2013年度
「リーマン面・不連続群論」研究集会

大阪大学

2013年 11月 9日（土）– 11月 11日（月）

プログラム＋アブストラクト＋講演スライド

Program

9 November (Saturday)

14:00–14:50 Lizhen Ji (University of Michigan)
Spines of Teichmuller spaces and symmetric spaces

15:00–15:50 Yoshihiko Shinomiya (Tokyo Institute of Technology)
Periodic points on Veech surfaces

16:00–16:50 Chikako Mese (Johns Hopkins University)
Harmonic maps in rigidity problems

Banquet

10 November (Sunday)

10:00–10:50 Makoto Masumoto (Yamaguchi University)
On the existence of holomorphic mappings of once-holed tori

11:00–11:50 Hideki Miyachi (Osaka University)
Rigidity of isometries on Teichmueller space at infinity

Lunch

14:00–14:50 Hiroshige Shiga (Tokyo Institute of Technology)
On deformations spaces of Kleinian groups

15:00–15:50 Yu Kawakami (Yamaguchi University)
On function-theoretic properties for Gauss maps of several classes of surfaces

16:00–16:50 Yuriko Umemoto (Osaka City University)
Growth rates of cocompact hyperbolic Coxeter groups and 2–Salem numbers

11 November (Monday)

10:00–10:50 Masanori Amano (Tokyo Institute of Technology)
On behavior of pairs of Teichmüller geodesic rays

11:00–11:50 Tanran Zhang (Tohoku University)
Uniformisation and description of a once-punctured annulus

Lunch

14:00–14:50 Ryosuke Mineyama (Osaka University)
Limit sets of Coxeter groups of type (n-1,1)

15:00–15:50 Ken'ichi Ohshika (Osaka University)
Primitive stable closed hyperbolic 3-manifolds

Abstract

Lizhen Ji (University of Michigan)
Spines of Teichmuller spaces and symmetric spaces
Abstract: Let T_g be the Teichmuller space of a compact surface S_g of genus g, and Mod_g the mapping class group of S_g. Then Mod_g acts properly on T_g, and the quotient $Mod_g\ T_g$ is the moduli space of compact Riemann surfaces of genus g. This action of Mod_g on T_g is an analogue of the action of an arithmetic subgroup Γ of a semisimple Lie group G on the associated symmetric space $X = G/K$, where K is a maximal compact subgroup of G.

A longstanding open problem concerns spines of T_g, i.e., equivariant deformation retracts of T_g with compact quotient by Mod_g and of dimension equal to the virtual cohomological dimension of Mod_g. Similarly, when Γ is a nonuniform arithmetic subgroup, existence of spines of X is also open in general.

In this talk, I will describe the history of these problems (for example, Thurston's attempt) and some recent results on them.

Yoshihiko Shinomiya (Tokyo Institute of Technology)
Periodic points on Veech surfaces
Abstract: We will discuss periodic points on Veech surfaces. A periodic point on a Veech surface is a point whose orbit under the affine group is finite. It is known that the number of periodic points on a non-arithmetic Veech surface is finite. We will give upper bounds of the numbers of periodic points depending only on the types of Veech surfaces and signatures of the Veech groups.

Chikako Mese (Johns Hopkins University)
Harmonic maps in rigidity problems
Abstract: We discuss harmonic maps into non-positively curved metric spaces (NPC spaces). Of particular interest is the regularity for these maps into special classes of spaces that include the Euclidean and Hyperbolic buildings and Weil-Petersson completion of Teichmuller space. As an application of the regularity theory, we study rigidity questions.

Makoto Masumoto (Yamaguchi University)
On the existence of holomorphic mappings of once-holed tori
Abstract: We address the existence problem of handle-preserving holomorphic mappings of once-holed tori into a given Riemann surface of positive genus. The once-holed tori allowing such mappings form a subset of the Teichmüller space of a once-holed torus. We are particularly interested in geometric properties of the set.

By a *once-holed torus* we mean a noncompact Riemann surface of genus one with exactly one (Kerékjártó-Stoïlow) boundary component. For example, the Riemann surface obtained from a compact Riemann surface of genus one, or a *torus*, by removing one point is a once-holed torus, which will be referred to as a *once-punctured torus*.

Let R be a Riemann surface of positive genus; it may be compact or the genus may be infinite. A *mark of handle* of R means an ordered pair $\chi = \{a, b\}$ of simple loops a and b on R whose intersection number $a \times b$ is equal to one. The pair $Y = (R, \chi)$ is said to be a Riemann surface *with marked handle*. Since the genus of R is positive, the surface has one or more handles. We choose just one of them and mark it with a pair of simple loops.

Let $Y' = (R', \chi')$, where $\chi' = \{a', b'\}$, be another Riemann surface with marked handle. If $f : R \to R'$ is continuous and maps a and b onto loops freely homotopic to a' and b' on R', respectively, then we say that f is a continuous mapping of Y into Y' and use the notation $f : Y \to Y'$. If $f : R \to R'$ possesses some additional properties, then $f : Y \to Y'$ is said to have the same properties. For example, if $f : R \to R'$ is conformal, that is, if $f : R \to R'$ is holomorphic and injective, then f is called a conformal mapping of Y into Y'.

A once-holed torus (resp. torus, once-punctured torus) with marked handle is usually called a marked once-holed torus (resp. marked torus, marked once-punctured torus). Let $\mathfrak{T}$ be the set of marked once-holed tori, where two marked once-holed tori are identified with each other if there is a conformal mapping of one *onto* the other.

We introduce a global coordinate system on $\mathfrak{T}$ as follows. For a marked once-holed torus $X = (T, \chi)$, where $\chi = \{a, b\}$, set $\Lambda(X) = (\lambda_1, \lambda_2, \lambda_3)$, where λ_1, λ_2 and λ_3 are the extremal lengths of the free homotopy classes of a, b and ab^{-1}, respectively. Then Λ defines an injective mapping of $\mathfrak{T}$ into $\mathbb{R}^3_+$, whose image is

$$\Lambda(\mathfrak{T}) = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3_+ \mid \xi_1^2 + \xi_2^2 + \xi_3^2 - 2(\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1) + 4 \leqq 0\}.$$

Identifying $\mathfrak{T}$ with $\Lambda(\mathfrak{T})$, we consider $\mathfrak{T}$ as a 3-dimensional real analytic manifold with boundary. A marked once-holed torus lies on the boundary if and only if it is a marked once-punctured torus.

As a set, $\mathfrak{T}$ is the union of the Teichmüller space of a once-punctured torus and the reduced Teichmüller space of a once-holed torus which is not a once-punctured torus. The real analytic structure on $\mathfrak{T}$ is compatible with the real analytic structures on those Teichmüller spaces. We will call $\mathfrak{T}$ the *Teichmüller space of a once-holed torus*.

Now, fix a Riemann surface Y_0 with marked handle. We are interested in the set $\mathfrak{T}_a[Y_0]$ (resp. $\mathfrak{T}_c[Y_0]$) of marked once-holed tori $X \in \mathfrak{T}$ for which there is a holomorphic (resp. conformal) mapping of X into Y_0. Clearly, $\mathfrak{T}_c[Y_0]$ is nonempty and included in $\mathfrak{T}_a[Y_0]$.

THEOREM 1. *The sets $\mathfrak{T}_a[Y_0]$ and $\mathfrak{T}_c[Y_0]$ are noncompact closed domains with Lipschitz boundary.*

Our next result is expressed in terms of another global coordinate system on $\mathfrak{T}$. Every marked once-holed torus is realized as a horizontal slit domain of a marked torus. To be more specific let $\mathbb{H}$ denote the upper half-plane. For any $\tau \in \mathbb{H}$ let G_τ be the additive group generated by 1 and τ, and set $T_\tau = \mathbb{C}/G_\tau$, which is a torus. The oriented segments $[0, 1]$ and $[0, \tau]$ are projected onto simple loops a_τ and b_τ on T_τ, respectively, which make a mark χ_τ of handle of T_τ. We set $X_\tau = (T_\tau, \chi_\tau)$. Let $\pi_\tau : \mathbb{C} \to T_\tau$ be the natural projection. Cutting T_τ along the image $\pi_\tau([0, s])$ of the segment $[0, s]$, where $0 \leqq s < 1$, we obtain a once-holed torus $T_\tau^{(s)} := T_\tau \setminus \pi_\tau([0, s])$. It is a horizontal slit domain of the torus T_τ. Note that $T_\tau^{(0)}$ is a once-punctured torus. Choose a mark $\chi_\tau^{(s)} = \{a_\tau^{(s)}, b_\tau^{(s)}\}$ of handle of $T_\tau^{(s)}$ so that the inclusion mapping $T_\tau^{(s)} \hookrightarrow T_\tau$ is a conformal mapping of $X_\tau^{(s)} := (T_\tau^{(s)}, \chi_\tau^{(s)})$ into X_τ. Then the correspondence $(\tau, s) \mapsto X_\tau^{(s)}$ is a homeomorphism of $\mathbb{H} \times [0, 1)$ onto $\mathfrak{T}$, whose restrictions to $\mathbb{H} \times (0, 1)$ and to $\mathbb{H} \times \{0\}$ are real analytic. Note that $1/\operatorname{Im}\tau$ is exactly the extremal length of the free homotopy class of $a_\tau^{(s)}$.

THEOREM 2_a. *There is a nonnegative real number $\lambda_a[Y_0]$ such that*

(i_a) *if* $\operatorname{Im}\tau \geqq 1/\lambda_a[Y_0]$, *then there are no holomorphic mappings of* $X_\tau^{(s)}$ *into* Y_0 *for any* $s \in [0,1)$, *while*

(ii_a) *if* $\operatorname{Im}\tau < 1/\lambda_a[Y_0]$, *then there are holomorphic mappings of* $X_\tau^{(s)}$ *into* Y_0 *for some* $s \in [0,1)$,

where $1/0 = +\infty$.

For the existence of conformal mappings of marked once-holed tori, we have the following theorem. It is quite similar to the previous theorem though the sign of equality does not appear in (i_c).

THEOREM 2_c. *There is a positive real number* $\lambda_c[Y_0]$ *such that*

(i_c) *if* $\operatorname{Im}\tau > 1/\lambda_c[Y_0]$, *then there are no conformal mappings of* $X_\tau^{(s)}$ *into* Y_0 *for any* $s \in [0,1)$, *while*

(ii_c) *if* $\operatorname{Im}\tau < 1/\lambda_c[Y_0]$, *then there are conformal mappings of* $X_\tau^{(s)}$ *into* Y_0 *for some* $s \in [0,1)$.

Finally, we evaluate the critical extremal lengths $\lambda_a[Y_0]$ and $\lambda_c[Y_0]$. Let $Y_0 = (R_0, \chi_0)$, where $\chi_0 = \{a_0, b_0\}$. Let $\lambda[Y_0]$ stand for the extremal length of the free homotopy class of a_0. If R_0 is not a torus, then it carries a hyperbolic metric. We denote by $l[Y_0]$ the length of the geodesic freely homotopic to a_0, where the curvature is normalized to be -1. If R_0 is a torus, then we define $l[Y_0] = 0$.

THEOREM 3. *It holds that* $\lambda_a[Y_0] = \dfrac{1}{\pi} l[Y_0]$ *and* $\lambda_c[Y_0] = \lambda[Y_0]$.

It follows that $\lambda_a[Y_0] < \lambda_c[Y_0]$ for any Y_0. Also, $\lambda_a[Y_0]$ is strictly positive unless Y_0 is a marked torus.

Hideki Miyachi (Osaka University)

Rigidity of isometries on Teichmueller space at infinity

Abstract: In this talk, I will give a rigidity result for isometries with respect to the Teichmueller distance on Teichmueller space of Riemann surfaces of analytically finite type. Indeed, we will provide mappings acting on Teichmueller space which are close to isometries at infinity, and discuss properties of the mappings. If time permits, we will re-prove Ivanov's theorem, which says that except for few cases, the isometry group of Teichmuller space is isomorphic to the extended mapping class group.

Hiroshige Shiga (Tokyo Institute of Technology)

On deformations spaces of Kleinian groups

Abstract: Let G be a non-elementary Kleinian group. We consider the space of quasiconformal deformations of G. The space has a natural complex structure and it is finite dimensional if G is finitely generated. In this talk, we consider complex analytic properties of the spaces, which are related to some results by Bers, Kra-Maskit and McMullen.

Yu Kawakami (Yamaguchi University)
On function-theoretic properties for Gauss maps of several classes of surfaces
Abstract: The aim of this talk is to reveal the geometric background of function-theoretic properties for Gauss maps of several classes of immersed surfaces in space forms (e.g. minimal surfaces in the Euclidean 3-space, flat surfaces in the hyperbolic 3-space etc.). For the purpose, we give an optimal curvature bound for a specified conformal metric on an open Riemann surface and give some applications.

Yuriko Umemoto (Osaka City University)
Growth rates of cocompact hyperbolic Coxeter groups and 2–Salem numbers
Abstract: The group generated by reflections with respect to facets of a Coxeter polytope in n–dimensional hyperbolic space $\mathbb{H}^n$ is called a hyperboric Coxeter group. By the results of Cannon, Wagreich and Parry, it is known that the growth rate of a cocompact Coxeter group in $\mathbb{H}^2$ and $\mathbb{H}^3$ is a Salem number. On the other hand, Kerada defined a j–Salem number, which is a generalization of a Salem number. In this talk, I will present that we realize infinitely many 2–Salem numbers as the growth rates of cocompact Coxeter groups in $\mathbb{H}^4$. Our Coxeter polytopes are constructed by successive gluing of Coxeter polytopes which we call Coxeter dominoes.

Masanori Amano (Tokyo Institute of Technology)
On behavior of pairs of Teichmüller geodesic rays
Abstruct: In this talk, we obtain the explicit limit value of the Teichmüller distance between two Teichmüller geodesic rays which are determined by Jenkins-Strebel differentials having a common end point in the augmented Teichmüller space. Furthermore, we also obtain a condition under which these two rays are asymptotic. This is the Teichmüller space varsion of a result of Farb and Masur for the moduli space.

Tanran Zhang (Tohoku University)
Uniformisation and description of a once-punctured annulus
Abstract: The Uniformisation Theorem shows that the universal covering space $\widetilde{X}$ of an arbitrary Riemann surface X is homeomorphic, by a conformal map $\mathfrak{m}$, to either the Riemann sphere $\hat{\mathbb{C}}$, the complex plane $\mathbb{C}$ or the unit disk $\mathbb{D}$. And then the fundamental group $\Pi_1(X)$ has a representation as a group G of conformal homeomorphisms of $\mathfrak{m}(\widetilde{X})$. This theorem also indicates that if $\widetilde{X}$ is homeomorphic to a proper subset of $\mathbb{C}$ with at least three boundary points, then $\widetilde{X}$ is conformally equivalent to a quotient space $\mathbb{D}/G$, where G is a torsion-free Fuchsian group that acts (discontinuously) on $\mathbb{D}$ (or $\mathbb{H}$). The group G is isomorphic to $\Pi_1(X)$. Hempel and Smith studied the hyperbolic Riemann surface model of the twice-punctured disk $\mathbb{D}\backslash\{p_1, p_2\}$ in 1980s. They estimated the hyperbolic density on it near aone puncture and considered the coalescing of the two punctures. Later on Beardon gave five different ways to uniformize $\mathbb{D}\backslash\{p_1, p_2\}$ in 2012. He investigated several conformal invariants to characterize $\mathbb{D}\backslash\{p_1, p_2\}$ considering the fundamental domain, symmetric collars and extremal length. We extend his work to the once-punctured annulus $A := \{z : 1/R < |z| < R\}\backslash\{a\}$, $R > 1$, $1/R < a < R$. We provide several parameter pairs to uniformize and characterize it. The main tools we use are Möbius transformations, covering space, homotopy classes and elliptic integrals.

References

1. A.F. Beardon, *On the geometry of discrete groups*, Graduate Texts in Mathematics, no. 91, Springer-Verlag, 1983.

2. A.F. Beardon, *The uniformisation of a twice-punctured disc*, Comput. Methods Funct. Theory **12** (2012), no. 2, 585–596.

3. J.A. Hempel and S.J. Smith, *Uniformization of the twice-punctured disc - problems of confluence*, Bull. Australian Math. Soc. **39** (1989), 369–387.

Ryosuke Mineyama (Osaka University)

Limit sets of Coxeter groups of type (n-1,1)

Abstract: Recentry Hohlweg, Labbe, Ripoll introduced a non-linear action of Coxeter groups to investigate asymptotic behavior of their roots. This turns out to be a discrete action on a CAT(0) space in the case that associating bilinear form of the Coxeter group has singnature (n-1,1). I am interested in how geometric aspects of Coxeter groups are mirrored on their limit sets. In this talk we discuss the existence of Cannon-Thurston maps from Gromov boundaries of Coxeter groups to their limit sets. If we have the time left, we observe a relationship between limit sets and sets of accumulation points of roots. This talk partially based on the joint work with Akihiro Higashitani and Norihiro Nakashima.

Ken'ichi Ohshika (Osaka University)

Primitive stable closed hyperbolic 3-manifolds

Abstract: This is joint work with Cyril Lecuire and Inkang Kim. We show that every Heegaard splitting with large Hempel distance and bounded combinatorics induces a primitive stable representation of a free group. This implies that every point on the boundary of the Schottky space can be approximated by unfaithful primitive stable representations corresponding to closed hyperbolic 3-manifolds.

Periodic points on Veech surfaces

Yoshihiko Shinomiya

Tokyo Institute of Technology

November 9, 2013

The purpose of this talk is to estimate the number of periodic points on non-arithmetic Veech surfaces.

Theorem

Let (X, u) be a non-arithmetic Veech surface of type (g, n). The number of periodic points of (X, u) is at most

$$2^{-26} d^{10} (\lambda\mu)^{-34} \left(\frac{1}{2}\lambda^6 \mu^6\right)^{2^{2d+3}}.$$

Here, $\Gamma(X, u)$ is the Veech group of (X, u), $d := 3g - 3 + n$, $\lambda := 2\exp(5d/e)$, and $\mu := \mathrm{Area}\,(\mathbb{H}/\Gamma(X, u))$.

If we have time, we apply this estimation to holomorphic families of Riemann surfaces induced by Teichmüller curves.

1. Introduction

Flat surfaces

Let X be a (connected) surface of finite type and C a finite subset of X. **A flat structure** u on X is an atlas of $X \setminus C$ such that, for coordinate neighborhoods $(U, z), (V, w) \in u$ with $U \cap V \neq \emptyset$, the transition function is of the form

$$w = \pm z + c$$

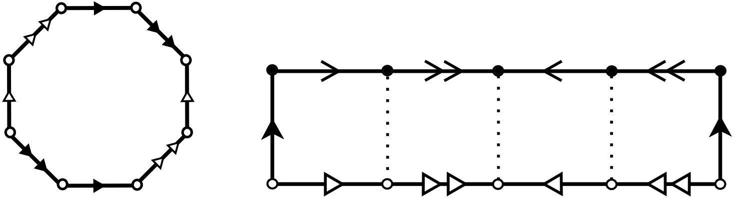
in $z(U \cap V)$ for some $c \in \mathbb{C}$.
The pair (X, u) is called **a flat surface** with singularities at C.

On flat surfaces, we can consider some notations in the Euclidean geometry: segments, their lengths or directions, area, etc. **A closed θ-geodesic** in (X, u) is a closed geodesic in (X, u) whose direction is $\theta \in [0, \pi)$ and which does not contain singularities.

We assume that the Euclidean area of (X, u) is finite.

Examples of flat surfaces

Typical examples of flat surfaces are tori. They have natural flat structures induced by universal coverings. Tori are flat surfaces with no singularities. Let us consider the following examples.



X_1 X_2

The surfaces X_1 and X_2 are of genus 2. We give flat structures u_1 and u_2 to X_1 and X_2 from Euclidean structures on the regular octagon and the rectangle, respectively. Then, the flat surface (X_1, u_1) has only one singularity corresponding to the vertices of the octagon. The singularities of the flat surface (X_2, u_2) are the points corresponding to the vertices of squares.

Let (X, u) be a flat surfaces with singularities at C. **An affine map** of (X, u) is a quasiconformal self-map h of X that satisfies $h(C) = C$ and, for coordinate neighborhoods $(U, z), (V, w) \in u$ with $h(U) \subset V$, the composition $w \circ h \circ z^{-1}$ is of the form

$$w \circ h \circ z^{-1} = Az + c$$

in $z(U) \subset \mathbb{C} = \mathbb{R}^2$ for some $A \in \mathrm{SL}(2, \mathbb{R})$ and $c \in \mathbb{C}$.
The affine group $\mathrm{Aff}^+(X, u)$ is the group of all affine maps of (X, u).

Take an affine map h of (X, u). For coordinate neighborhoods $(U, z), (V, w) \in u$ with $h(U) \subset V$, the derivative of the composition $w \circ h \circ z^{-1} = Az + c$ is the matrix $A \in \mathrm{SL}(2, \mathbb{R})$. The matrix A does not depend on the choice of coordinate neighborhoods up to the sign since transition functions of u are of the form $z \mapsto \pm z + c$. Thus, we have the homomorphism

$$D : \mathrm{Aff}^{+}(X, u) \to \mathrm{PSL}(2, \mathbb{R})$$

which maps each affine map h to its derivative $\pm A$.
The image $\Gamma(X, u) := \mathrm{Im}(D)$ of the homomorphism D is called **the Veech group of** (X, u).

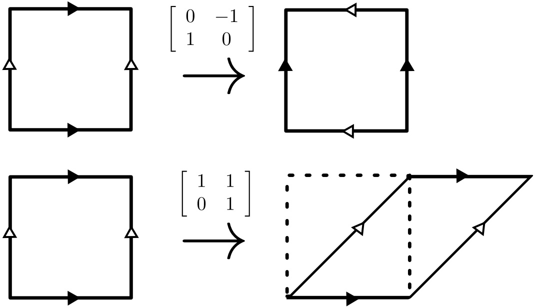
Theorem (Veech)

The Veech group $\Gamma(X, u)$ is a Fuchsian group.

Examples of Veech groups

Let (T, u_T) be the torus obtained from an unit square. Then, the Veech group $\Gamma(T, u_T)$ is $\mathrm{PSL}(2, \mathbb{Z}) = \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$.

We can see the actions of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ as follows.

