MODULUS TECHNIQUES IN GEOMETRIC FUNCTION THEORY

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ABSTRACT. This is an expository account on quasiconformal mappings and μ-conformal homeomorphisms with an emphasis on the role played by the modulus of an annulus or a semiannulus. In order that the reader gets acquainted with modulus techniques, we give proofs for some of typical and important results. We also include several recent results on μ-conformal homeomorphisms.

CONTENTS

1. INTRODUCTION

In geometric function theory, the (conformal) modulus of a ring (an annulus) is a key notion to analyze local behaviour of mappings. For instance, as we will see later, quasiconformal mappings can be characterized in terms of the moduli of annuli. In this survey, we exhibit techniques to derive useful properties of the mappings by observing the modulus change of annuli under homeomorphisms of a certain kind.

Basically, the same technique can be used in higher dimensions. We, however, restrict ourselves to the case of plane mappings for the sake of simplicity. The reader can consult a nice monograph [4] by Anderson, Vamanamurthy and Vuorinen for the information in higher dimensions. See also Ahlfors [1] and Lehto-Virtanen [25] for quasiconformal mappings, [6], [21], [22] for modern treatments of (possibly degenerate) Beltrami equations, and [26] for more recent and detailed information about modulus techniques.

A doubly connected domain D in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called a *ring* (domain) or an *annulus*. That is to say, a ring B is a connected open subset of $\widehat{\mathbb{C}}$ such that the complement of the complement $\widehat{\mathbb{C}} \setminus B$ of B consists of exactly two connected components, say, E_1 and E_2. We will say that B separates z_1 from z_2 when $z_1 \in E_1$

and $z_2 \in E_2$. To avoid an exceptional case, we will always assume that B is *not* a twice-punctured sphere (i.e., at least one of E_1 and E_2 is not a singleton). Then, B is known to be conformally equivalent to a round ring of the form $A_R = \{z \in \mathbb{C} : 1 < |z| < R\}$ for some $1 < R \le +\infty$. Note that the number R is uniquely determined for a given B. The quantity $\log R \in (0, +\infty]$ is called the (conformal) modulus of the ring B and will be denoted by $\operatorname{mod} B$. (It may be more natural to define the modulus to be $\frac{1}{2\pi} \log R$. In the present survey, however, we will not adopt this so that some results will take simpler forms.)

It is, however, not necessarily easy to evaluate or even estimate $\operatorname{mod} B$ because a conformal mapping between B and A_R cannot be given explicitly except for annuli of very special types. Therefore, it is desirable to have another expression of the modulus. Ahlfors and Beurling [3] introduced the concept of extremal length for a curve family. See also [1] or [2, Chap. 4] for details. As we will see below, this is quite a useful device to estimate the modulus of a ring.

Let Γ be a curve family in a Borel subset Ω of $\widehat{\mathbb{C}}$, that is, a collection of curves in Ω. (A curve is allowed to be broken into at most countable pieces in the most general situation. In the present survey, however, a curve will mean a continuous map from an interval into $\widehat{\mathbb{C}}$ for simplicity.) For a non-negative Borel function ρ on Ω, we consider the two quantities

$$L(\Gamma, \rho) = \inf_{\gamma \in \Gamma} \int_\gamma \rho(z)|dz|$$

and

$$\operatorname{Area}(\rho) = \iint_\Omega \rho(z)^2 dx dy,$$

where $z = x + iy$. Here, we define $\int_\gamma \rho(z)|dz|$ to be $+\infty$ when the integral cannot be computed appropriately (for instance, when the curve γ is not locally rectifiable whereas ρ is nonzero on it). See [27] or [30] for more precise definition. The extremal length of Γ, denoted by $\lambda(\Gamma)$, is defined by

$$\lambda(\Gamma) = \sup_{0 < \operatorname{Area}(\rho) < +\infty} \frac{L(\Gamma, \rho)^2}{\operatorname{Area}(\rho)}.$$

A function ρ for which the supremum is attained in the above will be called an extremal metric for the family Γ. Note that $\lambda(\Gamma)$ does not depend on the set Ω. In particular, we can take all non-negative Borel functions ρ on $\mathbb{C}$ with $0 < \operatorname{Area}(\rho) < \infty$ in the above definition. It should be noted that $\lambda(\Gamma') \ge \lambda(\Gamma)$ when $\Gamma' \subset \Gamma$.

The most important property of extremal length is conformal invariance. Let $f : \Omega \to \Omega'$ be a conformal homeomorphism and set $f(\Gamma) = \{f(\gamma) : \gamma \in \Gamma\}$. Then we have $\lambda(f(\Gamma)) = \lambda(\Gamma)$. This will be seen as a special case of a more general result (see Theorem 2.8 below).

For a ring B, we denote by Γ_B the collection of those closed curves in B whose winding number is ± 1 about the two components E_1 and E_2. In other words, choosing two points $\zeta_1 \in E_1$ and $\zeta_2 \in E_2$, a closed (oriented) curve γ in B is in Γ_B if and only if

$$\frac{1}{2\pi i} \int_\gamma \frac{z - \zeta_2}{z - \zeta_1} dz = \pm 1.$$

Also, we denote by Γ'_B the collection of open arcs in B joining the two boundary components of B. That is to say, an open arc γ in B is in Γ'_B if and only if $E_1 \cup \gamma \cup E_2$ is a closed connected set. Then we have the following (cf. Example 3 in Chapter 3D of [1]). Since it is a good exercise to check it as a warming-up, we include its proof here.

Lemma 1.1. *For a ring B in $\widehat{\mathbb{C}}$,*

$$\mod B = \frac{2\pi}{\lambda(\Gamma_B)} = 2\pi\lambda(\Gamma'_B).$$

Proof. By the conformal invariance, we may assume that B is a round ring of the form $A_R = \{1 < |z| < R\}$ for some $R > 1$.

Let ρ be a non-negative Borel function on B. For the circle $\gamma_r : \theta \mapsto re^{i\theta}$ $(1 < r < R)$, by the Cauchy-Schwarz inequality, we have

$$L(\Gamma_B, \rho)^2 \leq \left(\int_{\gamma_r} \rho(z)|dz|\right)^2 = \left(\int_0^{2\pi} \rho(z)rd\theta\right)^2$$
$$\leq \int_0^{2\pi} rd\theta \cdot \int_0^{2\pi} \rho(z)^2 rd\theta = 2\pi r\int_0^{2\pi} \rho(z)^2 rd\theta.$$

We now divide the above by r and then integrate in $1 < r < R$ to get

$$L(\Gamma_B, \rho)^2 \log R \leq 2\pi \int_1^R \int_0^{2\pi} \rho(z)^2 rd\theta dr = 2\pi\text{Area}(\rho),$$

and hence,

$$\frac{L(\Gamma_B, \rho)^2}{\text{Area}(\rho)} \leq \frac{2\pi}{\log R}.$$

Taking the supremum in ρ with $0 < \text{Area}(\rho) < +\infty$, we have

$$\lambda(\Gamma_B) \leq \frac{2\pi}{\log R}.$$

We next show the reverse inequality. Define ρ_0 by $\rho_0(z) = 1/|z|$ if $z \in B$ and $\rho_0(z) = 0$ otherwise. Since each $\gamma \in \Gamma_B$ has winding number 1 or -1 about the origin, writing $z = re^{i\theta}$, we have

$$2\pi = \left|\int_\gamma d\arg z\right| \leq \left|\int_\gamma \frac{dz}{z}\right| \leq \int_\gamma \frac{|dz|}{|z|} = \int_\gamma \rho_0(z)|dz|.$$

Hence, $L(\Gamma_B, \rho_0) \geq 2\pi$. Since $\text{Area}(\rho_0) = 2\pi \log R$, we have

$$\lambda(\Gamma_B) \geq \frac{L(\Gamma_B, \rho_0)^2}{\text{Area}(\rho_0)} \geq \frac{2\pi}{\log R}.$$

We have now proved that $\lambda(\Gamma_B) = 2\pi/\log R = 2\pi/\mod B$.

