DRAWING BERS EMBEDDINGS OF THE TEICHMÜLLER SPACE OF ONCE-PUNCTURED TORI

YOHEI KOMORI, TOSHIYUKI SUGAWA, MASAAKI WADA, AND YASUSHI YAMASHITA

Abstract. We present a computer-oriented method of producing pictures of Bers embeddings of the Teichmüller space of once-punctured tori. The coordinate plane is chosen in such a way that the accessory parameter is hidden in the relative position of the origin. Our algorithm consists of two steps. To each point in the coordinate plane, we first compute the corresponding monodromy representation by numerical integration along certain loops. Then we decide if the representation is discrete or not by applying the Jørgensen's theory on the quasifuchsian space of once-punctured tori.

1. Introduction

Let Γ be a Fuchsian group acting on the unit disk $\mathbb{D}$ uniformizing a Riemann surface, and $B_2(\mathbb{D},\Gamma)$ the complex Banach space of holomorphic quadratic differentials for Γ on $\mathbb{D}$ with finite norm. It is well known that the Teichmüller space $T(\Gamma)$ of Γ can be realized as a bounded contractible open set in $B_2(\mathbb{D},\Gamma)$ through the Bers embedding. Throughout the paper, the space $T(\Gamma)$ is understood as the image of the Bers embedding.

In 1972, Bers wrote ([Bers 1972] page 278, the notation was changed to adapt with ours):

> Unfortunately, there is no known method to decide whether a given $\phi \in B_2(\mathbb{D},\Gamma)$ belongs to $T(\Gamma)$. This is so even if $d = \dim B_2(\mathbb{D},\Gamma) < \infty$. Even the case $d = 1$ is untractable.

In what follows, we will assume that the quotient Riemann surface $\mathbb{D}/\Gamma$ is a once-punctured torus T so that the Teichmüller space $T(\Gamma)$ has complex dimension one. In this case, two elements $\alpha, \beta \in \Gamma$ are called standard generators if the oriented intersection number $i(\alpha,\beta)$ in D/Γ with respect to the orientation coming from the complex structure of D is equal to $+1$.

In this paper, we provide an algorithm of producing the picture of $T(\Gamma)$ or even the "discreteness locus" concerning the holonomy representations in $B_2(\mathbb{D},\Gamma)$, and present the pictures of $T(\Gamma)$ in $B_2(\mathbb{D},\Gamma)$ for several Γ's and explain our algorithm for producing such pictures. Then, we describe our experiments concerning an open problem posed by C. McMullen [McMullen 1996] on the self-similarity of Bers slices.

To describe the idea of the algorithm, let us recall some basic facts in Teichmüller theory. For every ϕ in $B_2(\mathbb{D},\Gamma)$, there exists a locally univalent meromorphic function f_ϕ on $\mathbb{D}$ with $\{f_\phi, z\} = \phi(z)$, where $\{f, \cdot\}$ is the Schwarzian derivative of f. The function f_ϕ is called a developing map of ϕ and unique up to post-composition of Möbius transformations. The homomorphism $\theta_\phi : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ defined by

$$f_\phi \circ \gamma = \theta_\phi(\gamma) \circ f_\phi, \quad \gamma \in \Gamma,$$

Date: 29 October, 2004.

is called the *holonomy representation* of Γ associated with $\phi \in B_2(\mathbb{D}, \Gamma)$ and unique up to Möbius conjugacy. Note that this homomorphism θ_ϕ is *type preserving* in the sense that $\mathrm{tr}[\theta_\phi(\alpha), \theta_\phi(\beta)] = -2$ for any standard generators α, β of Γ. We consider the set $K(\Gamma)$, that is defined as the totality of those $\phi \in B_2(\mathbb{D}, \Gamma)$ for which $\theta_\phi(\Gamma)$ is discrete in $\mathrm{PSL}(2, \mathbb{C})$, i.e., $\theta_\phi(\Gamma)$ is a Kleinian group. Then $T(\Gamma)$ is equal to the component of $\mathrm{Int}\, K(\Gamma)$ containing the origin [Shiga 1987]. The other components of $\mathrm{Int}\, K(\Gamma)$ than $T(\Gamma)$ are called *exotic*. As early as in 1969, Maskit [Maskit 1969] pointed out the existence of exotic components and, in recent years, many authors have been studying the structure of the set $K(\Gamma)$ (see, for instance, [Ito 2000]). Though Goldman [Goldman 1987] succeeded in enumerating all the components of $\mathrm{Int}\, K(\Gamma)$ in terms of integral measured foliations, the shape and the configuration of those components is still unclear.

We actually draw the picture of $K(\Gamma)$ in $B_2(\mathbb{D}, \Gamma)$ for the given group Γ. The algorithm involves the following two steps: For each element ϕ in $B_2(\mathbb{D}, \Gamma) \cong \mathbb{C}$, we

Step 1: compute the holonomy representation θ_ϕ and
Step 2: decide whether the image $\theta_\phi(\Gamma)$ is discrete in $\mathrm{PSL}(2, \mathbb{C})$.

These steps will be described in the following sections.

Remark 1.1. The first and second named authors proposed a different approach to draw pictures of Bers embedding in [Komori and Sugawa 2004]. One can find a numerical method which enables us to present

(1) the image of holonomy representation corresponding to a given cusp boundary point,
(2) generators of a Fuchsian group uniformizing a given once-punctured torus,
(3) values of the accessory parameter (see section 2.2) and
(4) pictures of pleating loci.

On the other hand, the present approach given here has the following merits:

(1) we do not have to calculate the accessory parameter to get the picture and
(2) we can draw the pictures of exotic components besides the Bers slice.

Remark 1.2. Our definition of the (Bers embedded) Teichmüller space is different from the standard one. In the standard definition, our space $T(\Gamma)$ is the Teichmüller space of the surface $\mathbb{D}^*/\Gamma$, the mirror image of $\mathbb{D}/\Gamma$, where $\mathbb{D}^*$ is the exterior of the unit disk $\mathbb{D}$.

2. Holonomy representation

In this section we will describe an algorithm which takes an element ϕ of $B_2(\mathbb{D}, \Gamma)$ as the input and returns a holonomy representation θ_ϕ as the output. To make our calculation easier, we will work with a 4-times punctured sphere. For a detailed exposition, see [Komori and Sugawa 2004].

2.1. **Commensurability relations.** Fix a pair of standard generators (α, β) of Γ. Then the once-punctured torus T admits an intermediate covering space, the complex plane $\mathbb{C}$ minus lattice points $L_\tau = \{m + n\tau; m, n \in \mathbb{Z}\}$ so that α and β correspond to the generators

$$z \to z + 1, \quad z \to z + \tau$$

for L_τ, where τ is a complex number with $\mathrm{Im}\, \tau > 0$.

