CARDIOIDS AND TEICHMÜLLER SPACES

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Abstract. In this note, we give an expository account on the role played by cardioids in both Teichmüller spaces and in complex dynamics. Especially, we will observe that the shape of the Bers embedding of the Teichmüller space of a once-punctured rectangular torus converges to a cardioid in the sense of Carathéodory when the base surface is pinched along the meridian. A more general and complete result will be included in a forthcoming paper [27] of the author.

1. Introduction

A cardioid is a plane curve similar to $\{e^{i\theta} - e^{2i\theta}/2 : \theta \in \mathbb{R}\}$, see Figure 1. In this note, however, a *cardioid* means the (open) Jordan region bounded by a cardioid curve. We will call the *standard cardioid* the domain bounded by the above curve and denote it by C_0. Note that the function $f_0(z) = z - z^2/2$ maps the unit disk univalently onto C_0. Conventionally, we denote by aC_0+b the image of C_0 under the similarity map $z \mapsto az+b$.

It is an accidental coincidence that the cardioid appears both in the Kleinian group theory and in complex dynamics. However, according to Sullivan's dictionary, the Bers embedding of the Teichmüller space of a Riemann surface (or a Fuchsian group) corresponds to the Mandelbrot set (or its generalization). Therefore, this coincidence might not be a surprising fact. As is well known, the Mandelbrot set contains a cardioid (the

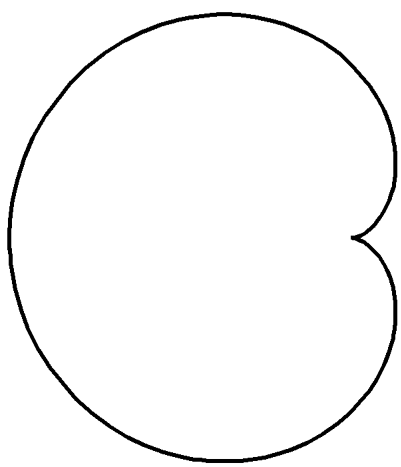

Figure 1. The cardioid

Date: November 13, 2007, `File: teich_HRI.tex`.

2000 *Mathematics Subject Classification.* Primary 30F60; Secondary 30C62, 37F45.

Key words and phrases. cardioid, Teichmüller space, holomorphic motion, once-punctured torus.

The author was partially supported by the JSPS Grant-in-Aid for Scientific Research (B), 17340039.

so-called main cardioid) as a connected component of its interior. On the other hand, as Kalme [12] pointed out, the cardioid appears in the universal Teichmüller space in a natural way.

In this note, we point out similarities between the cardioid in the Mandelbrot set and cardioids in the Bers embedded Teichmüller spaces. Our emphasis will be put on the role played by holomorphic motions to give a unified aspect on these matters. Moreover, we will observe that the shape of a one-dimensional Teichmüller space tends to that of a cardioid when the base Riemann surface goes to the boundary of the moduli space, at least in a special case. It appears that this is a phenomenon which has not been observed in the literature.

Acknowledgements. The author would like to thank the organizers of the International Workshop on Teichmüller Theory and Moduli Problems held at the Harish-Chandra Research Institute, Allahabad, India, for giving him a chance to consider the present topic seriously. Without their kind invitation to the workshop, this work would not have been carried out due to the author's busyness and laziness.

Finally, he thanks the referee for careful reading and for constructive suggestions.

2. Holomorphic motions

We briefly summarize the basic properties of holomorphic motions for convenience of the reader.

Let E be a subset of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and D be a domain (or a complex manifold in general) with base point t_0. A holomorphic motion of E over (D, t_0) is a map $H : D \times E \to \widehat{\mathbb{C}}$ satisfying the following three conditions:

(i) $H_t = H(t, \cdot) : E \to \widehat{\mathbb{C}}$ is injective for each $t \in D$,
(ii) $H(\cdot, z) : D \to \widehat{\mathbb{C}}$ is holomorphic for each $z \in E$, and
(iii) $H(t_0, z) = z$ for $z \in E$.

Mañé-Sad-Sullivan [15] revealed the following remarkable facts about holomorphic motions over the unit disk (see also [3]): Let H be a holomorphic motion of E over $(\mathbb{D}, 0)$. Then

(a) H extends to a holomorphic motion of $\overline{E}$ uniquely in such a way that the extended $H : \mathbb{D} \times \overline{E} \to \widehat{\mathbb{C}}$ is (jointly) continuous, and
(b) H_t is $|t|$-quasiconformal on each component of $\operatorname{Int} \overline{E}$ for each $t \in \mathbb{D}$.

Here and hereafter, $\mathbb{D}$ denotes the unit disk $\{z \in \mathbb{C} : |z| < 1\}$. Note that we do not assume any continuity in the variable z for the holomorphic motion $H(t, z)$. For a constant $0 \le k < 1$, a sense-preserving homeomorphism $f : \Omega \to \Omega'$ between domains in $\widehat{\mathbb{C}}$ is called k-quasiconformal if f has locally integrable partial derivatives on $\Omega \setminus \{\infty, f^{-1}(\infty)\}$ (in the sense of distribution) such that $|f_{\bar{z}}| \le k|f_z|$ holds a.e.

Slodkowski [25] proved another remarkable fact: *every holomorphic motion H of E over the unit disk can be extended to that of $\widehat{\mathbb{C}}$. In particular, H_t is a restriction of a $|t|$-quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ for each $t \in \mathbb{D}$.*

Readers interested in holomorphic motions may consult recent textbooks on Teichmüller theory such as Gardiner-Lakic [5] and Hubbard [10] for details.