Similarly, we can show the second formula. Indeed, for an admissible ρ and the radial segment $\delta_\theta : r \mapsto re^{i\theta}$, we have

$$L(\Gamma'_B, \rho)^2 \leq \left(\int_{\delta_\theta} \rho(z)|dz| \right)^2 = \left(\int_1^R \rho(re^{i\theta}) dr \right)^2$$

$$\leq \int_1^R \frac{dr}{r} \cdot \int_1^R \rho(re^{i\theta})^2 r dr.$$

Integrating in $0 < \theta < 2\pi$, we obtain

$$2\pi L(\Gamma'_B, \rho)^2 \leq \log R \cdot \int_0^{2\pi} \int_1^R \rho(re^{i\theta})^2 r dr d\theta = \log R \cdot \mathrm{Area}(\rho),$$

and hence,

$$\lambda(\Gamma'_B) \leq \frac{\log R}{2\pi}.$$

Also, since $|dz|/|z| \geq dr/r$ for $z = re^{i\theta}$, we have for the above ρ_0, the inequality

$$\int_\gamma \rho_0(z)|dz| \geq \int_\gamma \frac{dr}{r} = \log R$$

holds for each $\gamma \in \Gamma'_B$. Therefore, $L(\Gamma'_B, \rho_0) \geq \log R$ and

$$\lambda(\Gamma'_B) \geq \frac{L(\Gamma'_B, \rho_0)^2}{\mathrm{Area}(\rho_0)} \geq \frac{\log R}{2\pi}.$$

We now complete the proof of $\lambda(\Gamma'_B) = \frac{1}{2\pi} \log R = \frac{1}{2\pi} \bmod B$. $\qquad \square$

The above estimations are typical cases of the so-called length-area method.

2. Differential calculus

We summarize very basics on differential calculus necessary for developments of the theory of quasiconformal mappings or (degenerate) Beltrami equations.

For simplicity, we may assume, for a while, that f is smooth enough on an open set in $\mathbb{C}$. However, definitions below may be extended for more general f as long as they make sense.

Complex partial derivatives of a (complex-valued) function f are defined by

$$f_z = \partial f = \frac{1}{2}(f_x - i f_y) \quad \text{and} \quad f_{\bar{z}} = \bar{\partial} f = \frac{1}{2}(f_x + i f_y),$$

where $f_x = \partial f/\partial x$ and $f_y = \partial f/\partial y$ for $z = x + iy$.

The Jacobian J_f of $f = u + iv$ can be expressed by

$$J_f = u_x v_y - u_y v_x = |f_z|^2 - |f_{\bar{z}}|^2.$$

Note that, if $J_f(z_0) \neq 0$, f is locally univalent at z_0 and, if $J_f(z_0) > 0$ in addition, f is orientation-preserving at z_0. We also note that $J_f(z_0) > 0$ is equivalent to $|f_{\bar{z}}(z_0)| < |f_z(z_0)|$.

The complex dilatation μ_f of f is defined by

$$\mu_f = \frac{f_{\bar{z}}}{f_z}.$$

Note that $|\mu_f| < 1$ if f is an orientation-preserving (local) diffeomorphism. It is often convenient to use the related quantity

$$K_f(z) = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|},$$

which is sometimes called the pointwise maximal dilatation of f. In applications, it is important to notice the formula

$$\mu_{\varphi \circ f \circ \psi} = \mu_f \circ \psi \cdot \frac{\overline{\psi'}}{\psi'}$$

and, in particular, $|\mu_{\varphi \circ f \circ \psi}| = |\mu_f| \circ \psi$ and $K_{\varphi \circ f \circ \psi} = K_f \circ \psi$, for non-constant holomorphic functions φ and ψ with $\psi'(z) \neq 0$. The quantity K_f is obtained by discarding the information about the argument of μ_f. Therefore, it is sometimes necessary to look at a more refined quantity. Andreian Cazacu [5] introduced the notion of directional dilatations, which were effectively used by Reich and Walczak [29], Lehto [23] and later by Brakalova and Jenkins [8], [13], Brakalova [10], [11], Gutlyanskiĭ, Martio, Vuorinen and the author [17], [18]. We now give a definition of it. Let μ be a (Borel measurable, complex-valued) function on an open set Ω in $\mathbb{C}$ with $|\mu| < 1$ and fix a point $z_0 \in \mathbb{C}$ (not necessarily in Ω). Then, we set

$$D_{\mu, z_0}(z) = \frac{\left| 1 - \mu(z) \frac{\bar{z} - \bar{z}_0}{z - z_0} \right|^2}{1 - |\mu(z)|^2} = \frac{|1 - e^{-2i\theta} \mu(z)|^2}{1 - |\mu(z)|^2}$$

for $z \in \Omega$, where $\theta = \arg(z - z_0)$. It is easy to check the inequalities

$$\frac{1}{K_\mu(z)} \leq D_{\mu, z_0}(z) \leq K_\mu(z), \quad z \in \Omega,$$

for $K_\mu = (1 + |\mu|)/(1 - |\mu|)$.

Let f be a function with $f_z(z) \neq 0$ on an open set Ω and $\mu = \mu_f$. For a fixed point $z_0 \in \Omega$, we write $z = z_0 + re^{i\theta}$ in polar coordinates. Then, by the chain rule, the partial derivatives of f with respect to θ and r are computed as

$$f_\theta = \frac{\partial z}{\partial \theta} f_z + \frac{\partial \bar{z}}{\partial \theta} f_{\bar{z}} = ire^{i\theta} f_z - ire^{-i\theta} f_{\bar{z}},$$

$$f_r = \frac{\partial z}{\partial r} f_z + \frac{\partial \bar{z}}{\partial r} f_{\bar{z}} = e^{i\theta} f_z + e^{-i\theta} f_{\bar{z}}.$$

It is easy to verify the following formulae:

(2.1) $$|f_\theta(z)|^2 = r^2 D_{\mu, z_0}(z) J_f(z)$$

and

$$|f_r(z)|^2 = D_{-\mu, z_0}(z) J_f(z).$$

Hence, D_{μ, z_0} and $D_{-\mu, z_0}$ are sometimes called the *angular dilatation* and the *radial dilatation* of f (or μ) at z_0, respectively.

When the theory of quasiconformal mappings was initiated by Grötzsch and Teichmüller, only smooth ones were considered. Later, however, it was recognized that we need to relax the smoothness (regularity) assumption to guarantee a normality of the class of (suitably normalized) K-quasiconformal mappings. Nowadays, it is standard to use a Sobolev space setting for that.

For C^1 functions f on an open set Ω in $\mathbb{C}$, the Sobolev norm with exponent $p \geq 1$ is defined to be

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|f_x\|_{L^p(\Omega)} + \|f_y\|_{L^p(\Omega)}.$$

The completion of the set $\{f \in C^1(\Omega) : \|f\|_{W^{1,p}} < \infty\}$ with respect to this norm is called the Sobolev space with exponent p and denoted by $W^{1,p}(\Omega)$. We remark that the completion is realized in the Lebesgue space $L^p(\Omega)$ and the partial derivatives are understood in the sense of distributions. We also denote by $W^{1,p}_{\mathrm{loc}}(\Omega)$ the set of measurable functions f on Ω such that $f|_{\Omega_0} \in W^{1,p}(\Omega_0)$ for every relatively compact, open subset Ω_0 of Ω.

Here is another relating concept. A continuous function f on an open set $\Omega \subset \mathbb{C}$ is called *ACL* (Absolutely Continuous on Lines) if for every closed rectangle $R = \{x + iy : a \leq x \leq b, c \leq y \leq d\}$ in Ω, the function $f(x + iy)$ is absolutely continuous in $a \leq x \leq b$ for almost every $y \in [c, d]$ (with respect to the 1-dimensional Lebesgue measure) and absolutely continuous in $c \leq y \leq d$ for almost every $x \in [a, b]$. We note that for such a function f we can define the partial derivatives f_x, f_y and, therefore, $f_z, f_{\bar{z}}$ as well, as Borel measurable functions a.e. on Ω. The definition of ACL functions seems to depend strongly on the coordinates. For instance, it is not clear that $f(e^{i\theta}(x + iy))$ is again ACL for an ACL function f. However, we do not need to worry about it, when partial derivatives are locally integrable in Ω. For the proof of the next result, the reader is referred, for instance, to [16, §4.9.2] or [32, Theorem 2.1.4].

Lemma 2.2. *Let f be a continuous function on an open set $\Omega \subset \mathbb{C}$. If f is ACL and the partial derivatives of f are integrable in Ω, then $f \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Conversely, every function f in $W^{1,1}_{\mathrm{loc}}(\Omega)$ is ACL in Ω. Moreover, their partial derivatives as ACL functions are same as distributional derivatives.*

The following property was discovered by Gehring and Lehto (cf. [1] or [25]) and often very useful.

