NORMAL FAMILIES AND QUASICONFORMAL MAPPINGS

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Abstract. In this note, basic theory of normal families of holomorphic functions and preliminaries of quasiconformal mappings in the complex plane. Especially, our focus will be on the Ahlfors five island theorem and its proof by following the idea of Bergweiler. The contents can also be used to course lectures on preparation for complex dynamics and some exercises are provided for the readers.

1. Introduction

1.1. **Complex Dynamics.** The theory of it complex dynamics is to study the dynamical behaviour of the orbits $z_n = f(z_{n-1})$ $(n = 1, 2, 3, \dots)$ with $z_0 = z \in \mathbb{C}$ of a point $z \in \mathbb{C}$ (or $z \in \widehat{\mathbb{C}}$) under the iteration of an entire function $f : \mathbb{C} \to \mathbb{C}$ (or a rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, respectively). The theory was initiated by the French mathematicians Fatou and Julia in the early 20th century. The domain $\mathbb{C}$ (or $\widehat{\mathbb{C}}$) is divided into two parts; the stable part and chaotic part. The stable part is the open set where the iteration family $f^{\circ n} = f \circ f \circ \cdots \circ f$ (n-times), (n=1,2,3,...) is normal and called the *Fatou set* and the chaotic part is its complement and called the *Julia set*. Though the theory was almost forgotten for a long time, the theory was revived around 1980 by the discovery of the Mandelbrot set together with the rapid development of computing technologies. Since then, the theory is still growing steadily up to now.

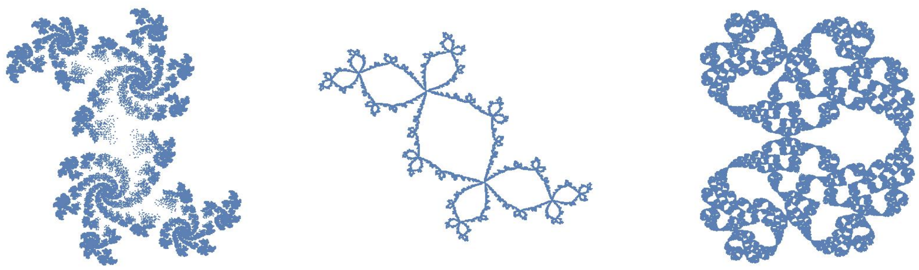

Figure 1. The Julia sets of $z^2 + 0.37(1 + i)$, $z^2 - 0.12 + 0.74i$, $z(z^2 + 2)/(z^3 - 1)$

The theory is greatly attractive and fascinating but not necessarily easy to understand with enough rigor. The main reason is perhaps due to the fact that the theory becomes fully available after the technical notions of *normal families* and *quasiconformal mappings*. The primary aim of the present note is to give the reader such backgrounds which will be useful to learn the theory of complex dynamics. Note that we do not touch the theory of complex dynamics in the sequel. If the reader got interests in the theory, he or she should consult a suitable textbook on complex dynamics. We give some hints for the references in the last section.

1.2. **Organization and history of the present note.** Section 2 gives basic knowledge about the hyperbolic and the spherical metrics, which will be key notions to understand normality of families of holomorphic functions. In Section 3, we discuss normal and compactness properties of a family of holomorphic functions on a domain and summarize basic results. Section 4 devotes to the brief introduction of plane quasiconformal mappings. Section 5 deals with useful results in the value distribution theory. We will give several exercises for the reader in Section 6 and some information about further readings in Section 7. The present note is by no means self-contained but the author tried to give proofs for simple statements so that the reader can understand the mechanism by which the theory works.

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2. Metric

2.1. **Metric and distance.** Let Ω be a subdomain of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ or, more generally, a Riemann surface. A continuous differential form $\rho(z)|dz|$ on Ω is called a *conformal metric* on Ω (in a weak sense) if the density $\rho(z)$ is positive for each point in Ω except for a discrete set. If $\rho(z)$ is always positive, then $\rho(z)|dz|$ is a conformal metric in the usual sense. When a conformal metric ρ is given for Ω, a distance on Ω can be associated to ρ in the following manner:

$$\delta_\rho(z, w) = \inf_\gamma \int_\gamma \rho(\zeta)|d\zeta|,$$

where the infimum is taken over all the rectifiable curves γ joining z and w within Ω. The distance $\delta_\rho(z, w)$ is called the *induced distance* of ρ.

Let $f : \Omega_0 \to \Omega$ be a non-constant holomorphic map. Then the pull-back of ρ under f is given by

$$f^*\rho(z)|dz| = \rho(f(z))|f'(z)||dz|.$$

Note that $f^*\rho$ is a conformal metric on Ω_0 while the quantity $\delta_\rho(f(z), f(w))$ is not necessarily a distance on Ω_0. The following is obvious but useful below.

2.2. **Lemma.**

$$\delta_\rho(f(z), f(w)) \le \delta_{f^*\rho}(z, w), \quad z, w \in \Omega_0.$$

2.3. **Hyperbolic metric.** The hyperbolic (or Poincaré) metric $\rho_{\mathbb{D}}(z)|dz|$ on the unit disk $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ is defined by

$$\rho_{\mathbb{D}}(z) = \frac{1}{1 - |z|^2}.$$

Then the induced distance (called the hyperbolic distance) takes the form

$$h_{\mathbb{D}}(z,w) = \operatorname{arctanh}\left|\frac{z-w}{1-\bar{z}w}\right|,$$

where $\operatorname{arctanh} t = (1/2)\log((1+t)/(1-t))$. For a general domain $\Omega \subset \widehat{\mathbb{C}}$ with $\#\partial\Omega \geq 3$, the hyperbolic metric $\rho_\Omega(z)|dz|$ on it is defined so that $f^*\rho_\Omega = \rho_{\mathbb{D}}$ holds for a holomorphic universal cover $f : \mathbb{D} \to \Omega$ of Ω. A crucial fact is that a domain $\Omega \subset \widehat{\mathbb{C}}$ with $\#\partial\Omega \leq 2$ does not carry the hyperbolic metric, namely, it admits no holomorphic universal cover from the unit disk.

The Schwarz-Pick lemma yields the useful contraction property $f^*\rho_\Omega \leq \rho_{\Omega_0}$ for any holomorphic maps $f : \Omega_0 \to \Omega$. The similar inequality $h_\Omega(f(z), f(w)) \leq h_{\Omega_0}(z,w)$ also holds, where h_Ω denotes the hyperbolic distance on Ω induced by $\rho_\Omega(z)|dz|$. Note that the hyperbolic distance is complete.

2.4. **Spherical metric.** The spherical metric $\sigma(z)|dz|$ on the Riemann sphere $\widehat{\mathbb{C}}$ is defined by

$$\sigma(z) = \frac{1}{1+|z|^2}.$$

This is nothing but the induced metric from the Euclidean metric on $\mathbb{R}^3$ when $\widehat{\mathbb{C}}$ is embedded as the sphere $\{(x_1,x_2,x_3); x_1^2+x_2^2+(x_3-1/2)^2 = 1/2^2\}$ via the stereographic projection (see Exercise 2). Therefore, the induced distance between two points is the length of the shorter arc of the great circle passing through those two points. Due to the simplicity, we prefer to the use of the chordal distance rather than the arc distance. The chordal distance is given by

$$d^\#(z,w) = \frac{|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}}$$

for $z, w \in \mathbb{C}$ while the arc distance is given by $2\arctan(d^\#(z,w)/2)$. When either z or w is the point at infinity, the distance is given by an obvious limiting process.

2.5. **Spherical derivative.** Recall the fact that a meromorphic function on a domain can be regarded as a holomorphic map from the domain into the Riemann sphere. Let f be a meromorphic function on a domain $\Omega \subset \mathbb{C}$. Then the density of the pull-back of the spherical metric under f is called the *spherical derivative* of f and denoted by $f^\#$:

$$f^\#(z) = \frac{|f'(z)|}{1+|f(z)|^2}.$$

3. Compactness properties of a family of holomorphic functions

In this section, we see fundamental properties of limit functions of locally uniformly convergent sequence of holomorphic maps. The Weierstrass double series theorem implies that the limit function of such a sequence is necessarily holomorphic, too. The key tool is the argument principle here. After then, we discuss normality of a family of meromorphic or analytic functions in the sense of Montel. This concept is indispensable to develope the theory of complex dynamics.