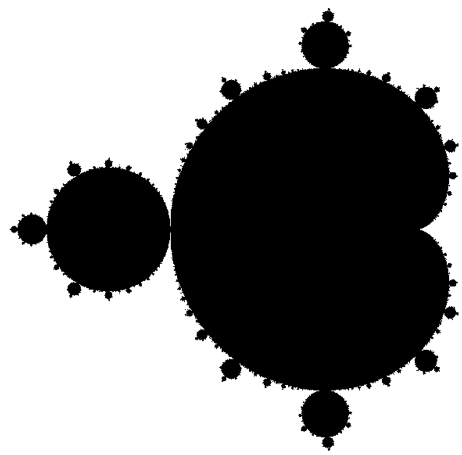

FIGURE 2. The Mandelbrot set

3. Cardioid in Mandelbrot set

Even polynomials of the simple form $P_c(z) = z^2 + c$ are not fully understood in the context of iterations, that is, in the theory of complex dynamics. We denote by P_c^n the n-th iterate of P_c, namely, $P_c^1 = P_c$ and $P_c^n = P_c^{n-1} \circ P_c$ for $n = 2, 3, \ldots$. Here c is a complex parameter. The boundary of the set $K_c = \{z \in \mathbb{C} : \{P_c^n(z)\}_{n=1,2,\ldots} \text{ is bounded }\}$ is called the Julia set and denoted by J_c. The Mandelbrot set M is defined to be the set of $c \in \mathbb{C}$ for which J_c is connected (see Figure 2). It is known that J_c is either connected or a Cantor set and that $c \in M$ iff $0 \in K_c$.

An *attracting fixed point* of P_c is a point $\alpha \in \mathbb{C}$ such that $P_c(\alpha) = \alpha$ and $|P_c'(\alpha)| < 1$. It is easily seen that a neighbourhood of an attracting fixed point of P_c is contained in K_c. In particular, $c \in M$. A fixed point α of P_c must satisfy the relation $P_c(\alpha) - \alpha = \alpha^2 - \alpha + c = 0$. If we set $\lambda = P_c'(\alpha) = 2\alpha$, then $c = \lambda/2 - (\lambda/2)^2$. We now conclude that P_c has an attracting fixed point iff $c \in D = \{\lambda/2 - (\lambda/2)^2 : \lambda \in \mathbb{D}\}$. Therefore, D is contained in the Mandelbrot set. Indeed, it is easy to see that D is a component of the interior of M and $D = (1/2)C_0$. The component D is called the main cardioid of M.

We now see that the Julia set J_c varies holomorphically (in a sense) on the parameter $c \in D$. A point z is called *periodic* for P_c if $P_c^n(z) = z$ for an integer n and it is called *repelling* further if $|(P_c^n)'(z)| > 1$. It is well known that the set of repelling periodic points of P_c is a dense subset of J_c. It is easy to see that $R_0 = \{\mathrm{e}^{2\pi\mathrm{i}\theta} : 2^n\theta \in \mathbb{Z} \text{ for some } n \in \mathbb{N}\}$ is the set of repelling fixed points of $P_0(z) = z^2$. Since repelling periodic points do not collide for $c \in D$, for each $z_0 \in R_0$ one can take a repelling periodic point z_c of P_c, $c \in D$, in such a way that z_c is a holomorphic function of $c \in D$ assuming z_0 for $c = 0$. In this way, we obtain a function $H_c(z_0) = z_c$ on R_0 for each $c \in D$. The same reasoning as above yields that the function H_c is injective on R_0. Thus the map $H : (c, z) \mapsto H_c(z)$ gives a holomorphic motion of R_0 over $(D, 0)$. Since R_0 is dense in $\partial\mathbb{D}$, by the Mañé-Sad-Sullivan theorem, the holomorphic motion of R_0 extends to that of $\partial\mathbb{D}$ which will be still denoted by the same symbol $H(c,z)=H_c(z)$. Thus, one has the relation $J_c=H_c(\partial\mathbb{D})$ for $c\in D$. By the Slodkowski theorem, the map H_c can be extended to a quasiconformal map of $\widehat{\mathbb{C}}$ and thus $J_c=H_c(\partial\mathbb{D})$ is known to be a quasidisk for $c\in D$. For details and interesting figures, see Astala and Martin [2].

It is known that near the cusp point $1/4$ of the main cardiod D rather complicated phenomena of phase transition are observed, which are called the parabolic implosion. See [4] for details and references.

4. Bers embedding of Teichmüller space

We give minimal basics in the theory of Teichmüller spaces to state and to prove our result. For details, see [18] or [11].

In the following, we denote by $\mathbb{H}$ the upper half-plane $\{z\in\mathbb{C}:\ \operatorname{Im} z>0\}$ and by $\mathbb{H}^*$ the lower half-plane $\{z\in\mathbb{C}:\ \operatorname{Im} z<0\}$. Let Γ be a Fuchsian group acting on $\mathbb{H}$, namely, Γ is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$. We denote by $\mathrm{Belt}(\mathbb{H},\Gamma)$ the space of Beltrami coefficient for Γ on $\mathbb{H}$, more precisely,

$$\mathrm{Belt}(\mathbb{H},\Gamma)=\{\mu\in L^\infty(\mathbb{H}):\|\mu\|_\infty<1,\ (\mu\circ\gamma)\frac{\overline{\gamma'}}{\gamma'}=\mu \text{ for all } \gamma\in\Gamma\}.$$

For $\mu\in\mathrm{Belt}(\mathbb{H},\Gamma)$, we denote by f^μ the quasiconformal map f of $\widehat{\mathbb{C}}$ which is determined by the Beltrami equations $f_{\bar z}=\mu f_z$ in $\mathbb{H}$ and $f_{\bar z}=0$ in $\mathbb{H}^*$ and normalization conditions $f(0)=0, f(1)=1$ and $f(\infty)=\infty$. Note that $f^\mu\Gamma(f^\mu)^{-1}$ is a Kleinian group acting on $f^\mu(\mathbb{H})$ and $f^\mu(\mathbb{H}^*)$ properly discontinuously.