We observe that the mapping $z + L_\tau \mapsto 2z + L_\tau$ induces an unbranched covering of the 4-times punctured torus $\widetilde{T} = (\mathbb{C} - \frac{1}{2}L_\tau)/L_\tau$ onto T. We now choose a 4-times punctured sphere $S = \widehat{\mathbb{C}} - \{0, 1, \infty, \lambda\}$ so that T and S have the common covering space $\widetilde{T}$. Set $e_1 = \wp(1/2),\ e_2 = \wp(\tau/2),\ e_3 = \wp((1+\tau)/2)$ and

$$\lambda = \frac{e_3 - e_2}{e_1 - e_2},$$

where $\wp$ is the Weierstrass $\wp$-function with period lattice L_τ. Then a covering projection π of $\widetilde{T}$ onto S is given by

$$\pi(z + L_\tau) = \frac{\wp(z) - e_2}{e_1 - e_2}.$$

Note that $\lambda = \lambda(\tau)$ is known to be an elliptic modular function.

The canonical projection $\mathbb{D} \to \mathbb{D}/\Gamma = T$ induces the universal cover $\tilde{q} : \mathbb{D} \to \widetilde{T}$. Let Γ_S be the covering group of the universal covering projection $p = \pi \circ \tilde{q}$ of $\mathbb{D}$ onto S. Note that we have $B_2(\mathbb{D}, \Gamma_S) = B_2(\mathbb{D}, \Gamma)$ (see [Komori and Sugawa 2004]).

Let $B_2(S)$ be the Banach space of (hyperbolically) bounded holomorphic quadratic differentials on S. By definition, the spaces $B_2(\mathbb{D}, \Gamma_S)$ and $B_2(S)$ are isomorphic via the pull-back $p^* : B_2(S) \to B_2(\mathbb{D}, \Gamma_S)$ defined by $p^*\psi = \psi \circ p \cdot (p')^2$. The rational function

$$\psi_0(z) = \frac{1}{z(z-1)(z-\lambda)} \tag{1}$$

gives a non-trivial bounded quadratic differential $\psi_0(z)dz^2$, which forms a basis of the Banach space $B_2(S)$ since $\dim B_2(S) = 1$. Therefore each element $\phi \in B_2(\mathbb{D}, \Gamma) = B_2(\mathbb{D}, \Gamma_S)$ can be written as $\phi = t\phi_0$, where t is a complex number and $\phi_0 = p^*(\psi_0)$.

2.2. **The monodromy of a 4-times punctured sphere.** Now, for each $\phi = t\phi_0$, consider the developing map $f_\phi : \mathbb{D} \to \widehat{\mathbb{C}}$. Our idea is to compute f_ϕ on S instead of $\mathbb{D}$.

For this purpose, we take the branch P of p^{-1} around $p(0)$ so that $P(p(0)) = 0$ and put $g(z) := f_\phi(P(z))$. Then we have

$$\{g, z\} = \{f_\phi, P(z)\}(P'(z))^2 + \{P, z\} = t\psi_0(z) + \{P, z\}. \tag{2}$$

For $\{P, z\}$ in (2) we use the next lemma:

Lemma 2.1 ([Forsyth 1902] Ch. X, p. 492)**.** *The Schwarzian derivative of P is of the form*

$$\{P, z\} = \frac{1}{2z^2} + \frac{(1-\lambda)^2}{2(z-1)^2(z-\lambda)^2} + \frac{c(\lambda)}{z(z-1)(z-\lambda)}. \tag{3}$$

on S, where $c(\lambda)$ is a constant determined by λ and called the accessory parameter.

By the above lemma and (2), $\{g, z\}$ is globally defined on $\widehat{\mathbb{C}} - \{0, 1, \infty, \lambda\}$. Combining (1), (2) and (3), the equation to solve is

$$2y'' + \left(\frac{1}{2z^2} + \frac{(1-\lambda)^2}{2(z-1)^2(z-\lambda)^2} + \frac{t'}{z(z-1)(z-\lambda)}\right) y = 0, \tag{4}$$

where we have set $t' = t + c(\lambda)$. As is well known, $\{y_1/y_0, z\} = \{g, z\}$ holds. Hence, $f_\varphi = M \circ (y_1/y_0) \circ p$ around the origin for some Möbius transformation M.

We now describe how to compute the monodromy. Let γ_S be an element of Γ_S. We start with the pair (y_0, y_1) of fundamental solutions of (4) determined by the initial conditions

$$\begin{cases} y_0(z_0) & = 1 \\ y_0'(z_0) & = 0 \end{cases} \quad \text{and} \quad \begin{cases} y_1(z_0) & = 0 \\ y_1'(z_0) & = 1 \end{cases}$$

at a fixed point z_0 in S. Then we continue them analytically along a closed path in S corresponding to γ_S. Returning to the starting point, we will arrive with a new pair of solutions (Y_0, Y_1). However, these new solutions must be linear combinations of the original solutions. Thus we have

$$Y_0 = Dy_0 + Cy_1, \quad Y_1 = By_0 + Ay_1,$$

for some complex numbers A, B, C and D. We define

$$\tilde{\theta}_\psi(\gamma_S) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$$

for each $\gamma_S \in \Gamma_S$. We note that, by the monodromy theorem, the matrix is independent of the particular choice of the path corresponding to γ_S.

Since $f_\phi \circ \gamma_S$ corresponds to

$$\frac{Y_1}{Y_0} = \frac{A(y_1/y_0) + B}{C(y_1/y_0) + D},$$

we obtain the following lemma.

Lemma 2.2. *The monodromies θ_ϕ and $\tilde{\theta}_\psi$ are essentially the same. More precisely, on $\Gamma \cap \Gamma_S$, θ_ϕ is equal to the* $\mathrm{PSL}(2, \mathbb{C})$ *representation induced by $\tilde{\theta}_\psi$ up to Möbius conjugacy.*

So we can calculate θ_ϕ on S by means of (4). The reader can find a reason why the holonomy representation of Γ_S takes the values in $\mathrm{SL}(2, \mathbb{C})$ in [Komori and Sugawa 2004, Remark 4.1].

2.3. **Markov triples.** Though α and β are in Γ, they are not in Γ_S on which $\tilde{\theta}_\psi$ is defined. In other words, α and β do not correspond to the closed paths in S. So we need a little more calculation to end this section.

Let A and B be the matrices in $\mathrm{SL}(2, \mathbb{C})$ such that $\pm A = \theta_\phi(\alpha)$ and $\pm B = \theta_\phi(\beta)$ in $\mathrm{PSL}(2, \mathbb{C})$. Set $x = \operatorname{tr} A$, $y = \operatorname{tr} B$ and $z = \operatorname{tr} AB$. The triple (x, y, z) is well defined up to changing the signs of any two entries. It determines θ_ϕ up to conjugacy in $\mathrm{PSL}(2, \mathbb{C})$. In the next section, this holonomy is represented by using Jørgensen's normalization and denoted by $\theta_{x,y,z}$. Since our homomorphism is type preserving, the well-known trace identity $2 + \operatorname{tr}[X, Y] = (\operatorname{tr} X)^2 + (\operatorname{tr} Y)^2 + (\operatorname{tr} XY)^2 - \operatorname{tr} X \operatorname{tr} Y \operatorname{tr} XY$ implies the relation

$$x^2 + y^2 + z^2 = xyz. \tag{5}$$

Conversely, given any triple (x, y, z) satisfying (5), we can reconstruct the image of the group Γ up to conjugacy. We call such a triple of complex numbers a *Markov triple*.