3.1. **Hurwitz's theorem.** *Let f_n, $n = 1, 2, \dots,$ be a locally uniformly convergent sequence of univalent meromorphic functions on a domain Ω. Then the limit f of the sequence is also univalent unless it is a constant.*

Proof. On the contrary, we assume that f is non-constant and that there are two points z_1 and z_2 in Ω with $z_1 \neq z_2$ such that $f(z_1) = f(z_2) =: w_0$. We take a smooth Jordan domain Ω_0 with $z_1, z_2 \in \Omega_0$ so that $\overline{\Omega}_0 \subset \Omega$. Since the set of zeros of $f - w_0$ is discrete, we can choose Ω_0 so further that $f - w_0 \neq 0$ on $\partial\Omega_0$. Set $m = \min\{|f(z) - w_0|; z \in \partial\Omega_0\} (> 0)$. We may also assume that f is a bounded holomorphic function on Ω_0. Since f_n converges to f uniformly on Ω_0, there is an integer n_0 such that $|f_n - f| < m/2$ on Ω_0 for $n \geq n_0$. By construction, we now see that $|f - w_0 - (f_n - w_0)| < |f - w_0|$ on $\partial\Omega_0$. Rouche's theorem implies that the number of zeros of $f_n - w_0$ in Ω_0 is same as that of $f - w_0$, which is at least two. This contradicts the univalence of f_n. $\square$

The same argument works in the proof of the following assertion.

3.2. **Lemma.** *Let f_n be a locally uniformly convergent sequence of holomorphic maps from a domain Ω into another domain $D \subset \widehat{\mathbb{C}}$. If $f_n(z_0)$ approaches to a point $w_0 \in \partial D$ for some $z_0 \in \Omega$, then f_n converges to w_0 locally uniformly in Ω.*

3.3. **Slight generalization of locally uniform convergence.** In practice, we encounter the situation that the domain where the function f_n is defined may change for different n's. We can formulate the concept of locally uniform convergence even for the case.

Suppose that meromorphic functions $f_n : \Omega_n \to \widehat{\mathbb{C}}$, $n = 1, 2, \dots,$ and $f : \Omega \to \widehat{\mathbb{C}}$ are given. The sequence f_n is said to converge to f locally uniformly in Ω if for every compact subset K of Ω there exists an integer k such that $K \subset \Omega_n$ for $n \geq k$ and that f_n, $n = k, k+1, \dots,$ converges to f uniformly on K.

If we generalize the notion of locally uniform convergence in this way, the same thing can be said as in the above.

3.4. **Normality.** Let Ω be a subdomain of $\widehat{\mathbb{C}}$. Let (X, d) be a complete metric space and denote by $C(\Omega, X)$ the set of continuous functions from Ω into X. We give to $C(\Omega, X)$ the compact-open topology, in other words, the topology of locally uniform convergence. A subset $\mathcal{F}$ of $C(\Omega, X)$ is called *normal* if the closure of $\mathcal{F}$ in $C(\Omega, X)$ is compact. Since $C(\Omega, X)$ is metrizable (see Exercise 5), $\mathcal{F}$ is normal if and only if any sequence of maps in $\mathcal{F}$ has a locally uniformly convergent subsequence.

3.5. **Equicontinuity.** A family $\mathcal{F} \subset C(\Omega, X)$ is said to be *equicontinuous* on a set $E \subset \Omega$ if, for any number $\varepsilon > 0$, there exists a number $\delta > 0$ such that $d(f(z), f(w)) < \varepsilon$ whenever $z, w \in E$ satisfy $d^{\#}(z, w) < \delta$ and $f \in \mathcal{F}$. Also, $\mathcal{F}$ is called *locally equicontinuous* on Ω if it is equicontinuous on each compact subset of Ω.

By using these notions, we can characterize the normality in more comprehensive terms.

3.6. **Arzelá-Ascoli theorem.** *A family $\mathcal{F} \subset C(\Omega, X)$ is normal if and only if the following two conditions are satisfied:*

(i) *$\mathcal{F}$ is locally equicontinuous on Ω, and*
(ii) *for each $z \in \Omega$ the set $\{f(z); f \in \mathcal{F}\}$ is relatively compact in X.*

The proof uses a standard diagonal process. See, for instance, [2] or [18].

3.7. **Lemma (Normality is a local property).** *Let $\mathcal{F}$ be a subset of $C(\Omega, X)$. Suppose that, for each point $z \in \Omega$, there is an open neighbourhood V of z in Ω so that $\mathcal{F}$ is normal on V. Then $\mathcal{F}$ is normal on the whole Ω.*

Proof. Use the diagonal process to extract a convergent subsequence from a given sequence in $\mathcal{F}$. □

3.8. **Normality of holomorphic or meromorphic functions.** A family $\mathcal{F}$ of meromorphic functions on a fixed domain Ω is said to be *normal* as meromorphic functions if $\mathcal{F}$ is normal as a subset of $C(\Omega, \widehat{\mathbb{C}})$, in other words, if any sequence of functions in $\mathcal{F}$ has a subsequence which converges locally uniformly to either a meromorphic function or ∞. In what follows, we will simply say that $\mathcal{F}$ is normal if $\mathcal{F}$ is normal as meromorphic functions if no confusion occurs.

A family of holomorphic functions on a fixed domain Ω is said to be normal as holomorphic functions if the family is normal as a subset of $C(\Omega, \mathbb{C})$, where $\mathbb{C}$ is equipped with the Euclidean metric.

The following criterion is classical.

3.9. **Theorem (Montel's theorem).** *A family $\mathcal{F}$ of holomorphic functions on Ω is normal as holomorphic functions if and only if it is locally uniformly bounded.*

Proof. By Cauchy's integral formula, locally uniform boundedness implies local equicontinuity. Then use the Arzelá-Ascoli theorem. We now show the converse. If $\mathcal{F}$ is not locally uniformly bounded, then there exist a point $z_0 \in \Omega$ and a sequence f_n in $\mathcal{F}$ such that $f_n(z_0) \to \infty$. Lemma 3.2 now implies that f_n converges to ∞ locally uniformly. This implies that $\mathcal{F}$ is not normal as holomorphic functions. □

3.10. **Theorem.** *A family $\mathcal{F}$ of meromorphic functions on a domain Ω is normal if and only if for every $z_0 \in \Omega$ there is a neighbourhood U of z_0 such that either $|f| < 2$ in U or $|f| > 1/2$ in U holds for each $f \in \mathcal{F}$.*

Proof. Note that if a subdomain U is such as above then $\mathcal{F}$ is normal in U by Montel's theorem. Thus, the "if" part is a simple consequence of Lemma 3.7. We now show the "only if" part. Assume that $\mathcal{F}$ is normal and fix a point $z_0 \in \Omega$. Take a number ε with $0 < \varepsilon < d^{\#}(1, 2) = d^{\#}(1, 1/2)$. Then the equicontinuity of $\mathcal{F}$ guarantees the existence of a number $\delta > 0$ so that $d^{\#}(f(z), f(z_0)) < \varepsilon$ whenever $d^{\#}(z, z_0) < \delta$ and $f \in \mathcal{F}$. Let now $U = \{z; d^{\#}(z, z_0) < \delta\}$. Then either $|f| < 2$ in U or $|f| > 1/2$ in U holds according to the cases $|f(z_0)| \le 1$ and $|f(z_0)| \ge 1$. □

The following result gives an extremely weak sufficient condition for normality.