Lemma 2.3. *Let f be a continuous open mapping of a domain Ω into $\mathbb{C}$. Suppose that f has the partial derivatives f_x and f_y a.e. in Ω. Then f is totally differentiable at almost every point in Ω.*

We recall here that f is called totally differentiable at z_0 if

$$f(z_0 + z) = f(z_0) + Ax + By + o(|z|)$$

as $z = x + iy \to 0$ for some constants $A, B \in \mathbb{C}$.

We are now ready to give an analytical definition of quasiconformal mappings.

Definition 2.4. Let Ω and Ω' be domains in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and let $K \geq 1$ be a constant. A homeomorphism $f : \Omega \to \Omega'$ is called K-quasiconformal if $f \in W^{1,1}_{\mathrm{loc}}(\Omega \setminus \{\infty, f^{-1}(\infty)\})$ and

$$|f_{\bar{z}}| \leq k|f_z| \quad \text{a.e. in } \Omega,$$

where $k = (K - 1)/(K + 1)$.

If we do not care about K, the mapping f is simply called *quasiconformal*. The next result is fundamental in the study of quasiconformal mappings.

Theorem 2.5 (The measurable Riemann mapping theorem)**.** *Let μ be a complex-valued measurable function on $\mathbb{C}$ with $\|\mu\|_\infty < 1$. Then there exists a quasiconformal mapping $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ satisfying*

$$(2.6) \qquad\qquad\qquad f_{\bar{z}} = \mu f_z$$

a.e. in $\mathbb{C}$. Moreover, such an f is unique up to post-composition with a Möbius transformation.

The equation in (2.6) is called the *Beltrami equation.* The condition $\|\mu\|_\infty < 1$ implies a uniform ellipticity of the equation. It is, however, occasionally necessary to consider the degenerate case when $|\mu| < 1$ a.e. but $\|\mu\|_\infty = 1$. Such a case occurs, for instance, in the study of planar harmonic mappings, transonic gas dynamics and parabolic bifurcations of a complex dynamics.

A measurable function μ on Ω is called a *Beltrami coefficient* if $|\mu| < 1$ a.e. on Ω. For such a μ, a homeomorphism $f : \Omega \to \Omega'$ is called μ-conformal if $f \in W^{1,1}_{\text{loc}}(\Omega)$ and $f_{\bar{z}} = \mu f_z$ a.e. in Ω. In the degenerate case, we should note that a homeomorphic solution might not exist and, even if it exists, the uniqueness assertion (the Stoïlow property) might not be true.

We also have the following result concerning quasiconformal mappings.

Lemma 2.7. *Let f be a quasiconformal mapping of a domain Ω. Then $f_z(z) \neq 0$ a.e. in Ω.*

Therefore, we can define the complex dilatation $\mu_f = f_{\bar{z}}/f_z$ as a Borel measurable function on Ω for a quasiconformal mapping of Ω and K-quasiconformality is characterized by $K_f = (1 + |\mu_f|)/(1 - |\mu_f|) \leq K$ a.e. in Ω.

The extremal length is important in connection with quasiconformal mappings.

Theorem 2.8. *Let $f : \Omega \to \Omega'$ be a K-quasiconformal mapping and Γ be a curve family in Ω. Then*

$$\frac{\lambda(\Gamma)}{K} \leq \lambda(f(\Gamma)) \leq K\lambda(\Gamma).$$

Proof. For a non-negative Borel function ρ on Ω, we define ρ' so that the formula

$$\rho = \rho' \circ f \cdot (|f_z| - |f_{\bar{z}}|)$$

is valid on Ω. If we write $w = f(z)$, then we have $dw = f_z dz + f_{\bar{z}} \bar{z}$ so that $|dw| \geq (|f_z| - |f_{\bar{z}}|)|dz|$. Thus, for any $\gamma \in \Gamma$, we have

$$\int_\gamma \rho(z)|dz| = \int_\gamma \rho'(w)(|f_z| - |f_{\bar{z}}|)|dz| \leq \int_{f(\gamma)} \rho'(w)|dw|.$$

Hence,

$$L(\Gamma, \rho) \leq L(f(\Gamma), \rho').$$

On the other hand, by using the inequality

$$J_f = K_f \cdot (|f_z| - |f_{\bar{z}}|)^2 \leq K(|f_z| - |f_{\bar{z}}|)^2,$$

we observe

$$\text{Area}(\rho) = \iint_\Omega \rho'(f)^2 (|f_z| - |f_{\bar{z}}|)^2 dxdy$$

$$\geq \frac{1}{K} \iint_\Omega \rho'(f)^2 J_f dxdy$$

$$= \frac{1}{K} \iint_{\Omega'} \rho'(w)^2 dudy = \frac{\text{Area}(\rho')}{K}.$$

Thus,

$$\frac{L(\Gamma, \rho)}{\text{Area}(\rho)} \leq K \cdot \frac{L(f(\Gamma), \rho')}{\text{Area}(\rho')}.$$

Taking the supremum in ρ, we get $\lambda(\Gamma) \leq \lambda(f(\Gamma))$. The other inequality can be obtained by applying f^{-1} to the first one. $\qquad \square$

Corollary 2.9. *The extremal length is conformally invariant.*

We end this section by summarizing basic properties of quasiconformal mappings.

Lemma 2.10.

(1) *1-quasiconformal mapping is nothing but a conformal mapping.*
(2) *The inverse mapping of a K-quasiconformal mapping is again K-quasiconformal.*
(3) *The composition of a K-quasiconformal mapping with a K'-quasiconformal mapping is a KK'-quasiconformal mapping.*
(4) *Let f and g be quasiconformal mappings of a domain Ω. If $\mu_f = \mu_g$ on Ω then $g = \varphi \circ f$ for a conformal mapping $\varphi : f(\Omega) \to g(\Omega)$ (the Stoïlow property).*
(5) *If a sequence of K-quasiconformal mappings f_n of a domain Ω converges locally uniformly on Ω to a homeomorphism f of Ω, then f is also K-quasiconformal.*

3. ROUND RINGS

A subset B_0 of a ring B is called a *subring* when B_0 is a ring with $\Gamma_{B_0} \subset \Gamma_B$. Then, by the monotonicity of extremal length, $\lambda(\Gamma_B) \leq \lambda(\Gamma_{B_0})$. Therefore, we have $\text{mod } B_0 \leq \text{mod } B$. It is also possible to show that the inequality is strict unless $B_0 = B$.

For $z_0 \in \widehat{\mathbb{C}}$ and $0 < r_1 < r_2 < +\infty$, we set

$$A(z_0; r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$$

if $z_0 \in \mathbb{C}$ and

$$A(\infty; r_1, r_2) = A(0, 1/r_2, 1/r_1)$$

if $z_0 = \infty$. A ring is said to be *round* (and centered at z_0) if it is of the form $A(z_0; r_1, r_2)$.

Obviously, a ring does not necessarily contain a round subring. However, this is true if the modulus is large enough. This sort of result was first proved by Teichmüller. A prototype of the following result was shown by Herron-Liu-Minda [20]. Avkhadiev and Wirths finally obtained the following sharp form.

Lemma 3.1 (Avkhadiev-Wirths [7])**.** *Let B be a ring in $\mathbb{C}$ with $\text{mod } B > \pi$ which separates a given point $z_0 \in \mathbb{C}$ from ∞. Then there is a round subring A of B centered*

at z_0 such that $\operatorname{mod} A \geq \operatorname{mod} B - \pi$. *The constant π cannot be replaced by any smaller number.*

For convenience of the reader, we give an outline of the proof.