Noting that f^μ is conformal on $\mathbb{H}^*$, we define a holomorphic function $\Phi(\mu)$ on $\mathbb{H}^*$ by $\Phi(\mu)=S_{f^\mu|\mathbb{H}^*}$, where S_f stands for the Schwarzian derivative of f :

$$S_f=\left(\frac{f''}{f'}\right)'-\frac{1}{2}\left(\frac{f'}{f}\right)^2=\frac{f'''}{f'}-\frac{3}{2}\left(\frac{f'}{f}\right)^2.$$

It is convenient to measure the Schwarzian derivative by the norm

$$\|\varphi\|_{\mathbb{H}^*}=\sup_{z\in\mathbb{H}^*}(-2\operatorname{Im} z)^2|\varphi(z)|.$$

We denote by $B_2(\mathbb{H}^*)$ the Banach space consisting of analytic functions φ on $\mathbb{H}^*$ with $\|\varphi\|_{\mathbb{H}^*}<\infty$. Let

$$B_2(\mathbb{H}^*,\Gamma)=\{\varphi\in B_2(\mathbb{H}^*):(\varphi\circ\gamma)(\gamma')^2=\varphi \text{ for all } \gamma\in\Gamma\},$$

which is a closed subspace of $B_2(\mathbb{H}^*)$.

It is known that $\Phi(\mu)\in B_2(\mathbb{H}^*,\Gamma)$ for $\mu\in\mathrm{Belt}(\mathbb{H},\Gamma)$ and that the image $\Phi(\mathrm{Belt}(\mathbb{H},\Gamma))$ coincides with the Bers embedding of the Teichmüller space of Γ. We set $\mathrm{Teich}(\Gamma)=\Phi(\mathrm{Belt}(\mathbb{H},\Gamma))$ and we identify it with the Teichmüller space of Γ (or equivalently, of the orbifold $\mathbb{H}/\Gamma$). It is known that $\mathrm{Teich}(\Gamma)$ is a bounded contractible domain in $B_2(\mathbb{H}^*,\Gamma)$. The map $\Phi_\Gamma=\Phi:\mathrm{Belt}(\mathbb{H},\Gamma)\to\mathrm{Teich}(\Gamma)$ is called the Bers projection and known to be a holomorphic split submersion. In the case when Γ is the trivial group 1, the set $\mathrm{Teich}(1)$ is called the *universal Teichmüller space.*

We denote by $o(\Gamma)$ and $i(\Gamma)$ the outer and inner radii of the Teichmüller space $\mathrm{Teich}(\Gamma)$, in other words, $o(\Gamma)$ is the smallest number r with $\mathrm{Teich}(\Gamma)\subset\{\varphi\in\mathrm{B}_2(\mathbb{H}^*,\Gamma):\|\varphi\|_{\mathbb{H}^*}<r\}$ and $i(\Gamma)$ is the largest number r with $\mathrm{Teich}(\Gamma)\supset\{\varphi\in\mathrm{B}_2(\mathbb{H}^*,\Gamma):\|\varphi\|_{\mathbb{H}^*}<r\}$.

It is well known that $i(1) = 2$ and $o(1) = 6$ for trivial group 1, and hence, $2 \le i(\Gamma) \le o(\Gamma) \le 6$ for an arbitrary Fuchsian group Γ unless $\mathrm{Teich}(\Gamma)$ is a singleton. For a cofinite Fuchsian group Γ, it is also known that $i(\Gamma) > 2$ (cf. [20]) and $o(\Gamma) < 6$ (due to Sekigawa [24]). Furthermore, Nakanishi and Yamamoto [22] and Nakanishi and Velling [21] gave characterizing conditions for Γ to satisfy $o(\Gamma) = 6$ and $i(\Gamma) = 2$, respectively. It is remarkable that their conditions are coincident. In particular, $o(\Gamma) = 6$ iff $i(\Gamma) = 2$. One can see that the present investigation lie in the same line as above-mentioned studies.

5. Kalme's observation

Kalme [12] made an interesting observation on a special holomorphic family of quasiconformal maps of the Riemann sphere. Following [12], we present some facts related to our investigation.

For a complex number $\alpha \in \mathbb{C}$, consider the function

$$F_\alpha(z) = z^\alpha = e^{\alpha \log z}$$

on $\mathbb{H}^*$, where the branch of $\log z$ is taken so that $-\pi < \arg z = \operatorname{Im} \log z < 0$ for $z \in \mathbb{H}^*$. Since $F'_\alpha(z) = \alpha z^{\alpha-1} \neq 0$, the map F_α is locally univalent for $\alpha \neq 0$. A characterization of univalence of F_α is known.

Lemma 1 (Royster [23]). *Let $\alpha \in \mathbb{C} \setminus \{0\}$. Then, F_α is univalent in $\mathbb{H}^*$ if and only if either $|\alpha - 1| \le 1$ or $|\alpha + 1| \le 1$.*

An easy computation gives

$$S_{F_\alpha}(z) = \frac{1-\alpha^2}{2z^2} = 2(1-\alpha^2)\varphi_0^*(z),$$

where $\varphi_0^* \in B_2(\mathbb{H}^*)$ is given by $\varphi_0^*(z) = 1/(4z^2)$. Note that $\|\varphi_0^*\|_{\mathbb{H}^*} = 1$.