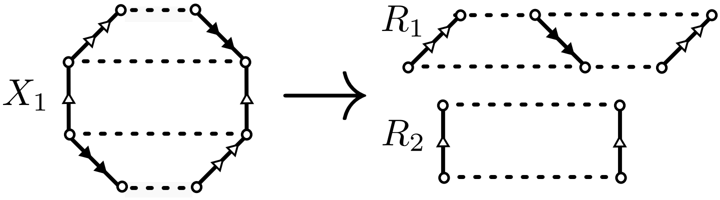


The action of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is the Dehn twist along a horizontal closed curve of (T, u_T).

Examples of Veech groups

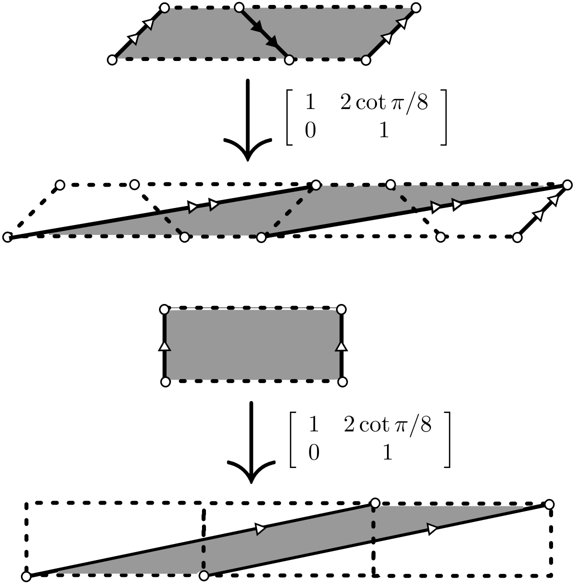
Let (X_1, u_1) be the flat surface obtained from a regular octagon. Then, $\Gamma(X_1, u_1) = \left\langle \begin{bmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{bmatrix}, \begin{bmatrix} 1 & 2\cot\pi/8 \\ 0 & 1 \end{bmatrix} \right\rangle$.

The action of $\begin{bmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{bmatrix}$ is a rotation. To see the action of $\begin{bmatrix} 1 & 2\cot\pi/8 \\ 0 & 1 \end{bmatrix}$, we cut X_1 along the horizontal segments connecting the singularity. Then, X_1 is decomposed into two cylinders R_1 and R_2.

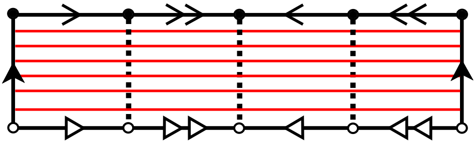


The action of $\begin{bmatrix} 1 & 2\cot\pi/8 \\ 0 & 1 \end{bmatrix}$ is the composition of the right hand Dehn twist along a core curve of R_1 and the square of the right hand Dehn twist along a core curve of R_2.

Examples of Veech groups



As we saw in the previous example, some flat surfaces can be decomposed into cylinders. A direction $\theta \in [0, \pi)$ is said to be **a Jenkins-Strebel direction** of a flat surface (X, u) if almost all points of X lie in closed θ-geodesics.
If θ is a Jenkins-Strebel direction, (X, u) is decomposed into cylinders foliated by closed θ-geodesics. The cylinders are called **the cylinder decomposition of** (X, u) **by the direction** θ.
The boundaries of these cylinders consist of segments of direction θ connecting singularities.



The directions $\theta = 0, \frac{\pi}{4}$ and $\frac{\pi}{2}$ are Jenkins-Strebel directions.

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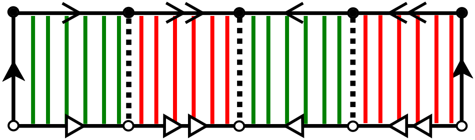

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Jenkins-Strebel direction

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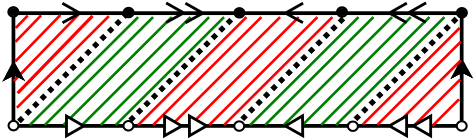


The directions $\theta = 0, \frac{\pi}{4}$ and $\frac{\pi}{2}$ are Jenkins-Strebel directions.

A flat surface (X, u) is called **a Veech surface** if its Veech group $\Gamma(X, u)$ is a lattice in $\mathrm{PSL}(2, \mathbb{R})$, that is, the orbifold $\mathbb{H}/\Gamma(X, u)$ has finite area. We classify Veech surfaces by their Veech groups.

Let Γ_1 and Γ_2 be Fuchsian groups. The group Γ_1 is said to be **commensurable** with Γ_2 if there exists $A \in \mathrm{PSL}(2, \mathbb{R})$ such that $A\Gamma_1 A^{-1} \cap \Gamma_2$ is a finite index subgroup of $A\Gamma_1 A^{-1}$ and Γ_2.

A Veech surface (X, u) is **arithmetic** if the Veech group $\Gamma(X, u)$ is commensurable with $\mathrm{PSL}(2, \mathbb{Z})$, and is **non-arithmetic** if $\Gamma(X, u)$ is not commensurable with $\mathrm{PSL}(2, \mathbb{Z})$.

Theorem (Gutkin-Judge)

Let (X, u) be a Veech surface. The Veech surface (X, u) is arithmetic if and only if (X, u) is obtained by gluing finitely many copies of a parallelogram by their parallel sides.

We consider periodic points of Veech surfaces (X, u). A point $z \in X$ is called **a periodic point** of (X, u) if its $\mathrm{Aff}^+(X, u)$-orbit $\mathrm{Aff}^+(X, u)\{z\}$ is finite. The cardinal of $\mathrm{Aff}^+(X, u)\{z\}$ is called **the period of** z. Denote by $P(X, u)$ the set of all periodic points of (X, u).

Theorem (Gutkin-Hubert-Schmidt)

If (X, u) is arithmetic, then $P(X, u)$ is dense in X. If (X, u) is non-arithmetic, then $P(X, u)$ is finite.

Gutkin, Hubert and Schmidt gave upper bounds of the numbers of periodic points of non-arithmetic Veech surfaces depending only on parameters of two cylinder decompositions. For compact non-arithmetic Veech surfaces, Möller gave upper bounds which depend only on genera.

2. Main result and proof

Main result

We give upper bounds depending only on types of surfaces and signatures of Veech groups. The basic idea is due to Gutkin, Hubert and Schmidt.

Let (X, u) be a non-arithmetic Veech surface of type (g, n). Set $d := 3g - 3 + n$, $\lambda := 2\exp(5d/e)$, and $\mu := \mathrm{Area}\,(\mathbb{H}/\Gamma(X, u))$. Here,

$$\mathrm{Area}\,(\mathbb{H}/\Gamma(X, u)) = 2\pi \left(2p - 2 + \sum_{i=1}^{k} (1 - \frac{1}{\nu_i}) \right).$$

if $\Gamma(X, u)$ is a Fuchs group of signature $(p, k : \nu_1, \cdots, \nu_k)$ $(\nu_i \in \{2, 3, \cdots, \infty\})$.

Theorem (S)

The number of periodic points of (X, u) is at most

$$2^{-26} d^{10} (\lambda\mu)^{-34} \left(\frac{1}{2} \lambda^6 \mu^6 \right)^{2^{2d+3}}.$$

Proof

We show that if (X, u) has a point whose period is sufficiently large, (X, u) is arithmetic.

Let $(X, u = \{(U_\lambda, z_\lambda)\})$ be a Veech surface and $A \in \mathrm{GL}(2, \mathbb{R})$. We can define a new flat structure $A \circ u = \{(U_\lambda, A \circ z_\lambda)\}$. Then, $\mathrm{Aff}^+(X, A \circ u) = \mathrm{Aff}^+(X, u)$ as subgroups of $\mathrm{Homeo}^+(X)$, $P(X, A \circ u) = P(X, u)$ and the Veech group $\Gamma(X, A \circ u)$ coincides with $A\Gamma(X, u)A^{-1}$.

It is known that the set of Jenkins-Strebel directions of (X, u) is dense in $[0, \pi)$. We assume that $\theta = 0$ is a Jenkins-Strebel direction of (X, u). Veech showed that $\Gamma(X, u)$ contains an element of the form $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ with $b > 0$. Taking conjugation, we may assume that $\Gamma(X, u)$ contains $B := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and it is primitive.

Theorem (S)

Let Γ be a lattice Fuchsian group. If Γ contains $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ as a primitive element, then there exists $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$ such that

$$1 \leq c < \mathrm{Area}(\mathbb{H}/\Gamma).$$

Choose $A_0 := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(X, u)$ such that $1 \leq c < \mu = \mathrm{Area}(\mathbb{H}/\Gamma(X, u))$. Conjugating by $\begin{bmatrix} 1 & -a/c \\ 0 & 1 \end{bmatrix}$, we may assume that

$$A_0 = \begin{bmatrix} 0 & -1/c \\ c & d \end{bmatrix}, B_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Choose $h, h_{B_0} \in \mathrm{Aff}^+(X, u)$ such that $D(h) = A_0$, $D(h_{B_0}) = B_0$. Let $R_1, \cdots, R_l$ be the cylinder decomposition of (X, u) by the direction $\theta = 0$ and $C_1, \cdots, C_l$ their core curves. The cylinders $h(R_1), \cdots, h(R_l)$ are the cylinder decomposition by $\theta = \frac{\pi}{2}$.

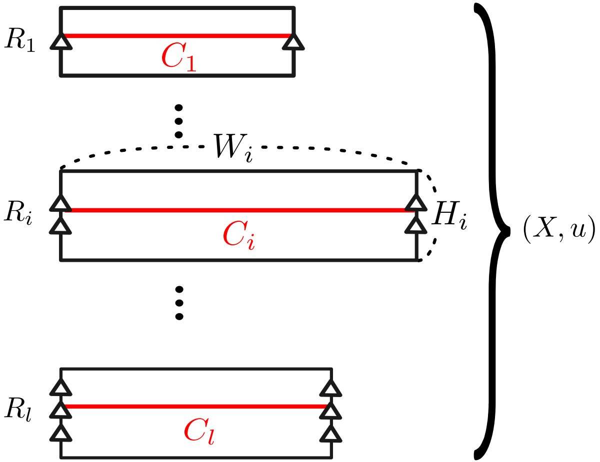
<u>Fact</u> For a closed curve C of X, let τ_C be the Dehn twist along C. There exists $\alpha < \lambda = 2\exp(5d/e)$ such that

$$\begin{aligned} h_{B_0}^{\alpha} &= \tau_{C_1}^{N_1} \circ \cdots \circ \tau_{C_l}^{N_l}, \\ h \circ h_{B_0}^{\alpha} \circ h^{-1} &= \tau_{h(C_1)}^{N_1} \circ \cdots \circ \tau_{h(C_l)}^{N_l}. \end{aligned}$$

Let W_i, H_i be the circumference and height of R_i, respectively. We have

$$W_i/H_i = \alpha/N_i \in \mathbb{Q}.$$

Direction $\theta = 0, \frac{\pi}{2}$ give cylinder decompositions. The affine map $h^{\alpha}_{B_0}$ is a composition of Dehn twists along C_i's. The affine map $h \circ h^{\alpha}_{B_0} \circ h^{-1}$ is a composition of Dehn twists along $h(C_j)$'s.



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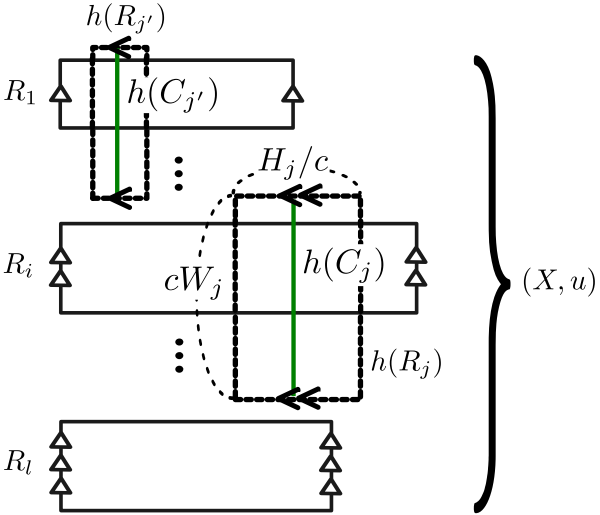


Proposition 1

We have

(1) $1 \leq N_i < (\lambda\mu)^2$ for $i \in \{1, \cdots, l\}$,

(2) $0 \leq i(C_i, h(C_j)) < (\lambda\mu)^2$ for $i, j \in \{1, \cdots, l\}$,

(3) $cW_i/H_j < (\lambda\mu)^2$ if $i(C_i, h(C_j)) \neq 0$.

Set $h_B := h_{B_0}^{\alpha}$, $h_A := h \circ h_{B_0}^{\alpha} \circ h^{-1}$, $B := D(h_B)$, $A := D(h_A)$ and $G := \langle h_A, h_B \rangle$. A point $z \in X$ is said to be **a B-periodic point** if the cardinal $\sharp \langle h_B \rangle \{z\}$ is finite. The cardinal $\sharp \langle h_B \rangle \{z\}$ is called **the B-period** of z. Denote by P_n^B the set of points of X whose B -periods are less than or equal to n. We define **A-periodic points**, **G-periodic points**, their **periods**, P_n^A and P_n^G as well.

Note that periodic points of (X, u) are G-periodic points.

For $B = \left[\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right]$, let us consider the set P_n^B in torus (T, u_T) case. Let $z = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in T$ be a B-periodic point of B-period n. We assume that $0 \leq x, y < 1$. Since $B^m \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} x + mNy \\ y \end{smallmatrix}\right)$, we have

$$mNy \notin \mathbb{N} \text{ for } m \in \{1, \cdots, n-1\} \text{ and } nNy \in \mathbb{N}.$$

Thus, $Ny = \frac{s}{n} + t$ for some $1 \leq s \leq n-1$ with $\gcd(s, n) = 1$ and $t \in \{0, \cdots, N-1\}$. This implies that $y \in \mathbb{Q}$ and the set of points whose B-periods are n consists of $N\phi(n)$ horizontal closed curves. Here,

$$\phi(n) = \sharp\{s \in \mathbb{N} : 1 \leq s \leq n-1,\ \gcd(s, n) = 1\}$$

is Euler's totient function. Setting $\Phi(n) = \sum_{m=1}^{n} \phi(m)$, the set P_n^B consists of $N\Phi(n)$ horizontal closed curves.
By the same argument as above, y-coordinate of a B-periodic point $z \in R_i$ satisfies $y/H_i \in \mathbb{Q}$.

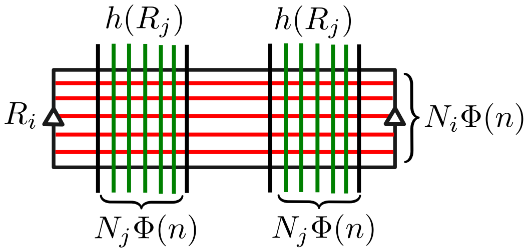
Lemma 1

Let $\beta := \frac{1}{4}\, d^2(\lambda\mu)^6$. We have

$$\sharp P_n^G \le \sharp\left(P_n^A \cap P_n^B\right) < \beta n^4.$$

Proof By definition, $P_n^G \subset P_n^A \cap P_n^B$. The above observation gives

$$\sharp\left(P_n^A \cap P_n^B\right) = \sum_{1 \le i,j \le l} i(C_i, h(C_j)) N_i \Phi(n) N_j \Phi(n) < \beta n^4. \ \square$$



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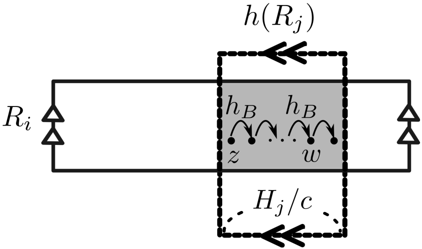
Lemma 2

Let $\mathcal{O} = G\{z\}$ be a finite G-orbit. If $\sharp\mathcal{O} \geq \beta n^4$, $\mathcal{O}$ contains a point whose A-period or B-period is greater than n.

Lemma 3

Let $\mathcal{O}$ be a finite G-orbit. Suppose that $\mathcal{O}$ contains a point z whose B-period is greater than $\frac{1}{2}(\lambda\mu)^4 n^2$. If $z \in R_i \cap h(R_j)$, there exists a point $w \in \langle h_B \rangle \{z\} \cap R_i \cap h(R_j)$ whose A-period is greater than or equal to n.



Proof The set $P^A_{n-1} \cap h(R_j)$ consists of $N_i\Phi(n-1)$ vertical closed geodesics. Since the distance between z and $h_B(z)$ is less than $W_i/\frac{1}{2}(\lambda\mu)^4 n^2$, we have

$$\sharp\left(\langle h_B \rangle \{z\} \cap h(R_j)\right) > \frac{1}{2}(\lambda\mu)^4 n^2 H_j/cW_i > N_i\Phi(n-1).$$

Thus, we obtain the claim. $\square$

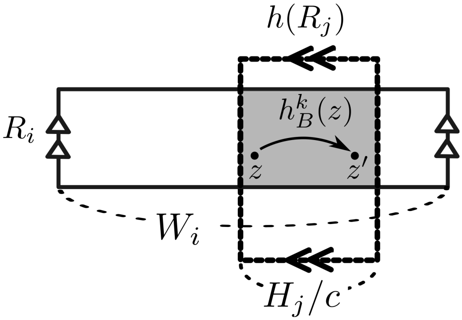
Lemma 4

Let $f(x) = \frac{1}{2}(\lambda\mu)^4 x^2$. Let $\mathcal{O}$ be a finite G-orbit. Assume that $\sharp\mathcal{O} > \beta\left(f^{2d-1}(n)\right)^4$. Each horizontal cylinder R_i contains a point whose B-period is greater than or equal to n.

Recall that W_i and H_i are the circumference and height of the horizontal cylinder R_i, respectively. The circumference and height of the vertical cylinder $h(R_j)$ are cW_j and H_j/c.

Lemma 5

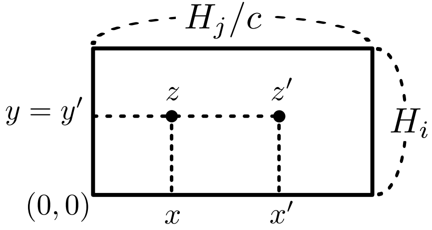
Suppose that $R_i \cap h(R_j) \neq \emptyset$. Let L be a connected component of $R_i \cap h(R_j)$. If $\overline{L}$ contains two G-periodic points z and z' with $\langle h_B \rangle \{z\} = \langle h_B \rangle \{z'\}$, then $cW_i/H_j \in \mathbb{Q}$.



Proof Let us identify $L = (0, H_j/c) \times (0, H_i)$, $z = \begin{pmatrix} x \\ y \end{pmatrix}$ and $z' = \begin{pmatrix} x' \\ y' \end{pmatrix}$. As $h_B^k(z) = z'$ for some k, we have $y' = y$ and $x' = x + k\alpha y + N_i W_i$. Since z and z' are G-periodic points, $y/H_i \in \mathbb{Q}$, cx/H_j, $cx'/H_j \in \mathbb{Q}$. Then,

$$\mathbb{Q} \ni \frac{c\,(x' - x)}{H_j} = \frac{(k\alpha y + N_i W_i)}{H_j} = \frac{cW_i}{H_j}\left(k\alpha \frac{y}{H_i}\frac{H_i}{W_i} - N_i\right).$$

As $W_i/H_i \in \mathbb{Q}$, we obtain the claim. $\square$



Proposition 2

Let $\mathcal{O}$ be a finite G-orbit. If $\sharp\mathcal{O} > \beta\left(f^{2d-1}\left((\lambda\mu)^2\right)\right)^4$, we have the following :

(1) $cW_i/H_j \in \mathbb{Q}$ if $R_i \cap h(R_j) \neq \emptyset$,

(2) $W_i/W_{i'} \in \mathbb{Q}$ for any $i, i' \in \{1, \cdots, l\}$,

(3) $H_i/H_{i'} \in \mathbb{Q}$ for any $i, i' \in \{1, \cdots, l\}$.

<u>Proof</u> By Lemma 4, every cylinder R_i contains a point z_i whose B-period is greater than $(\lambda\mu)^2$. The distance between z_i and $h_B(z_i)$ is less than $W_i/(\lambda\mu)^2$. If $R_i \cap h(R_j) \neq \emptyset$, each connected component L is a rectangle with width H_j/c. By Proposition 1-(3), we have $W_i/(\lambda\mu)^2 < H_j/c$. Thus, $\bar{L}$ contains two point in a the same B-orbit. From Lemma 4, we obtain (1).

If R_i $R_{i'}$ intersect with common $h(R_j)$,

$$\frac{W_i}{W_{i'}} = \frac{cW_i}{H_j} \cdot \frac{H_j}{cW_{i'}} \in \mathbb{Q}.$$

As X is connected, we obtain (2).
The equation

$$\frac{H_i}{H_{i'}} = \frac{H_i}{W_i} \cdot \frac{W_i}{W_{i'}} \cdot \frac{W_{i'}}{H_{i'}}$$

implies (3). $\square$

Proposition 3

If the Veech surface (X, u) has a point whose G-period is greater than $\beta\left(f^{2d-1}\left((\lambda\mu)^2\right)\right)^4$, then (X, u) is arithmetic.

Proof By Proposition 2, replacing u with some flat structure $A \circ u$, we may assume that W_i and H_i are integers and $c \in \mathbb{Q}$. If $c = m/n$ for some $n, m \in \mathbb{Z}_{>0}$, then (X, u) is realized by gluing finitely many squares whose side length is $1/m$. By the theorem of Gutkin-Hubert-Schmidt, (X, u) is arithmetic. $\square$

By Proposition 3, the periods of periodic points of the non-arithmetic Veech surface (X, u) are at most $\beta\left(f^{2d-1}\left((\lambda\mu)^2\right)\right)^4$. Applying Lemma 1, the number of periodic points is at most $\beta^5\left(f^{2d-1}\left((\lambda\mu)^2\right)\right)^{16}$.

3. Application to Teichmüller curves

Hereafter, we assume $3g - 3 + n > 0$. **A Teichmüller curve** $f : C \to \mathcal{M}(g,n)$ is a holomorphic local isometry from a hyperbolic Riemann surface C of finite type into the moduli space $\mathcal{M}(g,n)$ equipped with the Teichmüller distance.

Proposition

Let $f : C \to \mathcal{M}(g,n)$ be a Teichmüller curve. Given a base point t_0 of C. There exists a Veech surface (X,u) of type (g,n), a branched covering $\phi : C \to C_0 := \mathbb{L}/\Gamma(X,u)$ and an injective holomorphic local isometry $f_0 : C_0 \to \mathcal{M}(g,n)$ with the following properties:

(1) $f = f_0 \circ \phi$,

(2) $f(t_0) = (X,u)$ as Riemann surfaces,

(3) for each $t \in C$, there exists $A_t \in \mathrm{SL}(2,\mathbb{R})$ such that $f(t) = (X, A_t \circ u)$ as Riemann surfaces.