Proof. Let E_1 and E_2 be the connected components of $\widehat{\mathbb{C}} \setminus B$ with $z_0 \in E_1$ and $\infty \in E_2$. We take a point $z_1 \in E_1$ so that $|z_1 - z_0| = \max_{z \in E_1} |z - z_0|$. We may assume that $z_0 = 0$ and $z_1 = -1$. We claim now that $A(0; 1, R) \subset B$ for $R = \exp(\operatorname{mod} B - \pi)$. Suppose, to the contrary, that $A(0; 1, R) \setminus B \neq \emptyset$. Since $E_1 \subset \{|z| \leq 1\}$, E_2 must intersect $A(0; 1, R)$. Therefore, we can take a point $w_0 \in E_2$ so that $|w_0| = R$. Then, Teichmüller's lemma (cf. [1]) implies that $\operatorname{mod} B \leq \operatorname{mod} D_R$, where D_R is the Teichmüller ring $\mathbb{C} \setminus ([-1, 0] \cup [R, +\infty))$. We now use the inequality $\operatorname{mod} D_R < \log R + \pi$ (see [7]) to obtain

$$\operatorname{mod} B \leq \operatorname{mod} D_R < \log R + \pi = \operatorname{mod} B,$$

which is impossible. The sharpness can be seen at the extremal example $B_0 = \mathbb{C} \setminus ([-1, 0] \cup [1, +\infty))$, which satisfies $\operatorname{mod} B_0 = \pi$ (see Corollary in §4.11 of [2]). Thus we are done. $\qquad\square$

Remark 3.2. The authors of [18] wrote that the Herron-Liu-Minda theorem can read the constant $\pi^{-1} \log 2(1 + \sqrt{2}) = 0.50118\dots$ works in the above lemma instead of π. But, this was not very correct because there was confusion with the definition of the modulus of a ring. The constant should be $2 \log 2(1 + \sqrt{2}) = 3.1490\dots$ and the point z_0 should be taken in ∂B. See also [31] for related estimates.

As a consequence of the last lemma, we get information about the size of the bounded connected component of a ring in $\mathbb{C}$.

Corollary 3.3. *Let B be a ring in $\mathbb{C}$ and E_1 be the bounded component of $\mathbb{C} \setminus B$. Then*

$$\operatorname{diam} E_1 \leq e^{\pi - \operatorname{mod} B} \operatorname{diam} B.$$

Proof. If $\operatorname{mod} B \leq \pi$, then the inequality clearly holds. Thus we can assume that $\operatorname{mod} B > \pi$. Take a round subring $A = A(z_0; r_1, r_2)$ of B so that $\operatorname{mod} A \geq \operatorname{mod} B - \pi$. Then

$$\operatorname{diam} E_1 \leq 2r_1 = \frac{r_1}{r_2} \cdot 2r_2 = e^{-\operatorname{mod} A} \operatorname{diam} A \leq e^{\pi - \operatorname{mod} B} \operatorname{diam} B.$$

$\qquad\square$

In a similar way, we can prove the following form (cf. [18, Lemm 2.8]).

Lemma 3.4 ([19, Theorem 2.8]). *Let B be a ring in $\mathbb{C}$ and let E_1 and E_2 be the bounded and unbounded connected components of $\mathbb{C} \setminus B$, respectively. If $\operatorname{mod} B > \pi$, then the inequality*

$$\sup_{z \in E_1} |z - z_0| \leq \operatorname{dist}(z_0, E_2) \exp(\pi - \operatorname{mod} S)$$

holds for any point $z_0 \in \overline{E_1}$.

These results fit Euclidean geometry and the point at infinity plays a special role. It may be sometimes useful to have a spherical variant, where the point at infinity is no longer special. The spherical (chordal) distance is defined by

$$d^\sharp(z, w) = \frac{|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}}$$

for $z, w \in \widehat{\mathbb{C}}$ and the spherical diameter of a set E will be denoted by $\mathrm{diam}^\sharp E$. Similarly, the spherical distance between two sets E_1, E_2 will be denoted by $\mathrm{dist}^\sharp(E_1, E_2)$.

Let B be a ring with E_1 and E_2 as the connected components of its complement. Then the inequality

$$\min\{\mathrm{diam}^\sharp E_1, \mathrm{diam}^\sharp E_2\} \leq \frac{\pi}{\sqrt{2 \, \mathrm{mod}\, B}}$$

holds (Lehto-Virtanen [25, Lemma I.6.1]). When $\mathrm{mod}\, B$ is large, however, the following result (see [18]) gives a better bound.

Lemma 3.5. *Let B be a ring in $\widehat{\mathbb{C}}$ and let E_1 and E_2 be the connected components of $\widehat{\mathbb{C}} \setminus B$. Then the inequality*

$$\min\{\mathrm{diam}^\sharp E_1, \mathrm{diam}^\sharp E_2\} \leq C_1 e^{-\frac{1}{2} \mathrm{mod}\, B}$$

holds, where C_1 is an absolute constant.

We remark that the constant can be taken as $C_1 = 2e^{\pi/2} = 9.6209\ldots$ As we noted above, the value of C_1 given in [18, Lemma 2.6] is not correct. This lemma follows from the next elementary result [18, Lemma 2.7]. Here, a *circular domain* means a simply connected domain in $\widehat{\mathbb{C}}$ bounded by a circle or a line.

Lemma 3.6. *Let A be a ring in $\widehat{\mathbb{C}}$ whose complement consists of disjoint closed circular domains E_1 and E_2. Then*

$$\min\{\mathrm{diam}^\sharp E_1, \mathrm{diam}^\sharp E_2\} \leq \frac{1}{\cosh(\frac{1}{2} \mathrm{mod}\, A)}.$$

Equality holds if and only if $\mathrm{diam}^\sharp E_1 = \mathrm{diam}^\sharp E_2$ and if the spherical centers of E_1 and E_2 are antipodal.

4. Length-area method

Reich and Walczak [29] gave an efficient method to estimate the modulus of the image of a ring under quasiconformal mappings in terms of its directional dilatations. The following variant of Reich-Walczak inequality can be found, for example, in [18].

Theorem 4.1. *Let μ be a Beltrami coefficient on a domain Ω in $\mathbb{C}$ and $f : \Omega \to \Omega'$ be a μ-conformal homeomorphism. Suppose that $D_{\mu, z_0}(z)$ is locally integrable in a round ring $A = A(z_0; r_1, r_2) \subset \Omega$. Then*

$$(4.2) \qquad \int_{r_1}^{r_2} \frac{dr}{r \psi_\mu(r, z_0)} \leq \mathrm{mod}\, f(A),$$

where

$$\psi_\mu(r, z_0) = \frac{1}{2\pi} \int_0^{2\pi} D_{\mu, z_0}(z_0 + re^{i\theta})d\theta.$$

Proof. We may assume that $z_0 = 0$ and $\Omega = A = A(0; 1, R)$. By post-composing a conformal mapping, we may further assume that $A' = f(A) = A(0; 1, R')$. Since

$$\iint_E J_f(z)dxdy \le \iint_{f(E)} dudv < +\infty$$

for a compact subset E of A, we have $J_f \in L^1_{\mathrm{loc}}(A)$. (In the classical case, indeed equality holds. For a detailed proof, see [1] or [25].)

Denote by γ_r the circle $|z| = r$. Then the assumption $f \in W^{1,1}_{\mathrm{loc}}(A)$ together with the Gehring-Lehto theorem (Lemma 2.3) implies that, for almost all $r \in (1, R)$, f is absolutely continuous on γ_r and totally differentiable at every point in γ_r except for a set of linear measure 0. By Fubini's theorem, we observe that $D_{\mu, 0}$ and J_f are integrable on γ_r for almost all $r \in (1, R)$. For such an r, we have

$$2\pi \le \int_{\gamma_r} |d \arg f| \le \int_{\gamma_r} \frac{|df(z)|}{|f(z)|} = \int_0^{2\pi} \frac{|f_\theta(re^{i\theta})|}{|f(re^{i\theta})|}d\theta.$$

We use the Cauchy-Schwarz inequality and (2.1) to obtain

$$(2\pi)^2 \le r^2 \int_0^{2\pi} D_{\mu, 0}(re^{i\theta})d\theta \int_0^{2\pi} \frac{J_f}{|f|^2}(re^{i\theta})d\theta,$$

and hence

$$\frac{2\pi}{r\psi_\mu(r)} \le r \int_0^{2\pi} \frac{J_f}{|f|^2}(re^{i\theta})d\theta$$

for almost all $r \in (1, R)$, where $\psi_\mu(r) = \psi_\mu(r, 0)$. Integrating both sides with respect to r from 1 to R, we obtain

$$2\pi \int_1^R \frac{dr}{r\psi_\mu(r)} \le \int_1^R \int_0^{2\pi} \frac{J_f}{|f|^2}rd\theta dr = \iint_A \frac{J_f dxdy}{|f|^2}$$
$$\le \iint_{A'} \frac{dudv}{|w|^2} = 2\pi \log R' = 2\pi \bmod A'$$

and thus arrive at the required inequality in (4.2). $\square$

The next inequality can also be proved by replacing the curve family of circles by that of radial segments joining the two boundary components of A in the above proof.

