Lindelöf theorems for monotone Sobolev functions

Yoshihiro Mizuta
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1 Introduction

Let $\mathbf{R}^n$ $(n \geq 2)$ denote the n-dimensional Euclidean space. We use the notation $\mathbf{D}$ to denote the upper half space of $\mathbf{R}^n$, that is,

$$\mathbf{D} = \{x = (x_1, ..., x_{n-1}, x_n) : x_n > 0\}.$$

Denote by $B(x, r)$ the open ball centered at x with radius r, and set $\sigma B(x, r) = B(x, \sigma r)$ for $\sigma > 0$ and $S(x, r) = \partial B(x, r)$.

Let μ be a (Radon) measure on $\mathbf{R}^n$ satisfying the doubling condition :

$$\mu(2B) \leq M\mu(B)$$

for every ball $B \subset \mathbf{R}^n$. If $1 < p < \infty$ and G is a bounded open set in $\mathbf{R}^n$, then we define the (p, μ)-capacity by

$$\mathrm{cap}_{p,\mu}(E; G) = \inf \int_G |f(y)|^p d\mu(y),$$

where the infimum is taken over all Borel measurable functions f such that

$$\int_G |x - y|^{1-p} f(y) dy \geq 1 \qquad \text{for all } x \in E.$$

We write $\mathrm{cap}_{p,\mu}(E) = 0$ if $\mathrm{cap}_{p,\mu}(E \cap G; G) = 0$ for all bounded open set G in $\mathbf{R}^n$.

Let $w \in L^1_{loc}(\mathbf{R}^n)$ be nonnegative. For $q > 1$, we say that $w \in A_q$ if there exists a constant $C > 0$ such that

$$\sup_B \left(\fint_B w(y) dy \right) \left(\fint_B w(y)^{1/(1-q)} dy \right)^{q-1} < C,$$

where the supremum is taken over all balls B in $\mathbf{R}^n$.

A continuous function u on $\mathbf{D}$ is called monotone in the sense of Lebesgue (see [5]) if for every relatively compact open set $G \subset \mathbf{D}$,

$$\max_{\overline{G}} u = \max_{\partial G} u \qquad \text{and} \qquad \min_{\overline{G}} u = \min_{\partial G} u.$$

If u is a monotone Sobolev function on $\mathbf{D}$ and $p > n-1$, then

$$|u(x) - u(x')| \leq Mr \left(\frac{1}{r^n} \int_{B(y,2r)} |\nabla u(z)|^p dz \right)^{1/p} \tag{1}$$

for all $x, x' \in B(y,r)$, whenever $B(y,2r) \subset \mathbf{D}$ (see [6, Theorem 1] and [4, Theorem 2.8]). For further results of monotone functions, we refer to [3], [7], [13] and [15].

Manfredi-Villamor [8] proved the following result.

THEOREM A. *Let $n-1 < p \leq n$ and w be an A_q weight for some $1 < q < p/(n-1)$. Let u be a monotone Sobolev function on the unit ball $\mathbf{B}$ satisfying*

$$\int_{\mathbf{B}} |\nabla u(z)|^p w(z) dz < \infty. \tag{2}$$

Then, for each $\varepsilon > 0$, there exists an open set U in $\mathbf{R}^n$ satisfying $cap_{p,w}(U) < \varepsilon$ such that for every $x_0 \in \partial \mathbf{B} \setminus U$, if we have a curve $\gamma \subset \mathbf{B}$ ending at x_0 with

$$\lim_{x \to x_0, x \in \gamma} u(x) = \alpha,$$

then it follows that u has a nontangential limit α at x_0.

Our aim in this talk is to improve their result.

2 Nontangential limits

Let μ be a measure on $\mathbf{R}^n$ satisfying the doubling condition and

$$\frac{\mu(B')}{\mu(B)} \geq M \left(\frac{\operatorname{diam}(B')}{\operatorname{diam}(B)} \right)^s \tag{3}$$

for all $B' = B(\xi', r')$ and $B = B(\xi, r)$ with $\xi', \xi \in \partial \mathbf{D}$ and $B' \subset B$, where $s > 1$ and $\operatorname{diam}(B)$ denotes the diameter of B.

A pair $(u, g) \in L^1_{loc}(\mathbf{D}; \mu) \times L^p_{loc}(\mathbf{D}; \mu)$ is called monotone if u is continuous on D and

$$|u(x) - u_B| \leq Mr \left(\frac{1}{\mu(\sigma B)} \int_{\sigma B} g(z)^p d\mu(z) \right)^{1/p} \tag{4}$$

for every $x \in B$ with $\sigma B \subset \mathbf{D}$, where $\sigma > 1$, $B = B(y, r)$ and

$$u_B = \fint_B u(y)d\mu(y) = \frac{1}{\mu(B)} \int_B u(y)d\mu(y).$$

Our first aim is to establish the nontangential limit result.