Let $f : C \to \mathcal{M}(g, n)$ be a Teichmüller curve. We can construct a holomorphic family of Riemann surfaces (M, π, C) so that the fiber X_t over $t \in C$ is the Riemann surface $f(t)$.
Let $\phi : C_\phi \to C$ be a finite unbranched holomorphic covering. Then, $f \circ \phi : C_\phi \to \mathcal{M}(g, n)$ is also a Teichmüller curve. Let $(M_\phi, \pi_\phi, C_\phi)$ be the holomorphic family corresponding to $f \circ \phi$.

Theorem (S)

(1) *Holomorphic sections of $(M_\phi, \pi_\phi, C_\phi)$ do not intersect each other. Given a base point $t_0 \in C_\phi$. Let (X, u) be the Veech surface corresponding to $f \circ \phi(t_0)$. For a holomorphic section $s : C_\phi \to M_\phi$, $s(t_0)$ is a periodic point of (X, u).*

(2) *Let $d = 3g - 3 + n$. Assume that C is of type (p, k). The number of holomorphic sections of $(M_\phi, \pi_\phi, C_\phi)$ is at most*

$$32\pi \deg(\phi)(2p - 2 + k)d^2 \left\{ 2d + 3 \exp\left(\frac{5}{e}d\right) \right\}.$$

This bound tends to infinity as $\deg(\phi) \to \infty$.

Applying the main theorem, we obtain upper bounds of the numbers of holomorphic sections which depend only on g, n and the topological type of C.

Theorem (S)

Let $f : C \to \mathcal{M}(g,n)$ be a Teichmüller curve corresponding to a non-arithmetic Veech surface (X,u). Assume that C is a Riemann surface of type (p,k). For any finite unramified holomorphic covering $\phi : C_\phi \to C$, the number of holomorphic sections of $(M_\phi, \pi_\phi, C_\phi)$ is at most

$$2^{-26} d^{10} (\lambda\mu)^{-34} \left(\frac{1}{2}\lambda^6\mu^6\right)^{2^{2d+3}} .$$

Here, $d = 3g-3+n$, $\lambda = 2\exp(5d/e)$ and $\mu = 2\pi(2p-2+k)$.

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On the Existence of Holomorphic Mappings of Once-Holed Tori

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Outline

Outline

Planar Riemann surfaces

General uniformization theorem

Every Riemann surface of genus zero is conformally embedded into the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

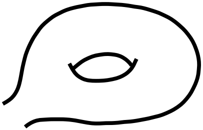
- Function theory on Riemann surfaces of genus zero is essentially part of function theory on plane domains.
- The core of the theory of Riemann surfaces should be occupied by Riemann surfaces of positive genus, or those with handles.

What are the simplest nonplanar Riemann surfaces?

Definition

A once-holed torus is an open Riemann surface of genus **1** with exactly one boundary component.

- "open" = "noncompact"



- Once-holed tori are the simplest among open Riemann surfaces of positive genus.

Are there holomorphic mappings?

Let
R_0 be a Riemann surface of positive genus, and
T be a once-holed torus.

Naive question

Are there "non-degenerate" holomorphic mappings $T \longrightarrow R_0$ or holomorphic mappings $T \longrightarrow R_0$ "preserving handles"?

- What does "preserving handles" mean?

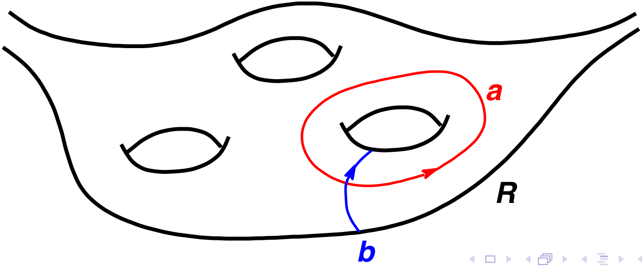
Mark of handle

Let $\boldsymbol{R}$ be a Riemann surface of positive genus.

Definition

A mark of handle of $\boldsymbol{R}$ is an ordered pair $\chi = \{\boldsymbol{a}, \boldsymbol{b}\}$ of simple loops on $\boldsymbol{R}$ such that $\boldsymbol{a} \times \boldsymbol{b} = \mathbf{1}$.

- A mark of handle specifies a handle of $\boldsymbol{R}$.



Riemann surface with marked handle

Definition

A Riemann surface with marked handle is a pair $Y = (R, \chi)$, where R is a Riemann surface of positive genus and χ is a mark of handle of R.

Let $Y_j = (R_j, \chi_j)$, $j = 1, 2$, be Riemann surfaces with marked handle, where $\chi_j = \{a_j, b_j\}$.

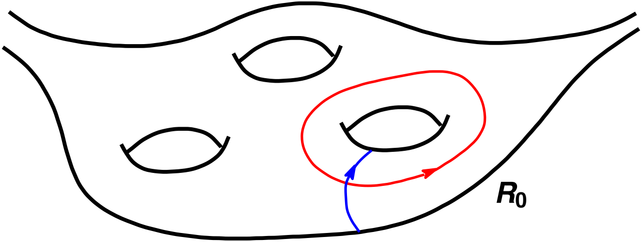
Definition

$f : Y_1 \longrightarrow Y_2$: holomorphic (resp. conformal)
$\Leftrightarrow$ (i) $f : R_1 \longrightarrow R_2$: holomorphic (resp. conformal)
(ii) $f_*(a_1) \sim a_2$, $f_*(b_1) \sim b_2$ ($\sim$ means "free homotopy")

- conformal = “holomorphic" & “injective"

Problem

Fix a Riemann surface $Y_0 = (R_0, \chi_0)$ with marked handle.



Problem

Determine the set of marked once-holed tori $X = (T, \chi)$ for which there is a holomorphic mapping $X \to Y_0$.

Space of marked once-holed tori

- Let $\mathfrak{T}$ denote the set of marked once-holed tori, where two marked once-holed tori are identified if there is a conformal mapping of one onto the other.
- As a set, $\mathfrak{T}$ is the union of
  the Teichmüller space of a once-punctured torus
  and
  the reduced Teichmüller space of a once-holed torus
  that is not a once-punctured torus.

Problems (revised)

- Let Y_0 be a Riemann surface with marked handle.

Definition

$\mathfrak{T}_a[Y_0] = \{X \in \mathfrak{T} \mid \exists$ holomorphic mapping $X \to Y_0\}$,
$\mathfrak{T}_c[Y_0] = \{X \in \mathfrak{T} \mid \exists$ conformal mapping $X \to Y_0\}$.

- "a" = "analytic", and "c" = "conformal".

Problems (revised)

$\mathfrak{T}_a[Y_0] = ?, \quad \mathfrak{T}_c[Y_0] = ?$

Remark

$\emptyset \neq \mathfrak{T}_c[Y_0] \subset \mathfrak{T}_a[Y_0]$

Torus case

Example

If Y_0 is a marked torus, then

$$\mathfrak{T}_a[Y_0] = \mathfrak{T}$$

by the Behnke-Stein theorem, while

$$\mathfrak{T}_c[Y_0] \neq \mathfrak{T}.$$

Outline

Mapping $\Lambda : \mathfrak{T} \longrightarrow \mathbb{R}_+^3$

Definition

For $\boldsymbol{X} = (\boldsymbol{T}, \chi) \in \mathfrak{T}$, $\chi = \{\boldsymbol{a}, \boldsymbol{b}\}$, define

$$\Lambda(\boldsymbol{X}) = (\lambda_1, \lambda_2, \lambda_3),$$

where λ_1, λ_2, λ_3 are the extremal lengths of the free homotopy classes of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{ab}^{-1}$, respectively.

- Λ defines a mapping of $\mathfrak{T}$ into $\mathbb{R}_+^3$, where $\mathbb{R}_+ = [0, +\infty)$.

Global coordinate system on $\mathfrak{T}$

Proposition

The mapping $\Lambda : \mathfrak{T} \to \mathbb{R}^3_+$ *is injective with image*

$$\Lambda(\mathfrak{T}) = \{\xi \in \mathbb{R}^3_+ \mid Q(\xi) + 4 \leqq 0\},$$

where

$$Q(\xi_1, \xi_2, \xi_3) = \xi_1^2 + \xi_2^2 + \xi_3^2 - 2(\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1).$$

- Identifying $\mathfrak{T}$ with $\Lambda(\mathfrak{T})$, we consider $\mathfrak{T}$ as a 3-dimensional real analytic manifold with boundary.
- The eigenspaces of the coefficient matrix of the quadratic form Q are the line $\xi_1 = \xi_2 = \xi_3$ and the plane $\xi_1 + \xi_2 + \xi_3 = 0$.

Once-holed torus case

- $\Lambda : \mathfrak{T} \longrightarrow \mathbb{R}^3_+$,
  $Q(\xi_1, \xi_2, \xi_3) = \xi_1^2 + \xi_2^2 + \xi_3^2 - 2(\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1)$.

Example

If $Y_0 \in \mathfrak{T}$, then

$$\Lambda(\mathfrak{T}_c[Y_0]) = \{\xi \in \mathbb{R}^3_+ \mid Q(\xi - \xi_0) \leqq 0 \text{ and } Q(\xi) \leqq Q(\xi_0)\},$$

where $\xi_0 = \Lambda(Y_0)$.

- $\Lambda(\mathfrak{T}_c[Y_0])$ is a cone with vertex at ξ_0.

First result

- Let Y_0 be a Riemann surface with marked handle.

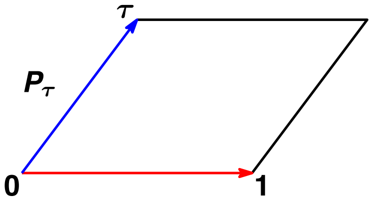
Theorem 1

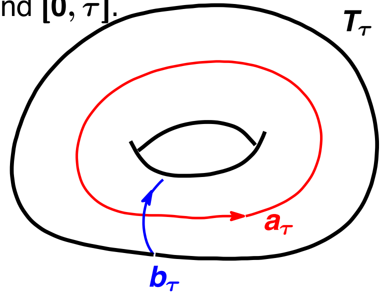
The sets $\mathfrak{T}_a[Y_0]$ and $\mathfrak{T}_c[Y_0]$ are noncompact closed domains with Lipschitz boundary, and are retracts of $\mathfrak{T}$.

- A subset A of a topological space X is called a retract of X if there is a continuous map $r : X \longrightarrow A$ such that $r(a) = a$ for any $a \in A$.

Canonical construction of marked tori

Let $\mathbb{H}$ be the upper half plane: $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathbf{Im}\,\tau > 0\}$.
For $\tau \in \mathbb{H}$ let
$\boldsymbol{P_\tau}$: the parallelogram with vertices $\mathbf{0}, \mathbf{1}, \tau + \mathbf{1}, \tau$,
$\boldsymbol{T_\tau}$: the torus obtained from $\boldsymbol{P_\tau}$ by identifying the opposite sides,
$\chi_\tau = \{\boldsymbol{a_\tau}, \boldsymbol{b_\tau}\}$, where $\boldsymbol{a_\tau}$ and $\boldsymbol{b_\tau}$ are the projections of $[\mathbf{0}, \mathbf{1}]$ and $[\mathbf{0}, \tau]$.



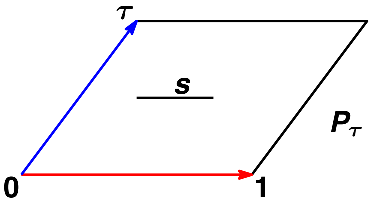


Horizontal slit tori

For $\tau \in \mathbb{H}$ and $s \in [0, 1)$ let

$T_\tau^{(s)}$: the once-holed torus obtained from T_τ by deleting a horizontal segment of length s,

$\chi_\tau^{(s)}$: the mark of handle of $T_\tau^{(s)}$ induced by the embedding $T_\tau^{(s)} \longrightarrow T_\tau$.



Another global coordinate system on $\mathfrak{T}$

- Set $X_\tau^{(s)} = (T_\tau^{(s)}, \chi_\tau^{(s)})$ for $(\tau, s) \in \mathbb{H} \times [0, 1)$.

Proposition

The correspondence $(\tau, s) \mapsto X_\tau^{(s)}$ *is a homeomorphism of* $\mathbb{H} \times [0, 1)$ *onto* $\mathfrak{T}$.

- The restrictions of the homeomorphism to $\mathbb{H} \times (0, 1)$ and to $\mathbb{H} \times \{0\}$ are real-analytic.
- The extremal length of the free homotopy class of $a_\tau^{(s)}$ is exactly $1/\operatorname{Im}\tau$, where $\chi_\tau^{(s)} = \{a_\tau^{(s)}, b_\tau^{(s)}\}$.

Second results

Theorem 2_a

There exists $\lambda_a[Y_0] \in [0, +\infty)$ such that:
(i) If $\operatorname{Im} \tau \geqq 1/\lambda_a[Y_0]$, then $X_\tau^{(s)} \notin \mathfrak{T}_a[Y_0]$ for any $s \in [0, 1)$.
(ii) If $\operatorname{Im} \tau < 1/\lambda_a[Y_0]$, then $X_\tau^{(s)} \in \mathfrak{T}_a[Y_0]$ for some $s \in [0, 1)$.

- If Y_0 is a marked torus, then $\lambda_a[Y_0] = 0$.

Theorem 2_c

There exists $\lambda_c[Y_0] \in (0, +\infty)$ such that:
(i) If $\operatorname{Im} \tau > 1/\lambda_c[Y_0]$, then $X_\tau^{(s)} \notin \mathfrak{T}_c[Y_0]$ for any $s \in [0, 1)$.
(ii) If $\operatorname{Im} \tau < 1/\lambda_c[Y_0]$, then $X_\tau^{(s)} \in \mathfrak{T}_c[Y_0]$ for some $s \in [0, 1)$.

- It follows from $\mathfrak{T}_a[Y_0] \supset \mathfrak{T}_c[Y_0]$ that $\lambda_a[Y_0] \leqq \lambda_c[Y_0]$.

Third result

- Let $Y_0 = (R_0, \chi_0)$, where $\chi_0 = \{a_0, b_0\}$.

Notations

- $\ell[Y_0]$: the length of the hyperbolic geodesic on R_0 freely homotopic to a_0
  If R_0 is a torus, then define $\ell[Y_0] = 0$.
- $\lambda[Y_0]$: the extremal length of the free homotopy class of a_0

Theorem 3

$$\lambda_a[Y_0] = \frac{1}{\pi}\ell[Y_0], \text{ and } \lambda_c[Y_0] = \lambda[Y_0].$$

Outline

1. Motivation and problem
2. Results
3. Proof of Theorem 2

Order

Definition (Order)

For $X, X' \in \mathfrak{T}$,

$$X \preceq X' \Leftrightarrow \exists \text{ a conformal mapping } X \to X'$$

- $X \preceq X' \Leftrightarrow X \in \mathfrak{T}_c[X'] \Leftrightarrow \mathfrak{T}_c[X] \subset \mathfrak{T}_c[X']$

Proposition

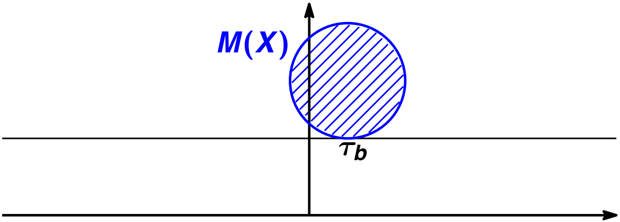
$(\mathfrak{T}, \preceq)$ *is an ordered set.*

Which torus accepts a given once-holed torus?

For $X \in \mathfrak{T}$ let $M(X) = \{\tau \in \mathbb{H} \mid X \in \mathfrak{T}_c[X_\tau]\}$,
where $X_\tau = (T_\tau, \chi_\tau)$ (the marked torus of modulus τ).

Proposition (Shiba, 1987)

- *$M(X)$ is a closed disk (or a point) in $\mathbb{H}$.* the moduli disk of X
- *If τ_b is the bottom point of $M(X)$, then $X = X^{(s)}_{\tau_b}$ for some s.*



Order-reversing isomorphism

Let $\mathfrak{D}$ be the set of closed disks in $\mathbb{H}$,
where a singleton is regarded as a closed disk of radius $\mathbf{0}$.

Proposition

The correspondence $\boldsymbol{X} \mapsto \boldsymbol{M}(\boldsymbol{X})$ defines an order-reversing isomorphism between the ordered sets $(\mathfrak{T}, \preceq)$ and $(\mathfrak{D}, \subset)$.

- For $\boldsymbol{X}, \boldsymbol{X}' \in \mathfrak{T}$,

$$\boldsymbol{X} \preceq \boldsymbol{X}' \Leftrightarrow \boldsymbol{M}(\boldsymbol{X}) \supset \boldsymbol{M}(\boldsymbol{X}').$$

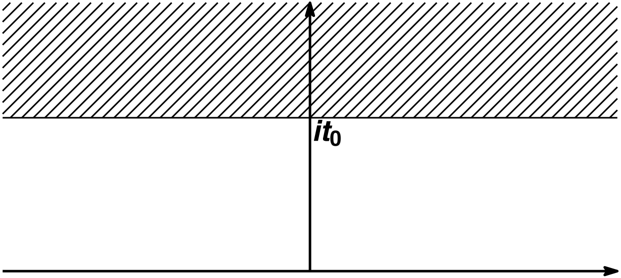
- For any $\boldsymbol{\Delta} \in \mathfrak{D}$ there is $\boldsymbol{X} \in \mathfrak{T}$ such that $\boldsymbol{M}(\boldsymbol{X}) = \boldsymbol{\Delta}$.

Essential part of the second result

Let $\mathfrak{T}[Y_0] = \mathfrak{T}_a[Y_0]$ or $\mathfrak{T}[Y_0] = \mathfrak{T}_c[Y_0]$.

Theorem 2

There exists $t_0 \in [0, +\infty)$ *such that:*
(i) *If* $\mathrm{Im}\,\tau > t_0$, *then* $X_\tau^{(s)} \notin \mathfrak{T}[Y_0]$ *for any* $s \in [0, 1)$.
(ii) *If* $\mathrm{Im}\,\tau < t_0$, *then* $X_\tau^{(s)} \in \mathfrak{T}[Y_0]$ *for some* $s \in [0, 1)$.



Proof of Theorem 2

$\mathfrak{T}[Y_0] = \mathfrak{T}_a[Y_0]$ or $\mathfrak{T}_c[Y_0]$

- Set
  $t_0 = \sup\{\mathrm{Im}\,\tau \mid \tau \in \mathbb{H}$ and $X_\tau^{(s)} \in \mathfrak{T}[Y_0]$ for some $s\}$.
- If $\mathrm{Im}\,\tau > t_0$, then $X_\tau^{(s)} \notin \mathfrak{T}[Y_0]$ for any s.

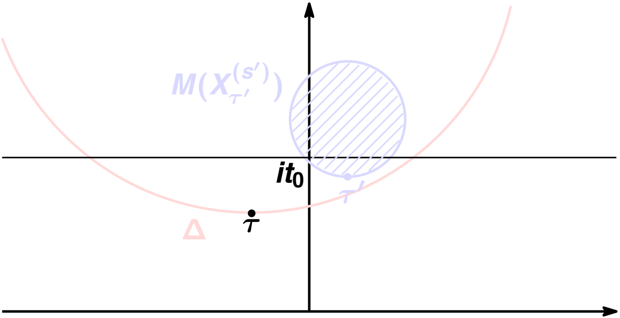
Observation

For $X, X' \in \mathfrak{T}$,

$$(X \preceq X' \text{ and } X' \in \mathfrak{T}[Y_0]) \Rightarrow X \in \mathfrak{T}[Y_0]$$

Proof of Theorem 2

- If $\mathbf{Im}\,\tau < t_0$, then $X_{\tau'}^{(s')} \in \mathfrak{T}[Y_0]$ for some τ' and s' with $\mathbf{Im}\,\tau' > \mathbf{Im}\,\tau$.
- $\exists \Delta \in \mathfrak{D}$ s.t. $\Delta \supset M(X_{\tau'}^{(s')})$ and τ is the bottom of Δ.



Proof of Theorem 2

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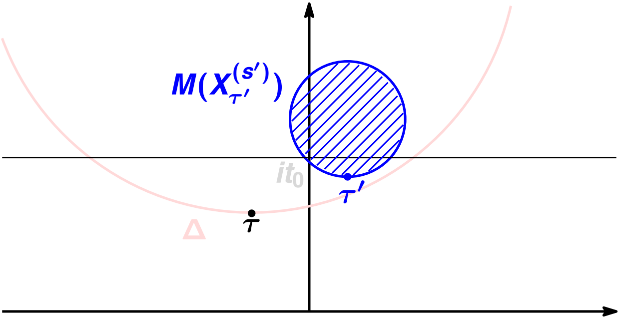

Proof of Theorem 2

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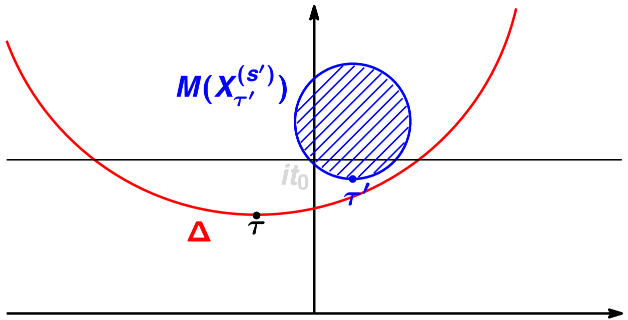
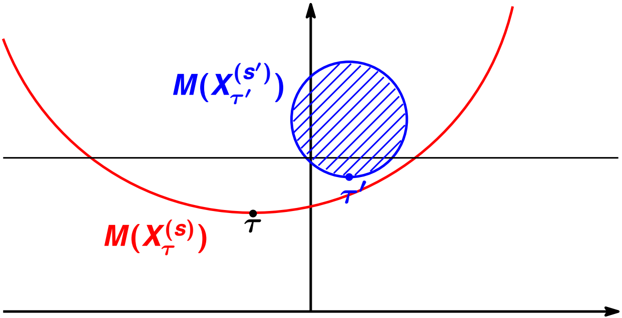


Proof of Theorem 2

- $\Delta = M(X_{\tau}^{(s)})$ for some s.
- Then $M(X_{\tau}^{(s)}) \supset M(X_{\tau'}^{(s')})$ and hence $X_{\tau}^{(s)} \preceq X_{\tau'}^{(s')}$.
- Since $X_{\tau'}^{(s')} \in \mathfrak{T}[Y_0]$, we have $X_{\tau}^{(s)} \in \mathfrak{T}[Y_0]$.