Theorem 4.3. *Let μ be a Beltrami coefficient on a domain Ω in $\mathbb{C}$ and $f : \Omega \to \Omega'$ be a μ-conformal homeomorphism. Suppose that $D_{-\mu, z_0}(z)$ is locally integrable in a round ring $A = A(z_0; r_1, r_2) \subset \Omega$ Then*

$$(4.4) \qquad \bmod f(A) \le \left[\int_0^{2\pi} \frac{d\theta}{\varphi_\mu(\theta, z_0)} \right]^{-1},$$

where

$$\varphi_\mu(\theta, z_0) = \int_{r_1}^{r_2} D_{-\mu, z_0}(z_0 + re^{i\theta}) \frac{dr}{r}.$$

5. APPLICATION TO MODULUS OF CONTINUITY

As a special case of Theorem 2.8, we have the inequalities

$$(5.1) \qquad \frac{1}{K} \operatorname{mod} B \leq \operatorname{mod} f(B) \leq K \operatorname{mod} B$$

for a K-quasiconformal mapping f of a domain Ω and a ring B in Ω. It is a remarkable fact that the converse is also true. In other words, if a sense-preserving homeomorphism $f : \Omega \to \Omega'$ satisfies (5.1) for any ring $B \subset \Omega$, then f is K-quasiconformal (see [25]).

As a simple application of results in the previous section, let us see that (5.1) leads to the well-known (local) Hölder continuity of a K-quasiconformal mapping.

Since we are dealing with local property, we may assume that f is a K-quasiconformal mapping of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ into itself. Take two points z_0, z_1 in the smaller disk $|z| < 1/2$ and consider the round ring $A = A(z_0; r, \frac{1}{2}) \subset \mathbb{D}$ for $0 < r < 1/2$. Then, by (5.1), we have

$$(5.2) \qquad \operatorname{mod} f(A) \geq \frac{\operatorname{mod} A}{K} = \frac{1}{K} \log \frac{1}{2r}.$$

Then, by Corollary 3.3, we have

$$\operatorname{diam} E_1 \leq e^{\pi - \operatorname{mod} f(A)} \operatorname{diam} f(A),$$

where E_1 is the bounded component of $\mathbb{C} \setminus f(A)$. Since $\operatorname{diam} f(A) \leq \operatorname{diam} \mathbb{D} = 2$, the above inequality together with (5.2) implies

$$\operatorname{diam} E_1 \leq 2e^{\pi + (\log 2r)/K} = Cr^{1/K},$$

where $C = 2^{1+1/K} e^\pi$. When $|z_1 - z_0| < 1/2$, we take $r = |z_1 - z_0|$ to obtain

$$|f(z_1) - f(z_0)| \leq C|z_1 - z_0|^{1/K}.$$

If we estimate $\operatorname{mod} f(A(z_0; r, r_0))$ from below in terms of r, then we could obtain a modulus of continuity estimate for f at z_0 in the same way.

For a function f defined in a neighbourhood of a point $z_0 \in \mathbb{C}$, its modulus of continuity at z_0 is defined by

$$\delta_f(z_0; r) = \sup_{|z - z_0| \leq r} |f(z) - f(z_0)|$$

for a sufficiently small $r > 0$. For instance, f is continuous at z_0 if and only if $\delta_f(z_0; r) \to 0$ as $r \to 0+$ and f is Hölder continuous with exponent α at z_0 if and only if $\delta_f(z_0; r) = O(r^\alpha)$.

We now have the following, whose proof can be done in the same way as above.

Theorem 5.3. *Let f be an injective continuous map of the disk $\mathbb{D}(z_0, r_0)$ into the disk $|w| < M$ and h is a non-negative function on $(0, +\infty)$. If the inequality $\operatorname{mod} f(A) \geq h(\operatorname{mod} A)$ holds for a ring A of the form $A(z_0; r, r_0)$, $0 < r < r_0$, then*

$$\delta_f(z_0; r) \leq 2M \exp\left(\pi - h(\log \tfrac{r_0}{r})\right).$$

An estimate of $\mathrm{mod}\, f(A)$ can be obtained by the Reich-Walczak theorem as stated in the previous section. We also have a spherical variant of this sort of result by using Lemma 3.5 instead of Corollary 3.3.

Finally, we state a normality criterion for a family of homeomorphisms of the Riemann sphere described by a modulus condition as in the following. This sort of result was used by Lehto [24] and played an important role in the proof of existence theorems of solutions of degenerate Beltrami equations in Brakalova-Jenkins [12] and Gutlyanskiĭ-Martio-Sugawa-Vuorinen [18]. The following form is found in [18].

Theorem 5.4. *Let $\rho(z,r,R)$ be a non-negative function in $(z,r,R) \in \widehat{\mathbb{C}} \times (0,+\infty) \times (0,+\infty)$ with $r < R$ such that $\rho(z,r,R) \to +\infty$ as $r \to 0+$ for fixed z and R. Then the set $\mathcal{H}_\rho$ of orientation-preserving self-homeomorphisms f of $\widehat{\mathbb{C}}$ satisfying $f(0) = 0, f(1) = 1, f(\infty) = \infty$ and*

$$\mathrm{mod}\, f(A(z_0; r, R)) \geq \rho(z_0, r, R)$$

for every $(z_0, r, R) \in \widehat{\mathbb{C}} \times (0,+\infty) \times (0,+\infty)$ with $r < R$, is compact in the topology of uniform convergence with respect to the spherical metric. Moreover, for each $R > 0$, there exists a constant $C = C(R, \rho) > 0$ depending only on R and ρ such that

$$|f(z_1) - f(z_2)| \leq C \exp(-\rho(z_0, r_1, r_2)), \quad z_1, z_2 \in \mathbb{D}(z_0, r_1)$$

for $|z_0| \leq R$ and $0 < r_1 < r_2 < R$.

Here, $\mathbb{D}(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$.

6. Modulus of semiannulus

It is well known that a quasiconformal map of the unit disk onto itself has a homeomorphic extension to the boundary. But, this is no longer true for general homeomorphisms of the unit disk.

Let $\Theta(r)$ be a real-valued continuous function in $0 < r < 1$ which has no finite limit as $r \to 1-$. Then the mapping $f : \mathbb{D} \to \mathbb{D}$ defined by

(6.1) $$f(re^{i\theta}) = re^{i(\Theta(r)+\theta)}$$

is homeomorphic but has no continuous extension to the boundary. In the next section, we give a characterization of self-homeomorphisms of the unit disk which has a homeomorphic extension to the boundary. To control the boundary behaviour, we need a notion corresponding to the "half" of a ring. In order to distinguish the genuin boundary, which is part of the boundary of the original ring, from the new boundary, which is the relative boundary in the original ring, we adopt a tactical definition for that. Thus, we use a different term from "ring".

A subset S of $\widehat{\mathbb{C}}$ is called a *semiannulus* if it is homeomorphic to

$$T_R = \{z \in \mathbb{C} : 1 \leq |z| \leq R, \mathrm{Im}\, z > 0\}$$

for some $R \in (1, +\infty)$. The two simple arcs in the boundary of S which correspond to $\{|z| = 1, \mathrm{Im}\, z > 0\}$ and $\{|z| = R, \mathrm{Im}\, z > 0\}$ are called the sides of S. A basic account of semiannulus is given in [19]. Here, we give a more detailed account of it to complement the paper [19].

A semiannulus S in a plane domain D is said to be *properly embedded* in D if $S \cap K$ is compact whenever K is a compact subset of D. When D is simply connected, $D \setminus S$ consists of exactly two connected components (cf. [28, Prop. 2.12]).

Unlike the case of rings, a semiannulus is not necessarily mapped to the standard one T_R conformally inside. Therefore, in order to define the modulus of a semiannulus S, we should take another way. Let Γ_S be the collection of open arcs in Int S dividing the two sides of S and Γ'_S be that of closed arcs in S joining the two sides of S. We define the modulus of S by

$$\operatorname{mod} S = \frac{\pi}{\lambda(\Gamma_S)}$$

so that $\operatorname{mod} T_R = \log R$ (see Lemma 6.5 below).

First, we give a topological criterion for a semiannulus to have a positive modulus. Let us recall the notion of prime ends due to C. Carathéodory (see [28] for details).