Thus it suffices to compute x and y. Again by the trace identity $\operatorname{tr} X \operatorname{tr} Y = \operatorname{tr} XY + \operatorname{tr} XY^{-1}$, we have

$$x = \sqrt{\operatorname{tr} \tilde{\theta}_\psi(\alpha^2) + 2}, \quad y = \sqrt{\operatorname{tr} \tilde{\theta}_\psi(\beta^2) + 2}.$$

Now we can calculate $\tilde{\theta}_\psi(\alpha^2)$ and $\tilde{\theta}_\psi(\beta^2)$ by solving equation (4) because α^2 and β^2 are in Γ_S.

2.4. **Technical remarks.** A simple closed loop in S separating $\{0,1\}$ from $\{\infty,\lambda\}$ and that separating $\{0,\lambda\}$ from $\{1,\infty\}$ with two intersection points correspond to α^2 and β^2, respectively, with suitably chosen orientations. Practically, we choose polygonal curves with a common end point as such loops. For each oriented line segment of such curves, we solve the differential equation (4) in a numerical way and find the transition matrix of it along the segment. Then the ordered products of the transition matrices corresponding to the polygonal curves are representatives of α^2 and β^2 in $\mathrm{SL}(2,\mathbb{C})$ (see [Komori and Sugawa 2004] for details). Here, we may think that a value of the parameter t' is given in (4) instead of t so that we are free from the value of the accessory parameter $c(\lambda)$.

3. Jørgensen's theory to decide discreteness

The input of the algorithm of this section is a Markov triple and the output is the answer "discrete", "indiscrete" or "undecided".

The general idea is to try to construct a Ford fundamental region of the given Markov triple. If the image of the corresponding holonomy representation is indiscrete, the term "Ford fundamental region" does not make sense and our process of constructing it will fail. Then we will search for evidence of its indiscreteness.

This algorithm is based on Jørgensen's theory on once-punctured tori [Jørgensen 2003]. An exposition of this theory with proofs is in preparation [Akiyoshi et al.]. This algorithm may not halt in a finite time for some inputs. For example, if $\mathbb{H}^3/\theta_{x,y,z}(\Gamma)$ is geometrically infinite or a $\mathbb{Z}$-covering space of a punctured torus bundle over the circle, our algorithm will not stop in a finite time. In practice, we will stop our calculation at a certain time and give the answer "undecided".

3.1. **Notation.** Let T be a once-punctured torus. We fix standard generators α, β of the fundamental group of T. Let θ be a type preserving $\mathrm{PSL}(2,\mathbb{C})$ homomorphism of $\pi_1(T)$. Then θ can be specified by the Markov triple $x = \operatorname{tr}\theta(\alpha)$, $y = \operatorname{tr}\theta(\alpha\beta)$ and $z = \operatorname{tr}\theta(\beta)$ up to conjugation in $\mathrm{PSL}(2,\mathbb{C})$. We denote this representation by $\theta_{x,y,z}$.

Recall that a *slope* in T is the isotopy class of an essential simple closed curve on T. By choosing a basis of $H_1(T;\mathbb{Z})$, a slope is represented by a number in $\mathbb{Q}\cup\{1/0=\infty\}$. To fix our notation, we choose α and β as the basis so that the slope of α and β are $1/0$ and $0/1$ respectively. For a slope $q \in \mathbb{Q}\cup\{1/0\}$, set $S_q = \{g \in \pi_1(T) \mid \text{slope of } g = q\}$. Note that $\alpha \in S_{1/0}, \beta \in S_{0/1}$ and $\alpha\beta \in S_{1/1}$. We identify the set of slopes as a subset of $\partial\mathbb{H}^2$. Two rational numbers p/q and r/s are *Farey neighbors* if $|ps-qr| = 1$. By joining all pairs of Farey neighbors by geodesics, we get the *Farey tessellation* of $\mathbb{H}^2$ by ideal triangles. Note that the slopes of α, β and $\alpha\beta$ form an ideal triangle of the above tessellation. By taking the dual graph of this triangulation, we have a trivalent graph Σ properly embedded in $\mathbb{H}^2$. For each vertex v in Σ we can associate a subset S_v of $\pi_1(T)$ by

$$S_v = S_{q_1} \cup S_{q_2} \cup S_{q_3},$$

where slopes $q_1, q_2, q_3 \in \mathbb{Q}\cup\{\infty\}$ are the ideal vertices of the triangle in Farey tessellation which is dual to v. Set $I_v = \{\text{isometric hemisphere of } g \mid g \in S_v\}$.

Jørgensen's theory on punctured tori claims that if the image of the holonomy representation $\theta_{x,y,z}$ is discrete, then there is a path P in Σ which depends on (x, y, z) such that the boundary of the Ford region is given by $\bigcup_{v\in P} I_v$. After the Jørgensen's normalization, which will be introduced in subsection 3.2, we can define a direction of "upward" / "downward" in P. We will say that some vertex $v' \in P$ is upper/lower neighbor of $v \in P$ if v' is adjacent to v and the direction from v to v' is upward/downward. We will also use terms like "upper end point" / "lower end point" of P for end points of P.

In the next subsection, we recall Jørgensen's description. It describes the Ford region for a given discrete representation $\theta_{x,y,z}(\Gamma)$ with $v_0 \in P$ where $v_0 \in \Sigma$ be the dual of $1/0, 0/1$ and $1/1$. After this subsection, we will describe our algorithm.

3.2. **Jørgensen's description of the Ford region.** The Ford region of $\theta_{x,y,z}$ is defined (if the image of $\theta_{x,y,z}$ is discrete) to be the set of points lying above the isometric hemispheres of all elements in $\theta_{x,y,z}(\Gamma)$ not fixing ∞. Recall that the isometric hemisphere $I(A)$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$ with $A(\infty) \neq \infty$ is the hemisphere in $\mathbb{H}^3$ with radius $1/|c|$ centered at $-d/c \in \mathbb{C} = \partial\mathbb{H}^3 - \{\infty\}$. In order to obtain a fundamental region for $\theta_{x,y,z}(\Gamma)$, we have to take the intersection of this Ford region with some fundamental region for the stabilizer of ∞.

Now let (x, y, z) be a Markov triple. We can reconstruct $\theta_{x,y,z}$ by using Jørgensen's normalization [Jørgensen 2003]:

$$\theta_{x,y,z}(\alpha) = \frac{1}{x}\begin{pmatrix} xy - z & y/x \\ xy & z \end{pmatrix}, \quad \theta_{x,y,z}(\beta) = \frac{1}{x}\begin{pmatrix} xz - y & -z/x \\ -xz & y \end{pmatrix}. \tag{6}$$

Then we can check that

$$\theta_{x,y,z}(\alpha\beta) = \begin{pmatrix} x & -1/x \\ x & 0 \end{pmatrix}, \quad \theta_{x,y,z}(K) = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} \quad \text{where } K = [\alpha, \beta]. \tag{7}$$

