0<\alpha<1$. To this end, we recall the proof of Theorem B. Set $p_\beta = T^\beta$ and

$$h_\beta(z) = p_\beta(z) + \frac{zp_\beta'(z)}{p_\beta(z)} = \left(\frac{1+z}{1-z}\right)^\beta + \frac{2\beta z}{1-z^2}$$

for $0<\beta<1$. Mocanu showed that $h_\beta$ is univalent in $\mathbb{D}$ and $\gamma(\beta)$ is obtained as the minimum of $(2/\pi)\arg h_\beta(e^{i\theta})$ over $0<\theta<\pi$.

Suppose now that $\beta^*(\alpha)$ and $\gamma^{-1}(\alpha)$ are the same number, say $\beta$, for some $0<\alpha<1$. Then, Proposition 3.1 implies that a point in the boundary of $q_\alpha(\mathbb{D})$ has argument $\pi\beta/2$. Therefore, if we set $\omega = p_\beta^{-1} \circ q_\alpha$, $|\omega(z_0)| = 1$ holds for some $z_0 = e^{i\theta},\ \theta \in (0,\pi)$. Since $\omega(0) = 0$ and $\omega(z)/z$ is not constant, Lemma 2.1 implies that $m = z_0\omega'(z_0)/w_0 > 1$, where

$w_0 = \omega(z_0)$. The relation (3.1) now turns to

$$\begin{aligned} T^\alpha(z_0) &= p_\beta(w_0) + \frac{z_0 p_\beta'(w_0)\omega'(z_0)}{p_\beta(w_0)} \\ &= p_\beta(w_0) + \frac{m w_0 p_\beta'(w_0)}{p_\beta(w_0)} \\ &= -(m-1)p_\beta(w_0) + m h_\beta(w_0). \end{aligned}$$

Since $\pi > \arg h_\beta(w_0) \ge \pi\gamma(\beta)/2 = \pi\alpha/2$ and $\arg p_\beta(w_0) = \pi\beta/2$, we have

$$\frac{\pi\alpha}{2} < \arg\big\{ -(m-1)p_\beta(w_0) + m h_\beta(w_0)\big\} < \frac{\pi\beta}{2} + \pi.$$

This is, however, impossible because $\arg T^\alpha(z_0) = \pi\alpha/2$. Thus we have shown that $\beta^*(\alpha) < \gamma^{-1}(\alpha)$.

The proof of Theorems 1.1 and 1.2 is now complete.

## 4. Concrete bounds for the order of strong starlikeness

In the present section, we propose elementary bounds for the quantities $\beta^*(\alpha)$ and $\kappa^*(\alpha)$ for certain $\alpha$. For $\alpha \in (0,1), u \in (0,1), v \in (0,+\infty), c \in (0,1]$, we consider the function

$$q_{\alpha,u,v,c}(z) = \frac{(1+v)u(1+cz)^\alpha + (1-u)v(1-z)^\alpha}{u(1+cz)^\alpha + v(1-z)^\alpha}.$$

We further set

$$h_{\alpha,u,v,c}(z) = q_{\alpha,u,v,c}(z) + \frac{z q_{\alpha,u,v,c}'(z)}{q_{\alpha,u,v,c}(z)}.$$

Then our theorem is now stated as follows.

**Theorem 4.1.** *Let $\alpha \in (0,1), u \in (0,1), v \in (0,+\infty)$, and $c \in (0,1]$. The function $q = q_{\alpha,u,v,c}$ is univalent in $\mathbb{D}$ and the image $q(\mathbb{D})$ is a convex subdomain of the right half-plane. Moreover, $h = h_{\alpha,u,v,c}$ is univalent, and if an analytic function $p$ in $\mathbb{D}$ with $p(0) = 1$ satisfies $p(z) + zp'(z)/p(z) \prec h(z)$, then $p(z) \prec q(z)$.*

The following lemma will be needed to prove the theorem and it may be of independent interest.

**Lemma 4.2.** *Let $\alpha$ be a real number with $0 < \alpha < 1$ and let $a, b, c, d$ be non-negative numbers with $ad - bc \ne 0$. If $q = (aT^\alpha + b)/(cT^\alpha + d)$, the function $zq'(z)/q(z)$ is starlike and, in particular, univalent in $\mathbb{D}$. Here, $T^\alpha(z) = ((1+z)/(1-z))^\alpha$.*

*Proof.* Set $\varphi(z) = zq'(z)/q(z)$, $\psi(z) = z\varphi'(z)/\varphi(z)$ and $p = T^\alpha$. We have to show that $u := \operatorname{Re}\psi > 0$ on $\mathbb{D}$. First of all, we have expressions

$$\varphi(z) = \frac{zp'(z)}{p(z)} \cdot \frac{(ad-bc)p(z)}{(ap(z)+b)(cp(z)+d)} = \frac{2\alpha z}{1-z^2} \cdot \frac{(ad-bc)p(z)}{(ap(z)+b)(cp(z)+d)}, \tag{4.1}$$

and

$$\psi(z) = \frac{1+z^2}{1-z^2} + \frac{zp'(z)}{p(z)} - \frac{azp'(z)}{ap(z)+b} - \frac{czp'(z)}{cp(z)+d}$$
$$= \frac{1+z^2}{1-z^2} + \frac{2\alpha z}{1-z^2} \cdot \frac{bd - acp(z)^2}{(ap(z)+b)(cp(z)+d)} \tag{4.2}$$