An interesting fact is that the holomorphic motion $f_t = F_{1-t}$ ($|t| < 1$) of $\mathbb{H}^*$ can be extended to that of the Riemann sphere in an explicit way:

$$f_t(z) = \begin{cases} z\bar{z}^{-t} & z \in \overline{\mathbb{H}} \\ z^{1-t} & z \in \mathbb{H}^*. \end{cases}$$

Note that f_t is normalized and has the Beltrami coefficient

$$\mu_{f_t}(z) = -t\frac{z}{\bar{z}} = -t\frac{|\varphi_0(z)|}{\varphi_0(z)}$$

for $z \in \mathbb{H}$, where $\varphi_0(z) = \overline{\varphi_0^*(\bar{z})} = 1/(4z^2)$ for $z \in \mathbb{H}$. This is of the form of a Teichmüller differential on $\mathbb{H}$ and, by definition of Φ, one has the relation

$$\Phi(-t\mu_0) = 2(1-(1-t)^2)\varphi_0^* = g_0(t)\varphi_0^*, \tag{5.1}$$

where

$$\mu_0 = \frac{|\varphi_0|}{\varphi_0} \quad \text{and} \quad g_0(t) = 4\left(t - \frac{t^2}{2}\right).$$

We can now see that the intersection of $\mathrm{Teich}(1)$ with the linear span of φ_0^* is precisely the cardioid $4C_0$ (times φ_0^*).

Hille [9] observed that for $\varepsilon \in \mathbb{R} \setminus \{0\}$, $F_{i\varepsilon}$ is a universal covering projection onto an annulus, and hence, is never univalent whereas $S_{F_{i\varepsilon}}(z) = (1+\varepsilon^2)/(2z^2)$ and thus the norm

$\|S_{F_{i\varepsilon}}\|_{\mathbb{H}^*} = 2(1+\varepsilon^2)$ tends to 2 as $\varepsilon \to 0$. We point out that the point $S_{F_{i\varepsilon}}$ approaches to the inward cusp of the above cardioid $4C_0\varphi_0^*$ from the outside.

6. One-dimensional Teichmüller spaces.

From now on, we restrict ourselves to one-dimensional Teichmüller spaces. For simplicity, we assume Fuchsian groups to be torsion-free. Then $\dim \mathrm{Teich}(\Gamma) = 1$ iff the signature of Γ is (0,4) or (1,1). Since to each Fuchsian group Γ of signature (0,4), there corresponds a Fuchsian group Γ' of signature (1,1) such that $\mathrm{Teich}(\Gamma) = \mathrm{Teich}(\Gamma')$ (cf. [13]), without loss of generality we may further assume that Γ is of signature (1,1). Then the quotient Riemann surface $\mathbb{H}/\Gamma$ is represented by a once-punctured torus of the form $T = (\mathbb{C}\setminus\Omega)/\Omega$, where

$$\Omega = \langle \omega_1, \omega_2 \rangle_{\mathbb{Z}} = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$$

and $\omega_1, \omega_2 \in \mathbb{C}$ with $\mathrm{Im}\,(\omega_2/\omega_1) > 0$. We also write $\mathrm{Teich}(T)$ for $\mathrm{Teich}(\Gamma)$ when $T = \mathbb{H}/\Gamma$.

In recent years, much progress has been made in the study of one-dimensional Teichmüller spaces. For instance, Minsky [16] proved that the Bers embedding of a one-dimensional Teichmüller space is a Jordan domain. By using Minsky's method, Miyachi [17] showed that the one-dimensional Teichmüller space is "cusp-shaped" at every boundary point corresponding to a cusp. Goodman [6] even observed a spiraling shape of the boundary of the one-dimensional Teichmüller space. Computer graphics of the Bers embeddings are presented by [13] and [14]. Also, the inner and outer radii of the Teichmüller space of a once-punctured square torus are numerically computed in [26].

In [27], the following result is proved.

Theorem 2. *The Bers embedding* $\mathrm{Teich}(T)$ *of the Teichmüller space of a once-punctured torus* T *converges to a cardioid in the sense of Carathéodory* (*under suitable indentification of* $B_2(T) = B_2(\mathbb{D}, \Gamma)$ *with* $\mathbb{C}$), *when* T *goes to infinity in the moduli space*

There is some technicality with the suitable choice of the basis of the vector space $B_2(T)$ so that one can identify $B_2(T)$ with $\mathbb{C}$. Therefore, in this note, we restrict ourselves to the case when T is a rectangular torus for the sake of simplicity. See [27] for the general case.

Let Ω_λ be the lattice generated by $\lambda > 0$ and $\mathrm{i} = \sqrt{-1}$ over $\mathbb{Z}$, namely, $\Omega_\lambda = \{m\lambda + n\mathrm{i} : m, n \in \mathbb{Z}\}$. We set $T_\lambda = (\mathbb{C} \setminus \Omega_\lambda)/\Omega_\lambda$. We denote by $\pi_\lambda(\zeta) = [\zeta]$ the canonical projection $\mathbb{C} \setminus \Omega_\lambda \to T_\lambda$.