Proof of Theorem 2

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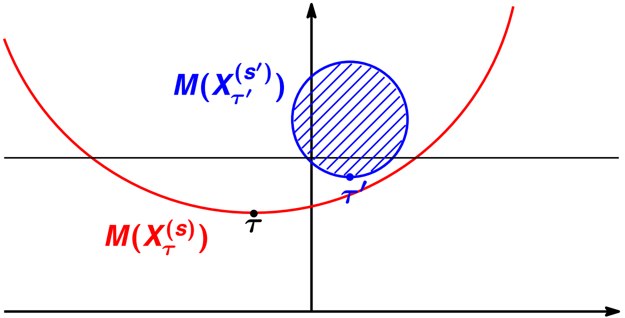


Concluding Remark

- The above reasoning works for any subset $\mathfrak{T}_0$ of $\mathfrak{T}$ with the property described in the observation: For $X, X' \in \mathfrak{T}$,

$$(X \preceq X' \ \& \ X' \in \mathfrak{T}_0) \Rightarrow X \in \mathfrak{T}_0$$

Example

- The set of $X \in \mathfrak{T}$ for which there is a holomorphic mapping $f : X \to Y_0$ with $\sup_q \# f^{-1}(q) \leqq \nu$, where ν is a given positive integer.
- The set of $X \in \mathfrak{T}$ for which there is a K-quasiconformal mapping $X \to Y_0$, where $K > 1$ is fixed.

On deformation spaces of Kleinian groups

Nov. 10, 2013
Conference on Riemann surfaces and discontinuous groups

Hiroshige Shiga
Tokyo Institute of Technology

G_0 : 有限生成 Klein群

$\Lambda(G_0)$: G_0 の limit set. $\#\Lambda(G_0)=\infty$ とする.

$w:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ qc. G_0-equivariant.

i.e. $\forall g\in G_0$ に対し $wgw^{-1}\in PSL(2,\mathbb{C})$.

$w_1, w_2:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ qcs G_0-equiv.

$w_1 \sim w_2 \overset{\text{def}}{\iff}$ w_1 と w_2 が同じ iso. をみちび

く. up to $PSL(2,\mathbb{C})$

$D(G_0)=\{[w]\mid w:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ qc. G_0-equiv.$\}$

$\cap$

$\mathrm{Hom}(G_0, PSL(2,\mathbb{C}))/\sim$

(Maskit, Kra-Maskit)

$D(G_0)$ は complex manifold, 正則凸.

Ex. $\Gamma_0 := G_0$ Fuchs群 のとき,

$D(\Gamma_0) \cong T(\Gamma_0) \times T(\overline{\Gamma_0})$

$\Lambda(G_0)$: G_0 の limit set

$\Omega(G_0) := \hat{\mathbb{C}} \setminus \Lambda(G_0)$.

<u>Thm</u> (Ahlfors の有限性定理)

$$\Omega(G_0)/G_0 = \coprod^{k} R_i$$

R_i : 有限型 (hyperbolic) Riemann面.

Thm 1

Γ_0: Fuchs群 $\Rightarrow$ $D(\Gamma_0)$ は H^∞-凸.

Thm 2

$\Omega(G_0)$ が単連結でない成分を持っていたとする. $\Rightarrow$ $D(G_0)$ は H^∞-凸ではない.

Thm 3.

$D(G_0)$ 上 Teichmüller distance と Kobayashi distance は等しい.

Cor. G_0: as in Thm 2.

$\Rightarrow$ $D(G_0)$ 上 Carathéodory distance と Kobayashi dist. は異なる.

§ Thm 1 の証明

FACT M : cpx mfd.

$\forall p, q \in M$

Δ上の hyp. distance　holo.

$$C_M(p,q) := \sup\{\rho(f(p), f(q)) \mid f : M \to \Delta = \{|z|<1\}\}$$

C_M が complete ならば M は H^∞-convex

C_M ： M の Carathéodory pseudo distance という.

$\mathcal{O}(M)$: M 上の holo. functions 全体.

$H^\infty(M)$: M 上の bounded holo. functions 全体.

$\mathcal{O} \subset \mathcal{O}(M)$ に対して, M が凸 ($\mathcal{O}$-convex)

$\overset{\text{def}}{\Longleftrightarrow}$ ${}^\forall K \Subset M$ に対して, $\hat{K}_{\mathcal{O}}$ が compact

ここに $\hat{K}_{\mathcal{O}} = \{p \in M \mid |f(p)| \leq \|f\|_{K,\infty} \ \forall f \in \mathcal{O}\}$

$\mathcal{O}=\mathcal{O}(M)$ のとき，M を正則凸とよぶ．

$\mathcal{O}=H^\infty(M)$ のとき H^∞-convex という．

$\mathcal{O}_1\subset\mathcal{O}_2\subset\mathcal{O}(M)$ のとき．

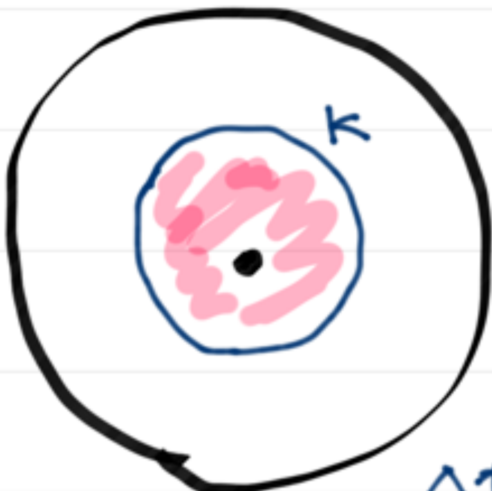
$\mathcal{O}_1$-convex $\Rightarrow$ $\mathcal{O}_2$-convex.

$D\subset\mathbb{C}^n$ (domain) のとき，

D が正則凸 $\Longleftrightarrow$ D が domain of holomorphy

(Oka)

$\Delta^*\subset\mathbb{C}$　正則凸



$\widehat{K}_{H^\infty(\Delta^*)}$

$^\forall f\in H^\infty(\Delta^*)\Longrightarrow$

$^\exists\tilde{f}\in H^\infty(\Delta)$

s.t. $\tilde{f}|_{\Delta^*}=f$

Δ^* は H^∞-凸ではない．

$D(\Gamma_0)$ で $C_{D(\Gamma_0)}$ が complete をいえばよい

$D(\Gamma_0) \simeq T(\Gamma_0) \times T(\overline{\Gamma_0})$

・$C_{T(\Gamma_0)}$ は complete $\Longrightarrow$ $D(\Gamma_0)$ 上も complete.

§ Thm 2 の証明

Thm 2 の証明には次を示す:

$\exists\, h : \Delta^* \longrightarrow D(G_0)$ holo. map s.t.

$z \to 0$ のとき, $h(z) \longrightarrow \partial D(G_0)$

h の構成: ← 準連結でない成分の存在を使う.

$\Omega(G_0)$の を Ω_0/G_0上
ある成分Ω_0 で考える.

Dehn twist w_0 を考える.

$= R$

q_α : α についての R 上の Jenkins-Strebel 微分

$\to \Delta \ni \lambda \longmapsto \lambda \dfrac{\overline{q_\alpha}}{|q_\alpha|}$: Beltrami 微分

$\to {}^\exists H : \Delta \to T(R)$ 1-1 holo. isometric

$\tilde{h} \rightsquigarrow \downarrow D(G_0)$

Δ

${}^\exists\gamma$: para

$\gamma(0) : 0 \to \gamma(0)$ H で Dehn twist

$\hat{h} : \Delta/\langle\gamma\rangle \longrightarrow D(G_0)$ とみなせる.

Δ^* の 0 へ近づくと stretch がーつくる $\Leftrightarrow$ α をつぶーす

cocompactな双曲Coxeter群のgrowth rateと 2−Salem数

(Growth rates of cocompact hyperbolic Coxeter groups and 2−Salem numbers)

「リーマン面・不連続群論」研究集会（大阪大学）
2013年11月10日

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（大阪市立大学大学院理学研究科　後期博士課程３年）

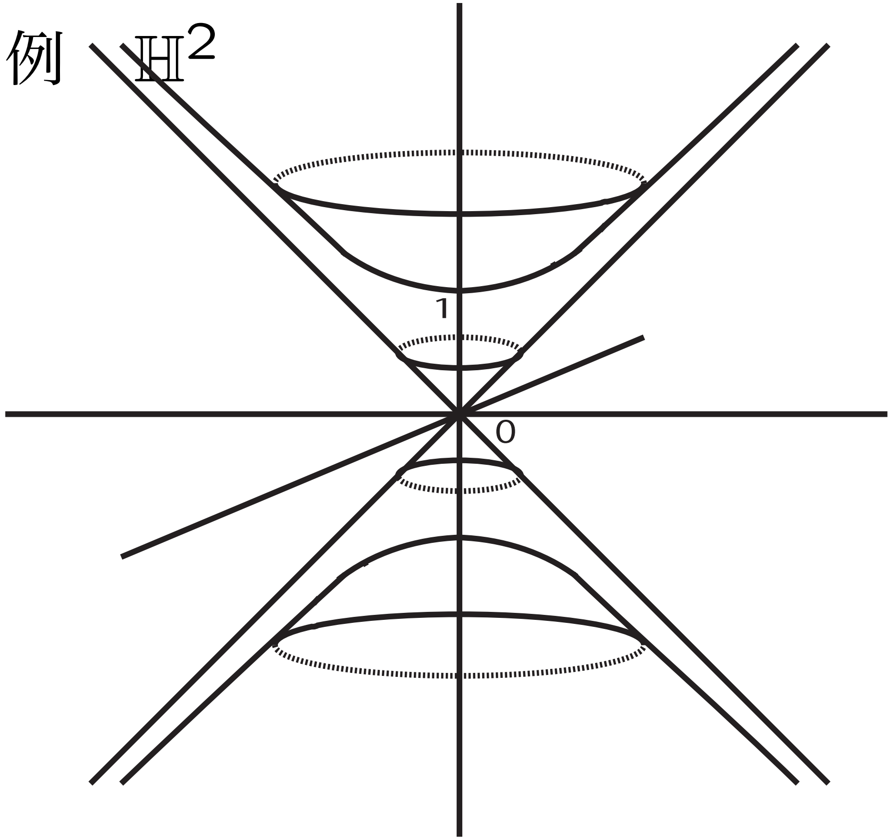
1. Introduction

$\mathbb{R}^{n,1} := (\mathbb{R}^{n+1}, \circ)$, $x \circ y := x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}$

$\mathbb{H}^n := \{x = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n,1} \,|\, x \circ x = -1, x_{n+1} > 0\}$

$(\mathbb{H}^n, d_{\mathbb{H}})$ を $\boldsymbol{n}$**次元双曲空間** という。

例 $\mathbb{H}^2$

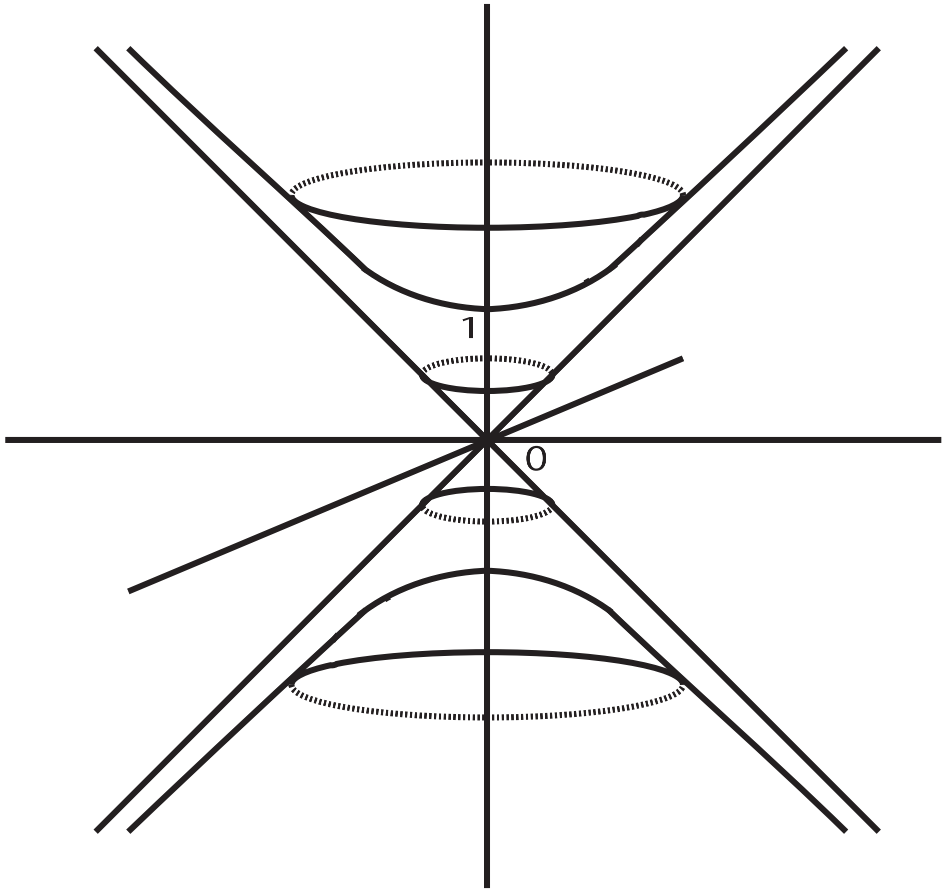


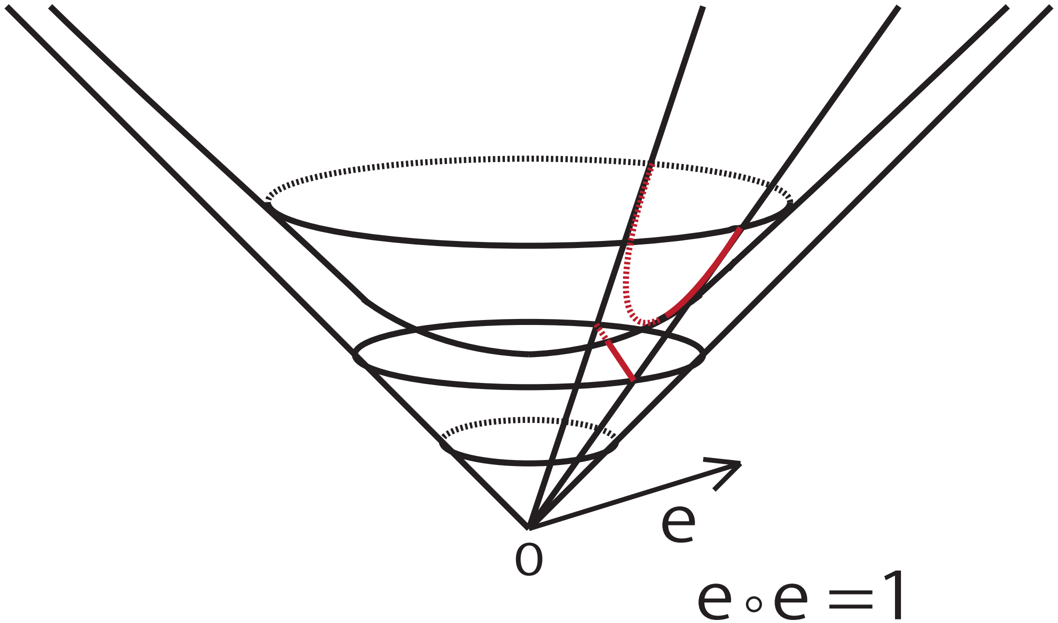
$e \in \mathbb{R}^{n,1}$, $e \circ e = 1$, $\quad e^{\perp} := \{x \in \mathbb{R}^{n,1} \,|\, x \circ e = 0\}$

$\Rightarrow e^{\perp}$: $\mathbb{R}^{n,1}$ の n 次元部分空間 $(e^{\perp} \cap \mathbb{H}^n \neq \emptyset)$

$H := e^{\perp} \cap \mathbb{H}^n$: $\mathbb{H}^n$**の超平面**

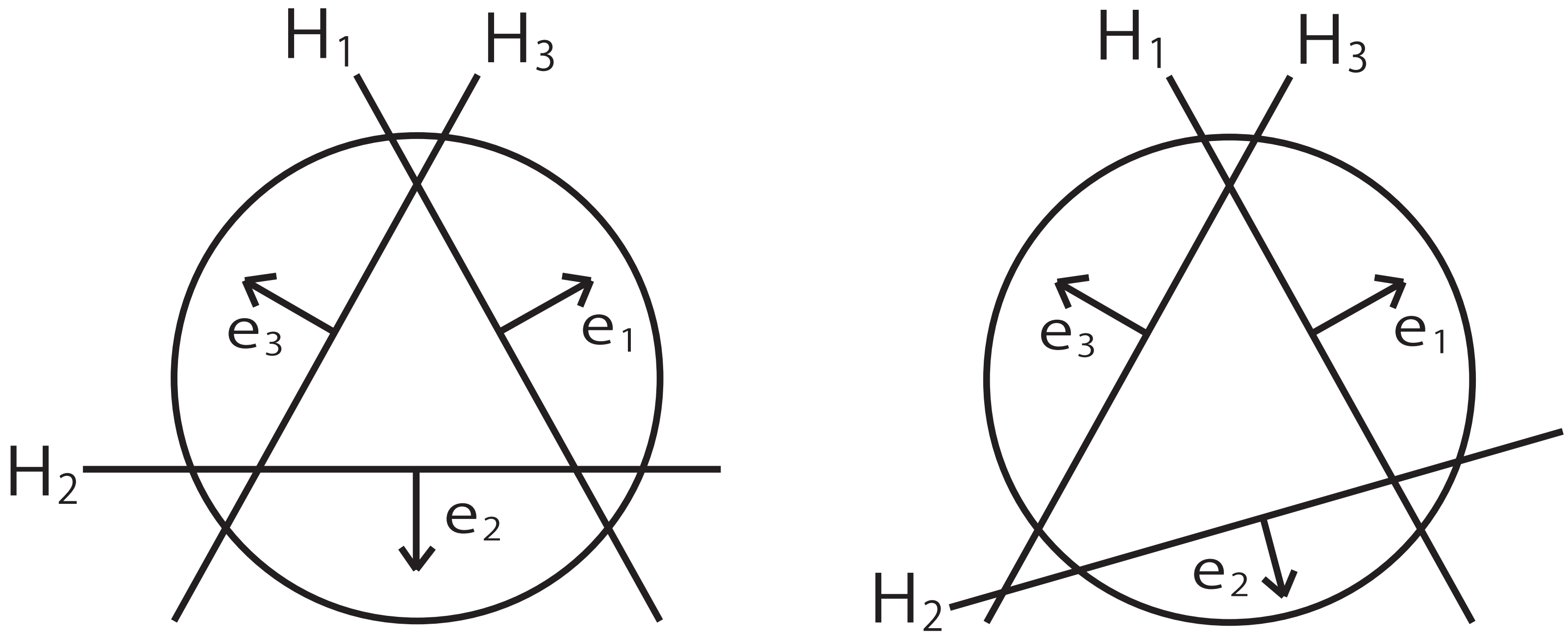
$H^{-} := \{x \in \mathbb{H}^n \,|\, x \circ e \leq 0\}$: $\mathbb{H}^n$**の半空間**





$P \subset \mathbb{H}^n$ が $\mathbb{H}^n$**の凸多面体** であるとは、

$P = \bigcap_{i=1}^{m} H_i^-$ 　かつ　($\mathbb{H}^n$ の) 内点を持つ　ことをいう。



1. $|e_i \circ e_j| < 1 \Leftrightarrow H_i$ と H_j は $\mathbb{H}^n$ 内で交わる

2. $|e_i \circ e_j| \geq 1 \Leftrightarrow H_i$ と H_j は $\mathbb{H}^n$ 内で交わらない

1 のとき、$\cos\theta_{ij} = -e_i \circ e_j$ を満たす $\theta_{ij} \in [0, \pi)$ を、

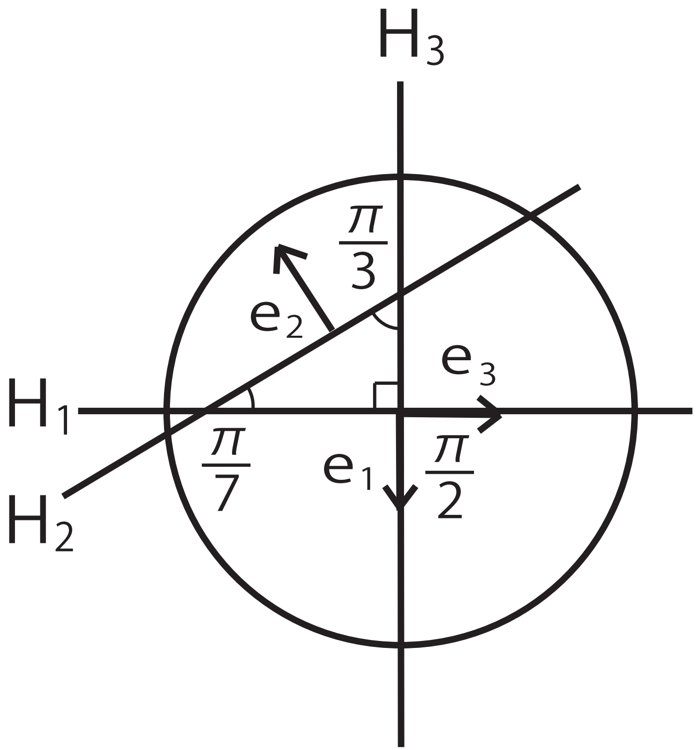
$\boldsymbol{H_i}$**と**$\boldsymbol{H_j}$**のなす**$\boldsymbol{P}$**の面角**　という。

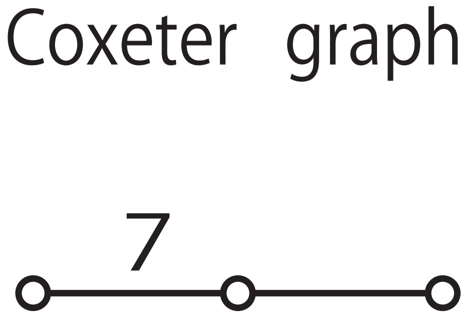
$P=\bigcap_{i=1}^{m} H_i^{-} \subset \mathbb{H}^n$: **Coxeter多面体**