The isometric hemispheres of α, $\alpha\beta$ and β are centered at $-z/xy$, 0 and y/zx with radii $1/y$, $1/x$ and $1/z$ respectively. It is easy to see that the isometric hemispheres of α^{-1}, $(\alpha\beta)^{-1}$ and $\beta^{-1}K^{-1}$ are the translated images of the above three hemispheres by $z \mapsto z + 1$. Since $\theta_{x,y,z}(\Gamma)$ contains the action $\theta_{x,y,z}(K)$ of translation $z \mapsto z + 2$, we have a bi-infinite sequence of translated images of the above three isometric hemispheres with symmetry of translation by one. Thus, we have a sequence of isometric hemispheres

$$\begin{aligned} &\ldots, I_{-4} = I(\alpha^{-1}K), I_{-3} = I((\alpha\beta)^{-1}K), I_{-2} = I(\beta^{-1}), I_{-1} = I(\alpha), I_0 = I(\alpha\beta), \\ &\qquad I_1 = I(\beta), I_2 = I(\alpha^{-1}), I_3 = I((\alpha\beta)^{-1}), I_4 = I(\beta^{-1}K^{-1}), I_5 = I(\alpha K^{-1}), \ldots \end{aligned} \tag{8}$$

See Figure 1. Note that $I_{n+3} = \sqrt{\theta_{x,y,z}(K)}(I_n)$ for any $n \in \mathbb{Z}$ where $\sqrt{\theta_{x,y,z}(K)}$ is the translation $z \mapsto z+1$. The group elements which correspond to I_{3n}, I_{3n+1} and I_{3n+2} belong to $S_{\alpha\beta}$, S_β and S_α respectively. Set $I_{1/1} := \{I_{3n}\}_{n\in\mathbb{Z}}$, $I_{0/1} := \{I_{3n+1}\}_{n\in\mathbb{Z}}$, $I_{1/0} := \{I_{3n+2}\}_{n\in\mathbb{Z}}$. $\{I_n\}_{n\in\mathbb{Z}}$ is equal to I_{v_0} as a set. We denote by L_{v_0} the polyline of infinite length given by connecting the centers of I_n and I_{n+1} for each $n \in \mathbb{Z}$.

Since we made an assumption that $v_0 \in P$ at the end of 3.1, it follows from Jørgensen's theory that we have:

(C1) Consecutive isometric hemispheres intersect with each other.

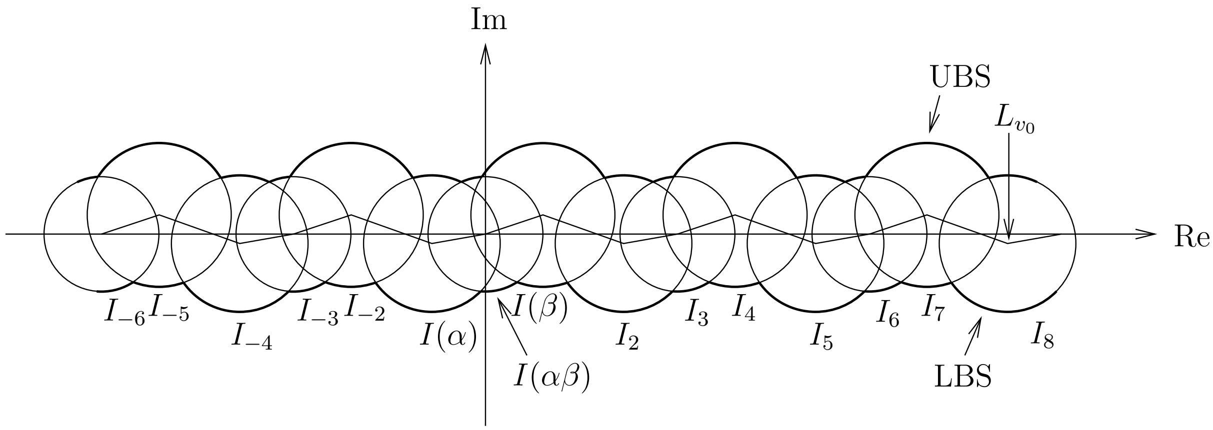

FIGURE 1. Isometric hemispheres

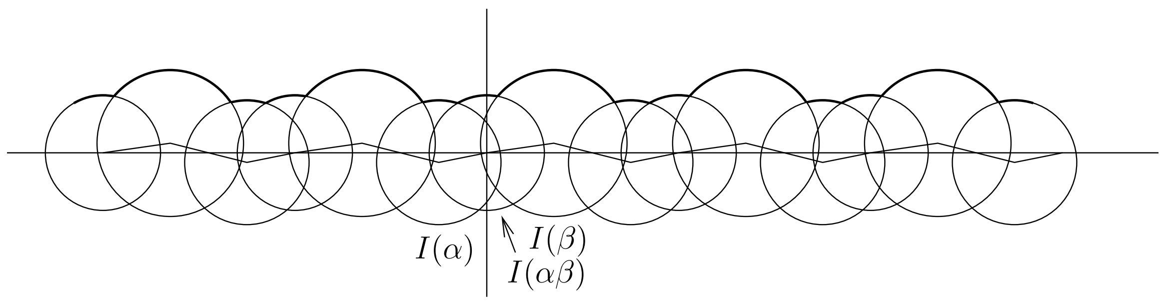

FIGURE 2. (S1)

(C2) Polyline L_{v_0} has no self-intersection.

So we have two sequences of sub-arcs of $\partial I_n \subset \mathbb{C}$ — upper boundary sequence UBS and lower boundary sequence LBS. See Figure 1.

For UBS and LBS, the set of subarcs can be divided into three groups — for those which come from $I_{1/0}, I_{0/1}$ and $I_{1/1}$. Let us consider UBS. We have three cases:

(S1) All the groups of subarcs $I_{1/0}, I_{0/1}$ and $I_{1/1}$ appear in the sequence. (Figure 2)
(S2) Only two groups of subarcs appear in the sequence and one group, say $I_{1/0}$, does not. (Figure 3)
(S3) Only one group, say $I_{0/1}$, appears in the sequence. (Figure 4)

The method to find upper neighbor and decide whether it is upper end point or not is as follows.

In case (S1), v_0 is the upper end point and there is no upper neighbor vertex for v_0. Next, suppose that UBS is of case (S2) and, for Faray triangle $\triangle q_1q_2q_3$ which is dual to v_0, only slope q_1 does not appear in UBS. There is a unique Faray triangle $\triangle q_2q_3q_4$ which is adjacent to $\triangle q_1q_2q_3$ along the geodesic connecting q_2 and q_3 and let v' be the dual vertex of it. Then v' is the upper neighbor of v_0. In case (S3), one of three vertices adjacent to v_0 is the upper neighbor. The role of choice is written in [Wada].

For LBS and lower neighbor, the rule is the same.