Take a holomorphic universal covering projection p_λ of the upper half-plane $\mathbb{H}$ onto $\mathbb{C} \setminus \Omega_\lambda$ so that $p_\lambda(\mathrm{i}) = a_\lambda$ and $p'_\lambda(\mathrm{i})/\mathrm{i} > 0$. Then $q_\lambda = \pi_\lambda \circ p_\lambda$ is a holomorphic universal covering projection of $\mathbb{H}$ onto T_λ. Let Γ_λ be the covering transformation group of $q_\lambda : \mathbb{H} \to T_\lambda$. Note that $\mathbb{H}/\Gamma_\lambda = T_\lambda$.

Let $\tilde{\varphi}_\lambda$ be the pullback of the quadratic differential $\mathrm{d}\zeta^2$ on $\mathbb{C} \setminus \Omega_\lambda$ under the map p_λ, namely, $\tilde{\varphi}_\lambda = (p'_\lambda)^2$. Further let

$$\varphi_\lambda = (\|\tilde{\varphi}_\lambda\|_{\mathbb{H}})^{-1}\tilde{\varphi}_\lambda.$$

Then, by definition, $\varphi_\lambda \in B_2(\mathbb{H}, \Gamma_\lambda)$ and $\|\varphi_\lambda\|_{\mathbb{H}} = 1$.

Set also

$$\varphi_\lambda^*(z) = \overline{\varphi_\lambda(\bar{z})}.$$

Then $\varphi_\lambda^* \in B_2(\mathbb{H}^*, \Gamma_\lambda)$ and $\|\varphi_\lambda^*\|_{\mathbb{H}^*} = 1$.

Since $d\zeta^2$ on $\mathbb{C} \setminus \Omega_\lambda$ projects to a nontrivial holomorphic quadratic differential on T_λ via π_λ, the vector space $B_2(\mathbb{H}^*, \Gamma_\lambda)$ is spanned by φ_λ^*. Let

$$U_\lambda = \{w \in \mathbb{C} : w\varphi_\lambda^* \in \mathrm{Teich}(\Gamma_\lambda)\}.$$

Then, our result can be stated as in the following.

Theorem 3. *The domain U_λ converges to the cardioid $4C_0$ in the sense of Carathéodory as $\lambda \to 0+$.*

Since U_λ is a simply connected bounded domain (indeed, a Jordan domain by Minsky's theorem), there exists a conformal homeomorphism g_λ of the unit disk $\mathbb{D}$ onto U_λ such that $g_\lambda(0) = 0$ and $g_\lambda'(0) > 0$. Recall that $g_0(t) = 2(1 - (t-1)^2)$ is a conformal map of $\mathbb{D}$ onto the cardioid $4C_0$. Then, the last theorem can be restated in the following way.

Theorem 4. *The conformal map g_λ converges to the map g_0 locally uniformly on $\mathbb{D}$.*

We illustrate the shape of the boundary of $\mathrm{Teich}(\Gamma_\lambda)$ for several values of λ (see Figures 3 through 6, and compare with Figure 7). We remark that the boundary curve is drawn very roughly near the "main" cusp when λ is small. This is simply because of our algorithm of computation.

7. Proof of Theorem 4

Let $\rho_{\mathbb{C}\setminus\Omega_\lambda}(\zeta)|d\zeta|$ and $\rho_{T_\lambda}([\zeta])|d\zeta|$ be the hyperbolic metrics on $\mathbb{C} \setminus \Omega_\lambda$ and T_λ, respectively. The density functions (the hyperbolic densities) $\rho_{\mathbb{C}\setminus\Omega_\lambda}(\zeta)|d\zeta|$ and $\rho_{T_\lambda}([\zeta])$ are

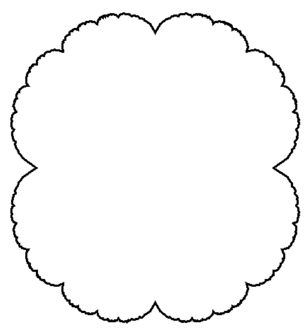

FIGURE 3. Bers embedding for the punctured square torus ($\lambda = 1$)

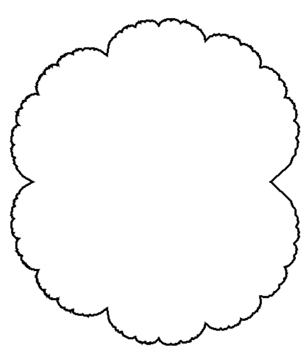

FIGURE 4. Bers embedding for small λ

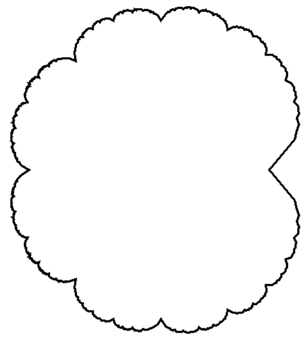

FIGURE 5. Bers embedding for very small λ

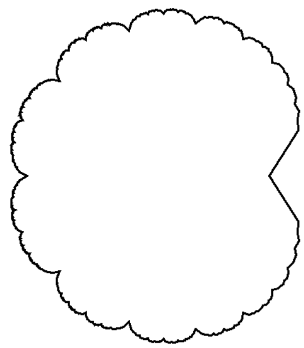

FIGURE 6. Bers embedding for very very small λ

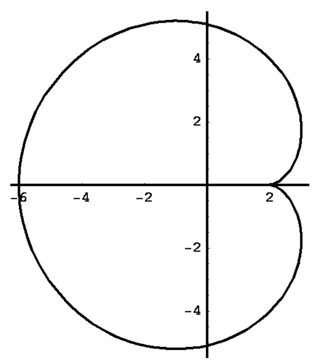

FIGURE 7. The cardioid $4C_0 = \{2(1-\alpha^2) : |\alpha - 1| < 1\}$.

characterized by the relations $\rho_{T_\lambda}([\zeta]) = \rho_{\mathbb{C}\setminus\Omega_\lambda}(\zeta)$ and $\rho_{\mathbb{C}\setminus\Omega_\lambda}(p_\lambda(z))|p'_\lambda(z)| = \rho_{\mathbb{H}}(z) = 1/(2\,\mathrm{Im}\, z)$. The following assertion can be found (in a more general form) in [26].