3.11. **Theorem (Montel's three point theorem).** *Let a, b and c be distinct three points in $\widehat{\mathbb{C}}$. The family $\mathcal{F}$ of meromorphic functions on a fixed domain Ω which omit these three values a, b and c is normal.*

Proof. Without loss of generality, we can assume that $\{a, b, c\} = \{0, 1, \infty\}$. Set $D = \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. Since normality is a local property (Lemma 3.7), we may also assume that Ω

is the unit disk $\mathbb{D}$. Let $f_n, n = 1, 2, \dots$, be a sequence of functions in $\mathcal{F}$. We now show that there is a locally uniformly convergent subsequence of f_n. If the set $\{f_n(0), n = 1, 2, \dots\}$ accumulates at a point in ∂D, Lemma 3.2 provides a desired subsequence. If not, we may further assume that $f_n(0)$ converges to a point w_0 in D. In particular, $h_D(f_n(0), w_0) < 1$ for sufficiently large n. Note that the hyperbolic disk $B_D(w_0, t) = \{w \in D; h_D(w_0, w) < t\}$ is bounded due to the completeness of the hyperbolic distance. The contraction property of the hyperbolic distance yields the inequality $h_D(f_n(z), f_n(0)) \le h_{\mathbb{D}}(z, 0) = \operatorname{arctanh}(|z|)$. Hence, $f_n(\mathbb{D}_r) \subset B_D(w_0, 1 + t)$, where $t = \operatorname{arctanh}(r)$, and therefore, the sequence f_n is uniformly bounded in $\mathbb{D}_r = \{|z| < r\}$. By Theorem 3.9, we finally choose a locally uniformly convergent subsequence. □

We remark that the final argument in the above proof is essentially same as the Schottky theorem.

3.12. **A simple proof of the great Picard theorem.** If we assume Montel's three point theorem, we can derive the great Picard theorem relatively easily from the little Picard theorem. Recall now these theorems.

The little Picard theorem: *Suppose that a meromorphic function f defined on the plane $\mathbb{C}$ omits at least three values in $\widehat{\mathbb{C}}$. Then f must be a constant.*
The great Picard thereom: *Suppose that a meromorphic function f defined on the punctured disk $\mathbb{D}^* = \{0 < |z| < 1\}$ omits at least three values in $\widehat{\mathbb{C}}$. Then the origin is either a pole of f or a removable singularity of f.*

The following proof is due to Montel.

Proof of the great Picard theorem. Suppose that a meromorphic function f on $\mathbb{D}^*$ omits three values, say w_1, w_2 and w_3. We consider the sequence f_n defined by $f_n(z) = f(z/n)$. Then, by Theorem 3.11, the sequence $f_n, n = k, k+1, \dots$, is normal on $|z| < k$. By the diagonal process, we can now take a subsequence f_{n_j} of f_n such that the sequence $f_{n_j}, j = k, k+1, \dots$, is uniformly convergent in $|z| \le k$. Let g be the limit function defined on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ of f_{n_j}. Then, by Lemma 3.2, g is either a constant function with value w_i for some i or a holomorphic map from $\mathbb{C}^*$ into $\widehat{\mathbb{C}} \setminus \{w_1, w_2, w_3\}$. In the latter case, however, g must be constant. Indeed, the function $g(e^z)$ is constant by the little Picard theorem. At any event, g must be a constant function, say 0. Consider now the small circles $\gamma_j = \{|z| = 1/n_j\}$. Since f_{n_j} converges to 0 uniformly on the unit circle, for every $\varepsilon > 0$, we have $|f| < \varepsilon$ on $\gamma_j,\ j \ge j_0$ for some j_0. The maximum modulus principle now yields that $|f| < \varepsilon$ on the annulus $1/n_{j+1} < |z| < 1/n_j$ for $j \ge j_0$. Hence, $|f| < \varepsilon$ in the neighbourhood of the origin. Riemann's removable singularity theorem implies the desired conclusion. □

We now show a very convenient necessary and sufficient condition for normality.

3.13. **Theorem (Marty's theorem).** *A family $\mathcal{F}$ of meromorphic functions on a domain $\Omega \subset \mathbb{C}$ is normal if and only if the spherical derivatives $f^{\#}$ of $f \in \mathcal{F}$ are locally uniformly bounded in Ω.*

Proof. Since the target space $\widehat{\mathbb{C}}$ is compact, normality is equivalent to local equicontinuity in this case by the Arzelá-Ascoli theorem.

First we show the "if" part. Let $z_0 \in \Omega$ be given and take a sufficiently small $r > 0$ so that $V = \{z \in \mathbb{C}; |z - z_0| \le r\} \subset \Omega$. Then there is a constant M such that $f^\# \le M$ on V for every $f \in \mathcal{F}$. By Lemma 2.2, for $z \in V$ and $f \in \mathcal{F}$ we have

$$d^\#(f(z), f(z_0)) \le \delta_{f^*\sigma}(z, z_0) \le M|z - z_0|.$$

This estimate implies local equicontinuity of $\mathcal{F}$ on Ω.

Next we show the "only if" part. By Theorem 3.10, for each point $z_0 \in \Omega$ there is a disk $V = \{z \in \mathbb{C}; |z - z_0| \le r\}$ such that either $|f| \le 2$ in V or $|f| \ge 1/2$ in V for every $f \in \mathcal{F}$. If $|f| \le 2$ holds in V, by Cauchy's estimate, one obtains the inequality $|f'(z)| \le 8/r$ in $|z - z_0| \le r/2$. Therefore, $f^\#(z) \le 8/r$ in $|z - z_0| \le r/2$. In the case when $|f| \ge 1/2$, the same inequality is obtained by considering $1/f$ instead of f above. □

The following characterization of non-normality is often used to deduce a deep connection between apparently different properties.

3.14. **Theorem (Zalcman's lemma).** *Let $\mathcal{F}$ be a family of meromorphic functions on a domain $\Omega \subset \mathbb{C}$. Then $\mathcal{F}$ is not normal if and only if there exist a sequence f_n of functions in $\mathcal{F}$, a sequence z_n of points in Ω tending to a point z_0 in Ω, a sequence ρ_n of positive numbers tending to 0 and a non-constant meromorphic function f on $\mathbb{C}$ whose spherical derivative is bounded such that $f_n(z_n + \rho_n z) \to f(z)$ locally uniformly in $\mathbb{C}$.*

The following proof is due to Bergweiler [7].

Proof. If f_n is locally uniformly convergent, then the limit of the functions $f_n(z_n + \rho_n z)$ must be constant in the above situation. Therefore, sufficiency of the above condition is clear.

We assume that $\mathcal{F}$ is not normal in order to show the converse direction. Then Marty's theorem implies that there exist a sequence f_n of functions in $\mathcal{F}$ and a sequence ζ_n of points in Ω tending to a point $\zeta_0 \in \Omega$ such that $f_n^\#(\zeta_n) \to \infty$. We may assume that $\zeta_0 = 0$ and $\overline{\mathbb{D}} \subset \Omega$. Choose $z_n \in \overline{\mathbb{D}}$ so that

$$M_n := \max_{|z|\le 1}(1 - |z|) f_n^\#(z) = (1 - |z_n|) f_n^\#(z_n)$$

and set $\rho_n = 1/f_n^\#(z_n)$. Since $M_n \ge (1 - |\zeta_n|) f_n^\#(\zeta_n)$, we see that $M_n \to \infty$ and hence that $\rho_n = (1 - |z_n|)/M_n \to 0$. Since $|z_n + \rho_n z| < 1$ for $|z| < M_n$, the function $g_n(z) = f_n(z_n + \rho_n z)$ is defined for $|z| < M_n$ and satisfies

$$g_n^\#(z) = \rho_n f_n^\#(z_n + \rho_n z) \le \frac{1 - |z_n|}{1 - |z_n + \rho_n z|} \le \frac{1 - |z_n|}{1 - |z_n| - \rho_n |z|} = \frac{1}{1 - |z|/M_n}$$

there. By Marty's theorem, the sequence $g_n,\ n = k, k+1, \dots,$ forms a normal family in $|z| < M_k$ for each k. Therefore, g_n has a subsequence which is locally uniformly convergent in $\mathbb{C}$. Replacing the original f_n by a suitable subsequence, we may assume that g_n converges to a meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ locally uniformly on $\mathbb{C}$ and that z_n tends to a point $z_0 \in \Omega$. Since $g_n^\#(0) = 1$ for all n, we have $f^\#(0) = 1$, and therefore, f is non-constant. Furthermore, by the above estimate, we obtain $f^\#(z) \le 1$ for all $z \in \mathbb{C}$. □

Normality of a sequence f_n of meromorphic functions in Ω does not imply convergence without extra assumptions. The following general property on sequences is useful to note.

3.15. **Lemma.** *Let $a_n, n = 1, 2, \dots,$ be a sequence of points in a metric space (X, d) and let $a \in X$. Suppose that every subsequence of a_n has a subsequence which converges to a. Then a_n itself converges to a.*

Proof. Suppose, on the contrary, that a_n does not converge to a. By definition, there are infinitely many n's so that $d(a_n, a) \geq \varepsilon_0$ for some fixed $\varepsilon_0 > 0$. If we take a subsequence from those n's, then it has no subsequence which converges to a. The contradiction now completes the proof. □

As an easy application of the above principle, we can show Vitali's theorem.