Let Ω be a simply connected hyperbolic domain. A simple open arc γ is called a *crosscut* of Ω if $\gamma \subset \Omega$ and if γ has an endpoint in $\partial\Omega$ in both directions. Then it is known that $\Omega \setminus \gamma$ consists of exactly two connected components (Proposition 2.12 in [28]). A sequence of crosscuts C_n $(n = 0, 1, 2, \dots)$ of Ω is called a *nullchain* if $\overline{C_n} \cap \overline{C_{n+1}} = \emptyset$ for $n = 0, 1, 2, \dots$, if C_n separates C_0 from C_{n+1} in Ω for $n = 1, 2, 3, \dots$ and if $\operatorname{diam}^\sharp C_n \to 0$ as $n \to \infty$. Two nullchains C_n, C'_n are defined to be equivalent if for a sufficiently large m, there is an n such that C_m separates C'_n from C'_0 and C'_m separates C_n from C_0 in Ω. The equivalence class of a nullchain of crosscuts is called a *prime end* of Ω. Let $P(\Omega)$ denote the set of all the prime ends of Ω. Note that $P(\Omega)$ is endowed with a natural topology. Carathéodory's main theorem asserts that a conformal homeomorphism between simply connected hyperbolic domains Ω and Ω' gives rise to a one-to-one correspondence (indeed a homeomorphism) between $P(\Omega)$ and $P(\Omega')$ in a natural way. Obviously, $P(\mathbb{D})$ can be identified with the topological boundary $\partial\mathbb{D}$ of the unit disk $\mathbb{D}$. In particular, $P(\Omega)$ is homeomorphic to the circle.

Let S be a semiannulus. Then Int S is a simply connected hyperbolic domain and thus the Riemann mapping theorem gives us a conformal homeomorphism $h : Int S \to \mathbb{D}$. The sides σ_1 and σ_2 of S correspond to disjoint open circular arcs O_1 and O_2 in $\partial\mathbb{D}$ under the mapping h and h extends to a homeomorphism of S onto $S' = \mathbb{D} \cup (O_1 \cup O_1)$. Let C_1 and C_2 be the connected components of $\partial\mathbb{D} \setminus (O_1 \cup O_2)$. We now observe that $h(\Gamma'_S)$ is the collection of closed curves joining O_1 and O_2 in S', in other words, $h(\Gamma'_S) = \Gamma'_{S'}$. In the same way, we can show that $h(\Gamma_S) = \Gamma_{S'}$. It is well known that $\lambda(\Gamma_{S'}) = +\infty$ if and only if $\lambda(\Gamma'_{S'}) = 0$ if and only if one of C_1 and C_2 reduces to a point. The last condition is equivalent to that one of the ends of S consists of one prime end. In this way, we can show the following criterion [19, Lemma 2.1].

Lemma 6.2. *Let S be a semiannulus. Then $\operatorname{mod} S = 0$ if and only if there exists a sequence of simple closed arcs $\gamma_n (n = 0, 1, 2, \dots)$ joining the sides of S such that $\operatorname{diam}^\sharp \gamma_n \to 0$ as $n \to \infty$.*

In particular, if $\operatorname{dist}^\sharp(\sigma_1, \sigma_2) > 0$ then $\operatorname{mod} S > 0$. The converse is, however, not true in general. For example, let $S = \{z : |z| < 3, \operatorname{Re} z \geq 0, , |z - 1| > 1, z \neq 0\}$. Then the sides consists of the two segments $\sigma_1 = (0, 3i)$ and $\sigma_2 = (0, -3i)$ and thus $\operatorname{dist}^\sharp(\sigma_1, \sigma_2) = \operatorname{dist}(\sigma_1, \sigma_2) = 0$. On the other hand, clearly $\operatorname{mod} S > 0$.

In a special but typical situation, the converse is also true.

Lemma 6.3. *Let S be a semiannulus embedded in a disk or a half-plane and let σ_1 and σ_2 be its sides. Then $\mathrm{mod}\, S > 0$ if and only if $\mathrm{dist}^\sharp(\sigma_1, \sigma_2) > 0$.*

Proof. The "if" part is immediate from Lemma 6.2. To show the "only if" part, we suppose that $\mathrm{dist}^\sharp(\sigma_1, \sigma_2) = 0$. Let Δ be the disk or half-plane in which S is properly embedded. We now take a sequence of open spherical geodesics γ_n with endpoints $z_n \in \sigma_1$ and $z'_n \in \sigma_2$ such that $\gamma_n \cap (\sigma_1 \cup \sigma_2) = \emptyset$ and that $\mathrm{diam}^\sharp \gamma_n \to 0$ as $n \to \infty$. Since Δ is convex in spherical geometry, $\gamma_n \subset \Delta$, and thus, $\gamma_n \cap \partial\Delta = \emptyset$. Therefore, $\gamma_n \cap \partial S = \emptyset$, which implies $\gamma_n \subset \mathrm{Int}\, S$. We now apply Lemma 6.2 to obtain $\mathrm{mod}\, S = 0$. $\square$

Let U_1 and U_2 be the two connected components $\mathbb{D} \setminus S$ for a semiannulus properly embedded in the unit disk $\mathbb{D}$. Let σ_1 and σ_2 be the sides of S with $\sigma_j \subset \overline{U_j}$, $j = 1, 2$. Then, by Lemma 6.3, if $\mathrm{mod}\, S > 0$, then $\mathrm{dist}(U_1, U_2) = \mathrm{dist}(\overline{U_1}, \overline{U_2}) = \mathrm{dist}(\sigma_1, \sigma_2) > 0$. Therefore, setting $\beta_j = \overline{U_j} \cap \partial\mathbb{D}$ $(j = 1, 2)$, one can see that $\mathrm{dist}(\beta_1, \beta_2) > 0$. In particular, $\partial\mathbb{D} \setminus (\beta_1 \cup \beta_2)$ consists of exactly two non-empty open circular arcs α_1, α_2.

Let S be a semiannulus properly embedded in the unit disk $\mathbb{D}$ and $f : T_R \to S$ be a homeomorphism. For a sufficiently small $\varepsilon > 0$, we set $W_\varepsilon = \{z \in T_R : \mathrm{Im}\, z \leq \varepsilon\}$ and consider a sort of cluster sets

$$I_S = \bigcap_{\varepsilon > 0} \overline{f(W_\varepsilon \cap \partial T_R)} \quad \text{and} \quad J_S = \bigcap_{\varepsilon > 0} \overline{f(W_\varepsilon)}.$$

Note that I_S and J_S do not depend on the particular choice of R and f. For instance, I_S is indeed the limit sets of the sides of S and J_S is the image of the two ends of S. By definition, it is clear that J_S is a (not necessarily disjoint) union of two closed intervals (possibly singletons) in $\partial\mathbb{D}$, and $J_S \setminus I_S$ is a union of two open (possibly empty) intervals. As a pathological example, we consider the semiannulus $S = f(\{z \in \mathbb{D} : |\mathrm{Re}\, z| \leq 1/2\})$ for a function given in (6.1) with $\limsup_{r \to 1^-} \Theta(r) = +\infty$. Then S is properly embedded in $\mathbb{D}$ and $I_S = J_S = \partial\mathbb{D}$. It is evident that $J_S \setminus I_S = \alpha_1 \cup \alpha_2$ when $\mathrm{mod}\, S > 0$. Conversely, when $\mathrm{mod}\, S = 0$, one of the ends must be degenerate by Lemma 6.3 and therefore, the set $J_S \setminus I_S$ cannot have more than one connected component. Thus, we have shown the following.

Lemma 6.4. *Let S be a semiannulus properly embedded in $\mathbb{D}$. Then $\mathrm{mod}\, S > 0$ if and only if $J_S \setminus I_S$ is a disjoint union of two non-empty open intervals in $\partial\mathbb{D}$.*

A semiannulus S is said to be *conformally equivalent* to T_R if there is a homeomorphism $f : S \to T_R$ which is conformal in $\mathrm{Int}\, S$. Then, a counterpart of Lemma 1.1 can be given in the following form.

Lemma 6.5. *Let S be a semiannulus and $R > 1$. Then S is conformally equivalent to the semiannulus T_R if and only if $\mathrm{mod}\, S = \log R$. Moreover, $\lambda(\Gamma_S) = 1/\lambda(\Gamma'_S)$.*

When S is properly embedded in $\mathbb{D}$ and $\mathrm{mod}\, S > 0$, we can construct a ring by reflecting S in the circle $|z| = 1$. More concretely, let

$$\hat{S} = \mathrm{Int}\, S \cup (\alpha_1 \cup \alpha_2) \cup \{1/\bar{z} : z \in \mathrm{Int}\, S\},$$

where α_1 and α_2 are defined above. By the symmetry principle (see [1, Chap. 1.E]), we have $\lambda(\Gamma'_{\hat{S}}) = \lambda(\Gamma'_S)/2$ and therefore $\operatorname{mod} S = \operatorname{mod} \hat{S}$. In this way, the theory of semiannulli can be reduced to that of rings.