Lemma 5. *The hyperbolic density $\rho_{T_\lambda}([\zeta])$ of T_λ takes its minimum at the point $[a_\lambda] = [(\lambda + \mathrm{i})/2]$.*

Corollary 6. *The supremum of $(2\,\mathrm{Im}\, z)^2|\varphi_\lambda(z)|$ over $z \in \mathbb{H}$ is attained at $z = \mathrm{i}$.*

Proof. Letting $\zeta = p_\lambda(z)$, we have the relation

$$(2\,\mathrm{Im}\, z)^2|\varphi_\lambda(z)| = \rho_{\mathbb{C}\setminus\Omega_\lambda}(\zeta)^{-2},$$

which is known to take its maximum at $\zeta = a_\lambda = p_\lambda(\mathrm{i})$ by the last lemma. □

By the normalization of φ_λ, we have thus the relation $4|\varphi_\lambda(\mathrm{i})| = \|\varphi_\lambda\|_{\mathbb{H}} = 1$. On the other hand, $\tilde{\varphi}_\lambda(\mathrm{i}) = p'_\lambda(\mathrm{i})^2 < 0$. Therefore, $\varphi_\lambda(\mathrm{i}) = \tilde{\varphi}_\lambda(\mathrm{i})/\|\tilde{\varphi}_\lambda\|_{\mathbb{H}} < 0$ and thus

$$\varphi_\lambda(\mathrm{i}) = -\frac{1}{4}. \tag{7.1}$$

We also note the following fact, which will be used later.

Lemma 7. *The hyperbolic density $\rho_{\mathbb{C}\setminus\Omega_\lambda}(\xi + \mathrm{i}/2)$ converges to $\pi/2$ uniformly in $\xi \in \mathbb{R}$ as $\lambda \to 0+$.*

Proof. Observe that $\mathbb{C} \setminus \Omega_\lambda$ converges to the parallel strip $S = \{\zeta : 0 < \operatorname{Im}\zeta < 1\}$ with respect to the point $\mathrm{i}/2$ in the sense of Carathéodory when $\lambda \to 0+$. Thus, the hyperbolic density $\rho_{\mathbb{C}\setminus\Omega_\lambda}$ converges to ρ_S locally uniformly on S (cf. [8]). Since $\rho_{\mathbb{C}\setminus\Omega_\lambda}(\xi + \mathrm{i}/2)$ is a periodic function of ξ with period λ and since $\rho_S(\xi + \mathrm{i}/2) = \pi/2$, the assertion follows. □

We now define the curves

$$\alpha_\lambda(s) = [a_\lambda + \lambda s] \ (0 \le s \le 1)$$
$$\beta_\lambda(s) = [a_\lambda + \mathrm{i}s] \ (0 \le s \le 1)$$

on T_λ, where we recall that $a_\lambda = (\lambda + \mathrm{i})/2$. Then, by the obvious symmetry of T_λ, these are simple hyperbolic geodesics of T_λ (see Figure 8).

We take the lifts $\hat{\alpha}_\lambda$ and $\hat{\beta}_\lambda$ of them starting at i via the covering projection $q_\lambda : \mathbb{H} \to T_\lambda$. We denote by A_λ and B_λ the elements of $\Gamma_\lambda \subset \mathrm{PSL}(2, \mathbb{R})$ corresponding to $\hat{\alpha}_\lambda$ and $\hat{\beta}_\lambda$, respectively. In other words, $\hat{\alpha}_\lambda(1) = A_\lambda(\hat{\alpha}_\lambda(0)) = A_\lambda(\mathrm{i})$ and $\hat{\beta}_\lambda(1) = B_\lambda(\hat{\beta}_\lambda(0)) = B_\lambda(\mathrm{i})$. See Figure 9.

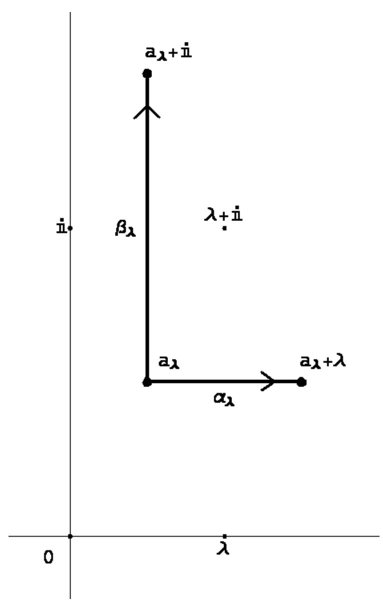

FIGURE 8. Geodesics on T_λ

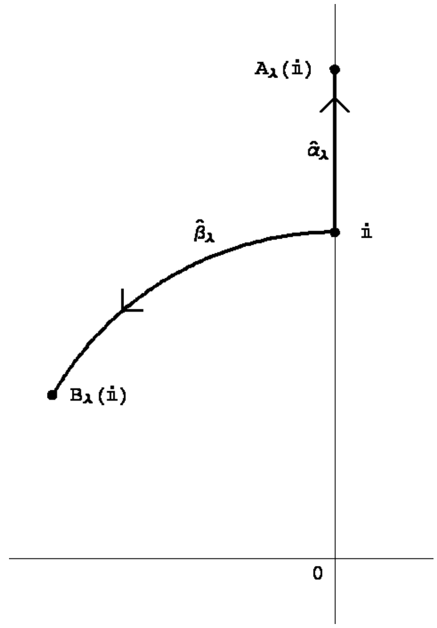

FIGURE 9. Lifts of curves on $\mathbb{H}$

Let σ and τ be the hyperbolic lengths of curves α_λ and β_λ, respectively. Then A_λ and B_λ can be represented by