3.16. **Theorem (Vitali's theorem).** *Suppose that a sequence $f_n, n = 1, 2, \dots,$ of meromorphic functions forms a normal family on a domain Ω. Assume that there is a subset W of Ω with accumulation points in Ω such that $f_n(z_0)$ converges for each $z_0 \in W$. Then f_n converges to a meromorphic function locally uniformly on Ω.*

Proof. We recall that the space $C(\Omega, \widehat{\mathbb{C}})$ with the topology of locally uniform convergence is metrizable. By hypothesis, f_n has a subsequence which converges to a meromorphic function f in $C(\Omega, \widehat{\mathbb{C}})$. We now show that f_n actually converges to f in $C(\Omega, \widehat{\mathbb{C}})$. Let f_{n_j} be any subsequence of f_n. Then, normality of $\{f_n\}$ implies that f_{n_j} has a convergent subsequence in $C(\Omega, \widehat{\mathbb{C}})$ with limit being g. By assumption, $f(z_0) = g(z_0)$ for each $z_0 \in W$. Now the identity theorem implies that $f = g$. Hence, Lemma 3.15 can be used to conclude the result. □

4. Plane quasiconformal mappings

4.1. **ACL functions.** A continuous function f defined in a domain $\Omega \subset \mathbb{C}$ is said to be *ACL* (absolutely continuous on lines) if for any closed rectangle $R = [a, b] \times [c, d]$ contained in Ω the function $f(x + iy)$ is absolutely continuous in $a \leq x \leq b$ for almost all $y \in [c, d]$ and absolutely continuous in $c \leq y \leq d$ for almost all $x \in [a, b]$.

Note that we can define the partial derivatives f_x and f_y a.e. in Ω for an ACL functions. Formally, we define

$$f_z = \frac{1}{2}(f_x - if_y) \quad \text{and} \quad f_{\bar{z}} = \frac{1}{2}(f_x + if_y).$$

The reader may feel dissatisfaction because the above definition seems to strongly depend on the coordinates. We try to give a more natural formulation under a mild extra assumption. (See also Theorem 4.4 below.)

Recall that a locally integrable function g is called a *distributional derivative* $\partial_x f$ of f in Ω if

$$\int_\Omega \varphi_x f dm = -\int_\Omega \varphi g dm$$

holds for every smooth function φ with compact support in Ω, where dm denotes the plane Lebesgue measure. Note that the smoothness requires only C^1 in this case. The distributional derivative $\partial_y f$ is also defined similarly.

4.2. **Lemma.** *Let $f : \Omega \to \mathbb{C}$ be a continuous function. Suppose that f is ACL and has locally integrable partial derivatives f_x, f_y in Ω. Then f_x and f_y are distributional derivatives $\partial_x f$ and $\partial_y f$ in Ω, respectively. Conversely, if f has locally integrable distributional derivatives in Ω, then f is ACL in Ω and $f_x = \partial_x f$ and $f_y = \partial_y f$ hold.*

Proof. First we show the first part. We need to show that

$$\int_\Omega \varphi_x f dm = -\int_\Omega \varphi f_x dm$$

for a smooth function φ with compact support in Ω. By using the partition of unity, we may assume that the support of φ lies in a closed rectangle $R = [a, b] \times [c, d] \subset \Omega$. By Fubini's theorem, we compute

$$\int_R (\varphi_x f + \varphi f_x) dm = \int_c^d \int_a^b (\varphi_x f + \varphi f_x) dxdy.$$

Because $\varphi_x f + \varphi f_x = (\varphi f)_x$, we have $\int_a^b (\varphi_x f + \varphi f_x) dx = [\varphi f]_a^b = 0$ for almost all $y \in [c, d]$. Hence, the desired identity has been shown. We can handle with f_y similarly.

Next we show the second part. Let $g = \partial_x f$. Suppose that a closed rectangle $R = [a, b] \times [c, d] \subset \Omega$ is given. Since $g \in L^1(R)$, by Fubini's theorem, there is a set E of full measure in $[c, d]$ so that $g(x + iy) \in L^1([a, b])$ for each $y \in E$. Set $R_\eta = [a, b] \times [c, \eta]$ for $c < \eta < d$. Assume that the distributional derivative $g = \partial_x f$ is locally integrable in Ω. Take $\varphi(x+iy) = \psi(x)\theta(y)$ as a test function, where smooth functions $\psi(x)$ and $\theta(y)$ have supports in $[a, b]$ and $[c, \eta]$, respectively. Then we have

$$\iint_{R_\eta} \psi'(x)\theta(y) f(x + iy) dxdy = -\iint_{R_\eta} \psi(x)\theta(y) g(x + iy) dxdy.$$

Letting $\theta(y)$ tend to 1 boundedly while $\psi(x)$ being fixed, we get

$$\int_c^\eta \int_a^b \psi'(x) f(x + iy) dxdy = -\int_c^\eta \int_a^b \psi(x) g(x + iy) dxdy.$$

Differentiating both sides with respect to η, we obtain

$$\int_a^b \psi'(x) f(x + iy) dx = -\int_a^b \psi(x) g(x + iy) dx \tag{4.1}$$

for almost all $y \in E$. The exceptional set in y here may depend on ψ. Nevertheless, we choose a common exceptional null set N for all $\psi \in C_0^1([a, b])$ because the space $C_0^1([a, b])$ is separable. Fix $\xi \in (a, b]$. By a suitable approximation, we can check that equation (4.1) still holds for the function ψ_n defined by $\psi_n(x) = n(x-a)$ for $a \le x \le a+1/n$, $\psi_n(x) = 1$ for $a+1/n \le x \le \xi-1/n$, $\psi_n(x) = n(\xi-x)$ for $\xi-1/n \le x \le \xi$ and $\psi_n(x) = 0$ otherwise, where n is a sufficiently large integer. Letting n tend to ∞, we finally obtain

$$f(a + iy) - f(\xi + iy) = -\int_a^\xi g(x + iy) dx$$

for every $\xi \in (a, b]$ and $y \in E \setminus N$. Therefore, $f(x + iy)$ is absolutely continuous in $a \le x \le b$ for every $y \in E \setminus N$ and the partial derivative f_x coincides with g. $\square$

4.3. **Definition of quasiconformal mappings.** Let $K \geq 1$ be a constant. A homeomorphism f from a domain $\Omega \subset \mathbb{C}$ onto another $\Omega' \subset \mathbb{C}$ is called *K-quasiconformal* if f is ACL in Ω and if there is a measurable function μ on Ω with $\|\mu\|_\infty \leq (K-1)/(K+1)$ such that

$$f_{\bar{z}}(z) = \mu(z) f_z(z) \tag{4.2}$$

holds a.e. in Ω.

For a proof of the following useful result, see [13].

4.4. **Theorem (Gehring-Lehto).** *Suppose that a continuous open mapping $f : \Omega \to \mathbb{C}$ has the partial derivatives f_x and f_y a.e. in Ω. Then f is totally differentiable at almost every point in Ω.*

4.5. **Equivalent definition of quasiconformality.** Let $f : \Omega \to \Omega'$ be an ACL homeomorphism. We consider the positive Borel measure $\lambda = \lambda_f$ on Ω defined by $\lambda(E) = m(f(E))$. Lebesgue's theorem gives a unique decomposition $\lambda = \lambda_{\mathrm{a}} + \lambda_{\mathrm{s}}$, where λ_{a} is the absolutely continuous part of λ and λ_{s} is the singular part of λ with respect to m. The Radon-Nikodym derivative of λ_{a} is given by

$$\frac{d\lambda_{\mathrm{a}}}{dm}(z_0) = \lim_{r\to 0} \frac{\lambda(B(z_0, r))}{\pi r^2}$$

for almost every $z_0 \in \Omega$, where $B(z_0, r) = \{z; |z - z_0| \leq r\}$. On the other hand, if f is totally differentiable at z_0, then clearly $\lambda(B(z_0, r))/(\pi r^2) \to J_f(z_0)$ as $r \to 0$, where J_f denotes the Jacobian of f, namely, $J_f = |f_z|^2 - |f_{\bar{z}}|^2$. Hence, by Gehring-Lehto theorem, we conclude that $d\lambda_{\mathrm{a}}/dm = J_f$ a.e. in Ω. Therefore, for a compact subset E of Ω, we have

$$\int_E J_f(z)dxdy = \lambda_{\mathrm{a}}(E) \leq \lambda(E) < \infty.$$

In particular, the Jacobian J_f is locally integrable.