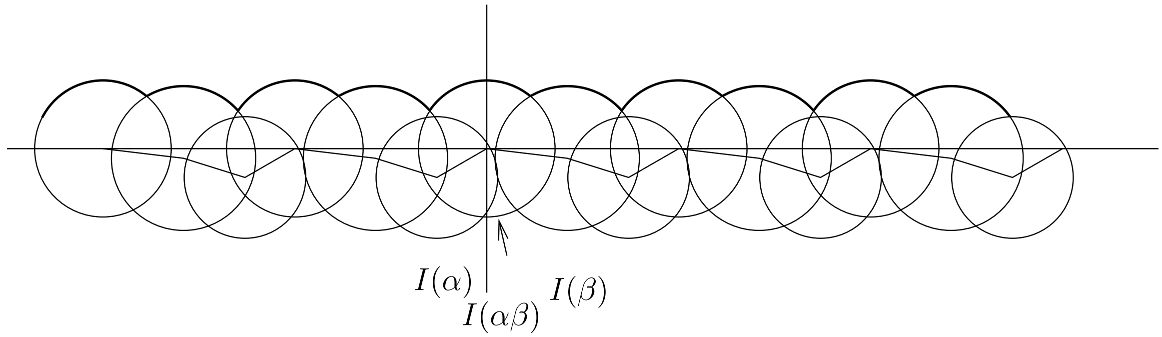

FIGURE 3. (S2)

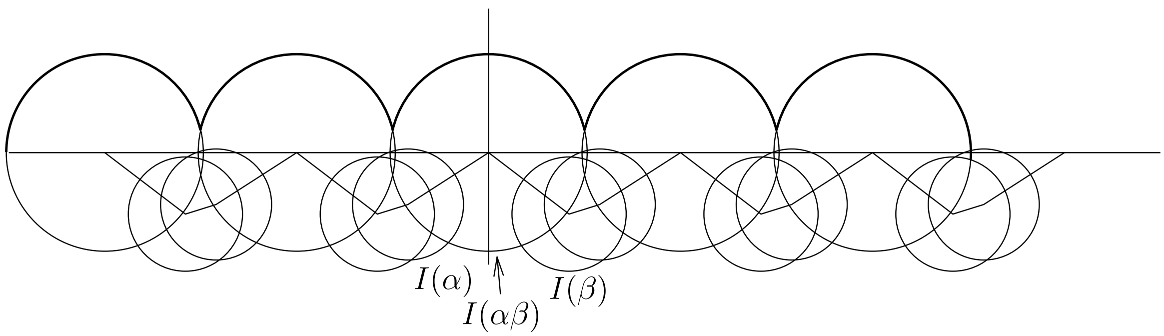

FIGURE 4. (S3)

For example, Figure 5-(a) depicts the case where both UBS and LBS of v_0 are of case (S2). The left hand side figure is the Faray diagram and its dual graph Σ. The right hand side is the picture of isometric hemispheres I_{v_0}. $I(\alpha)$ does not belong to UBS and the slope of α is $1/0$. In this case, v_1, which is the dual to the Faray triangle $0/1$, $1/1$ and $1/2$ is the upper neighbor of v_0. See Figure 5-(b).

Since UBS of v_1 is of case (S1), it is the upper end point.

If we carry out the same process for downward direction, we reach the vertex v_2 in Figure 5-(c) which turns out to be the lower end point. In this case, we conclude that the Jørgensen's path P is $v_1 v_0 v_2$.

3.3. **The algorithm.** In this subsection, we discuss the algorithm. Thus we do not assume that $v_0 \in P$. We also consider the condition for indiscreteness which we have not mentioned in the previous subsection.

Starting from v_0, we search Σ for Jørgensen's path. If we arrive at a new vertex in Σ, we get a new slope $q \in \mathbb{Q} \cup \{1/0\}$. Then we check the following Shimizu-Leutbecher's lemma for the elements of S_q and say "indiscrete" and stop the calculation if the condition is satisfied.

Lemma 3.1. *Suppose that a subgroup* Γ *of* $\mathrm{SL}(2,\mathbb{C})$ *contains* $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. *If there exists an element* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ *with* $0 < |c| < 1$, *then* Γ *is indiscrete.*

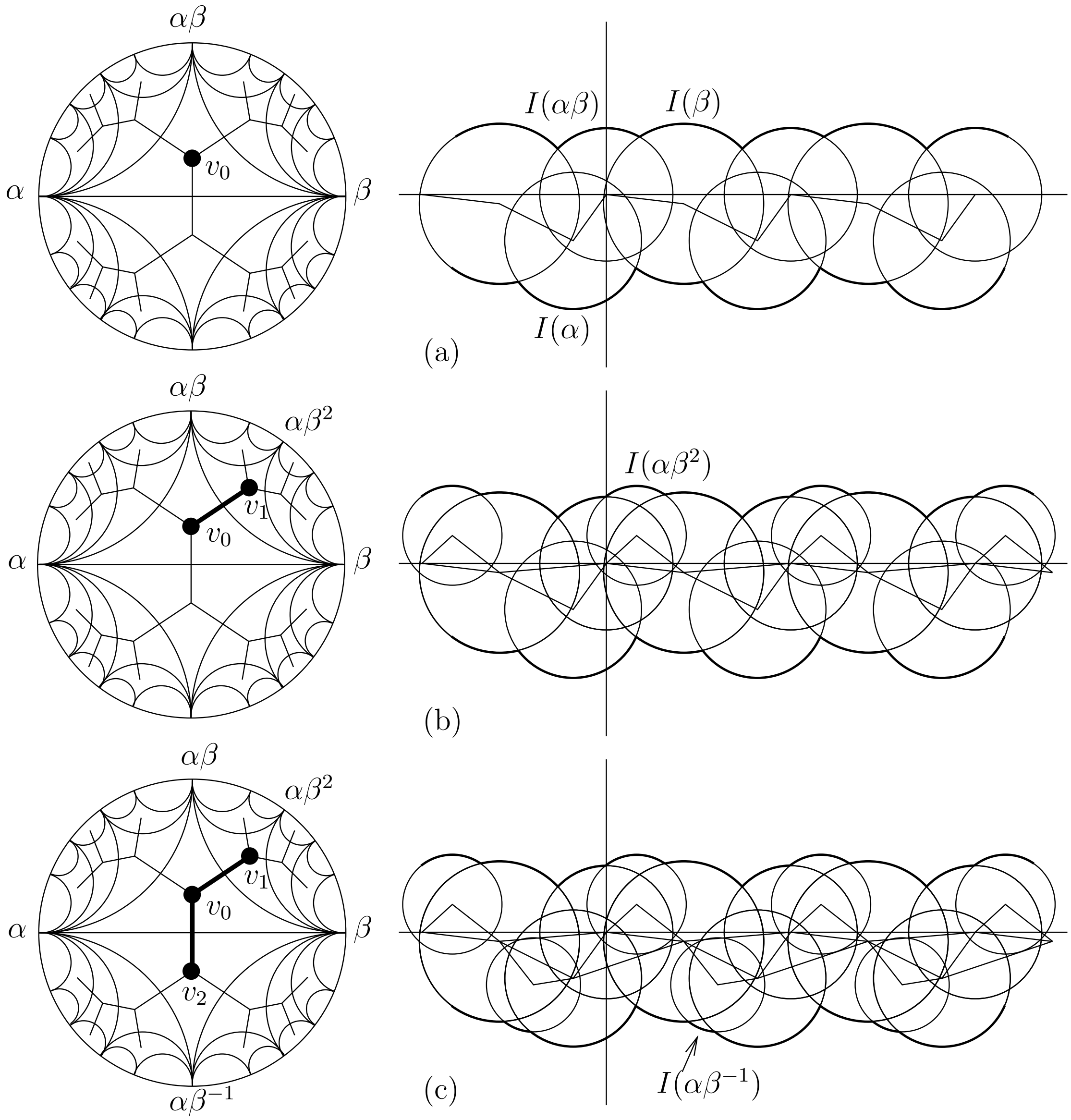

FIGURE 5. Markov triple $(x, y, z) = (2.536 - 1.115i, 2.616 - 0.645i, 2.203 + 0.660i)$. Left: Faray diagram and its dual graph Σ, Right: isometric hemispheres in upper half space model. (a) v_0: starting point (dual of $\triangle 01\infty$) (b) v_1: upper neighbor of v_0, top end point since it is of (S1) for UBS. (c) v_2: lower neighbor of v_0, lower end point since it is of (S1) for LBS. $v_2v_0v_1$: Jorgensen's Path. We conclude that $\theta_{(x,y,z)}$ is discrete.