(つまり、すべての面角が$\frac{\pi}{p}$, $p \in \mathbb{Z}_{\geq 2}$ の凸多面体)

Pから定まる鏡映群 (双曲Coxeter群) とは、Pの面を含む超平面に関する鏡映変換$S := \{s_1, ..., s_m\}$で生成される$\mathbb{H}^n$の等長変換部分群のことをいう。(ここで、$s_i(x) := x - 2\frac{x \circ e_i}{e_i \circ e_i} e_i$)

Pがcompactなとき、**cocompact**な群であるという。





(G, S) : 群とその有限生成系

$f_S(t) := \sum_{k \geq 0} a_k t^k$: $(\boldsymbol{G}, \boldsymbol{S})$ **の growth series**

$a_k := \#\{g \in G \mid g\text{の}S\text{による最短表示の長さが}k\}$

$\tau := \limsup_{k \to \infty} \sqrt[k]{a_k} = \dfrac{1}{R}$: $(\boldsymbol{G}, \boldsymbol{S})$ **の growth rate**

(Rは $f_S(t)$ の収束半径)

Coxeter群：

$G =< s_1, ..., s_m \,|\, s_i^2 = id, (s_i s_j)^{m_{ij}} = id \text{ if } i \neq j >$

$m_{ij} \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$

Coxeter系：(G, S)

2. Coxeter群のgrowth series

定理 1. [Steinberg 68]

(G,S) : **無限位数** の Coxeter群

(G_T,T) : $T\subset S$で生成される部分群

$f_S(t)$: (G,S)のgrowth series

$f_T(t)$: (G_T,T)のgrowth series

$\mathcal{F}=\{T\subset S\ :\ G_T$ は **有限位数** の部分群$\}$

このとき,

$$\boldsymbol{\frac{1}{f_S(t^{-1})}=\sum_{T\in\mathcal{F}}\frac{(-1)^{\#T}}{f_T(t)}}.$$

∞ $f_S(t)$ は **有理関数** $\boldsymbol{\dfrac{P(t)}{Q(t)}}$ の原点におけるベキ級数展開である。

growth rate $\tau:=\limsup_{k\to\infty}\sqrt[k]{a_k}\ =\frac{1}{R}$ は **実代数的整数**。

定理 2. [Solomon 66]

(G, S)： **有限位数** の Coxeter群

$f_S(t)$： (G, S) の growth series

このとき,

$$\boldsymbol{f_S(t) = \prod_{i=1}^{n} [m_i + 1]}.$$

ここで

$n = \#S$,

$[m] := 1 + t + \cdots + t^{m-1}$,

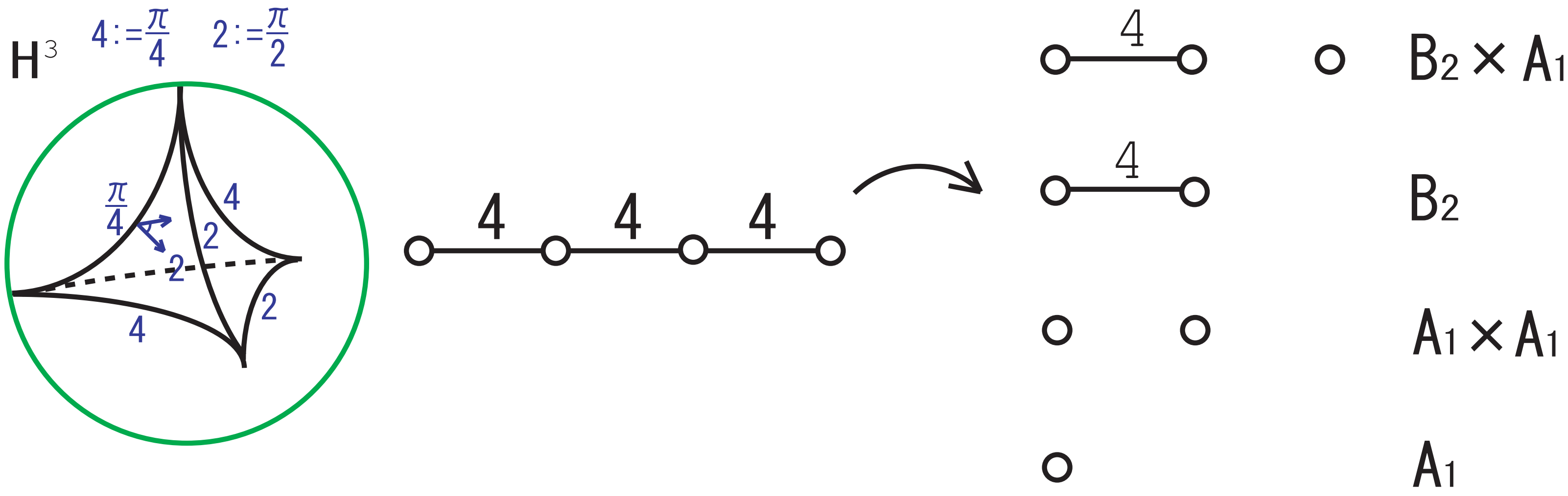
$1 = m_1 \leq m_2 \leq \cdots \leq m_n = h - 1$： (G, S) の exponents,

h： Coxeter element $s_{\sigma(1)} \cdots s_{\sigma(n)}$ の位数.

Graph	Exponents	$f_S(t)$
$A_{n\geq 1}$	$1, 2, \cdots, n$	$[2, 3, \cdots, n+1]$
$B_{n\geq 2}$	$1, 3, \cdots, 2n-1$	$[2, 4, \cdots, 2n]$
$D_{n\geq 4}$	$1, 3, \cdots, 2n-3, n-1$	$[2, 4, \cdots, 2n-2][n]$
E_6	$1, 4, 5, 7, 8, 11$	$[2, 5, 6, 8, 9, 12]$
E_7	$1, 5, 7, 9, 11, 13, 17$	$[2, 6, 8, 10, 12, 14, 18]$
E_8	$1, 7, 11, 13, 17, 19, 23, 29$	$[2, 8, 12, 14, 18, 20, 24, 30]$
F_4	$1, 5, 7, 11$	$[2, 6, 8, 12]$
H_3	$1, 5, 9$	$[2, 6, 10]$
H_4	$1, 11, 19, 29$	$[2, 12, 20, 30]$
$I_2(m)$	$1, m-1$	$[2, m]$

ここで $[m] := 1 + t + \cdots + t^{m-1}$, $[m, n] := [m][n]$.

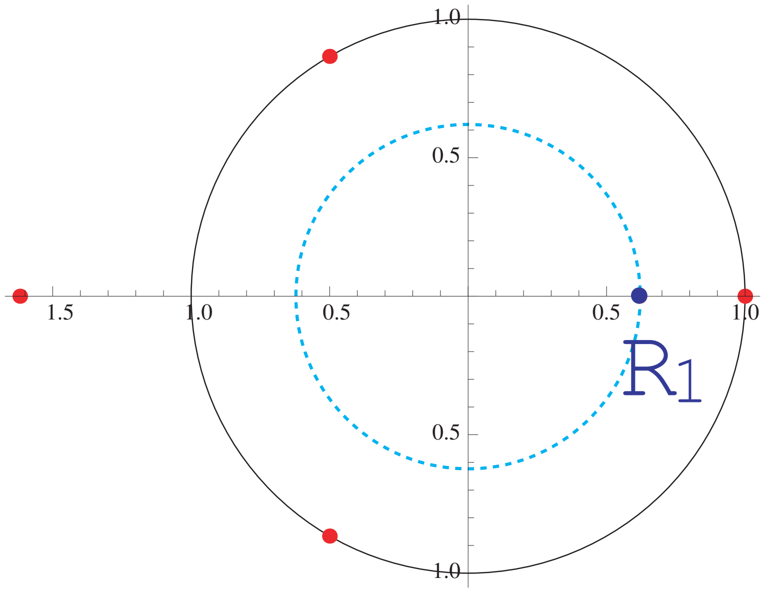
双曲Coxeter群のgrowth seriesの計算例



有限位数の部分群	growth series	number
$B_2 \times A_1$	$[2,4][2]$	2
B_2	$[2,4]$	3
$A_1 \times A_1$	$[2][2]$	3
A_1	$[2]$	4

$$[m] := 1 + t + \cdots + t^{m-1}$$

$$f_S(t) = \sum_{k=0}^{\infty} a_k t^k = \frac{(t+1)^2(t^3+t^2+t+1)}{(t-1)(t^2+t+1)(t^2+t-1)}$$



R_1 ： $f_S(t)$ の収束半径

3. 双曲Coxeter群のgrowth rate の数論的性質

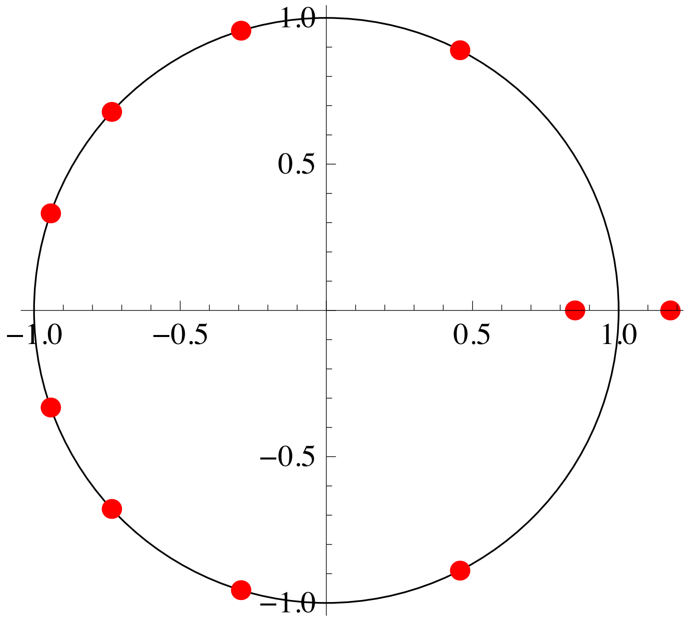
Coxeter多面体	compact	non-compact
$\mathbb{H}^2$	**Salem 数** (Cannon–Wagreich 92, Parry 93)	Pisot 数 (Floyd 92)
$\mathbb{H}^3$	**Salem 数** (Parry 93)	
$\mathbb{H}^4$	★**2–Salem 数**となる **無限系列**がある(U. 13)	

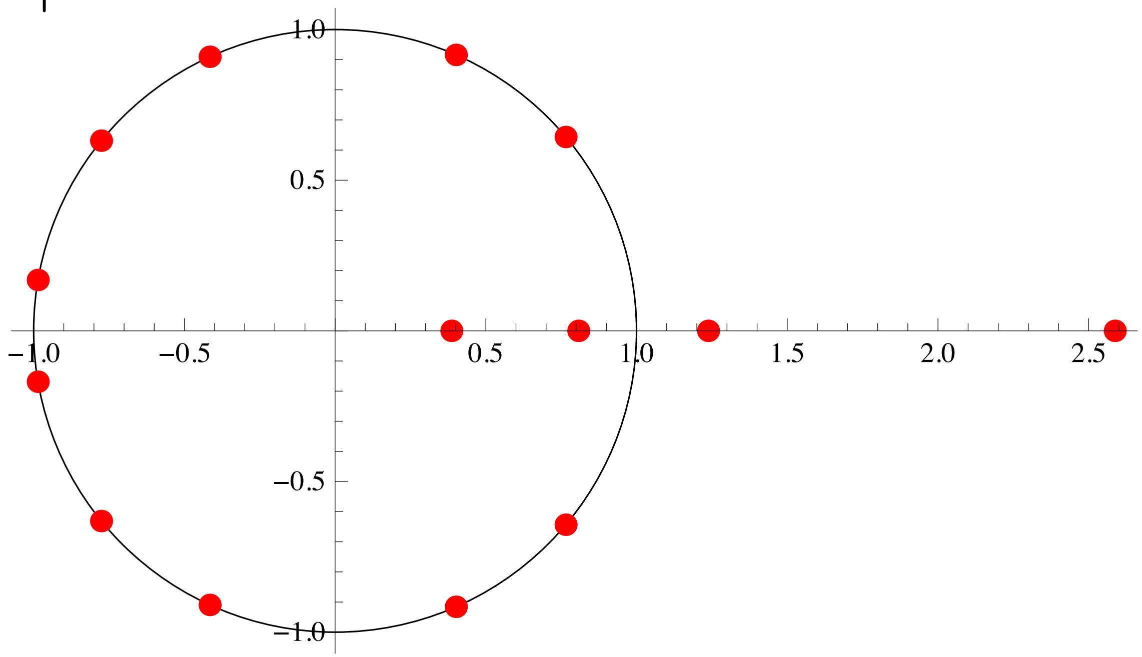
定義 1. α は **Salem数**

$\Leftrightarrow$ α は代数的整数，$\alpha > 1$，他の共役根 ω は $|\omega| \leq 1$ を満たす，ω のうち少なくとも一つは $|\omega| = 1$.

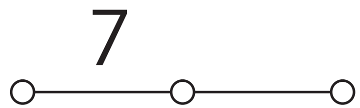
定義 2. α は **2-Salem数** [Samet 52, Kerada 95]

$\Leftrightarrow$ α は代数的整数, $|\alpha| > 1$, 他の共役根 β で $|\beta| > 1$ を満たすものがただ一つ，その他の共役根 ω はすべて $|\omega| \leq 1$ を満たす，ω のうち少なくとも一つは $|\omega| = 1$.



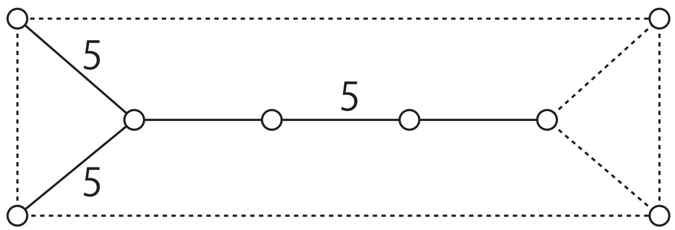


例 1 $\mathbb{H}^2$ の compact な Coxeter 三角形



$$f_S(t) = \frac{(t+1)^2(t^2+t+1)(t^6+t^5+t^4+t^3+t^2+t+1)}{t^{10}+t^9-t^7-t^6-t^5-t^4-t^3+t+1}.$$

例 2 $\mathbb{H}^4$ の compact な Coxeter 多面体



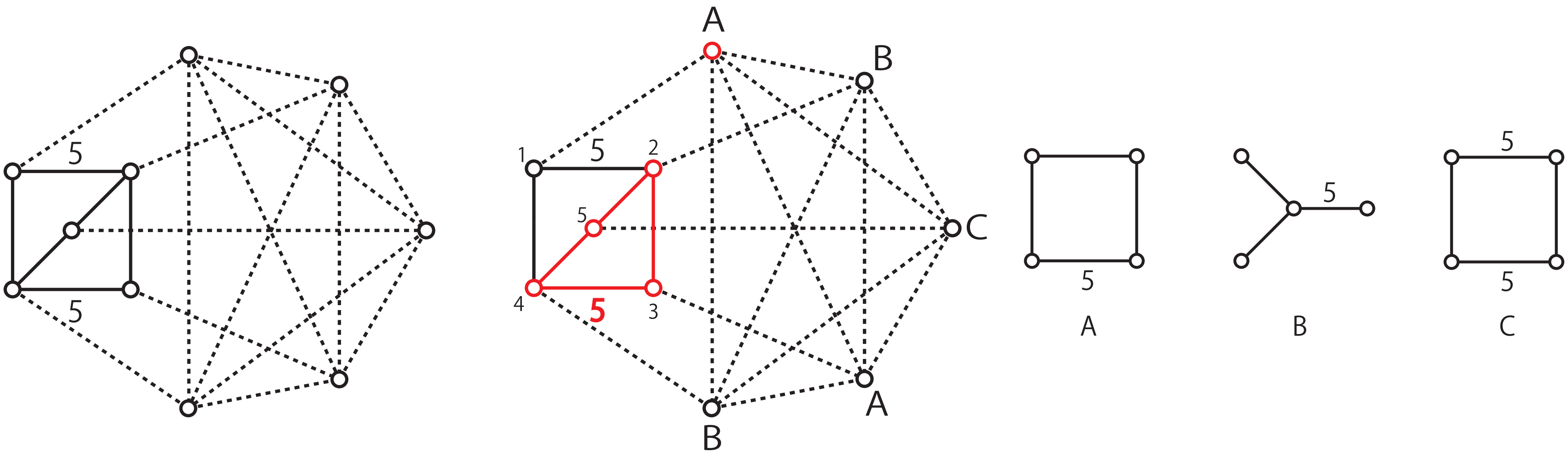
$$f_S(t) = \frac{(t+1)^4(t^2-t+1)(t^2+t+1)(t^4-t^3+t^2-t+1)(t^4+t^3+t^2+t+1)}{t^{16}-4t^{15}+t^{14}+t^{12}+t^{11}+2t^9+2t^7+t^5+t^4+t^2-4t+1}.$$

主結果 [U. 13 (to appear in Algebraic & Geometric Topology)]

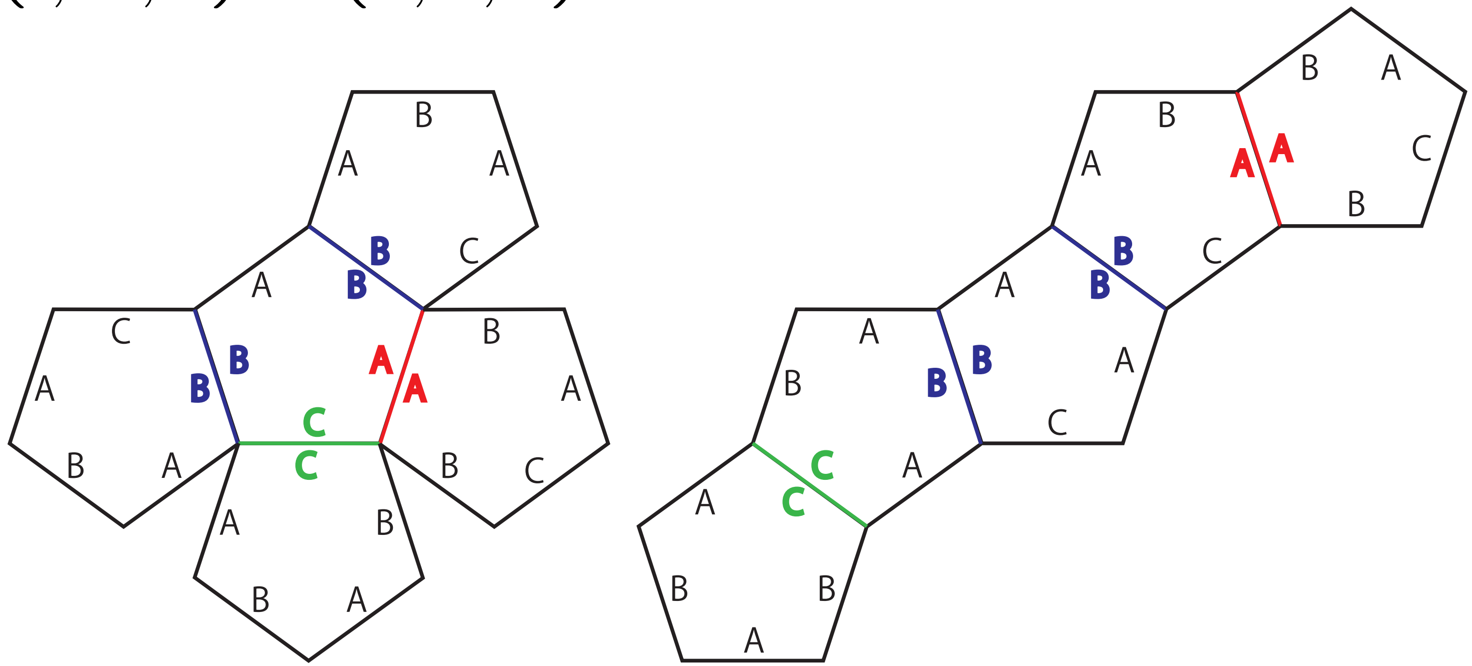
$T \subset \mathbb{H}^4$ ： 下のCoxeterグラフで表されるcompactなCoxeter 多面体 [Vinberg 85, Schlettwein 95]

$T_{\ell,m,n} \subset \mathbb{H}^4$ ： $n+1$個のTをorthogonal facet A でℓ回、Bでm回、Cで$n-\ell-m$回貼り合わせてできたcompactなCoxeter多面体

このとき，$\boldsymbol{n \equiv 1 \pmod 3}$**ならば，**$\boldsymbol{T_{0,n,n}, T_{n,0,n}}$**で定まる鏡映群の growth rate** $\boldsymbol{\tau_{0,n,n}, \tau_{n,0,n}}$ **は 2–Salem 数.**



例　$(l, m, n) = (1, 2, 4)$



$T_{\ell,m,n}$に関するgrowth function $W_{\ell,m,n}(t)$は次で与えられる

（[T. Zehrt–C. Zehrt 12]の系）：

$$\frac{1}{W_{\ell,m,n}(t)} = \frac{n+1}{W(t)} + \frac{t-1}{t+1}\left(\frac{\ell}{A(t)} + \frac{m}{B(t)} + \frac{n-\ell-m}{C(t)}\right).$$

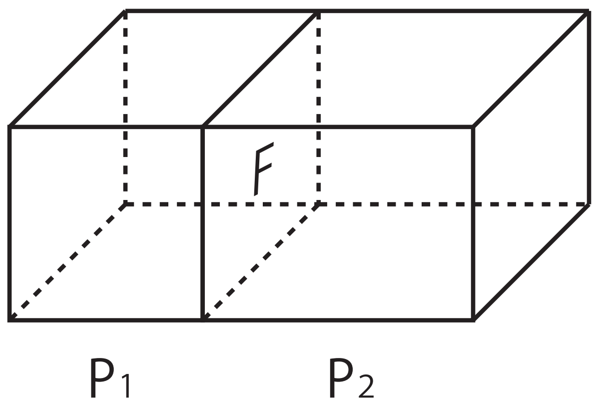
定理 3. [T. Zehrt–C. Zehrt 12]

$P_1, P_2 \subset \mathbb{H}^n$: 2つのCoxeter 多面体. orthogonal facet F を持つ.