If, in addition, f is K-quasiconformal, then $J_f = (1 - |\mu|^2)|f_z|^2 \geq (1 - k^2)|f_z|^2$, where $k = (K-1)/(K+1)$. Therefore local integrability of J_f implies local square integrability of f_z and hence $f_{\bar{z}}$. In this way, we have come to another definition of quasiconformal mappings.

A homeomorphism $f : \Omega \to \Omega'$ is K-quasiconformal if and only if f has locally integrable distributional derivatives f_z and $f_{\bar{z}}$ which satisfy (4.2) *for a measurable function μ on Ω satisfying $\|\mu\|_\infty \leq (K-1)/(K+1)$.*

The "if" part follows from Lemma 4.2. The "only if" part is a consequence of the above observation. Note that we can replace local integrability of f_z and $f_{\bar{z}}$ by local square integrability of them in the above characterization.

4.6. **Condition (N).** A homeomorphism $f : \Omega \to \Omega'$ is said to satisfy *condition (N)* if f preserves null sets, namely, if $m(f(E)) = 0$ for every Borel set $E \subset \Omega$ with $m(E) = 0$, where m denotes the plane Lebesgue measure. This condition is same as the absolute continuity of the measure λ defined in §4.5 with respect to the plane Lebesgue measure, in other words, $\lambda_{\mathrm{s}} = 0$. Note that a homeomorphism f satisfying condition (N) maps Lebesgue measurable sets to Lebesgue measurable sets.

We prepare a lemma for the possible use of a proof of condition (N).

4.7. **Lemma.** *Let Ω be a bounded domain with boundary of area zero. If a sequence Ω_n of domains is given in such a way that $\chi_n(z) \to \chi(z)$ as $n \to \infty$ for each point $z \in \mathbb{C} \setminus \partial\Omega$, where χ_n and χ denote the characteristic functions of the sets Ω_n and Ω. Then $m(\Omega_n) \to m(\Omega)$ as $n \to \infty$.*

Proof. By Lebesgue's convergence theorem, $m(\Omega_n) = \int_{\mathbb{C}} \chi_n dm \to \int_{\mathbb{C}} \chi dm = m(\Omega)$. □

4.8. **Theorem.** *A quasiconformal mapping $f : \Omega \to \Omega'$ satisfies condition (N) and*

$$\lambda_f(E) = \int_E J_f(z)dm(z) \tag{4.3}$$

for each Borel set $E \subset \Omega$.

Proof. Set $\lambda = \lambda_f$. Since the second part of the above statement implies that $\lambda_{\mathrm{s}} = 0$, it is enough to show (4.3). Let R be a closed rectangle contained in Ω such that f is absolutely continuous on the boundary of R. Note that $f(\partial R)$ is then rectifiable and, in particular, of area zero. By using mollifiers (smoothing operators), we may take a sequence f_n of C^1-functions in a fixed neighbourhood of R in such a way that f_n converges to f uniformly on R and satisfies $(f_n)_z \to f_z$ and $(f_n)_{\bar z} \to f_{\bar z}$ in $L^2(R)$. Then $\int_R J_{f_n} dm \to \int_R J_f dm$ as $n \to \infty$. On the other hand, since $\int_R J_{f_n} dm = m(f_n(R)) \to m(f(R)) = \lambda(R)$ by Lemma 4.7, we obtain $\int_R J_f dm = \lambda(R)$. Since every open set of Ω can be expressed as a countable disjoint union of such rectangles up to null sets, (4.3) is valid also for any open subset, and hence, for any Borel subset of Ω. □

4.9. **Remark.** By the standard approximation of a measurable function by simple functions, the relation in (4.3) can easily be strengthened to the formula

$$\int_\Omega \varphi(f(z))J_f(z)dm(z) = \int_{\Omega'} \varphi(w)dm(w)$$

for an integrable function φ on Ω', which is a generalization of a classical formula for the change of variables.

4.10. **Lemma (Chain rule).** *Let $f : \Omega \to \Omega'$ be a K-quasiconformal mapping with locally L^p derivatives for some $p \ge 2$ and $g : \Omega' \to \mathbb{C}$ be a continuous mapping with locally L^q derivatives for some $q > 1$ with $1/p + 1/q \le 1$. Then $g \circ f$ has locally L^r derivatives in Ω for $r = pq/(p + q - 2)$ and satisfies*

$$(g \circ f)_z = (g_z \circ f)f_z + (g_{\bar z} \circ f)\overline{f}_z \quad \text{and} \quad (g \circ f)_{\bar z} = (g_z \circ f)f_{\bar z} + (g_{\bar z} \circ f)\overline{f}_{\bar z} \tag{4.4}$$

and

$$\|(g \circ f)_z\|_{L^r(\Omega_0)} + \|(g \circ f)_{\bar z}\|_{L^r(\Omega_0)} \le M\|f_z\|^{1-2/q}_{L^p(\Omega_0)}(\|g_z\|_{L^q(f(\Omega_0))} + \|g_{\bar z}\|_{L^q(f(\Omega_0))}) \tag{4.5}$$

for each relatively compact subdomain Ω_0 of Ω, where M is a constant depending only on K.

Proof. Note that $|f_{\bar z}|^2 \le k^2|f_z|^2 \le (k^2/(1-k^2))J_f$ a.e., where $k=(K-1)/(K+1)<1$. First, assuming (4.4), we show inequality (4.5). By Hölder's inequality,

$$\int_{\Omega_0} |(g_z\circ f)f_z|^r dm \le \left(\int_{\Omega_0} |g_z\circ f|^q |f_z|^2 dm\right)^{r/q} \left(\int_{\Omega_0} |f_z|^p dm\right)^{1-r/q}.$$

Then, by Remark 4.9, we have

$$(1-k^2)\int_{\Omega_0} |g_z\circ f|^q|f_z|^2 dm \le \int_{\Omega_0} |g_z\circ f|^q J_f dm = \int_{f(\Omega_0)} |g_z|^q dm.$$

Similar estimates apply to other terms and (4.5) is obtained.

Next we prove (4.4). When g is smooth, Lemma 4.2 yields that $g\circ f$ has locally integrable derivatives satisfying (4.4). For a general g, we consider an approximating sequence g_n of g so that $\|(g_n)_z - g_z\|_{L^q(\Omega_0)} \to 0$ and $\|(g_n)_{\bar z} - g_{\bar z}\|_{L^q(\Omega_0)} \to 0$. Then, by (4.5), $(g_n\circ f)_z$ and $(g_n\circ f)_{\bar z}$ form Cauchy sequences in $L^r(\Omega_0)$. Those limits are easily seen to equal the distributional derivatives of $\partial_z(g\circ f)$ and $\partial_{\bar z}(g\circ f)$, respectively. Formulas in (4.4) also follow from this observation. □

4.11. **Composition of quasiconformal mappings.** Suppose that f and g are both quasiconformal in Lemma 4.10. Let $f_{\bar z} = \mu f_z$ and $g_{\bar z} = \nu g_z$ and adopt the (temporary) convention $\mu = 0$ on the set $\{z; f_z(z) = 0\}$. Then, by the chain rule (4.4), composition $h = f\circ g$ satisfies

$$h_{\bar z} = (g_z\circ f)\mu f_z + (\nu g_z)\circ f\cdot \bar f_z = (g_z\circ f)f_z\left[\mu + (\nu\circ f)\frac{\overline{f_z}}{f_z}\right]$$

$$h_z = (g_z\circ f)f_z + (\nu g_z)\circ f\cdot \overline{\mu f_z} = (g_z\circ f)f_z\left[1 + \bar\mu(\nu\circ f)\frac{\overline{f_z}}{f_z}\right].$$

Therefore, h satisfies the Beltrami equation $h_{\bar z} = \omega h_z$ with

$$\omega = \frac{\mu + (\nu\circ f)\frac{\overline{f_z}}{f_z}}{1+\bar\mu(\nu\circ f)\frac{\overline{f_z}}{f_z}}. \tag{4.6}$$

It is easy to check that $\|\omega\|_\infty \le (k_1+k_2)/(1-k_1k_2)$ if $\|\mu\|_\infty \le k_1$ and $\|\nu\|_\infty \le k_2$. Thus, we conclude that *the composition of K_1 and K_2-quasiconformal mappings is K_1K_2-quasiconformal.*

Note that $\omega = \mu$ if g is analytic, namely, if $g_{\bar z} = 0$.