A subset S_0 of a semiannulus S is called a *subsemiannulus* of S if S_0 is a semiannulus satisfying $\Gamma_{S_0} \subset \Gamma_S$. By definition, we have $\operatorname{mod} S_0 \leq \operatorname{mod} S$.

For $\zeta_1, \zeta_2 \in \partial\mathbb{D}$ we consider the Möbius transformation

$$L_{\zeta_1,\zeta_2}(z) = \frac{\zeta_2 + z}{\zeta_2 - z} - \frac{\zeta_2 + \zeta_1}{\zeta_2 - \zeta_1} = \frac{2\zeta_2(z - \zeta_1)}{(\zeta_1 - \zeta_2)(z - \zeta_2)}.$$

Note that $L = L_{\zeta_1,\zeta_2}$ maps $\mathbb{D}$ onto the right half-plane $\mathbb{H} = \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$ in such a way that $L(\zeta_1) = 0$ and $L(\zeta_2) = \infty$. For $0 < r_1 < r_2 < +\infty$, we set

$$T(\zeta_1, \zeta_2; r_1, r_2) = \mathbb{D} \cap L_{\zeta_1,\zeta_2}^{-1}(A(0, r_1, r_2)).$$

A semiannulus in $\mathbb{D}$ of this form will be called *canonical*. Note that

$$\operatorname{mod} T(\zeta_1, \zeta_2; r_1, r_2) = \operatorname{mod} \hat{T}(\zeta_1, \zeta_2; r_1, r_2) = \log \frac{r_2}{r_1}.$$

We also set

$$T(\zeta; r_1, r_2) = T(\zeta, -\zeta; r_1, r_2).$$

By using the reflection technique, we can immediately deduce the following from Lemma 3.1.

Lemma 6.6. *Let S be a semiannulus properly embedded in $\mathbb{D}$ with $\operatorname{mod} B > \pi$ and U_1 and U_2 be the two connected components of $\mathbb{D} \setminus S$. For given points $\zeta_j \in \partial\mathbb{D} \cap \partial U_j$ $(j = 1, 2)$, there exists numbers $0 < r_1 < r_2 < +\infty$ such that $T = T(\zeta_1, \zeta_2; r_1, r_2)$ is a subsemiannulus of S and $\operatorname{mod} T \geq \operatorname{mod} S - \pi$.*

We also have the following analog to Corollary 3.3.

Theorem 6.7. *Let S be a semiannulus properly embedded in $\mathbb{D}$ and U_1 and U_2 be the two connected components of $\mathbb{D} \setminus S$. Then*

$$\min\{\operatorname{diam} U_1, \operatorname{diam} U_2\} \leq C \exp(-\tfrac{1}{2} \operatorname{mod} S),$$

where $C = 4e^{\pi/2}$.

For the proof, we prepare a result which is a hyperbolic analog of Lemma 3.6 (cf. [19, Lemma 2.7]).

Lemma 6.8. *Let T be a canonical semiannulus properly embedded in $\mathbb{D}$ and let V_1 and V_2 be the connected components of its complement in $\mathbb{D}$. Then*

$$\min\{\operatorname{diam} V_1, \operatorname{diam} V_2\} \leq \frac{2}{\cosh(\frac{1}{2}\operatorname{mod} T)}.$$

Equality holds if and only if T is of the form $T(\zeta; r, 1/r)$ for some $\zeta \in \partial\mathbb{D}$ and $0 < r < 1$.

Proof. We denote by d_Ω the hyperbolic distance on a hyperbolic domain. Suppose that T is of the form $T(\zeta_1, \zeta_2; r_1, r_2)$. Let $L = L_{\zeta_1,\zeta_2} : \mathbb{D} \to \mathbb{H}$. Then the hyperbolic distance of

the hyperbolic half-planes V_1 and V_2 in $\mathbb{D}$ can be computed by

$$\delta := d_{\mathbb{D}}(V_1, V_2) = d_{\mathbb{H}}(L(V_1), L(V_2)) = \int_{r_1}^{r_2} \frac{dx}{2x} = \frac{1}{2} \log \frac{r_2}{r_1} = \frac{1}{2} \operatorname{mod} T.$$

Thus the problem now reduces to finding a configuration of two hyperbolic half-planes with a fixed hyperbolic distance such that the minimum of their Euclidean diameters is maximal (namely, the worst case). Such a configuration is attained obviously by the situation that $V_2 = -V_1$. By a suitable rotation, we may assume that $\zeta_1 = 1, \zeta_2 = -1$. Let $a > 0$ be the number determined by $V_1 \cap \mathbb{R} = (a, 1)$. Since 0 is the midpoint of the hyperbolic geodesic $[-a, a]$ joining V_1 and V_2, we have $\delta/2 = d_{\mathbb{D}}(0, a) = \operatorname{arctanh} a$ and $a = \tanh(\delta/2)$. The disk automorphism (hyperbolic isometry) $g(z) = (z + a)/(1 + az)$ maps the hyperbolic half-plane $\{z \in \mathbb{D} : \operatorname{Re} z > 0\}$ onto V_1. Therefore, we see that $g(i)$ and $g(-i)$ are the tips of V_1 and thus $\operatorname{diam} V_1 = |g(i) - g(-i)| = 2(1 - a^2)/(1 + a^2)$. Finally, we get the estimate in this case

$$\operatorname{diam} V_j = 2 \frac{1 - \tanh(\delta/2)^2}{1 + \tanh(\delta/2)^2} = \frac{2}{\cosh \delta}.$$

Since $\delta = \frac{1}{2} \operatorname{mod} T$, the estimate is now shown. The equality case is obvious from the above argument. $\qquad \square$

Proof of Theorem 6.7. When $\operatorname{mod} S \leq \pi$, the assertion trivially holds. We now suppose that $\operatorname{mod} S > \pi$. Then, by Lemma 6.6, we can take a canonical subsemiannulus T of S. Let V_1, V_2 be the two components of $\mathbb{D} \setminus T$ so that $U_j \subset V_j$ $(j = 1, 2)$. Since the boundary circular arc $\mathbb{D} \cap \partial V_j$ is perpendicular to $\partial \mathbb{D}$, at least one of V_j's, say, V_1 is contained in the half-plane of the form $\operatorname{Re} e^{i\theta} w > 0$. Then, as is easily checked, $\operatorname{diam} V_1 = |\xi - \eta|$, where ξ and η are the endpoints of the arc $\mathbb{D} \cap \partial V_1$. Since $\xi, \eta \in \overline{T}$, by the last lemma, we have

$$\min\{\operatorname{diam} U_1, \operatorname{diam} U_2\} \leq \operatorname{diam} U_1 \leq \operatorname{diam} V_1 \leq \frac{2}{\cosh(\frac{1}{2} \operatorname{mod} T)} < 4 \exp(-\tfrac{1}{2} \operatorname{mod} T).$$

$$\square$$

7. APPLICATION TO BOUNDARY EXTENSION

We are now in a position to state a criterion of extendibility of a homeomorphism of $\mathbb{D}$ to a boundary point (cf. [19, Prop. 3.1]).

Proposition 7.1. *Let $f : \mathbb{D} \to \mathbb{D}$ be a homeomorphism and let $\zeta \in \partial \mathbb{D}$. The mapping f extends continuously to ζ if*

$$\lim_{r \to 0+} \operatorname{mod} f(T(\zeta, \zeta'; r, R)) = +\infty$$

for some $\zeta' \in \partial$ with $\zeta' \neq \zeta$ and $R > 0$.