$W_1(t), W_2(t), F(t)$: P_1, P_2, F に関するgrowth function

このとき、**P_1とP_2をFで張り合わせることにより得られるCoxeter多面体から定まる鏡映群のgrowth function $W(t)$は以下で与えられる**:

$$\frac{1}{W(t)} = \frac{1}{W_1(t)} + \frac{1}{W_2(t)} + \frac{t-1}{t+1}\frac{1}{F(t)}.$$



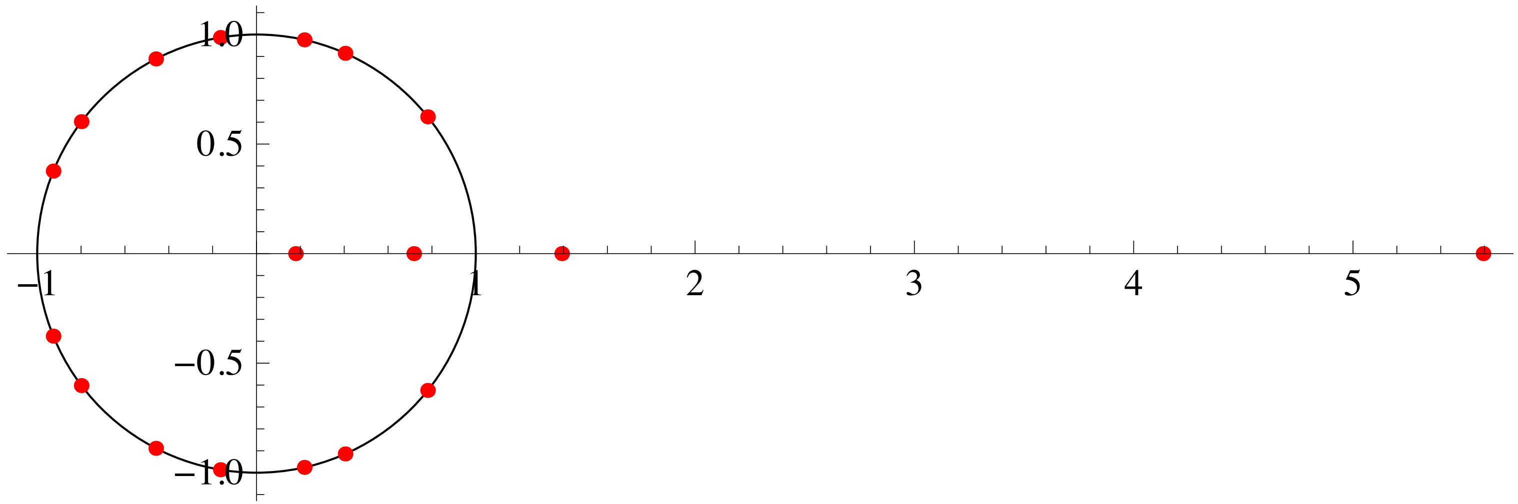
実際、$W_{\ell,m,n}(t) = \dfrac{P_{\ell,m,n}(t)}{Q_{\ell,m,n}(t)}$ は、

$$
\begin{aligned}
P_{\ell,m,n}(t) = {} & (t+1)^4(t^2+1)(t^2-t+1)(t^2+t+1) \\
& (t^4+t^3+t^2+t+1)(t^4-t^3+t^2-t+1)
\end{aligned}
$$

$$
\begin{aligned}
Q_{\ell,m,n}(t) = {} & t^{18} - (4n+6)t^{17} + (2n-m+3)t^{16} \\
& -(3n-m+\ell+5)t^{15} + (5n-3m+5)t^{14} \\
& -(n-4m+1)t^{13} + (8n-4m+\ell+9)t^{12} + (5m-\ell)t^{11} \\
& +(10n-5m+\ell+11)t^{10} - (2n-6m+2)t^9 \\
& +(10n-5m+\ell+11)t^8 + (5m-\ell)t^7 \\
& +(8n-4m+\ell+9)t^6 - (n-4m+1)t^5 + (5n-3m+5)t^4 \\
& -(3n-m+\ell+5)t^3 + (2n-m+3)t^2 - (4n+6)t + 1.
\end{aligned}
$$

主結果の証明

Step 1： $Q_{\ell,m,n}(t)$ は、単位円周上に14個の複素根、正の実軸上に4個の実根を持つことを示す。



$K_{\ell,m,n}(t) := (t+i)^{18} Q_{\ell,m,n}\left(\dfrac{t-i}{t+i}\right)$, $u := t^2$ とすると $K_{\ell,m,m}(u)$ は正の実根を7個、負の実根を2個持つ。

$\Leftrightarrow$ $Q_{\ell,m,n}(t)$ は 単位円周上に14個の根、実軸上に4個の実根を持つ。([Kempner 35, T. Zehrt–C. Zehrt 12]の系)

$$K_{\ell,m,n}(u) = 4\{(8n+8)u^9+(147n+45m+30\ell+207)u^8-(3068n+360m+160\ell+3148)u^7+(11256n+364m-184\ell+7208)u^6-(10124n-616m-480\ell-6724)u^5-(7162n+722m-532\ell+32018)u^4+(12268n+40m-96\ell+27964)u^3-(4608n-428m+120\ell+8528)u^2+(532n-168m+32\ell+836)u-(17n-13m+2\ell+21)\}$$

u	-41	-31	0	1/10	1/3	1/2	1	2	3	9
sign$(K_{\ell,m,n}(u))$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$

例えば、

$$\begin{aligned}\frac{1}{4}K_{\ell,m,n}(0) &= -21-2\ell+13m-17n \\ &\le -21-2\ell+13n-17n \\ &= -21-2\ell-4n<0\end{aligned}$$

Step 2：$Q_{0,n,n}(t)$，$Q_{n,0,n}(t)$ の $\mathbb{Z}$ 上既約性を示す。

1. $Q_{\ell,m,n}(t)$ は、既約でないならば、二つの偶数次の palindromic polynomial の積で表される。つまり、
(2次)(16次)，(4次)(14次)，(6次)(12次)，(8次)(10次) のパターンしかない。

2. $Q_{\ell,m,n}(t)$ は既約でないと仮定して、矛盾を導く。

Step 2

2. $Q_{\ell,m,n}(t)$ は既約でないと仮定して、矛盾を導く。

$Q_{\ell,m,n}(t) = (1 + at + bt^2 + at^3 + t^4)(1 + \sum_{k=1}^{7} c_k t^k + \sum_{k=1}^{6} c_{7-k} t^{k+7} + t^{14})$, $a, b, c_k \in \mathbb{Z}$

と仮定。

$$\begin{cases} c_1 = -a + (-6 - 4n) \\ c_2 = -ac_1 - b + (3 - m + 2n) \\ c_3 = -ac_2 - bc_1 - a + (-5 - \ell + m - 3n) \\ c_4 = -ac_3 - bc_2 - ac_1 - 1 + (5 - 3m + 5n) \\ c_5 = -ac_4 - bc_3 - ac_2 - c_1 + (-1 + 4m - n) \\ c_6 = -ac_5 - bc_4 - ac_3 - c_2 + (9 + \ell - 4m + 8n) \\ c_7 = -ac_6 - bc_5 - ac_4 - c_3 + (-\ell + 5m) \\ c_6 = -ac_7 - bc_6 - ac_5 - c_4 + (11 + \ell - 5m + 10n) \\ c_5 = -ac_6 - bc_7 - ac_6 - c_5 + (-2 + 6m - 2n) \end{cases}$$

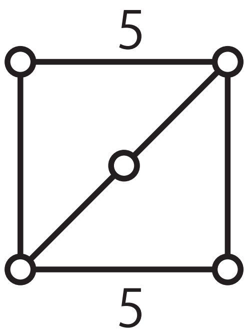
$$f_{\ell,m,n}(a,b) := -1 - a^8 - b^4 - m + a^7(-6-4n) + b^2(1+2m-3n) + a^6(-8+7b+m-2n) + b(\ell+m-n) + n + b^3(2-m+2n) + a^4(-11-15b^2+6m-11n+b(30-5m+10n)) + a(15+2\ell+2m+b(-46-8m-30n)+b^2(-15-3\ell+3m-9n)+9n+b^3(24+16n)) + a^2(2+10b^3-\ell+3m+b^2(-24+6m-12n)-5n+b(9-12m+21n)) + a^5(-29-\ell+m-19n+b(36+24n)) + a^3(13-2\ell+6m+b^2(-60-40n)+9n+b(68+4\ell-4m+44n))$$
$$= 0,$$

$$g_{\ell,m,n}(a,b) := 12 + a^7(2-b) + 2m + b^2(-23-4m-15n) + b^3(-5-\ell+m-3n) + 8n + b^4(6+4n) + a^5(12+6b^2-2m+b(-18+m-2n)+4n) + a^6(12+b(-6-4n)+8n) + b(15+2\ell+2m+9n) + a^3(6-10b^3-8m+b(-35+12m-23n)+14n+b^2(36-4m+8n)) + a(4b^4+2\ell+2m+b(4-\ell+7m-11n)+b^3(-14+3m-6n)-2n+b^2(10-10m+18n)) + a^4(34+2\ell-2m+b(-77-\ell+m-51n)+22n+b^2(30+20n)) + a^2(-46-8m+b^3(-36-24n)+b(-7-6\ell+10m-3n)-30n+b^2(87+3\ell-3m+57n))$$
$$= 0.$$

(a, b) mod 3

(a,b)	$f_{0,n,n}(a,b)$	$g_{0,n,n}(a,b)$	
$(0,0)$	-1		不可能
$(1,0)$	$1-n$	$-(1+n)$	不可能
$(-1,0)$	0	$1+n$	$n \equiv -1$のときのみ可能
$(0,1)$	1		不可能
$(0,-1)$	n	n	$n \equiv 0$のときのみ可能
$(1,1)$	0	$1+n$	$n \equiv -1$のときのみ可能
$(-1,1)$	0	$-n$	$n \equiv 0$のときのみ可能
$(1,-1)$	-1		不可能
$(-1,-1)$	0	n	$n \equiv 0$のときのみ可能

よって、$n \equiv 1 \pmod 3$ のとき、方程式$f_{0,n,n}(a,b) = g_{0,n,n}(a,b) = 0$を満たす整数の組$(a,b)$は存在せず、矛盾。

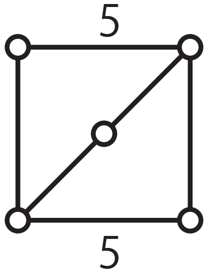


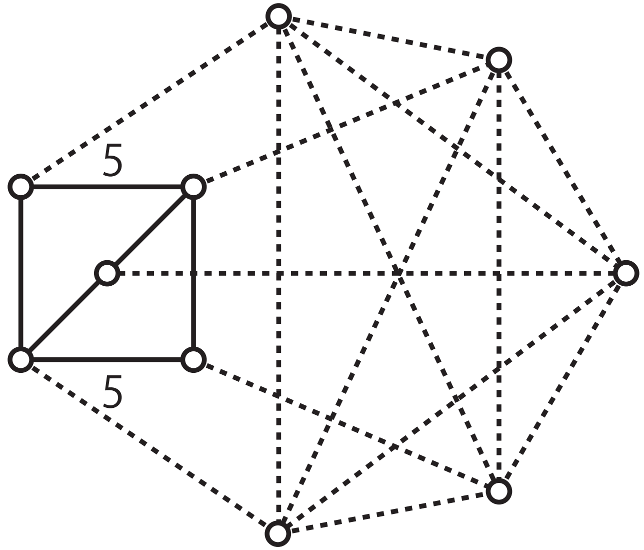
$$\begin{pmatrix} 1 & -\cos\frac{\pi}{5} & 0 & -\frac{1}{2} & 0 \\ -\cos\frac{\pi}{5} & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 & -\cos\frac{\pi}{5} & 0 \\ -\frac{1}{2} & 0 & -\cos\frac{\pi}{5} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \end{pmatrix} \sim \begin{pmatrix} \lambda_1 & 0 & & & 0 \\ 0 & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \lambda_4 & 0 \\ 0 & & & 0 & \lambda_5 \end{pmatrix}$$

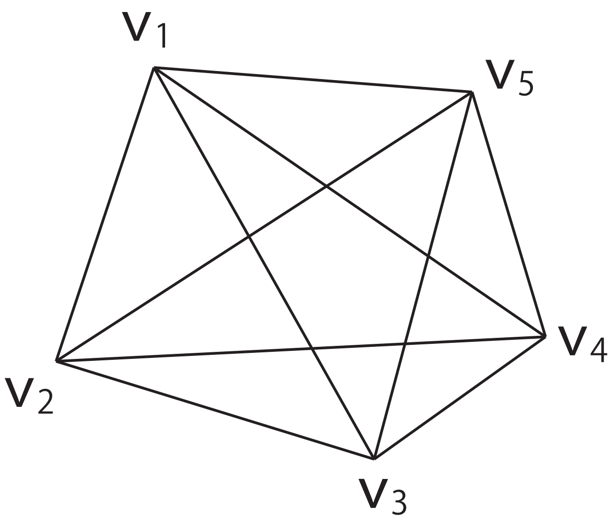
$\lambda_1, \ldots, \lambda_4 > 0, \lambda_5 < 0$ ($\boldsymbol{sig G = (4, 1)}$ とかく)

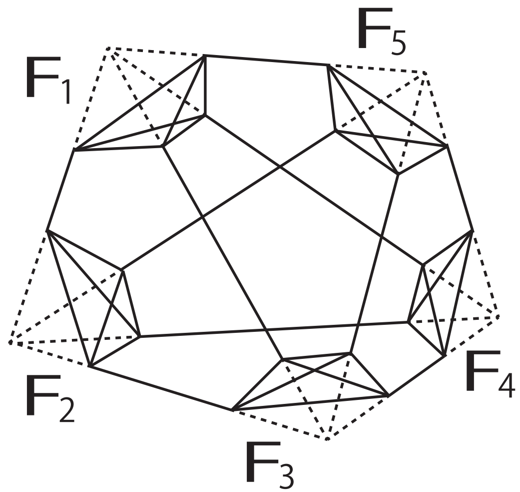
$\Rightarrow \mathbb{H}^4$**のCoxeter多面体**

Coxeter多面体 $P \subset \mathbb{H}^4$ の truncation (compact化)









On behavior of pairs of Teichmüller geodesic rays

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Nov. 11, 2013

Let X be a Riemann surfaces of type (g, n) with $3g - 3 + n > 0$ and $T(X)$ be the Teichmüller space of X.

Problem

Let $r(t)$, $r'(t)$ be Teichmüller geodesic rays on $T(X)$. Two rays $r(t)$, $r'(t)$ are **asymptotic** if there is a choice of base points $r(0)$, $r'(0)$ so that

$$\lim_{t \to \infty} d_{T(X)}(r(t), r'(t)) = 0.$$

We want conditions that $r(t)$, $r'(t)$ are asymptotic.

First, we express the main theorem. We will recall the definitions of terms after.
Let $r = r(t)$, $r' = r'(t)$ be Jenkins-Strebel rays on $T(X)$ starting at $p = [Y, f]$, $p' = [Y', f']$ and having unit norm Jenkins-Strebel differentials q, q' on Y, Y' respectively. We denote by $r(\infty)$, $r'(\infty)$ the end points of r, r' on the augmented Teichmüller space $\hat{T}(X)$ respectively.
We suppose that r, r' are similar, i.e., the Jenkins-Strebel differentials q, q' determine annuli which are generated by homotopy classes of simple closed curves $f(\gamma_1), \cdots, f(\gamma_k)$, $f'(\gamma_1), \cdots, f'(\gamma_k)$ respectively, where $\gamma_1, \cdots, \gamma_k$ are distinct and non-intersecting simple closed curves on X. Let m_j, m'_j be the corresponding moduli respectively for any $j = 1, \cdots, k$.

Theorem 1 ([Ama13])

If $r(\infty) = r'(\infty)$, then

$$\lim_{t\to\infty} d_{T(X)}(r(t), r'(t)) = \frac{1}{2}\log \max_{j=1,\cdots,k}\left\{\frac{m'_j}{m_j}, \frac{m_j}{m'_j}\right\}.$$

Corollary 2 ([Ama13])

For any two Jenkins-Strebel rays r, r', they are asymptotic if and only if r, r' are modularly equivalent and $r(\infty) = r'(\infty)$.

Farb and Masur showed the same result in the moduli space. [FM10]

Let X be a Riemann surface of type (g,n) with $3g-3+n>0$.

Definition (Teichmüller spaces)

$$T(X) := \{(Y,f) \mid Y : \text{a Riemann surface}, f : X \to Y : \text{a qc-mapping}\}/\sim,$$

$(Y_1,f_1) \sim (Y_2,f_2) :\Leftrightarrow$ There exists a conformal mapping $h : Y_1 \to Y_2$ which is homotopic to $f_2 \circ f_1^{-1}$.
We call $T(X)$ the **Teichmüller space** of X and denote by $[Y,f]$ the equivalence class of a pair (Y,f).

Definition (The Teichmüller distance)

The **Teichmüller distance** $d_{T(X)}$ is a complete distance on $T(X)$. This is defined the following formula. For any $p_1 = [Y_1, f_1], p_2 = [Y_2, f_2] \in T(X)$,

$$d_{T(X)}(p_1, p_2) := \frac{1}{2} \log \inf_h K(h),$$

where the infimum is taken over all qc-mappings $h : Y_1 \to Y_2$ homotopic to $f_2 \circ f_1^{-1}$ and $K(h)$ means the maximal dilatation of h.

Definition (Quadratic differentials)

A **holomorphic quadratic differential** q on X is represented locally by $q = q(z)dz^2$ where $q(z)$ is a holomorphic function of the local coordinate $z = x + iy$ on X. We allow holomorphic quadratic differentials to have simple poles at the punctures of X, then $\|q\| := \iint_X |q(z)|dxdy < \infty$. We call that q is of **unit norm** if $\|q\| = 1$.

Definition (q-coordinates)

A **critical point** of $q \neq 0$ is a zero of q or a puncture of X. A q**-coordinate** ζ on X is a local coordinate on $X - \{$critical points of $q\}$ such that $q = d\zeta^2$. For any two q-coordinates ζ_1, ζ_2 in a common neighborhood U, the equation $\zeta_2 = \pm\zeta_1 + c$ where $c \in \mathbb{C}$ holds, because $q = d\zeta_1^2 = d\zeta_2^2$.

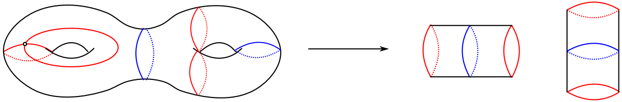
Definition (trajectories)

A **horizontal trajectory** of q is a maximal smooth path $z = \gamma(t)$ on X which satisfies $q(\gamma(t))\dot{\gamma}(t)^2 > 0$. A **critical trajectory** joins critical points of q. Let Γ_q be the set of all critical points and critical trajectories of q. For any component of $X - \Gamma_q$, there are the following two cases.

1. **annulus**: it is swept out by closed trajectories of q such that they are homotopic to each other. In this case, we call the homotopy class of the closed trajectory the **core curve** of the annulus.
2. **minimal domain**: it consists of infinitely many recurrent trajectories of q.

A quadratic differential q has finitely many critical points, then q has finitely many these domains.

If all components of $X - \Gamma_q$ are annuli, we call q a **Jenkins-Strebel differential** (J-S differential). In this case, the core curves which are determined by q are distinct and non-intersecting each other. After this, we treat only J-S differentials.



Definition (moduli of annuli)

For any J-S differential q, it generates finitely many annulus $\{A_j\}_{j=1,\cdots,k}$. Each annulus is conformally equivalent to the cylinder C_j which has the circumference a_j and the height b_j for any $j = 1, \cdots, k$. We set the modulus of the annulus A_j as $m_j = \frac{b_j}{a_j}$.

Definition (Teichmüller geodesic rays)

Let $p = [Y, f] \in T(X)$, q be a unit norm quadratic differential on Y. For any $t \in [0, \infty)$, we define the qc-mapping $g_t : Y \to Y_t$ by $z = x + iy \mapsto z_t = e^{-t}x + ie^{t}y$ and set $Y_0 = Y$ where z is the q-coordinate. The mapping $r : [0, \infty) \to T(X)$ which is defined by

$$r(t) := [Y_t, g_t \circ f]$$

satisfies $d_{T(X)}(r(t), r(s)) = |t - s|$ for any $t, s \in [0, \infty)$. We call r the **Teichmüller geodesic ray** on $T(X)$ starting at p and having q. If q is J-S, we call r the **Jenkins-Strebel ray** (J-S ray).

Now, let r, r' be two Teichmüller geodesic rays on $T(X)$ starting at $p = [Y, f]$, $p' = [Y', f']$ and having unit norm J-S differentials q, q' respectively.

Definition (J-S rays are similar)

J-S rays r, r' are called **similar** if there are distinct and non-intersecting homotopy classes of simple closed curves $\gamma_1, \cdots, \gamma_k$ on X such that q, q' have the core curves of annulus whose forms are $f(\gamma_1), \cdots, f(\gamma_k)$ on Y and $f'(\gamma_1), \cdots, f'(\gamma_k)$ on Y' respectively.

Definition (Modularly equivalent)

In this situation, the given rays r, r' are called **modularly equivalent** if there is $\lambda > 0$ such that $m'_j = \lambda m_j$ for any $j = 1, \cdots, k$.

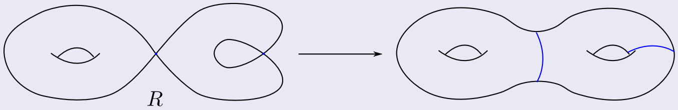
Definition (Asymptoticity)

We call that r, r' are **asymptotic** if there is a choice of initial points $r(0)$, $r'(0)$ such that $d_{T(X)}(r(t), r'(t)) \to 0$ as $t \to \infty$, in other words, for the given rays $r(t)$, $r'(t)$, there is $\alpha \in \mathbb{R}$ such that $d_{T(X)}(r(t), r'(t+\alpha)) \to 0$ as $t \to \infty$.

Definition (Riemann surfaces with nodes)

A connected Hausdorff space R is called a **Riemann surface of type** (g, n) **with nodes** if R satisfies the following two conditions:

1. Any $p \in R$ has a neighborhood which is homeomorphic to the unit disk $\mathbb{D}$ or the set $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < 1, |z_2| < 1, z_1 \cdot z_2 = 0\}$. (In the latter case, p is called a **node** of R. We allow R to have finitely many nodes.)
2. Any component of $R - \{\text{nodes of } R\}$ is a hyperbolic Riemann surface, and we get a Riemann surface of type (g, n) without nodes by opening each node of R.