The following result is very important to do almost everything with quasiconformal or quasiregular business. This is first established by Morrey in 1930s. Later, Bojarski observed that K-quasiconformal mapping has locally L^p-derivatives, where $p = p(K) > 2$ is a constant depending only on K. Recently, Astala proved that any number $p < 2K/(K-1)$ works, where $2K/(K-1)$ has been conjectured to be the best constant. The reader will find a self-contained proof of Theorems 4.12 and 4.19 in [4].

4.12. **Theorem (The measurable Riemann mapping theorem).** *Let μ be a complex valued measurable function on the complex plane with $\|\mu\|_\infty < 1$. Then there exists a unique normalized quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ satisfying $f_{\bar{z}} = \mu f_z$ a.e. in $\mathbb{C}$.*

It is usual to normalize f by $f(0) = 0$ and $f(1) = 1$. The above function f will be called the normalized μ-conformal homeomorphism of $\mathbb{C}$ and denoted by w^μ in the sequel.

4.13. **Theorem.** *The inverse of a K-quasiconformal mapping is also K-quasiconformal.*

From the analytic definition of quasiconformal mappings, it is not clear that the inverse of a quasiconformal mapping is again quasiconformal. Below we give a proof based on the measurable Riemann mapping theorem, although this fact is usually proved in the course of the proof of it as in [4]. There are several ways to show this claim, none of which seems easy to give a short proof in our framework. For instance, that is almost trivial if we adopt a geometric definition of quasiconformal mappings. However, it is not easy to prove the equivalence of those definitions.

Proof. Let $f : \Omega \to \Omega'$ be a K-quasiconformal mapping satisfying $f_{\bar{z}} = \mu f_z$, where μ is chosen so that $\mu = 0$ on the set where f_z vanishes and that $\sup_{z\in\Omega} |\mu(z)| = \|\mu\|_\infty$. Then define ν by

$$\nu = -\left(\frac{f_z}{\overline{f_z}} \cdot \mu\right) \circ f^{-1} \tag{4.7}$$

on Ω'. Note that ν is Lebesgue measurable by Theorem 4.8. (If μ is Borel measurable, then it is immediate to see that ν is Borel measurable without appealing to Theorem 4.8.) We extend ν to $\mathbb{C}$ by setting $\nu = 0$ off Ω. Then $\|\nu\|_\infty \le (K-1)/(K+1)$. Let $h : \mathbb{C} \to \mathbb{C}$ be a quasiconformal mapping with $h_{\bar{z}} = \mu h_z$ whose existence is guaranteed by Theorem 4.12. Then, by (4.6), we see that $(h \circ f)_{\bar{z}} = 0$ a.e. in Ω. Weyl's lemma implies that $\varphi = h \circ f$ is conformal in Ω. Hence, $f^{-1} = \varphi^{-1} \circ h$ is quasiconformal. □

Applying Theorem 4.8 to the function f^{-1}, we obtain the following.

4.14. **Corollary.** *The inverse of a quasiconformal mapping f satisfies condition (N). In particular, $|J_f| > 0$ a.e.*

The last assertion enables us to see that the coefficient $\mu(z)$ in (4.2) is determined by the function f in the sense of "almost everywhere" since $f_z \ne 0$ a.e. We call μ the *Beltrami coefficient* of f. Sometimes the Beltrami coefficient of f is denoted by μ_f. Note also that the Beltrami coefficient of f^{-1} is given by (4.7).

4.15. **Lemma (Stoïlow property).** *Let $f : \Omega \to \Omega'$ be a quasiconformal mapping satisfying $f_{\bar{z}} = \mu f_z$. Suppose that a continuous function $g : \Omega \to \mathbb{C}$ with locally square integrable derivatives also satisfies the Beltrami equation $g_{\bar{z}} = \mu g_z$ in Ω. Then there exists a holomorphic function $\varphi : \Omega' \to \mathbb{C}$ so that $g = \varphi \circ f$.*

Proof. More generally, if $g_{\bar{z}} = \nu g_z$, by combining (4.6) with (4.7), the Beltrami coefficient of $\varphi = g \circ f^{-1}$ is given by

$$\left(\mu_{g\circ f^{-1}}\right) \circ f = \frac{\nu - \mu}{1 - \bar{\mu}\nu} \frac{f_z}{\overline{f_z}}.$$

Thus, if $\nu = \mu$, we have $(\varphi)_{\bar{z}} = 0$. From Weyl's lemma, the conclusion follows. □

4.16. **Definition of quasiregular mappings.** A continuous function $g : \Omega \to \mathbb{C}$ with locally square integrable derivatives is called *quasiregular* if there exists a measurable function μ on Ω with $\|\mu\|_\infty < 1$ such that $g_{\bar{z}} = \mu g_z$ a.e. in Ω. By the above theorem, g is quasiregular if and only if g decomposes into the form $g = \varphi \circ f$, where $f : \Omega \to \Omega'$ is a quasiconformal mapping and $\varphi : \Omega' \to \mathbb{C}$ is a holomorphic function. Note that $g_z \neq 0$ a.e. in Ω, and hence, the coefficient μ is determined by g, unless g is constant.

For quasiregular mappings, we refer to [13] and [17].

4.17. **Continuity on Beltrami coefficients.** *Suppose that a sequence of measurable functions* $\mu_n,\ n = 1, 2, \ldots,$ *on* $\mathbb{C}$ *satisfies* $\|\mu_n\|_\infty \leq k(< 1)$ *for all* n *and* $\mu_n \to \mu$ *a.e. for some* μ. *Then the normalized* μ_n*-conformal homeomorphisms* $w^{\mu_n} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *converge to* w^μ *uniformly on* $\widehat{\mathbb{C}}$.

See, for instance, [4].

4.18. **Beltrami coefficients with parameters.** We often encounter the situation that the Beltrami coefficients in question have parameters. In practice, it is important to see dependence of the solutions of Beltrami equations on those parameters. A typical and important case is as follows. Let μ_t be a family of Beltrami coefficients on $\mathbb{C}$ parametrized by t over a domain $D \subset \mathbb{C}$. The family is said to be *holomorphic* if the mapping $t \mapsto \mu_t$ is holomorphic from D into the unit ball of the complex Banach space $L^\infty(\mathbb{C})$. In other words, for each $t_0 \in D$, the Beltrami coefficient μ_t is written in the form

$$\mu_t = \mu_{t_0} + (t - t_0)\nu + |t - t_0|\varepsilon_t, \tag{4.8}$$

where $\nu \in L^\infty(\mathbb{C})$ and $\|\varepsilon_t\|_\infty \to 0$ as $t \to t_0$.

We set

$$\theta^\omega(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{z(z-1)}{\zeta(\zeta - 1)(\zeta - z)} \omega(\zeta) d\xi d\eta$$

and $\theta^{\mu,\nu} = \theta^\omega \circ f$, where f is the normalized μ-conformal homeomorphism and

$$\omega = \left(\frac{\nu}{1 - |\mu|^2} \frac{f_z}{\overline{f_z}} \right) \circ f^{-1}.$$

Note that the quantity $\theta^{\mu,\nu}$ is linear in ν.

4.19. **Thoerem (Holomorphic dependence on parameters).** *Let* μ_t *be a holomorphic family of Beltrami coefficients over* D. *Then* $w^{\mu_t}(z)$ *is holomorphic in* $t \in D$ *for a fixed* $z \in \mathbb{C}$. *Moreover,* $\dot{w}^{\mu_{t_0}}(z) = \lim_{t \to t_0} (w^{\mu_t}(z) - w^{\mu_{t_0}}(z))/(t - t_0) = \theta^{\mu_{t_0},\nu}(z)$ *if* μ_t *has the expansion in* (4.8) *and the convergence is uniform on each compact set in* $\mathbb{C}$.

For the proof and more refined results, see [4].