$$A_\lambda = \pm \begin{pmatrix} e^\sigma & 0 \\ 0 & e^{-\sigma} \end{pmatrix} \quad \text{and} \quad B_\lambda = \pm \begin{pmatrix} \cosh\tau & \sinh\tau \\ \sinh\tau & \cosh\tau \end{pmatrix}.$$

Since the commutator $[A_\lambda, B_\lambda]$ must be parabolic, the following relation is required:

$$\sinh\sigma \sinh\tau = 1.$$

We remark that this sort of relation was already found by Hayman [7, §7].

We note the following fact.

Lemma 8. *The hyperbolic length σ of α_λ tends to 0 as $\lambda \to 0+$.*

Proof. By definition, σ can be computed by

$$\sigma = \int_0^\lambda \rho_{T_\lambda}([\xi + \mathrm{i}/2])d\xi = \int_0^\lambda \rho_{\mathbb{C}\setminus\Omega_\lambda}(\xi + \mathrm{i}/2)d\xi.$$

By Lemma 7, we see that $\sigma \sim \pi\lambda/2$ as $\lambda \to 0+$. In particular, the assertion follows. □

A crucial result is the following variant of a proposition due to T. Nakanishi [19, Proposition 3.1].

Lemma 9 (Nakanishi's lemma). *Let G_n be a sequence of Fuchsian groups acting on $\mathbb{H}$ each of which contains a hyperbolic element of the form $z \mapsto M_n z$ such that $M_n \to 1$ as $n \to \infty$. Further let φ_n be an element of $B_2(\mathbb{H}, G_n)$ such that φ_n converges locally uniformly to a holomorphic function φ_∞ on $\mathbb{H}$. Then $\varphi_\infty(z) = c/z^2$ for some constant c.*

Proof. Write $\varphi_n(z) = P_n(z)/z^2$ and $\varphi_\infty(z) = P_\infty(z)/z^2$. Then $P_n \to P_\infty$ locally uniformly. Since $\varphi_n(M_n z)M_n^2 = \varphi_n(z)$, we have $P_n(M_n^k z) = P_n(z)$ for any $k \in \mathbb{Z}$. Letting $n \to \infty$, we can see that P_∞ is constant along the positive imaginary axis, which implies that P_∞ is constant. □

Lemma 10. *The quadratic differential $\varphi_\lambda(z)$ converges to $\varphi_0(z) = 1/(4z^2)$ locally uniformly on $\mathbb{H}$ as $\lambda \to 0+$.*

Proof. Since $\|\varphi_\lambda\|_{\mathbb{H}} = 1$, the family $\{\varphi_\lambda\}$ is locally bounded on $\mathbb{H}$ and thus normal. Let λ_n be any sequence of positive numbers tending to 0 such that φ_{λ_n} converges locally uniformly on $\mathbb{H}$. Then Lemmas 8 and 9 imply that the limit function has the form c/z^2. On the other hand, by (7.1), the constant c must be $1/4$, which is independent of the sequence λ_n. Thus a standard argument gives us the local uniform convergence of φ_λ to $1/(4z^2)$. □

Let

$$\mu_\lambda = \frac{|\varphi_\lambda|}{\varphi_\lambda}.$$

Then $t\mu_\lambda \in \mathrm{Belt}(\mathbb{H}, \Gamma_\lambda)$ for every $t \in \mathbb{D}$. Since $\Phi(-t\mu_\lambda) \in B_2(\mathbb{H}^*, \Gamma_\lambda)$, we can write $\Phi(-t\mu_\lambda) = c\varphi_\lambda^*$ for some $c \in \mathbb{C}$ for each $t \in \mathbb{D}$. We define the function $h_\lambda : \mathbb{D} \to \mathbb{C}$ by the relation

$$\Phi(-t\mu_\lambda) = h_\lambda(t)\varphi_\lambda^*.$$

We now show the following.

Lemma 11. *The function h_λ is a conformal homeomorshism of $\mathbb{D}$ onto U_λ satisfying $h_\lambda(0) = 0$, $\overline{h_\lambda(t)} = h_\lambda(\bar{t})$, and, in particular, $h'_\lambda(0) \in \mathbb{R} \setminus \{0\}$.*

Proof. Since the map $t \mapsto \Phi_{\Gamma_\lambda}(-t\mu_\lambda)$ is of the form of a Teichmüller map, h_λ is known to be a proper holomorphic injection of $\mathbb{D}$ into U_λ (see, for instance, [18]). Since $\dim \mathrm{Teich}(\Gamma_\lambda) = 1$, this map is indeed a conformal homeomorphism.