Definition (Augmented Teichmüller spaces)

Let X be a Riemann surface of type (g, n) **without** nodes which satisfies $3g - 3 + n > 0$. We define the **augmented Teichmüller space** of X as follows.

$$\hat{T}(X) := \{(R, f) \mid R : \text{a Riemann surface of type } (g, n) \text{ with or without nodes, } f : X \to R : \text{a deformation } \}/\sim,$$

where the "deformation" is a mapping such that it contracts some disjoint loops on X to points (the nodes of R) and is a homeomorphism except on the loops. $(R_1, f_1) \sim (R_2, f_2) :\Leftrightarrow$ There is a biholomorphic mapping $h : R_1 \to R_2$ such that f_2 is homotopic to $h \circ f_1$.

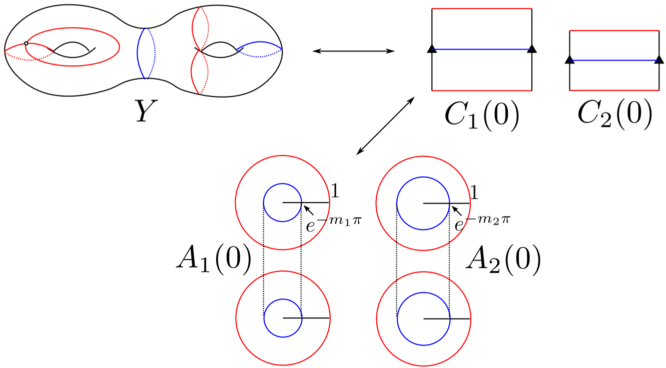
A homeomorphism $h : R_1 \to R_2$ is called biholomorphic if each restricted mapping of h which maps a component of $R_1 - \{\text{nodes of } R_1\}$ onto a component of $R_2 - \{\text{nodes of } R_2\}$ is biholomorphic. A topology on $\hat{T}(X)$ is defined by the following neighborhoods.

Definition (The neighborhood of a point on $\hat{T}(X)$)

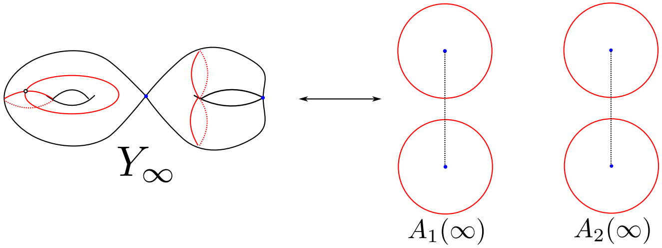
For any compact neighborhood V of the set of nodes in R and any $\varepsilon > 0$, a neighborhood $U_{V,\varepsilon}$ of a point $[R, f]$ is defined by

$$U_{V,\varepsilon} := \{[S, g] \in \hat{T}(X) \mid \text{there is a deformation } h : S \to R \text{ which is } (1+\varepsilon)\text{-quasiconformal on } h^{-1}(R - V) \text{ such that } f \text{ is homotopic to } h \circ g\}.$$

We consider the end point of a Jenkins-Strebel ray r starting at $r(0) = [Y, f]$ and having unit norm J-S differential q. First, we see that cylinders $\{C_j(0)\}_{j=1,\cdots,k}$ which are determined by q-coordinates on Y. Each $C_j(0)$ is transformed to $A_j(0)$ which is the pair of two ring domains $\{e^{-m_j\pi} \leq |z| < 1\}$ with the gluing, for any $j = 1, \cdots, k$.



The Teichmüller mapping $g_t : Y \to Y_t$ is represent to the form $z = re^{i\theta} \mapsto r^{e^{2t}} e^{i\theta}$in $A_j^l(0)$ for any $l = 1, 2$. We set the mapping $g_\infty : Y \to Y_\infty$ which maps $A_j^l(0)$ onto $\mathbb{D} = A_j^l(\infty) \cup \{pt\}$ by $z = re^{i\theta} \mapsto h_j(r)e^{i\theta}$, where $h_j : [\exp(-m_j\pi), 1) \to [0, 1)$ is an arbitrary monotone increasing diffeomorphism.



The Riemann surface with nodes Y_∞ is constructed by these disks $\{A_j^l(\infty) \cup \{pt\}\}_{j=1,\cdots,k}^{l=1,2}$ with the gluing, and we denote $[Y_\infty, g_\infty \circ f]$ by $r(\infty)$.

Theorem (cf. [HS07])

The Jenkins-Strebel ray $r(t) = [Y_t, g_t \circ f]$ converges to a point $r(\infty) = [Y_\infty, g_\infty \circ f]$ in $\hat{T}(X)$.

Let r, r' be two Jenkins-Strebel rays and we suppose that the rays are similar. We show the following.

Theorem 1 ([Ama13])

If $r(\infty) = r'(\infty)$, then

$$\lim_{t\to\infty} d_{T(X)}(r(t), r'(t)) = \frac{1}{2}\log \max_{j=1,\cdots,k}\left\{\frac{m'_j}{m_j}, \frac{m_j}{m'_j}\right\}.$$

First, we show that

$$\limsup_{t\to\infty} d_{T(X)}(r(t), r'(t)) \le \frac{1}{2}\log \max_{j=1,\cdots,k}\left\{\frac{m'_j}{m_j}, \frac{m_j}{m'_j}\right\}.$$

Lemma

Let us choose $0 < \varepsilon < 1$ arbitrary. Then, for any sufficiently large t, there is a quasiconformal mapping $F_t : Y_t \to Y'_t$ which is homotopic to $(g'_t \circ f') \circ (g_t \circ f)^{-1}$ such that the inequality $\lim_{t\to\infty} K(F_t) < \max_{j=1,\cdots,k}\left\{\frac{m'_j}{m_j}, \frac{m_j}{m'_j}\right\} + \varepsilon$ holds.

Since r, r' are similar, we notice that
$(g'_t \circ f') \circ (g_t \circ f)^{-1}(A_j(t)) \sim A'_j(t)$ on Y'_t for any $0 \le t \le \infty$.

Proof.
We set $M_j = \frac{m'_j}{m_j}$ for any $j = 1, \cdots, k$. By $r(\infty) = r'(\infty)$, there exists a biholomorphic mapping $h : Y_\infty \to Y'_\infty$ such that $h \circ g_\infty \circ f$ is homotopic to $g'_\infty \circ f'$. We can write

$$Y_\infty = \bigcup_{j=1}^{k} \overline{A_j^1(\infty)} \cup \overline{A_j^2(\infty)},$$
$$Y'_\infty = \bigcup_{j=1}^{k} \overline{A_j'^1(\infty)} \cup \overline{A_j'^2(\infty)},$$

where $A_j^l(\infty)$, $A_j'^l(\infty)$ are the punctured disks $\mathbb{D}^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ for any $j = 1, \cdots, k$ and $l = 1, 2$.

Now, we fix any $j = 1, \cdots, k$ and $l = 1, 2$. We set $h_j^l = h|_{A_j^l(\infty)} : A_j^l(\infty) \to h(A_j^l(\infty)) \subset Y'_\infty$. Since h is a biholomorphic mapping, then we can set $h_j^l(0) = 0$ and $\frac{dh_j^l(z)}{dz}\Big|_{z=0} \neq 0$. We describe
$h_j^l(z) = c_j^l z + c_{j,2}^l z^2 + \cdots = c_j^l z + \psi_j^l(z)$ where $c_j^l \neq 0$, $-\pi < \arg c_j^1 \leq \pi$ and $-\pi \leq \arg c_j^2 < \pi$.

We set $\delta_j(t) = \exp(-e^{2t} m_j \pi)$, $\delta'_j(t) = \exp(-e^{2t} m'_j \pi)$, for any $t \geq 0$. Then $\delta'_j(t) = \delta_j(t)^{M_j}$. After this, we assume that $A^l_j(t) = \mathbb{D}^* - \mathbb{D}_{\delta_j(t)} = \{z \in \mathbb{C} \mid \delta_j(t) \leq |z| < 1\}$ and $A'^l_j(t) = \mathbb{D}^* - \mathbb{D}_{\delta'_j(t)} = \{z \in \mathbb{C} \mid \delta'_j(t) \leq |z| < 1\}$ for any $t \geq 0$.

The Riemann surfaces Y_t, Y'_t are constructed by the domains $\{A^l_j(t)\}^{l=1,2}_{j=1,\cdots,k}$, $\{A'^l_j(t)\}^{l=1,2}_{j=1,\cdots,k}$ with the gluing respectively.

To obtain the mapping $F_t : Y_t \to Y'_t$, for sufficiently large t, we construct a quasiconformal mapping $F^l_{j,t} : A^l_j(t) \to h(A^l_j(t))$.

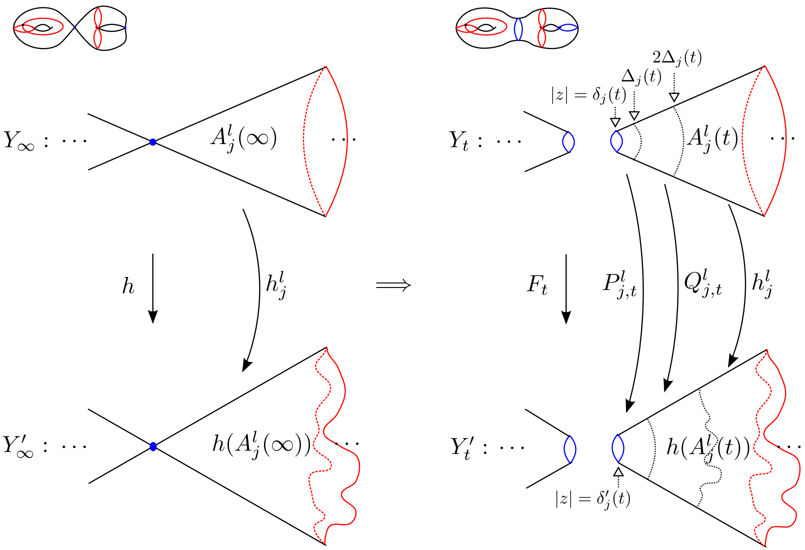
We consider the following three cases (1), (2) and (3).
(1) In the case of $M_j > 1$, we take X_j as

$$X_j < \frac{\log \frac{\varepsilon}{M_j+\varepsilon-1}}{\log M_j} < 0 \quad \Leftrightarrow \quad M_j^{X_j} < \frac{\varepsilon}{M_j+\varepsilon-1} < 1$$
$$\Leftrightarrow \quad \frac{M_j - M_j^{X_j}}{1 - M_j^{X_j}} < M_j + \varepsilon.$$

We take sufficiently large t such that the inequality $\delta_j(t)^{M_j} < |c_j^l|\delta_j(t)^{M_j^{X_j}}$ holds. We set $\Delta_j(t) = \delta_j(t)^{M_j^{X_j}}$. We construct $F_{j,t}^l$ by the following:

$$F_{j,t}^l(z) = \begin{cases} P_{j,t}^l(z) & (\delta_j(t) \le |z| \le \Delta_j(t)) & \text{(i)} \\ Q_{j,t}^l(z) & (\Delta_j(t) \le |z| \le 2\Delta_j(t)) & \text{(ii)} \\ h_j^l(z) & (2\Delta_j(t) \le |z| < 1) & \text{(iii)} \end{cases}$$

Upper estimate

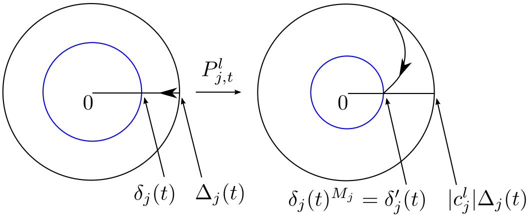


Upper estimate

(i) In $\delta_j(t) \le |z| \le \Delta_j(t)$, we set

$$P^l_{j,t}(z) = \Delta_j(t)^{\frac{1-M_j}{1-M_j^{X_j}}} \cdot {c^l_j}^{\frac{1}{1-M_j^{X_j}} + \frac{\log|z|}{\log\Delta_j(t) - \log\delta_j(t)}} \cdot |z|^{-\frac{1-M_j}{1-M_j^{X_j}}} \cdot z$$

which satisfies $P^l_{j,t}(z) = \delta_j(t)^{M_j - 1} \cdot z$ on $|z| = \delta_j(t)$, $P^l_{j,t}(z) = c^l_j z$ on $|z| = \Delta_j(t)$.



The mapping $P^l_{j,t}$ is conjugate to a one-to-one affine mapping by $\log z$. Then, $P^l_{j,t}$ is a qc-mapping, and its dilatation is the following:

$$K(P^l_{j,t}) = \frac{\left|\frac{\log c^l_j}{2(M_j^{X_j}-1)\log\delta_j(t)} + \frac{\alpha_j}{2} + 1\right| + \left|\frac{\log c^l_j}{2(M_j^{X_j}-1)\log\delta_j(t)} + \frac{\alpha_j}{2}\right|}{\left|\frac{\log c^l_j}{2(M_j^{X_j}-1)\log\delta_j(t)} + \frac{\alpha_j}{2} + 1\right| - \left|\frac{\log c^l_j}{2(M_j^{X_j}-1)\log\delta_j(t)} + \frac{\alpha_j}{2}\right|},$$

where $\alpha_j = -\dfrac{1-M_j}{1-M_j^{X_j}}$. We see that $(M_j^{X_j}-1)\log\delta_j(t) \to +\infty$ and

$$K(P^l_{j,t}) \to \frac{M_j - M_j^{X_j}}{1-M_j^{X_j}} < M_j + \varepsilon$$

as $t \to \infty$.

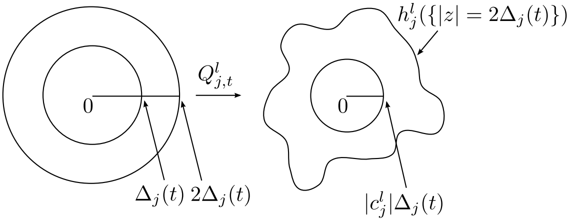
(ii) In $\Delta_j(t) \le |z| \le 2\Delta_j(t)$, we set

$$Q^l_{j,t}(z) = c^l_j z + \phi_{\Delta_j(t)}(|z|)\psi^l_j(z),$$

where $\phi_{\Delta_j(t)} : [\Delta_j(t), 2\Delta_j(t)] \to [0,1]$ is defined by

$$\phi_{\Delta_j(t)}(|z|) = \frac{|z|}{\Delta_j(t)} - 1.$$

This function satisfies $Q^l_{j,t}(z) = c^l_j z$ on $|z| = \Delta_j(t)$, $Q^l_{j,t}(z) = h^l_j(z)$ on $|z| = 2\Delta_j(t)$.



We consider the partial derivatives of $Q^l_{j,t}$,

$$\partial_{\bar z} Q^l_{j,t} = \frac{1}{2\Delta_j(t)} z^{\frac{1}{2}} \bar z^{-\frac{1}{2}} \psi^l_j(z),$$

$$\partial_z Q^l_{j,t} = c^l_j + \frac{1}{2\Delta_j(t)} z^{-\frac{1}{2}} \bar z^{\frac{1}{2}} \psi^l_j(z) + \phi_{\Delta(t)}(|z|) \frac{d\psi^l_j(z)}{dz}.$$

These partial derivatives are continuous in the domain. There is $C > 0$ such that $|\psi^l_j(z)| \le C\Delta_j(t)^2$ for sufficiently large t. We see that

$$\begin{aligned} \left| \frac{1}{2\Delta_j(t)} z^{\frac{1}{2}} \bar z^{-\frac{1}{2}} \psi^l_j(z) \right| &= \left| \frac{1}{2\Delta_j(t)} z^{-\frac{1}{2}} \bar z^{\frac{1}{2}} \psi^l_j(z) \right| \\ &= \frac{|\psi^l_j(z)|}{2\Delta_j(t)} \le \frac{C\Delta_j(t)}{2} \to 0 \end{aligned}$$

as $t \to \infty$. Then, $|\partial_{\bar z} Q^l_{j,t}| \to 0$, $|\partial_z Q^l_{j,t}| \to |c^l_j| \ne 0$ as $t \to \infty$. For sufficiently large t, $\operatorname{Jac} Q^l_{j,t} = |\partial_z Q^l_{j,t}|^2 - |\partial_{\bar z} Q^l_{j,t}|^2 \ne 0$. Hence, $Q^l_{j,t}$ is a local C^1-diffeomorphism.

In fact, $Q^l_{j,t}$ is a C^1-diffeomorphism. By the derivatives of $Q^l_{j,t}$, for sufficiently large t, it is a quasiconformal mapping such that its dilatation holds $K(Q^l_{j,t}) \to 1$ as $t \to \infty$.
(iii) In $2\Delta_j(t) \le |z| < 1$, $F^l_{j,t}(z) = h^l_j(z)$ and $K(h^l_j) = 1$.
Therefore, for sufficiently large t, we obtain the quasiconformal mapping $F^l_{j,t}$ such that

$$K(F^l_{j,t}) = \max\{K(P^l_{j,t}), K(Q^l_{j,t})\} \to \frac{M_j - M_j^{X_j}}{1 - M_j^{X_j}} < M_j + \varepsilon$$

as $t \to \infty$.

(2) In the case of $M_j < 1$, we take X_j as

$$X_j > \frac{\log \frac{M_j \varepsilon}{\frac{1}{M_j} - 1 + \varepsilon}}{\log M_j} > 2 \iff M_j^{X_j} < \frac{M_j \varepsilon}{\frac{1}{M_j} - 1 + \varepsilon} < M_j^2$$

$$\iff \frac{1 - M_j^{X_j}}{M_j - M_j^{X_j}} < \frac{1}{M_j} + \varepsilon.$$

We take sufficiently large t such that the inequality $\delta_j(t)^{M_j} < |c_j^l| \delta_j(t)^{M_j^{X_j}}$ holds. We also set $\Delta_j(t) = \delta_j(t)^{M_j^{X_j}}$, and also construct $F_{j,t}^l$ following.

$$F_{j,t}^l(z) = \begin{cases} P_{j,t}^l(z) & (\delta_j(t) \leq |z| \leq \Delta_j(t)) \\ Q_{j,t}^l(z) & (\Delta_j(t) \leq |z| \leq 2\Delta_j(t)) \\ h_j^l(z) & (2\Delta_j(t) \leq |z| < 1) \end{cases}$$

The functions $P^l_{j,t}$, $Q^l_{j,t}$ have the same notations as in the case of (1). The difference is only the dilatation of $P^l_{j,t}$. In this case,

$$K(P^l_{j,t}) \to \frac{1 - M_j^{X_j}}{M_j - M_j^{X_j}} < \frac{1}{M_j} + \varepsilon$$

as $t \to \infty$. Similarly as in the case of (1), for sufficiently large t, we obtain the quasiconformal mapping $F^l_{j,t}$ such that

$$K(F^l_{j,t}) = \max\{K(P^l_{j,t}), K(Q^l_{j,t})\} \to \frac{1 - M_j^{X_j}}{M_j - M_j^{X_j}} < \frac{1}{M_j} + \varepsilon$$

as $t \to \infty$.

(3) In the case of $M_j = 1$, we take sufficiently large t such that the inequality $\delta_j(t) < |c_j^l|\delta_j(t)^{\frac{1}{2}}$ holds and set $\Delta_j(t) = \delta_j(t)^{\frac{1}{2}}$. We set

$$F_{j,t}^l(z) = \begin{cases} P_{j,t}^l(z) = {c_j^l}^{2\left(1-\frac{\log|z|}{\log\delta_j(t)}\right)} z & (\delta_j(t) \le |z| \le \Delta_j(t)) \\ Q_{j,t}^l(z) & (\Delta_j(t) \le |z| \le 2\Delta_j(t)) \\ h_j^l(z) & (2\Delta_j(t) \le |z| < 1) \end{cases}$$

The function $Q_{j,t}^l$ is constructed similarly as in the case of (1). In this time, $K(P_{j,t}^l) \to 1$ as $t \to \infty$. The function $Q_{j,t}^l$ also satisfying $K(Q_{j,t}^l) \to 1$ as $t \to \infty$. Therefore, for sufficiently large t, $K(F_{j,t}^l) = \max\{K(P_{j,t}^l), K(Q_{j,t}^l)\} \to 1$ as $t \to \infty$.

Now, we can construct the quasiconformal mapping $F_t : Y_t \to Y_t'$ by gluing $\{F_{j,t}^l\}_{j=1,\cdots,k}^{l=1,2}$. For any mapping $F_{j,t}^l$, we can confirm the following.

1. Each h_j^l is homotopic to $(g_t' \circ f') \circ (g_t \circ f)^{-1}$ on $\{2\Delta_j(t) < |z| < 1\}$, since the mappings g_t, g_t' stretch the ring domains $A_j^l(0)$, $A_j'^l(0)$ along radial directions for any $0 \leq t \leq \infty$.
2. Each $Q_{j,t}^l$ satisfies $K(Q_{j,t}^l) \to 1$ as $t \to \infty$ and the domain $\{\Delta_j(t) < |z| < 2\Delta_j(t)\}$ has the constant modulus for any t. There is not a twist in this domain.
3. Each $P_{j,t}^l$ produces the twist of angle $\arg c_j^l$ in the domain $\{\delta_j(t) < |z| < \Delta_j(t)\}$ and satisfies $|\arg c_j^1 + \arg c_j^2| < 2\pi$, after the gluing of $A_j^1(t)$ and $A_j^2(t)$.

Therefore, for sufficiently large t, the mapping F_t **do not happen the Dehn twists** on $\{\delta_j(t) < |z| < 2\Delta_j(t)\}$ and is homotopic to $(g_t' \circ f') \circ (g_t \circ f)^{-1}$.

We conclude that

$$\lim_{t\to\infty} K(F_t) = \lim_{t\to\infty} \max_{j=1,\cdots,k,l=1,2} K(F^l_{j,t}) < \max_{j=1,\cdots,k} \left\{ M_j, \frac{1}{M_j} \right\} + \varepsilon.$$

□

Therefore, by this lemma, for any sufficiently large t, the inequality

$$\limsup_{t\to\infty} d_{T(X)}(r(t), r'(t)) \leq \lim_{t\to\infty} \frac{1}{2} \log K(F_t) < \frac{1}{2} \log \left(\max_{j=1,\cdots,k} \left\{ \frac{m'_j}{m_j}, \frac{m_j}{m'_j} \right\} + \varepsilon \right)$$

holds. Since ε is arbitrary, we are done.