5. The Ahlfors five island theorem

In this section, we give an exposition of the cerebrated Ahlfors five island theorem based on Bergweiler [7].

5.1. **Some terminology.** Let $f : \Omega \to \widehat{\mathbb{C}}$ be a meromorphic function. For a given Jordan domain D in $\widehat{\mathbb{C}}$, a connected component D_0 of $f^{-1}(D)$ is called a *simple island* over D if $f : D_0 \to D$ is a conformal homeomorphism.

5.2. **Theorem (Ahlfors five island theorem).** *Let $D_1, \ldots, D_5$ be Jordan domains in $\widehat{\mathbb{C}}$ whose closures are pairwise disjoint. Every non-constant meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ has a simple island over D_j for some $j = 1, \ldots, 5$.*

The following statement is also known (see Exercise 11).

5.3. **Theorem (Ahlfors).** *Let D_1, D_2, D_3 be bounded Jordan domains in $\mathbb{C}$ whose closures are pairwise disjoint. Every non-constant entire function $f : \mathbb{C} \to \mathbb{C}$ has a simple island over D_j for some $j = 1, 2, 3$.*

5.4. **Bergweiler's formulation.** In what follows, let $D_j,\ j = 1, 2, \ldots, q$, denote Jordan domains which have pairwise disjoint closures. We denote by $\mathcal{F}_{\mathrm{A}}(\Omega, \{D_j\}_{j=1}^q)$ the family of all meromorphic functions $f : \Omega \to \widehat{\mathbb{C}}$ which have no simple islands over D_j for any $j = 1, \ldots, q$. Then the Ahlfors five island theorem says that $\mathcal{F}_{\mathrm{A}}(\mathbb{C}, \{D_j\}_{j=1}^5)$ consists of only the constant functions.

Similarly, for given distinct points $a_1, \ldots, a_q$ in $\widehat{\mathbb{C}}$, let $\mathcal{F}_{\mathrm{N}}(\Omega, \{a_j\}_{j=1}^q)$ denote the family of all meromorphic functions $f : \Omega \to \widehat{\mathbb{C}}$ which have no simple a_j-points for every $j = 1, \ldots, q$. Then the values a_j are said to be *totally ramified.*

We consider now the following four assertions:

A1. *The family $\mathcal{F}_{\mathrm{A}}(\Omega, \{D_j\}_{j=1}^5)$ is normal for every domain $\Omega \subset \mathbb{C}$.*
A2. *The family $\mathcal{F}_{\mathrm{A}}(\mathbb{C}, \{D_j\}_{j=1}^5)$ consists of only the constant functions.*
N1. *The family $\mathcal{F}_{\mathrm{N}}(\Omega, \{a_j\}_{j=1}^5)$ is normal for every domain $\Omega \subset \mathbb{C}$.*
N2. *The family $\mathcal{F}_{\mathrm{N}}(\mathbb{C}, \{a_j\}_{j=1}^5)$ consists of only the constant functions.*

The second assertion is just a rephrase of the Ahlfors five island theorem. The last two assertions were proved by R. Nevanlinna in 1920's. Our aim in the rest of the present section is to give a proof for the above four assertions.

5.5. **A1 $\Rightarrow$ N1 and A2 $\Rightarrow$ N2.** For given $a_1, \ldots, a_5$, take a sufficiently small disks $D_1, \ldots, D_5$ such that $a_j \in D_j$. Because $\mathcal{F}_{\mathrm{A}}(\Omega, \{D_j\}_{j=1}^5) \supset \mathcal{F}_{\mathrm{N}}(\Omega, \{a_j\}_{j=1}^5)$, assertions N1 and N2 follows from A1 and A2, respectively.

5.6. **Bloch's principle.** Next we show equivalence of assertions X1 and X2 for X=A or N. This kind of equivalence is known as Bloch's principle which gives some heuristics to analogues between local properties of functions between global ones. See, for instance, [20] and [9].

For the sake of brevity, we shall use the symbol $\mathcal{F}(\Omega)$ to designate the family $\mathcal{F}_{\mathrm{A}}(\Omega, \{D_j\}_{j=1}^5)$ or $\mathcal{F}_{\mathrm{N}}(\Omega, \{a_j\}_{j=1}^5)$.

To deduce X2 from X1 is simple. Indeed, if $f \in \mathcal{F}(\mathbb{C})$ is non-constant, then the family of functions $f(nz),\ n = 1, 2, \ldots,$ is not normal at the origin. In order to deduce X1 from X2, we have just to use Zalzman's theorem, which ensures the existence of a non-constant function $f \in \mathcal{F}(\mathbb{C})$ under the hypothesis that $\mathcal{F}(\Omega)$ is not normal.

5.7. **N2 ⇒ A2.** We assume that there is a non-constant function f in $\mathcal{F}_{\mathrm{A}}(\mathbb{C}, \{D_j\}_{j=1}^5)$. We may assume that the closure of D_j does not contain ∞ for every j. Fix five distinct values $a_1, \dots, a_5 \in \mathbb{C}$ and consider the disks $\Delta_j(\varepsilon) = \{|z - a_j| < \varepsilon\}$ for $0 < \varepsilon < \min\{|a_j - a_k|; j \neq k\}$. It is obvious that there is a quasiconformal mapping $\psi_\varepsilon : \mathbb{C} \to \mathbb{C}$ such that $\psi_\varepsilon(D_j) \subset \Delta_j(\varepsilon)$ for all $j = 1, \dots, 5$. Let μ_ε be the Beltrami coefficient of the quasiregular mapping $\psi_\varepsilon \circ f$. Then, the measurable Riemann mapping theorem guarantees the existence of normalized μ_ε-conformal homeomorphism $\phi_\varepsilon : \mathbb{C} \to \mathbb{C}$. By construction, we see that $g_\varepsilon = \psi_\varepsilon \circ f \circ \phi_\varepsilon$ is a meromorphic function contained in $\mathcal{F}_{\mathrm{A}}(\mathbb{C}, \{\Delta_j(\varepsilon)\}_{j=1}^5)$.

We take now a sequence ε_n tending to zero. We may assume that the sequence g_{ε_n} is not normal, because otherwise one may replace it by $g_{\varepsilon_n}(M_n z)$ for a suitable sequence M_n tending to ∞. Zalzman's theorem yields now that, passing to a subsequence if necessary, $g_{\varepsilon_n}(z_n + \rho_n z)$ converges to a non-constant g in $\mathbb{C}$ for some sequences z_n and ρ_n. We see that $g \in \bigcap_{n=1}^\infty \mathcal{F}_{\mathrm{A}}(\mathbb{C}, \{\Delta_j(\varepsilon_n)\}_{j=1}^5) = \mathcal{F}_{\mathrm{N}}(\mathbb{C}, \{a_j\}_{j=1}^5)$. This violates the validity of assertion N2.

5.8. **Lemma (Schwarz lemma for square roots).** *Let F be a holomorphic function on the unit disk $\mathbb{D}$. Suppose that F has only multiple zeros and that $|F| < 1$ in $\mathbb{D}$. Then $|F'(0)|^2 \le 4|F(0)|$.*

If one could take a holomorphic square root G of F, the above inequality is nothing but the assertion $|G'(0)| \le 1$. The proof uses a result of Ahlfors on the ultrahyperbolic metrics [1].

Proof. By approximating $F(z)$ by $F(rz)$, we may assume that F is holomorphic on a neighbourhood of the closed unit disk and $|F| < 1$ there. Put

$$u(z) = \log \frac{|F'(z)|}{2\sqrt{|F(z)|}(1 - |F(z)|)} \quad \text{and} \quad v(z) = \log \frac{1}{1 - |z|^2}.$$

Note that $u(z) \to -\infty$ when z approaches to a zero of F with multiplicity at least three, while $u(z)$ is finite and smooth at any other points containing zeros of F with multiplicity two. Also note that $v(z) \to \infty$ as $|z| \to 1$. Therefore, the function $w = u - v$ takes its maximum at some point z_0 in $\mathbb{D}$, where w is smooth. Then $\Delta w(z_0) \le 0$. On the other hand, since $\Delta u = 4e^{2u}$ and $\Delta v = 4e^{2v}$, we see that $\Delta w(z_0) = 4(e^{2u(z_0)} - e^{2v(z_0)})$, and thus, $u(z_0) \le v(z_0)$. By the choice of z_0, we obtain $u(z) - v(z) = w(z) \le w(z_0) = u(z_0) - v(z_0) \le 0$. In particular, $u(0) \le v(0)$, which implies the desired inequality. □

5.9. **Proof of N1.** We assume that assertion N1 is false. Then, by Zalcman's theorem, there exists a non-constant $f \in \mathcal{F}_{\mathrm{N}}(\mathbb{C}, \{a_j\}_{j=1}^5)$ with bounded spherical derivative. We may assume that none of a_j's is ∞. Then consider the entire function

$$g(z) = \frac{f'(z)^2}{\prod_{j=1}^5 (f(z) - a_j)}.$$

Since $f^\#$ is bounded, g is small when f is large. In particular, g is non-constant and there is a sequence $z_n,\ n = 1, 2, \dots$, so that $g(z_n) \to \infty$, and hence $f(z_n)$ is bounded.