Since the radius of the isometric hemisphere for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $1/|c|$, it follows that, in our setting, if there exists an isometric hemisphere of radius > 1, then the group is indiscrete.

After starting from v_0, our first task is to search for a vertex which satisfies the condition (C1). For $v \in \Sigma$, where $v =$ dual of $\triangle q_1q_2q_3$, a simple calculation shows that (C1) is

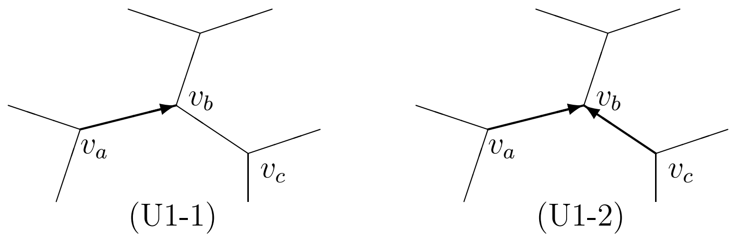

FIGURE 6. (U1) Suppose that, during the process, we have moved upward in Σ from v_a to v_b which turns out to be an upper most vertex. (U1-1) For v_b, v_a is the lower neighbor. (U1-2) For v_b, v_c is the lower neighbor (The direction of the arrows is from lower vertex to upper vertex)

equivalent to the triangle inequality for $|\tau(q_1)|, |\tau(q_2)|$ and $|\tau(q_3)|$, where $\tau(q) := \operatorname{tr} g$ with $g \in S_q$. So if v fails to satisfy (C1), one real number, say $|\tau(q_1)|$, is too large. Then we move to the adjacent vertex v' which is the dual to the Faray triangle of q_2, q_3 and the new slope — i.e., we discard the slope q_1. We repeat this process until we find a vertex which satisfies (C1).

Now suppose that we have found a vertex with (C1) satisfied. Next, we keep moving to the upper neighbor defined by the rule in the previous subsection until we must stop at some vertex v. We call this vertex v a *upper most vertex*, and we have two cases for v:

(U1) We stop because of case (S1).
(U2) We stop because v fails to satisfy (C2). (In this case, UBS and LBS are not well defined because we have used the condition (C2) to define UBS and LBS)

For later purpose, we define two subcases in case (U1). See figure 6

(U1-1) The lower neighbor vertex of v is where we come from.
(U1-2) The lower neighbor vertex of v is not where we come from.

We define the notions (D1), (D2), (D1-1) and (D1-2) for LBS in the same way. In case (U1), we change our direction and start moving to lower neighbor. In case (U2), we move to a neighbor by the rule we made by heuristics and consider this direction as "lower" and start moving to lower neighbor. In both two cases we keep moving in the direction of lower neighbor vertex in Σ.

For lower most vertex, we have the same cases (D1-1), (D1-2) and (D2) as above and again change our direction upward to move.

We continue this process for upper and lower directions alternately.

If we can find a path P in Σ such that the latest upper most vertex v_U is of case (U1-1) and the latest lower most vertex v_L (D1-1) and we can go from v_U to v_L by going downward and from v_L to v_U by upward, then this is the Jørgensen's path P. In this case, conditions for the Poincaré fundamental polygon theorem is satisfied and the output of our algorithm is "discrete". For detailed discussions of this Jørgensen's theory, see [Akiyoshi et al.].

4. Pictures

We present several pictures produced by our method in the following pages.

In Figure 8, $\lambda = 1/2$ and the corresponding once-punctured torus T is the square torus with one point removed. It is known that the accessory parameter $c(1/2)$ is equal to $1/2$,

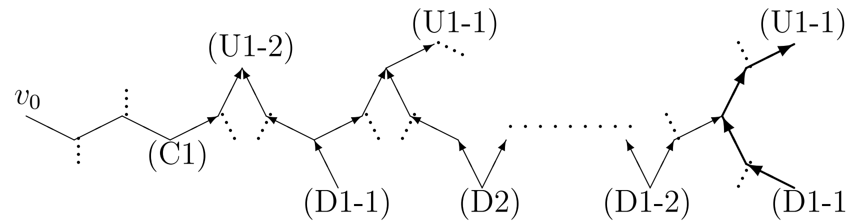

FIGURE 7. An example of the whole process. In Σ, starting from v_0, we search for a vertex with (C1) satisfied. Then we go upward until we reach at (U1) or (U2) vertex, say (U1-2). Then we go downward until (D1) or (D2), say (D1-1). We continue this alternating process of visiting vertices until we can find the Jørgensen's path as illustrated in thick arrows or some isometric hemisphere corresponding to the vertex at which we visit violates the Shimizu-Leutbecher condition.

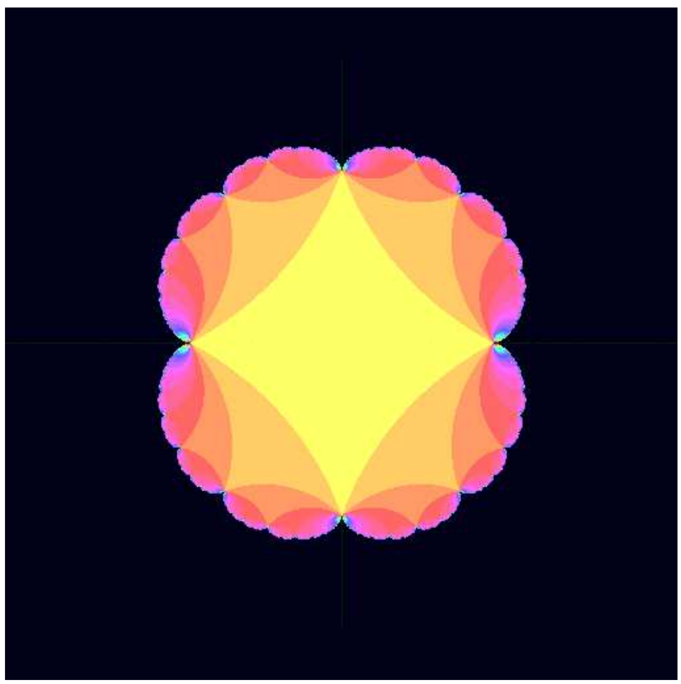

FIGURE 8. $\lambda = 1/2$, center$= 1/2$, range$= \pm 1/4$

and we take the center and the range to be $1/2$ and $\pm 1/4$ respectively. In the discreteness locus, a color is given according to the length of Jørgensen's path P mentioned in the previous section.

Figure 9 is a blowup of Figure 8. Many exotic components appear in this picture.