The inequality

$$\liminf_{t\to\infty} d_{T(X)}(r(t), r'(t)) \geq \frac{1}{2}\log \max_{j=1,\cdots,k}\left\{\frac{m_j'}{m_j}, \frac{m_j}{m_j'}\right\} \qquad (1)$$

is obtained by the result of Walsh [Wal12] and an easy calculation. $\square$

Remark

In Walsh's theorem, even if $\mathbf{r}(\infty) \neq \mathbf{r}'(\infty)$, the same inequality (1) also holds. Moreover, if r, r' are **not** similar, then

$$\liminf_{t\to\infty} d_{T(X)}(r(t), r'(t)) = +\infty.$$

Corollary 2 ([Ama13])

For any two Jenkins-Strebel rays r, r', they are asymptotic if and only if r, r' are modularly equivalent and $r(\infty) = r'(\infty)$.

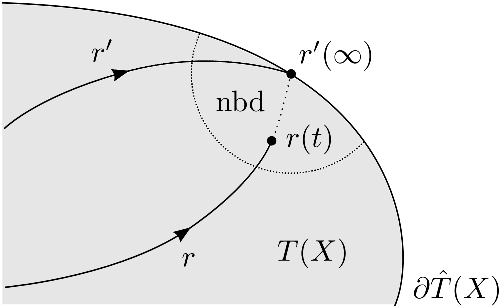
Proof.
Under the assumption of Theorem 1, if in addition the given rays r, r' are modularly equivalent, there is $\lambda > 0$ such that $m'_j = \lambda m_j$ for any $j = 1, \cdots, k$. Then, for $\alpha = -\frac{1}{2}\log\lambda$,

$$\begin{aligned} \lim_{t\to\infty} d_{T(X)}(r(t), r'(t+\alpha)) &= \frac{1}{2}\log \max_{j=1\cdots,k}\left\{\frac{e^{2\alpha}m'_j}{m_j}, \frac{m_j}{e^{2\alpha}m'_j}\right\} \\ &= \frac{1}{2}\log 1 = 0. \end{aligned}$$

This means that the rays r, r' are asymptotic.

Corollary 2

Conversely, if the rays r, r' are asymptotic, we can assume that $\lim_{t\to\infty} d_{T(X)}(r(t), r'(t)) = 0$ without loss of generality. The rays are similar and satisfy $m'_j = m_j$ for any $j = 1, \cdots, k$ by the previous remark and the inequality (1). Finally, we can obtain the equation $r(\infty) = r'(\infty)$. Indeed, for sufficiently large t, $r(t)$ is contained an arbitrary neighborhood of $r'(\infty)$. $\square$



Remark

Under the assumption of Theorem 1, the minimum of the limit value of the distance between the given rays $r(t)$, $r'(t)$ when we shift the initial points $r(0)$, $r'(0)$ is given by

$$\delta := \frac{1}{2}\left(\frac{1}{2}\log \max_{j=1,\cdots,k} \frac{m'_j}{m_j} + \frac{1}{2}\log \max_{j=1,\cdots,k} \frac{m_j}{m'_j}\right).$$

We notice that $\delta = 0$ if and only if r, r' are modularly equivalent.

By Theorem 1, we see that

$$\begin{aligned} \lim_{t\to\infty} d_{T(X)}(r(t), r'(t)) &= \frac{1}{2}\log \max_{j=1,\cdots,k}\left\{\frac{m'_j}{m_j}, \frac{m_j}{m'_j}\right\} \\ &\geq \delta. \end{aligned}$$

Proof.
The equality holds if we set

$$\beta = \frac{1}{4} \log \frac{\max\limits_{j=1,\cdots,k} \dfrac{m_j}{m'_j}}{\max\limits_{j=1,\cdots,k} \dfrac{m'_j}{m_j}}$$

and consider the rays $r(t)$, $r'(t+\beta)$. In this situation, we compute that

$$\max_{j=1,\cdots,k} \frac{e^{2\beta} m'_j}{m_j} = \max_{j=1,\cdots,k} \left\{ \frac{\sqrt{\max\limits_{j=1,\cdots,k} \dfrac{m_j}{m'_j}} \cdot m'_j}{\sqrt{\max\limits_{j=1,\cdots,k} \dfrac{m'_j}{m_j}} \cdot m_j} \right\} =$$

$$\sqrt{\max_{j=1,\cdots,k} \frac{m'_j}{m_j}} \cdot \sqrt{\max_{j=1,\cdots,k} \frac{m_j}{m'_j}} = \max_{j=1,\cdots,k} \frac{m_j}{e^{2\beta} m'_j}.$$

Therefore, we conclude that

$$\begin{aligned}
& \lim_{t\to\infty} d_{T(X)}(r(t), r'(t+\beta)) \\
= \quad & \frac{1}{2}\log \max_{j=1,\cdots,k}\left\{\frac{e^{2\beta}m'_j}{m_j}, \frac{m_j}{e^{2\beta}m'_j}\right\} \\
= \quad & \frac{1}{2}\left(\frac{1}{2}\log \max_{j=1,\cdots,k}\frac{m'_j}{m_j} + \frac{1}{2}\log \max_{j=1,\cdots,k}\frac{m_j}{m'_j}\right) \\
= \quad & \delta.
\end{aligned}$$

$\square$

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Uniformisation and description of a once-punctured annulus

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Background

Uniformisation Theorem

The universal covering space $\widetilde{X}$ of an arbitrary Riemann surface X is homeomorphic, by a conformal mapping φ, to either the Riemann sphere $\hat{\mathbb{C}}$, the complex plane $\mathbb{C}$ or the unit disk $\mathbb{D}$, and the fundamental group $\Pi_1(X)$ has a representation as a group G of conformal homeomorphisms of $\varphi(\widetilde{X})$.

Aims

- describe the once-punctured annulus in several different ways and give the connections between them
- consider the asymptotic behavior when the puncture is tending to the boundaries, or a boundary is shrinking to a point

Preliminary

- For a hyperbolic surface X, we choose the universal covering space $\widetilde{X}$ to be the upper half plane $\mathbb{H}$ or the unit disk $\mathbb{D}$.
- We identify a Möbius transformation

$$\phi(z) = \frac{az+b}{cz+d}, \ \ ad-bc=1, \ \ a,b,c,d \in \mathbb{C},$$

with the 2×2 complex matrix $\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{C})$ which is also denoted by ϕ, and define the *trace* of ϕ by $\mathrm{tr}\,\phi = \pm(a+d)$, so that $(\mathrm{tr}\,\phi)^2 = (a+d)^2$ is a conjugacy invariant. If ϕ is hyperbolic, the *translation length* of ϕ is defined by $T(\phi) = \inf_{z\in\mathbb{H}} \delta_{\mathbb{H}}(z, \phi(z))$ and it is known that $2\cosh(T(\phi)/2) = |\mathrm{tr}\,\phi|$.
- Let $\Omega \subseteq \mathbb{C}$ and Γ be a collection of finite unions of rectifiable curves in Ω. All of the metrics which are conformal with respect to the Euclidean metric can be defined in terms of a density $\varrho(z)|dz|$ where $\varrho(z)$ is a non-negative Borel measurable function on Ω. For $z = x+iy$, define $L(\gamma,\varrho) = \int_\gamma \varrho(z)|dz|$, $A(\Omega,\varrho) = \int_\Omega \varrho(z)^2 dxdy$, and $L(\Gamma,\varrho) = \inf_{\gamma\in\Gamma} L(\gamma,\varrho)$. Then the *extremal length* of Γ in Ω is given by

$$\lambda_\Omega(\Gamma) = \sup_\varrho \frac{L(\Gamma,\varrho)^2}{A(\Omega,\varrho)}.$$

Peripheral collars

Let γ be a simple closed geodesic on a hyperbolic surface X with hyperbolic length l. A *symmetric collar $C(\gamma)$ on X about γ of hyperbolic width w* is a doubly connected subdomain of X containing γ defined by $C(\gamma) = \{x \in X : \delta_X(x, \gamma) < w/2\}$, where δ_X is the hyperbolic distance on X. By a universal cover from $\mathbb{H}$ to X which lifts γ to the imaginary axis, a lift of the symmetric collar $C(\gamma)$ is the rigion in $\mathbb{H}$ given by $\{z : 1 < |z| < k^2, \frac{\pi}{2} - \theta < \arg z < \frac{\pi}{2} + \theta\}$, where $0 < \theta < \frac{\pi}{2}$, $\tan\theta = \sinh w$, and $k + k^{-1} = 2\cosh(l/2)$.

Collar Lemma (e.g. Keen, 1974)

With the same γ and θ, b as above, there is a symmetric collar $C(\gamma)$ on X about γ with the angular width θ satisfying

$$\tan\theta = \frac{2}{k - k^{-1}}.$$

If γ_1 and γ_2 are disjoint closed simple geodesics, the collars $C(\gamma_1)$ and $C(\gamma_2)$ are disjoint.

To obtain the maximal non-overlapped collar, we can extend one side of a symmetric collar about γ to the boundary, that means, the collar $C(\gamma)$ has a lift in the form $\widetilde{C}(\gamma) = \{z : 1 < |z| < k^2, \frac{\pi}{2} - \theta < \arg z < \pi\}$ in $\mathbb{H}$. We will refer to the collar $\widetilde{C}(\gamma)$ of such form as a *peripheral collar about γ with the angular width θ.*

Two free homotopy classes

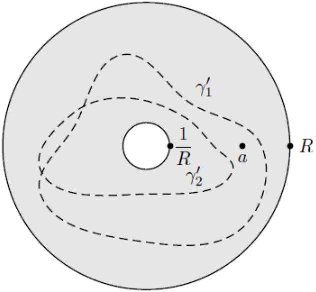


Figure: 1

After some rotations and scaler maps, we only need to consider the punctured annulus

$$A := \{z : 1/R < |z| < R\} \backslash \{a\},\ R > 1,\ 1/R < a < R.$$

We denote $B_1 := \{z : |z| = 1/R\}$, $B_2 := \{z : |z| = R\}$, and let C_1, C_2 be the free homotopy classes of the circles $\{z : |z| = r_1\}$, $\{z : |z| = r_2\}$ in A, respectively, where $a < r_1 < R$, $1/R < r_2 < a$. So C_1 separates $B_1 \cup \{a\}$ from B_2, C_2 separates $B_2 \cup \{a\}$ from B_1. Let γ_1, γ_2 be the hyperbolic geodesics in C_1, C_1.

Parameter pairs

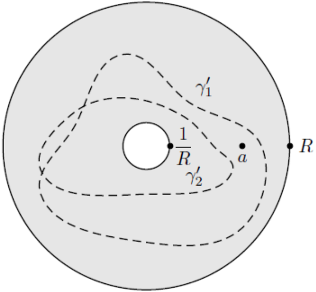


Figure: 1

The punctured annulus A can be described in the following ways.

- (k, r): from the generators of the covering group G
- (l_1, l_2): the hyperbolic lengths of geodesics γ_1 and γ_2
- (θ_1, θ_2): the angular widths of the maximal peripheral collars about γ_1 and γ_2
- (λ_1, λ_2): the extremal lengths of C_1 and C_2
- (R, a): the natural parameter pair

Fundamental domain

Lemma

Choose the covering group G of A to act on $\mathbb{H}$. Then there exist two real numbers k and r, $1 < r < k$, such that G is generated by a hyperbolic f and a parabolic g, where

$$f = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}, \quad g = \frac{1}{r-1} \begin{pmatrix} 2r & -(r+1) \\ r+1 & -2 \end{pmatrix}.$$

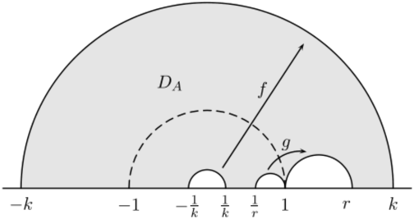


Figure: 2

Hyperbolic lengths

Theorem

In the punctured annulus A, for $l_1,\ l_2,\ k$ and r defined as above, we have

$$2\cosh(\frac{l_1}{2}) = k + \frac{1}{k}, \quad 2\cosh(\frac{l_2}{2}) = \frac{2}{r-1}(k - \frac{r}{k}).$$

The maximal peripheral collars

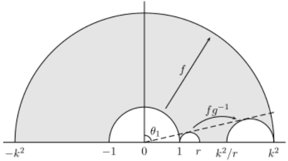
Theorem

Suppose that θ_1 and θ_2 are the angular widths of the maximal peripheral collars about γ_1, γ_2. Then we have

$$\cos\theta_1 = \frac{r-1}{r+1}, \quad \cos\theta_2 = \frac{t-1}{t+1} = \frac{2r(r+1)-2\delta}{\delta(r+1)-(r+1)^2}, \text{ where}$$

$$t = \frac{(r-1)(r+1+\delta)}{(r+3)\delta-(r+1)(3r+1)}, \quad \delta = k^2 + r - \sqrt{(k^2-1)(k^2-r^2)}$$

with k and r being the parameters of the generators of the covering group.



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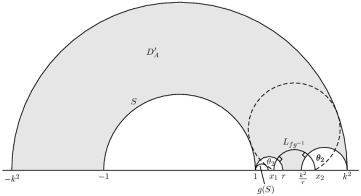

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Comparison with Collar Lemma

The collar defined by Collar Lemma is the minimum of the maximal peripheral collar supported by a hyperbolic transformation and it is smaller than the collar given above. We denote the angular widths of the collars defined by Collar Lemma about the axes of f and fg^{-1} by θ_1' and θ_2'. Then

$$\cos\theta_1' = \frac{k^2-1}{k^2+1}.$$

Then $\theta_1' < \theta_1$ by our theorem. From the symmetry we know $\theta_2' < \theta_2$ for fg^{-1}.

Collar Lemma (e.g. Keen, 1974)

With the same γ and θ, b as above, there is a symmetric collar $C(\gamma)$ on X about γ with the angular width θ satisfying

$$\tan\theta = \frac{2}{k-k^{-1}}.$$

If γ_1 and γ_2 are disjoint closed simple geodesics, the collars $C(\gamma_1)$ and $C(\gamma_2)$ are disjoint.

Comparison of C_1 and C_2

We can compare l_1 with l_2, θ_1 with θ_2 in terms of r and k. When $1 < r < 3$,

$$\begin{aligned} l_1 < l_2,\ \theta_1 > \theta_2, \qquad & \text{if } \ 1 < r < \sqrt{\frac{3r-1}{3-r}} < k, \\ l_1 = l_2,\ \theta_1 = \theta_2, \qquad & \text{if } \ 1 < r < \sqrt{\frac{3r-1}{3-r}} = k, \\ l_1 > l_2,\ \theta_1 < \theta_2, \qquad & \text{if } \ 1 < r < k < \sqrt{\frac{3r-1}{3-r}}; \end{aligned}$$

when $r \geq 3$, $l_1 > l_2$, $\theta_1 < \theta_2$. This corresponds that $\tan^2 \theta_i \sinh^2 \frac{l_i}{2} = 1$, $i = 1, 2$.

Corollary

In the punctured annulus $A = \{z : 1/R < |z| < R\}\backslash\{1\}$, the two parameters k and r satisfy $k^2 = \frac{3r-1}{3-r}$, and the covering group G of A is generated by

$$f(z) = \frac{3r-1}{3-r} z, \quad g(z) = \frac{2rz - (r+1)}{(r+1)z - 2},$$

where $1 < r < 3$ and r is related to R in some unknown way.

Hyperbolic lengths

Theorem

In the punctured annulus A, for $l_1,\ l_2,\ k$ and r defined as above, we have

$$2\cosh(\frac{l_1}{2}) = k + \frac{1}{k}, \quad 2\cosh(\frac{l_2}{2}) = \frac{2}{r-1}(k - \frac{r}{k}).$$

Hyperbolic lengths

Theorem

In the punctured annulus A, for $l_1,\ l_2,\ k$ and r defined as above, we have

$$2\cosh(\frac{l_1}{2}) = k + \frac{1}{k},\quad 2\cosh(\frac{l_2}{2}) = \frac{2}{r-1}(k - \frac{r}{k}).$$

Theorem

The parameters $l_1, l_2, \theta_1, \theta_2$ defined above satisfy

$$\cos\theta_1 = \frac{\sinh\frac{l_1}{2}}{\cosh\frac{l_1}{2} + \cosh\frac{l_2}{2}},\quad \cos\theta_2 = \frac{\sinh\frac{l_2}{2}}{\cosh\frac{l_1}{2} + \cosh\frac{l_2}{2}}.$$

Elliptic integrals and Jacobian elliptic functions

- Let
$$K(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}$$
with $0 < r < 1$ be *Legendre's complete elliptic integral of the first kind*. The parameter $r \in (0,1)$ is called the *modulus* and the *complementary modulus* of r is $r' = \sqrt{1-r^2}$, and denote $K'(r) = K(r') = K(\sqrt{1-r^2})$. We define the normalized quotient
$$\mu(r) = \frac{\pi}{2}\frac{K'(r)}{K(r)}$$
for $0 < r < 1$, then $\mu(r)$ is a strictly decreasing homeomorphism of the interval $(0,1)$ onto $(0,\infty)$ with limit values $\mu(0+) = \infty$, $\mu(1-) = 0$.

- Let
$$\mathrm{sn}(u,r) = \tau \quad \text{where} \quad u = \int_0^\tau \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}$$
be the *Jacobian elliptic sine function*. Two other functions can be then defined by $\mathrm{cn}(u,r) = \sqrt{1-\mathrm{sn}^2(u,r)}$, $\mathrm{dn}(u,r) = \sqrt{1-r^2\mathrm{sn}^2(u,r)}$.

Extremal lengths

Theorem

In the punctured annulus A, suppose that λ_1 and λ_2 are the extremal lengths of C_1 and C_2. Select a positive number q such that $\mu(q) = 4\log R$ and let $\mathcal{K} := K(q)$, $\mathcal{K}' := K'(q)$. Then

$$\lambda_1 = \frac{2\pi}{\mu(p_1)}, \quad \lambda_2 = \frac{2\pi}{\mu(p_2)},$$

where

$$p_1 = \frac{\sqrt{q}(\operatorname{dn} u_1 + 1)}{q + \operatorname{dn} u_1}, \quad p_2 = \frac{\sqrt{q}(\operatorname{dn} u_2 + 1)}{q + \operatorname{dn} u_2}$$

with

$$u_1 = \frac{2\mathcal{K}}{\pi}\log Ra, \quad u_2 = \frac{2\mathcal{K}}{\pi}\log\frac{R}{a},$$

and the Jacobian elliptic function dn *in p_1 and p_2 has the modulus $q' = \sqrt{1-q^2}$.*

Useful lemmas (1)

Lemma 1

For $0 < q < 1$ *let* $\mathcal{K} := K(q)$, $\mathcal{K}' := K'(q)$ *and select* $b = \exp(-\pi\mathcal{K}'/(4\mathcal{K}))$. *Then the conformal mappings* ω *and* σ *defined by*

$$\omega(z) = \sqrt{q}\,\mathrm{sn}\left(\frac{2i\mathcal{K}}{\pi}\mathrm{Log}\frac{z}{b} + \mathcal{K}, q\right), \quad \sigma(z) = \frac{z + \sqrt{q}}{\sqrt{q}z + 1}$$

are both unique up to rotations, where ω *takes the annulus* $b < |z| < 1$ *onto* $\mathbb{D} \setminus [-\sqrt{q}, \sqrt{q}]$, *and* σ *preserves* $\mathbb{D}$ *with* $\sigma(-1) = -1$, $\sigma(1) = 1$, $\sigma(-\sqrt{q}) = 0$.

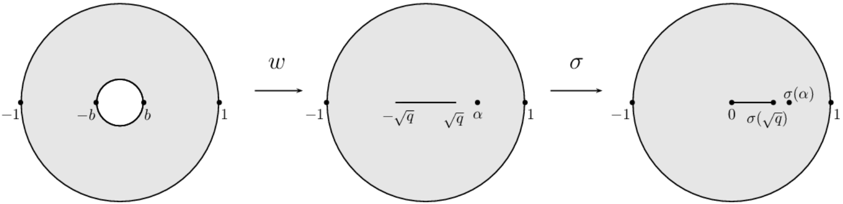


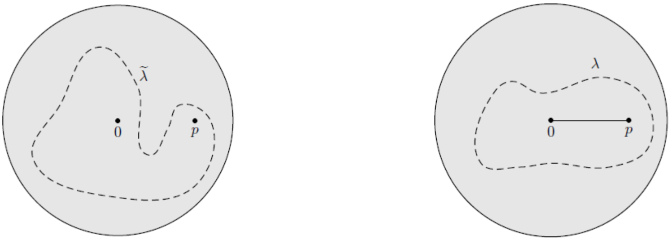
Figure: 5

Useful lemmas (2)

Lemma 2

Let $\widetilde{C}$ be the family of loops in $\mathbb{D}$ separating 0 and p from the unit circle $\partial\mathbb{D}$, $0 < p < 1$, and C be the family of loops in $\mathbb{D}$ separating the slit $(0, p)$ from $\partial\mathbb{D}$. Then the extremal lengths of $\widetilde{C}$ and C are

$$\lambda(\widetilde{C}) = \lambda(C) = \frac{2\pi}{\mu(p)}.$$



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Extremal cases

- R is fixed. When $a \to R$, $l_1 \to \infty$, and then $k \to +\infty$.
  When $a \to 1/R$, $l_2 \to \infty$, and then $r \to 1$.
- a is fixed and $R \to +\infty$. Then $k \to r \to 1$, so that $\lim_{k,\, r\to 1} \cosh(l_1/2) = 1$ and $\lim_{k,\, r\to 1} \cosh(l_2/2) = 1$.
- $a = 1$ and $R \to 1$. Then A is becoming a punctured domain shown in the figure below, which is conformally equivalent to an endless punctured stripe in the complex plane. So $k \to +\infty$ and $r \to 3$.

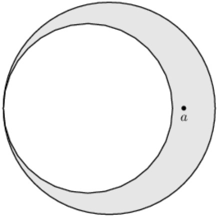

Figure: 8

Another model

We have a different uniformisation if taking the once-punctured annulus model as $A_1 = \{b^2 < |z| < 1\} \backslash \{x\}$, $0 < b^2 < x < 1$. With the same definitions of λ_1 and λ_2, when the puncture x is fixed and $b \to 0$, we have

$$\lambda_1 \to \frac{2\pi}{\mu(x)}, \quad \lambda_2 \to 0.$$

Lemma 2′

Let $\widetilde{C}$ be the family of loops in $\mathbb{D}$ separating 0 and p from the unit circle $\partial\mathbb{D}$, $0 < p < 1$. Then the extremal length of $\widetilde{C}$ is

$$\lambda(\widetilde{C}) = \frac{2\pi}{\mu(p)}.$$

Reference

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Thank you for your attention!