We consider the function $h_n(z) = f(z + z_n)$. Since $h_n^\#(z) = f^\#(z + z_n)$, the sequence h_n forms a normal family by Marty's theorem. Thus, we may assume that h_n converges to a meromorphic function $h : \mathbb{C} \to \widehat{\mathbb{C}}$ locally uniformly. Since $f(z_n)$ is a bounded sequence,

$h(0)$ is a finite value. If $h(0) \neq a_j$ for all j, then $g(z_n) \to h'(0)^2/\prod_{j=1}^5(h(0)-a_j) \neq \infty$, which is a contradiction. Thus, $h(0) = a_j$ for some j. On the other hand, the sequence $G_n(z) = g(z+z_n)$ of entire functions converges to $H(z) = h'(z)^2/\prod_{j=1}^5(h(z)-a_j)$ locally uniformly in the spherical metric. Since $G_n(0) \to \infty$, by Lemma 3.2, H must be identically ∞. Hence $h(z) \equiv a_j$.

Since $|h_n(z)-a_j| < 1$ on $\mathbb{D}$ for sufficiently large n, by Lemma 5.8, we obtain $|f'(z_n)|^2 \leq 4|f(z_n)-a_j|$ and hence

$$|g(z_n)| \leq \frac{4}{\prod_{k\neq j}|f(z_n)-a_k|},$$

which is a contradiction because $f(z_n) \to a_j$ and $g(z_n) \to \infty$.

6. Exercises

1. Show Lemma 2.2.
2. Give an explicit expression of the stereographic projection from $\mathbb{C}$ to the sphere $\{(x_1,x_2,x_3); x_1^2+x_2^2+(x_3-1/2)^2 = 1/2^2\}$. Using it, deduce that the induced metric on $\widehat{\mathbb{C}}$ from the Euclidean metric on $\mathbb{R}^3$ coincides with $\sigma(z)|dz|$.
3. Give a proof to Lemma 3.2.
4. Let $f_n,\ n=1,2,\dots,$ be a locally uniformly convergent sequence of meromorphic functions on Ω. Suppose that a sequence $g_n : D_n \to \widehat{\mathbb{C}},\ n=1,2,\dots,$ of meromorphic functions converges to $g : D \to \widehat{\mathbb{C}}$ locally uniformly in D and that $f_n(\Omega) \subset D_n$ for $n=1,2,\dots$. Prove that the composite functions $g_n \circ f_n$ converge to $g \circ f$ locally uniformly on Ω.
5. Show that the space $C(\Omega,X)$ introduced in §3.4 is metrizable in the following way. Let $\Omega_n,\ n=1,2,\dots,$ be an increasing sequence of relatively compact subdomains of Ω so that $\cup_{n=1}^\infty \Omega_n = \Omega$. Let δ_n be a pseudo-distance on $C(\Omega,X)$ defined by

   $$\delta_n(f,g) = \sup_{z\in\Omega_n} d(f(z),g(z))$$

   for $f,g \in C(\Omega,X)$. Then prove that

   $$\delta(f,g) = \sum_{n=1}^\infty 2^{-n}\frac{\delta_n(f,g)}{1+\delta_n(f,g)}$$

   gives a distance on $C(\Omega,X)$. Finally, check that the distance δ gives to $C(\Omega,X)$ the same topology as the compact-open topology.
6. Show that the group Möb of Möbius transformations is not normal in any subdomain of the Riemann sphere.
7. Fix three points z_1,z_2,z_3 of $\widehat{\mathbb{C}}$ and take a positive number $\delta > 0$. Is the family $\mathcal{F} = \{f \in \text{Möb}; \min\{d^\#(f(z_j),f(z_k)); j,k=1,2,3, j\neq k\} \geq \delta\}$ normal in $\widehat{\mathbb{C}}$?
8. Let $\mathcal{F}$ be a family of holomorphic functions on a domain Ω. If $\mathcal{F}$ is normal as holomorphic functions, then prove that the family $\mathcal{F}' = \{f'; f \in \mathcal{F}\}$ is normal, too. Can one say the same thing if one replaces "holomorphic" by "meromorphic" in the above?

9. In Theorem 3.13 we needed to assume the domain Ω to be a subdomain of $\mathbb{C}$. In the general case when $\Omega \subset \widehat{\mathbb{C}}$, it is natural to consider the "spherical density of spherical differential" given by
$$f^\flat(z) = \frac{(1+|z|^2)|f'(z)|}{1+|f(z)|^2}.$$
Deduce a criterion for normality similar to Marty's theorem.
10. Prove the following version of the Schwarz lemma: *Let* $F : \mathbb{D} \to \mathbb{D}^* = \mathbb{D} \setminus \{0\}$ *be a holomorphic map. Then* $|F'(0)| \le 2|F(0)| \log(1/|F(0)|)$ *holds.*
Hint: Use the hyperbolic metric.
11. Prove Theorem 5.3 by showing the following statement: Let $D_j,\ j = 1, 2, 3$, be Jordan domains and a_4 be a point in $\widehat{\mathbb{C}}$ such that any two of these have disjoint closures. Let $\mathcal{H}_{\mathrm{A}}(\Omega) = \mathcal{H}_{\mathrm{A}}(\Omega, D_1, D_2, D_3, a_4)$ denote the family of all meromorphic functions which has no simple island in Ω over D_j for all $j = 1, 2, 3$ and which omits the value a_4. Then $H_{\mathrm{A}}(\mathbb{C})$ contains only constant functions.
Hint: Letting $a_j \in D_j,\ j = 1, 2, 3$, and assuming $a_j \ne \infty,\ j = 1, 2, 3, 4$, consider the function
$$g(z) = \frac{f'(z)^4}{(f(z)-a_1)^2(f(z)-a_2)^2(f(z)-a_3)^2(f(z)-a_4)^3}.$$

7. References

7.1. **Complex Dyanmics.** It would be nice to refer the reader to several textbooks on the complex dynamics although this preliminary course will not treat it at all.

Beardon [6] and Steinmets [19] take analytic approach, which enables us to easily understand the contents. On the other hand, the lecture note [15] by Milnor has more geometric flavor.

The book [11] by Carleson and Gamelin is somewhat hard to read but useful even for experts. McMullen's book [14] gives us keen insights and provides the idea of renormalization. The book [16] by Morosawa, Nishimura, Taniguchi and Ueda deals also with entire functions and higher dimensional cases.

7.2. **Quasiconformal mappings.** Basic references are Lehto-Virtanen [13] and Ahlfors [3]. The outstanding paper [4] by Ahlfors and Bers is worth reading even though there are several misprints. The huge book [5] by Astala, Iwaniec and Martin contains many modern mathematical tools to develop the theory of plane quasiconformal mappings and their generalizations. Gutlyanskii, Ryazanov, Srebro and Yakubov [12] also covers degenerate cases of the Beltrami equation. Recently, a book [10] by Branner and Fagella was publised to serve as a textbook on quasiconformal surgery of complex dynamics.

7.3. **Basic materials.** Ahlfors' book [2] is an excellent textbook on complex analysis widely covering the necessary materials. In particular, as to the basic properties of normal families, the reader should consult it. The book [18] by Schiff is also a good source of the concept of normality. For basic properties of the hyperbolic metric, we refer to the book [1] by Ahlfors.

Concerning the Ahlfors five island theorem, articles [7] and [8] by Bergweiler provide a simple proof as well as references.

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