For Figure 10, $\lambda = 1/2 + \sqrt{3}/2i$ and T is a once-punctured torus with hexagonal symmetry. For the range of the parameter $t + c(\lambda)$, the center is $1/2 + 1/(2\sqrt{3})i$ and the range is $\pm 1/4$. Note that, to get the picture, we do not have to compute the exact value of the accessory parameter $c(\lambda)$ because it is hidden in the relative position of the origin.

Figure 11 is a blowup of Figure 10.

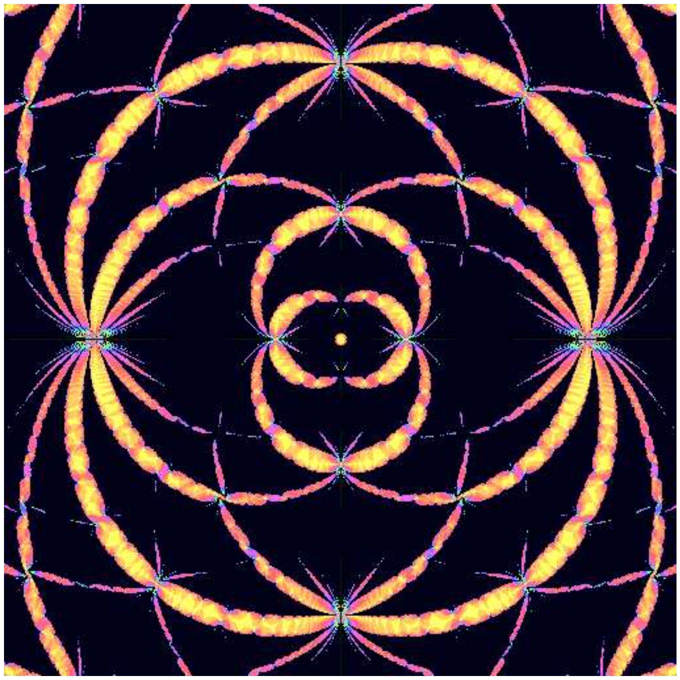

FIGURE 9. $\lambda = 1/2$, center$= 1/2$, range$= \pm 8$

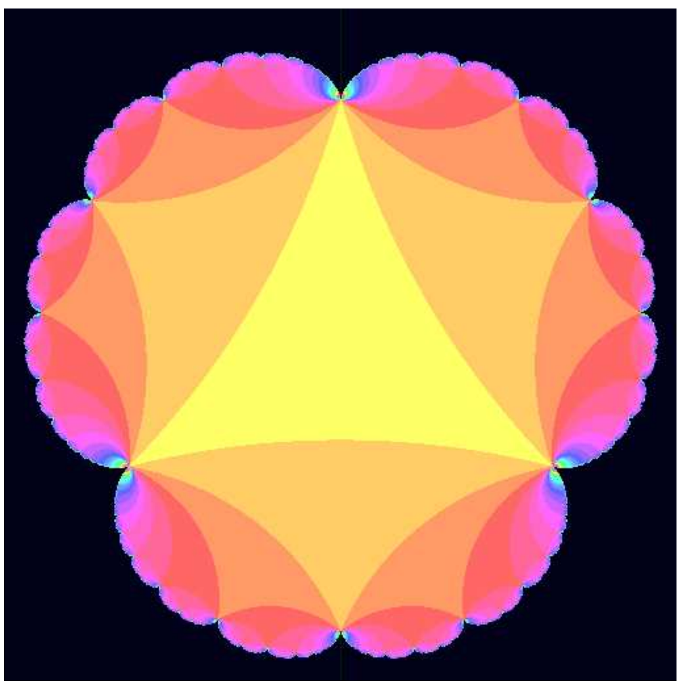

FIGURE 10. $\lambda = 1/2 + \sqrt{3}/2i$, center$= 1/2 + 1/(2\sqrt{3})\, i$, range$= \pm 1/4$

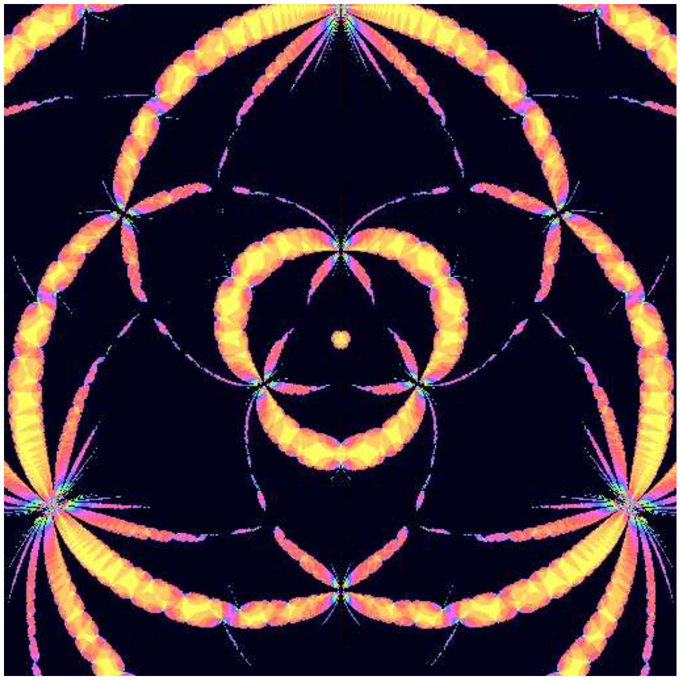

FIGURE 11. $\lambda = 1/2 + \sqrt{3}/2i$, center$= 1/2 + 1/(2\sqrt{3})\, i$, range$= \pm 8$

5. An experiment: Self-similarity of Bers slice

In [McMullen 1996, p. 178], McMullen asked "Is the boundary of a Bers slice self-similar?" and carried out a computer experiment for Maskit slice instead of Bers slice. His pictures of a part of Maskit boundary and its blowups suggest the affirmative answer for Maskit slice.

Motivated by his work, we have produced the following pictures.

Figure 12 depicts a part of the Bers slice of square torus ($\lambda = 1/2$). Figure 13 and Figure 14 are the blowups around the limit point $0.569645\cdots + 0.136675\cdots\, i$. Our conclusion is that this part of the boundary appears to have self-similarity around that point with scale factor about 4.8. That point also appears in [Sugawa 2002] as the farthest boundary point of the Teichmüller space of the once-punctured square torus from the origin and an observation was made there about the scale factor.

References

[Akiyoshi et al.] H. Akiyoshi, M. Sakuma, M. Wada and Y. Yamashita, Punctured torus groups and 2-bridge knot groups, In preparation.

[Bers 1972] L. Bers, Uniformization, moduli and Kleinian groups, *Bull. London Math. Soc.* **4** (1972), 257–300.

[Forsyth 1902] A. R. Forsyth, Theory of Differential equations Part III, Vol. IV, Cambridge University Press (1902).

[Goldman 1987] W. M. Goldman, Projective structures with Fuchsian holonomy, *J. Diff. Geom.* **25** (1987), 297–326.

[Ito 2000] K. Ito, Exotic projective structures and quasi-Fuchsian spaces, *Duke Math. J.* **105** (2000), 185–200.