THEOREM 1. Let (u, g) be a monotone pair. Define

$$E_1 = \{\xi \in \partial\mathbf{D} : \int_{B(\xi,1)\cap\mathbf{D}} |\xi - y|^{1-n} |\nabla u(y)| dy = \infty\} \tag{5}$$

and

$$E_2 \quad = \quad \{\xi \in \partial\mathbf{D} : \limsup_{r\to 0} \left(r^{-p}\mu(B(\xi, r))\right)^{-1} \int_{B(\xi,r)\cap\mathbf{D}} g(y)^p d\mu(y) > 0\}.$$

Then u has a nontangential limit at every $\xi \in \partial\mathbf{D} - (E_1 \cup E_2)$.

For a proof of Theorem 1, it suffices to note the following easy lemma.

LEMMA 1. If $B = B(a, r)$ is a ball and $B \subset G$, then

$$\mathrm{cap}_{p,\mu}(B; G) \leq M r^{-p} \mu(B).$$

REMARK 1. If

$$\int_{\mathbf{D}} |\nabla u(x)|^p d\mu(x) < \infty \tag{6}$$

and

$$\int_{\mathbf{D}} g(x)^p d\mu(x) < \infty, \tag{7}$$

then we see that $E_1 \cup E_2$ is of $\mathrm{cap}_{p,\mu}$-capacity zero.

REMARK 2. Let $1 < q < p/(n-1)$. Let w be a Muckenhoupt (A_q) weight, and define

$$d\mu(y) = w(y)dy.$$

If u is monotone in the sense of Lebesgue, then $(u, |\nabla u|)$ satisfies the monotonicity property (4) by applying Hölder's inequality to (1) with p replaced by p/q (see also Manfredi-Villamor [8]).

COROLLARY 1. *Let $n - 1 < p \leq n$ and w be an A_q weight for some $1 < q < p/(n-1)$. Let u be a monotone Sobolev function on $\mathbf{D}$ satisfying (2). Then there exists a set $E \subset \partial\mathbf{D}$ satisfying $\mathrm{cap}_{p,w}(E) = 0$ such that u has a nontangential limit at every $x_0 \in \partial\mathbf{D} \setminus E$.*

3 Lindelöf theorem

Now we show the Lindelöf theorem for monotone Sobolev functions.

THEOREM 2. *Let (u, g) be a monotone pair with g satisfying (7). Suppose $p > s - 1$, and set*

$$E = \{\xi \in \partial \mathbf{D} : \limsup_{r \to 0} \left(r^{-p}\mu(B(\xi, r))\right)^{-1} \int_{B(\xi,r)\cap \mathbf{D}} g(z)^p d\mu(z) > 0\}.$$

If $\xi \in \partial \mathbf{D} - E$ and there exists a curve γ in $\mathbf{D}$ tending to ξ along which u has a finite limit, then u has a nontangential limit at ξ.

REMARK 3. In Theorem2, if we consider

$$h(r) = \sup_{0<t<r, \xi \in \partial \mathbf{D}} r^{-p}\mu(B(\xi, r)),$$

then we see that $H_h(E) = 0$.

REMARK 4. Let $1 < q < p/(n-1)$. Let w be a Muckenhoupt (A_q) weight, and define

$$d\mu(y) = w(y)dy.$$

Then (3) holds for $s = nq$. In this case, however, we do not need the condition that $p > s - 1$.

COROLLARY 2. *Let $1 < q < p/(n-1)$ and w be a Muckenhoupt (A_q) weight. Suppose u is a monotone Sobolev function on $\mathbf{D}$ satisfying (6), and set*

$$E = \{\xi \in \partial \mathbf{D} : \limsup_{r \to 0} \left(r^{-p}w(B(\xi, r))\right)^{-1} \int_{B(\xi,r)\cap \mathbf{D}} |\nabla u(z)|^p w(z)dz > 0\}.$$

If $\xi \in \partial \mathbf{D} - E$ and there exists a curve γ in $\mathbf{D}$ tending to ξ along which u has a finite limit, then u has a nontangential limit at ξ.

Corollary 3. *Let u be a monotone Sobolev function on $\mathbf{D}$ satisfying*

$$\int_{\mathbf{D}} |\nabla u(z)|^p z_n^\alpha dz < \infty, \tag{8}$$

where $p > n-1$ and $0 \le n+\alpha-p < 1$. Define

$$E_{n+\alpha-p} = \{\xi \in \partial\mathbf{D} : \limsup_{r\to 0} r^{p-\alpha-n} \int_{B(\xi,r)\cap\mathbf{D}} |\nabla u(z)|^p z_n^\alpha dz > 0\}.$$

If $\xi \in \partial\mathbf{D} - E_{n+\alpha-p}$ and there exists a curve γ in $\mathbf{D}$ tending to ξ along which u has a finite limit, then u has a nontangential limit at ξ.

Remark 5. We know that $E_{n+\alpha-p}$ has $(n+\alpha-p)$-dimensional Hausdorff measure zero, and hence it is of $C_{1-\alpha/p,p}$-capacity zero; for these results, see Meyers [9, 10] and the author's book [13].

Remark 6. Let $w(y) = |y_n|^\alpha$ and $q > 1$. Then $w \in A_q$ if and only if $-1 < \alpha < q-1$. In this case, Corollary 2 may not imply Corollary 3 when $n \ge 3$.

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The Division of Mathematical and Information Sciences
Faculty of Integrated Arts and Sciences
Hiroshima University
Higashi-Hiroshima 739-8521, Japan