At this stage, we can see that $\varphi$ and $\psi$ are both analytic in $\mathbb{D}$. Since $\psi$ is analytically continued up to the boundary of $\mathbb{D}$ except for the points $z = \pm 1$, the harmonic function $u$ has a harmonic continuation across the curve $\partial\mathbb{D} \setminus \{\pm 1\}$. Therefore, in order to prove positivity of $u$, by the minimum principle, it is enough to see the following three properties:

(i) $u$ is symmetric in the real axis, namely, $u(\bar{z}) = u(z)$ for $z \in \mathbb{D}$,
(ii) $u(e^{i\theta}) \geq 0$ for $\theta \in (0, \pi)$, and
(iii) $\liminf_{z\to\pm1} u(z) \geq 0$.

Property (i) is straightforward to see. We proceed to property (ii). For $\theta \in (0, \pi)$, setting $\omega = p(e^{i\theta})$, we have

$$\psi(e^{i\theta}) = i\cot\theta + \frac{i\alpha}{\sin\theta} \cdot \frac{bd - ac\omega^2}{(a\omega+b)(c\omega+d)}$$

and, in particular,

$$u(e^{i\theta}) = -\frac{\alpha}{\sin\theta} \cdot \operatorname{Im} \frac{bd - ac\omega^2}{(a\omega+b)(c\omega+d)}.$$

We now have the following sequence of equivalent conditions:

$$\begin{aligned}
&u(e^{i\theta}) \geq 0 \\
\Leftrightarrow\ &\operatorname{Im} \frac{bd - ac\omega^2}{(a\omega+b)(c\omega+d)} \leq 0 \\
\Leftrightarrow\ &\operatorname{Im} \frac{(a\omega+b)(c\omega+d)}{bd - ac\omega^2} = \operatorname{Im} \frac{(ad+bc)\omega + 2bd}{bd - ac\omega^2} \geq 0 \\
\Leftrightarrow\ &\operatorname{Im} \left[ ((ad+bc)\bar{\omega} + 2bd)(ac\omega^2 - bd) \right] \geq 0 \\
\Leftrightarrow\ &\operatorname{Im} \left[ \omega((ad+bc)|\omega|^2 + bd(ad+bc) + 2abcd\omega) \right] \geq 0.
\end{aligned}$$

The last condition can be verified by $\arg\omega = \pi\alpha/2$ and

$$\arg((ad+bc)|\omega|^2 + bd(ad+bc) + 2abcd\omega) \leq \arg\omega = \pi\alpha/2.$$

We finally show property (iii). Observe first

$$\begin{aligned}
\psi(z) &= \frac{1+z^2}{1-z^2} - \frac{2\alpha z}{1-z^2} \cdot \left(1 - \frac{b}{ap(z)+b} - \frac{d}{cp(z)+d}\right) \\
&= \alpha \cdot \frac{1-z}{1+z} + (1-\alpha) \cdot \frac{1+z^2}{1-z^2} + \frac{2b\alpha z}{(1-z^2)(ap(z)+b)} + \frac{2d\alpha z}{(1-z^2)(cp(z)+d)}.
\end{aligned}$$

Since

$$\begin{aligned}\mathrm{Re}\,\frac{z}{(1-z^2)(ap(z)+b)} &= \mathrm{Re}\,\frac{(z-\bar{z}|z|^2)(\overline{ap(z)}+b)}{|1-z^2|^2|ap(z)+b|^2} \\ = \frac{1-|z|^2}{|1-z^2|^2}&\left[\frac{b\mathrm{Re}\,z}{|ap(z)+b|^2}+\frac{a\mathrm{Re}\,z\cdot\mathrm{Re}\,p(z)}{|ap(z)+b|^2}+a\cdot\frac{1+|z|^2}{1-|z|^2}\cdot\frac{\mathrm{Im}\,z\cdot\mathrm{Im}\,p(z)}{|ap(z)+b|^2}\right]\end{aligned}$$

and $\mathrm{Im}\,z\cdot\mathrm{Im}\,p(z)\ge 0$, we can easily show the inequality

$$\liminf_{z\to 1}\mathrm{Re}\,\frac{z}{(1-z^2)(ap(z)+b)}\ge 0.$$

Similarly, we have

$$\liminf_{z\to 1}\mathrm{Re}\,\frac{z}{(1-z^2)(cp(z)+d)}\ge 0.$$

Thus we obtain the inequality

$$\liminf_{z\to 1}u(z)=\liminf_{z\to 1}\mathrm{Re}\,\psi(z)\ge 0.$$

In order to show $\liminf_{z\to -1}u(z)\ge 0$, letting $\psi=\psi_{a,b,c,d}$, we observe the relation $\psi_{a,b,c,d}(-z)=\psi_{b,a,d,c}(z)$ by the second expression of $\psi$ in (4.2). Thus, the case can be reduced to the above by interchanging $a,c$ and $b,d$.