By the symmetry of Ω_λ, we have the relation $\overline{p_\lambda(-\bar{z}) - a_\lambda} + a_\lambda = p_\lambda(z)$. Therefore, $\varphi_\lambda = (p'_\lambda)^2$ satisfies $\overline{\varphi_\lambda(-\bar{z})} = \varphi_\lambda(z)$. Hence, $\overline{\mu_\lambda(-\bar{z})} = \mu_\lambda(z)$. This implies $\overline{\Phi(\bar{t}\mu_\lambda)(-\bar{z})} = \Phi(t\mu_\lambda)(z)$, which is equivalent to $\overline{h_\lambda(\bar{t})\varphi_\lambda^*(-\bar{z})} = h_\lambda(t)\varphi_\lambda^*(z)$. Since $\overline{\varphi_\lambda^*(-\bar{z})} = \varphi_\lambda^*(z)$, we have the relation $\overline{h_\lambda(t)} = h_\lambda(\bar{t})$. □

Remark. The quasiconformal deformation of T_λ corresponding to $\Phi(-t\mu_\lambda)$ can be constructed explicitly. Indeed, define a holomorphic motion H of $\mathbb{C}$ over $(\mathbb{D}, 0)$ by

$$H(t, \zeta) = H_t(\zeta) = \frac{\zeta - t\bar{\zeta}}{1+t}, \quad t \in \mathbb{D},\ \zeta \in \mathbb{C}.$$

Note that $H_t(\Omega_\lambda) = \Omega_{\frac{1-t}{1+t}\lambda}$. Thus, H_t induces a quasiconformal map $T_\lambda \to T_{\frac{1-t}{1+t}\lambda}$. Since the Beltrami coefficient of H_t is $-t$, its lift to $\mathbb{H}$ via p_λ has the Beltrami coefficient $-t\overline{p'_\lambda}/p'_\lambda = -t|\varphi_\lambda|/\varphi_\lambda = -t\mu_\lambda$.

By Lemma 10, we see that the Teichmüller differential $\mu_\lambda = |\varphi_\lambda|/\varphi_\lambda$ coverges pointwise to $\mu_0 = |\varphi_0|/\varphi_0 = z/\bar{z}$. We now use the following basic property of quasiconformal maps. It can be read from a paper of Ahlfors and Bers [1], but we give a proof for convenience.

Proposition 12. *Let μ_n be a sequence of measurable functions on $\mathbb{C}$ such that $\|\mu_n\|_\infty \le 1$ and $\mu_n \to \mu$ a.e. Then the normalized $t\mu_n$-quasiconformal map $f^{t\mu_n}(z)$ converges to $f^{t\mu}(z)$ locally uniformly in $(t,z) \in \mathbb{D} \times \mathbb{C}$.*

Proof. We set $f_n(t,z) = f^{t\mu_n}(z)$ and $f_\infty(t,z) = f^{t\mu}(z)$. Suppose that real numbers $\varepsilon > 0, 0 < k < 1, T > 0$ are given. By Theorem 8 in [1], the function $f_n(t,z)$ is uniformly continuous in the parameter t for $n = 1, 2, \dots, \infty$ and $|z| \le T$. Hence, there exists a $\delta > 0$ such that $|f_n(z,t) - f_n(z,t')| < \varepsilon$ whenever $n = 1, 2, \dots, \infty,\ |z| \le T$ and $|t - t'| < \delta$. We choose finitely many points $t_1, \dots, t_m$ in the disk $|t| \le k$ so that $\min\{|t - t_j|, j = 1, \dots, m\} < \delta$ for every t with $|t| \le k$. Theorem 9 in [1] implies that $f_n(t_j, z) \to f_\infty(t_j, z)$ as $n \to \infty$ uniformly on $|z| \le T$ for each j. Thus, there exists a number N such that $|f_n(t_j, z) - f_\infty(t_j, z)| < \varepsilon$ holds whenever $j = 1, \dots, m, |z| \le T$ and $n \ge N$.

For each t with $|t| \le k,\ z$ with $|z| \le T$ and $n \ge N$, choosing j so that $|t - t_j| < \delta$, we have

$$\begin{aligned} &|f_n(t,z) - f_\infty(t,z)| \\ \le\ &|f_n(t,z) - f_n(t_j,z)| + |f_n(t_j,z) - f_\infty(t_j,z)| + |f_\infty(t_j,z) - f_\infty(t,z)| \\ \le\ &3\varepsilon. \end{aligned}$$

Thus the required assertion has been proved. □

By the above proposition, $f^{-t\mu_\lambda}(z)$ converges to $f^{-t\mu_0}(z)$ locally uniformly in $(t,z) \in \mathbb{D} \times \mathbb{C}$ as $\lambda \to 0+$. Since $f^{-t\mu_\lambda}$ is analytic in $\mathbb{H}^*$, the Weierstrass double series theorem implies that

$$\Phi(-t\mu_\lambda)(z) = S_{f^{-t\mu_\lambda}}(z) \to S_{f^{-t\mu_0}}(z) = \Phi(-t\mu_0)(z)$$

locally uniformly in $(t,z) \in \mathbb{D} \times \mathbb{H}^*$. By definition of h_λ and (5.1), the above convergence is equivalent to the locally uniform convergence

$$h_\lambda(t)\varphi_\lambda^*(z) \to g_0(t)\varphi_0^*(z).$$

Since $\varphi_\lambda^* \to \varphi_0^*$ by Lemma 10, we now see that $h_\lambda \to g_0$ locally uniformly on $\mathbb{D}$. In particular, $h_\lambda'(0) \to g_0'(0) = 4$. Since $h_\lambda'(0)$ is continuous in λ and, by Lemma 11, assumes non-zero real numbers, we also see that $h_\lambda'(0) > 0$ for all $\lambda > 0$. By the uniqueness of the Riemann mapping function and by Lemma 11, we obtain $g_\lambda = h_\lambda$. Thus Theorem 4 has been proved.

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