Proof. Let U_r be the connected component of $\mathbb{D} \setminus T(\zeta, \zeta'; r, R)$ containing ζ in the boundary for $0 < r < R$ and let V_R be the other one, which does not depend on r. Then the family

of the sets U_r, $0 < r < R$, constitutes a fundamental system of neighbourhoods of ζ. Theorem 6.7 now yields

$$\min\{\operatorname{diam} f(U_r), \operatorname{diam} f(V_R)\} \le C \exp(-\tfrac{1}{2} \operatorname{mod} f((T(\zeta, \zeta'; r, R)))).$$

By assumption, the last term tends to 0 as $r \to 0+$. Since $\operatorname{diam} f(V_R)$ is a fixed number, this implies that $\operatorname{diam} \overline{f(U_r)} \to 0$ as $r \to 0$. Therefore, the intersection $\bigcap_{0<r<R} \overline{f(U_r)}$ consists of a single point. We can now assign this point as the extended value of f at ζ so that f has a continuous extension to ζ. □

We remark that the converse is not true in the last proposition. Indeed, consider the homeomorphism $f : \mathbb{D} \to \mathbb{D}$ determined by $f(\bar{z}) = \overline{f(z)}$, $z \in \mathbb{D}$ and

$$f(re^{i\theta}) = r \exp(i(\theta/\pi)^{-\log(1-r)}), \quad 0 \le \theta \le \pi, \ 0 < r < 1.$$

Then, by construction, f extends to 1 continuously by setting $f(1) = 1$. However, since $f(re^{i\theta}) \to 1$ as $r \to 1-$ for any fixed θ with $|\theta| < \pi$, the converse of the proposition does not hold (see the proof of the next theorem).

If the assumption of the last proposition is true for every point $\zeta \in \partial\mathbb{D}$, then the converse actually holds. The next theorem is due to Brakalova [9], though her formulation is slightly different. Note that, earlier than it, Jixiu Chen, Zhiguo Chen and Chengqi He [14] proved a similar result in a special situation (see also the proof of Lemma 2.3 in [15]).

Theorem 7.2 (Brakalova [9]). *A homeomorphism $f : \mathbb{D} \to \mathbb{D}$ admits a homeomorphic extension to $\overline{\mathbb{D}}$ if and only if for each $\zeta \in \partial\mathbb{D}$,*

$$\lim_{r \to 0+} \operatorname{mod} f(T(\zeta, \zeta'; r, R)) = +\infty$$

for some $\zeta' \in \partial\mathbb{D}$, $\zeta' \ne \zeta$ and $R = R(\zeta) > 0$.

Proof. By Proposition 7.1, f can be extended continuously to every boundary point. It is almost immediate to see that the extended mapping $\tilde{f} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is indeed continuous.

We next show that $\tilde{f}$ is injective. Suppose, to the contrary, that $\tilde{f}(\zeta_1)$ and $\tilde{f}(\zeta_2)$ are the same point, say, ω_0, for some $\zeta_1, \zeta_2 \in \partial\mathbb{D}$ with $\zeta_1 \ne \zeta_2$. We may assume that $\bar{\zeta}_1 = \zeta_2$. Consider the semiannulus $T = T(1, -1; r, R)$, $0 < r < R$, where $R = |(\zeta_1 - 1)/(\zeta_1 + 1)|$. Then, the outer side σ of T lands at ζ_1 and ζ_2. By assumption, $f(z)$ tends to the point ω_0 when z approaches ζ_j ($j = 1, 2$) along σ in both directions. In particular, $f(T)$ is enclosed by the Jordan curve $f(\sigma) \cup \{\omega_0\}$. Therefore, $I_T = J_T = \{\omega_0\}$ and so $J_T \setminus I_T = \emptyset$. Lemma 6.4 now implies that $\operatorname{mod} f(T) = 0$ for any $0 < r < R$, which contradicts the assumption of the theorem. Thus, we have shown that $\tilde{f}$ is injective.

Since $\overline{\mathbb{D}}$ is a compact Hausdorff space, the inverse mapping $\tilde{f}^{-1}$ is also continuous. Therefore, $\tilde{f} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is a homeomorphism. □

From the proof, obviously we can replace "some $\zeta' \in \partial\mathbb{D}$, $\zeta' \ne \zeta$" by "every $\zeta' \in \partial\mathbb{D}$, $\zeta' \ne \zeta$" in Theorem 7.2. Choosing $\zeta' = -\zeta$ and performing the Möbius transformation $L(z) = i(1+z)/(1-z)$, we can translate the above theorem into a result on the upper half-plane.

Theorem 7.3. *A homeomorphism f of the upper half-plane $\mathbb{H}$ admits a homeomorphic extension to $\overline{\mathbb{H}}$ if and only if for each $a \in \partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$,*

$$\lim_{r \to 0+} \mod f(\overline{A(a, r, R)} \cap \mathbb{H}) = +\infty$$

for some $R = R(a) > 0$.

We recall that $A(\infty; r, R)$ is defined to be $A(0; 1/R, 1/r)$.

Brakalova and Jenkins [13] proved the following.

Theorem 7.4 (Brakalova-Jenkins [13])**.** *Let f be a sense-preserving self-homeomorphism of the upper half-plane $\mathbb{H}$ and satisfies the equation $f_{\bar{z}} = \mu f_z$ a.e. Suppose that $f(z) \to \infty$ if and only if $z \to \infty$ in $\mathbb{H}$ and that*

$$\iint_{A(t;r,R) \cap \mathbb{H}} \frac{|\mu(z)|^2 + |\mathrm{Re}\, \frac{\bar{z}-t}{z-t}\mu(z)|}{1 - |\mu(z)|^2} \frac{dxdy}{|z-t|^2}$$

has a finite limit as $r \to 0+$ for every $t \in \mathbb{R}$ and some $R = R(t) > 0$. Then f extends to a homeomorphism of $\overline{\mathbb{H}}$ in such a way that the boundary function $f(t)$ is differentiable and

$$f'(t) = \lim_{z \to a \ in \ \mathbb{H}} \frac{f(z) - f(t)}{z - t} > 0, \quad t \in \mathbb{R}.$$

Moreover if the convergence in the above is locally uniform for $t \in \mathbb{R}$, then f' is continuous on $\mathbb{R}$.

In this theorem, the behaviour of the function at ∞ is assumed. It may be, however, more natural to describe the assumptions in terms of μ only. Gutlyanskiĭ, Sakan and the author [19] refined this result in the following form. To state it, we introduce the quantity (cf. [17])

$$Q_\mu(r, R) = \frac{1}{\pi \log(R/r)} \iint_{A(0;r,R) \cap \mathbb{H}} \frac{D_{\mu,0}(z)}{|z|^2} dxdy,$$

which is regarded as the average of $D_{\mu,0}$ over $A(0; r, R) \cap \mathbb{H}$ with respect to the measure $|z|^{-2}dxdy$.

Theorem 7.5 ([19])**.** *Let μ be a measurable function on the upper half-plane $\mathbb{H}$ with $|\mu| < 1$ a.e. Assume that the following conditions are satisfied for some positive constants M and R_0 :*

(1) $\displaystyle \lim_{R \to +\infty} \frac{Q_\mu(r, R)}{\log R} = 0,$

(2) $\displaystyle \iint_{A(t;r,R_0) \cap \mathbb{H}} \frac{|\mu(z)|^2}{1 - |\mu(z)|^2} \frac{dxdy}{|z-t|^2}$ *converges as $r \to 0+$ for each $t \in \mathbb{R}$, and*

(3) $\mathrm{Re} \displaystyle \iint_{A(t;r,R_0) \cap \mathbb{H}} \frac{\mu(z)}{(z-t)^2} \frac{dxdy}{1 - |\mu(z)|^2}$ *converges as $r \to 0+$ for $t \in \mathbb{R}$ locally uniformly.*

Suppose that there exists a μ-conformal self-homeomorphism f of $\mathbb{H}$. Then it extends to a homeomorphism of $\overline{\mathbb{H}}$ onto itself. Furthermore, if we normalize f so that $f(\infty) = \infty$, then the boundary function $f(t)$ has a non-vanishing continuous derivative on $\mathbb{R}$.

For the proof, we need the following fundamental estimates of the modulus change of semiannuli under a μ-conformal homeomorphism f.

$$\frac{\operatorname{mod} T}{Q_\mu(r, R)} \leq \operatorname{mod} f(T),$$

where $T = A(0; r, R) \cap \mathbb{H}$ and

$$-\frac{1}{\pi} \iint_T \frac{D_{-\mu,t}(z) - 1}{|z - t|^2} dx dy \leq \operatorname{mod} T - \operatorname{mod} f(T) \leq \frac{1}{\pi} \iint_T \frac{D_{\mu,t}(z) - 1}{|z - t|^2} dx dy,$$

where $T = A(t; r, R) \cap \mathbb{H}$, $t \in \mathbb{R}$. The first one is a semiannulus version of the inequality (2.5) in [17] and the second one is a sort of distortion estimate of the modulus (cf. Corollary 2.13 in [17]). See [19] for details.

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