The proof is now complete. □

*Proof of Theorem 4.1.* Firstly, we note that $q=q_{\alpha,u,v,c}$ can be written as $p\circ\omega$. Here, $p=q_{\alpha,u,v,1}$ and the map $\omega:\mathbb{D}\to\mathbb{D}$ is given by

$$\omega(z)=\omega_c(z)=\frac{(1+c)z}{2-(1-c)z}=\frac{az}{1-bz}, \tag{4.3}$$

where $a=(1+c)/2$ and $b=(1-c)/2$. Note here that $a+b=1$. The function $p$ can be written in the form $L\circ T^\alpha$, where $L$ is the Möbius transformation given by

$$L(z)=\frac{(1+v)uz+(1-u)v}{uz+v},$$

which maps the right half-plane $\mathbb{H}$ onto the disk with diameter $(1-u,1+v)$ in such a way that $L(0)=1-u, L(1)=1$ and $L(\infty)=1+v$. In particular, $p$ is convex. Since $\Delta=\omega(\mathbb{D})$ is a disk contained in $\mathbb{D}$, by Lemma 2.3, the image $p(\Delta)=q(\mathbb{D})$ is a convex subdomain of the right half-plane.

We next show that $Q(z)=zq'(z)/q(z)$ is starlike. Lemma 4.2 implies that $P(z)=zp'(z)/p(z)$ is starlike. The relation $q=p\circ\omega$ now yields

$$Q(z)=P(\omega(z))\frac{z\omega'(z)}{\omega(z)}=\frac{P(\frac{az}{1-bz})}{1-bz},$$

By Lemma 2.2, we conclude that $Q$ is starlike.

Since $q$ has positive real part, all the assumptions in Theorem D are fulfilled. Hence, the assertions in the theorem now follow. □

We now set

$$\begin{aligned}\beta(\alpha,u,v,c) &= \sup_{z\in\mathbb{D}} \frac{2}{\pi}|\arg q_{\alpha,u,v,c}(z)|,\\ \kappa(\alpha,u,v,c) &= \sup_{z\in\mathbb{D}} \left|\frac{q_{\alpha,u,v,c}(z)-1}{q_{\alpha,u,v,c}(z)+1}\right| \quad \text{and}\\ \Gamma(\alpha,u,v,c) &= \inf_{0<\theta<\pi} \frac{2}{\pi}\arg h_{\alpha,u,v,c}(e^{i\theta})\end{aligned}$$

for $\alpha, u \in (0,1), v \in (0,\infty)$ and $c \in (0,1]$. Here, the argument is taken to be the principal value. As a corollary of Theorem 4.1, we have

**Corollary 4.3.** *Let $\beta = \beta(\alpha,u,v,c), \kappa = \kappa(\alpha,u,v,c)$ and $\gamma = \Gamma(\alpha,u,v,c)$ for $\alpha, u \in (0,1), v \in (0,\infty)$ and $c \in (0,1]$. If $\gamma > 0$, then $\mathscr{K}_\gamma \subset \mathscr{S}^*_\beta \cap \mathscr{S}^*[\kappa]$. In particular, $\beta^*(\gamma) \le \beta$ and $\kappa^*(\gamma) \le \kappa$.*

**Example.** We try to estimate $\beta^*(1/2)$ with the aid of Mathematica. By numerical experiments, we found that the choice $\alpha = 0.4731, u = 0.9285, v = 4.2506, c = 0.9285$ yields $\Gamma(\alpha,u,v,c) \approx 1/2$ and $\beta(\alpha,u,v,c) \approx 0.32104$. Therefore, we obtain numerically, $\beta^*(1/2) < 0.3211$.

Mocanu's theorem, in turn, gives the estimate $\beta^*(1/2) \le \gamma^{-1}(1/2) \approx 0.35046$. On the other hand, by numerically solving the differential equation (3.1), we obtain an experimental value $\beta^*(1/2) \approx 0.309$, though we do not know how reliable it is.

We next try to estimate $\kappa^*(1/2)$. For $\alpha = 1/2, u = 0.95, v = 3.47, c = 0.49$, we obtain $\Gamma(\alpha,u,v,c) \approx 1/2$ and $\kappa(\alpha,u,v,c) \approx 0.634$. Therefore, $\kappa^*(1/2) < 0.635$. By a numerical computation, we have an experimental value $\kappa^*(1/2) \approx 0.613$.

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