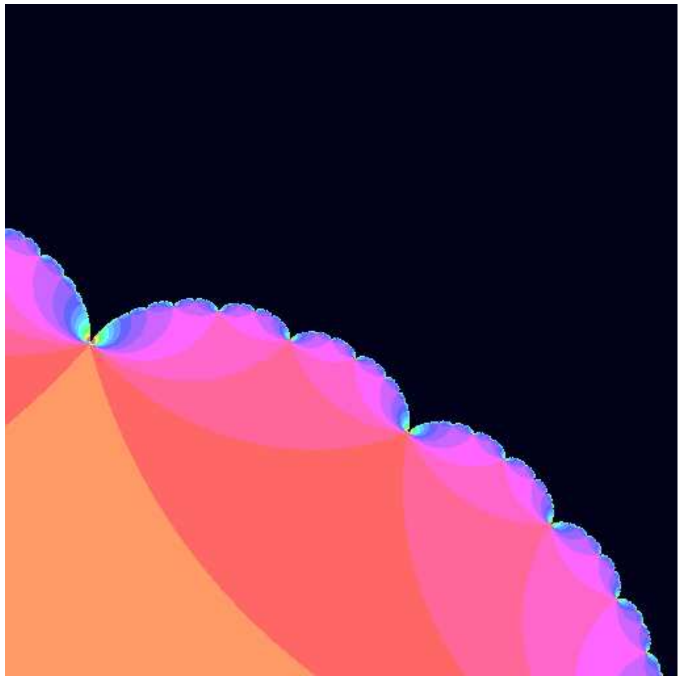

Figure 12. center $= 0.569645 + 0.136675\, i$, range=$\pm 0.0192$

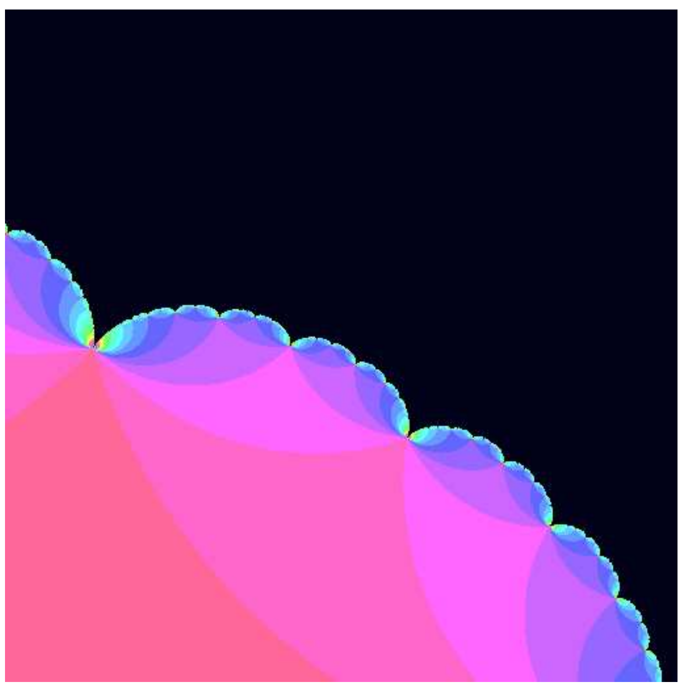

Figure 13. center $= 0.569645 + 0.136675\, i$, range=$\pm 0.004$

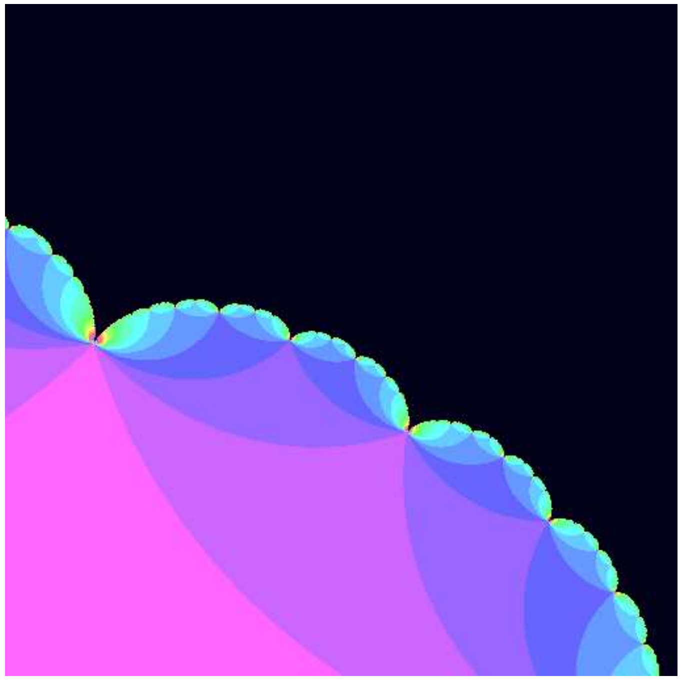

FIGURE 14. center $= 0.569645 + 0.136675\,i$, range$=\pm 0.000833$

[Jørgensen 2003] T. Jørgensen, On pairs of once-punctured tori, *London Math. Soc. Lec. Notes* **299** (2003), 183–207.

[Hempel 1988] J. A. Hempel, On the uniformization of the n-punctured sphere, *Bull. London Math. Soc.* **20** (1988), 97–115.

[Komori and Sugawa 2004] Y. Komori, T. Sugawa, Bers embedding of the Teichmüller space of a once-punctured torus, *Conform. Geom. Dyn.* **8** (2004), 115–142.

[Kra 1989] I. Kra, Accessory parameters for punctured spheres, *Trans. Amer. Math. Soc.* **313**:2 (1989), 589–617.

[Maskit 1969] B. Maskit, On a class of Kleinian groups, *Ann. Acad. Sci. Fenn. Ser. A* **442** (1969), 1–8. (1969).

[McMullen 1996] C. McMullen, Renormalization and 3-Manifolds which fiber over the Circle, *Ann. of Math. Stud.* **142** (1996).

[Shiga 1987] H. Shiga, Projective structures on Riemann surfaces and Kleinian groups, *J. Math. Kyoto Univ.* **27**:3 (1987), 433–438.

[Shiga and Tanigawa 1999] H. Shiga, and H. Tanigawa, Projective structures with discrete holonomy representations, *Trans. Amer. Math. Soc.* **351** (1999), 813–823.

[Sugawa 2002] T. Sugawa, Estimates of hyperbolic metric with applications to Teichmüller spaces, Kyungpook Math. J. **42** No.1 (2002), 51–60.

[Sugawa Web site] `http://www.cajpn.org/complex/Bers/` .

[Wada] M. Wada, OPTi's algorithm for Discreteness Determination, preprint.

[Yamashita Web site] `http://vivaldi.ics.nara-wu.ac.jp/~yamasita/Slice/` (The program and pictures).

Department of Mathematics, Osaka City University, Osaka, 558-8585, Japan
E-mail address: komori@sci.osaka-cu.ac.jp

Department of Mathematics, Graduate School of Science, Hiroshima University, Higashi-Hiroshima, 739-8526, Japan
E-mail address: sugawa@math.sci.hiroshima-u.ac.jp

Department of Information and Computer Sciences, Nara Women's University, Nara, 630-8506, Japan
E-mail address: wada@ics.nara-wu.ac.jp

Department of Information and Computer Sciences, Nara Women's University, Nara, 630-8506, Japan
E-mail address: yamasita@ics.nara-wu.ac.jp