Non-linearizability of cubic-perturbed analytic germs at irrationally indifferent fixed points

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Abstract

In this paper, we consider the non-linearizability of analytic germs with irrationally indifferent fixed points. Assume that an analytic germ f has an irrationally indifferent fixed point at the origin and its multiplier satisfies the Tortrat condition, which is a generalization of the Cremer condition, of degree three. Then a cubic perturbation of f is non-linearizable at the origin if this perturbation is large enough.

1 Introduction

For an analytic germ f at the origin with a fixed point $f(0) = 0$, we call $\lambda := f'(0)$ the multiplier of f at the origin. If $\lambda = \exp(2\pi i\alpha)$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the origin is said to be an irrationally indifferent fixed point of f. The linearization problem of f at the origin is whether there exists a holomorphic local change of coordinate $z = h(w)$ with $h(0) = 0$ and $h'(0) \neq 0$ which conjugates f to the linear map $w \mapsto \lambda w$. If such h exists, f is called (analytically) *linearizable* at the origin.

In this paper, we consider the non-linearizability of analytic germs at irrationally indifferent fixed points.

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From now on, we always assume $\lambda := \exp(2\pi i\alpha)$ for $\alpha \in (\mathbb{R}\setminus\mathbb{Q}) \cap (0, +1)$. For a precise analysis of α, we consider the continued fraction expansion of α:

$$\alpha = \frac{1}{a_1 +} \frac{1}{a_2 + \cdots}$$

where the a_j is uniquely determined positive integers and also consider the n-th convergent of it:

$$\frac{p_n}{q_n} = \frac{1}{a_1 +} \frac{1}{a_2 + \cdots +} \frac{1}{a_n}$$

where $(p_n, q_n) = 1$ and $p_n, q_n > 0$. We set for $d \geq 2$,

$$\mathcal{R}_{\lambda,d} := \{f;\ \text{rational map of degree } d,\ f(0) = 0 \text{ and } f'(0) = \lambda\} \text{ and} \tag{1}$$

$$\mathcal{P}_{\lambda,d} := \{f \in \mathcal{R}_{\lambda,d}; f \text{ is a polynomial}\}. \tag{2}$$

Brjuno showed in [2] that if α satisfies the Brjuno condition

$$\sum_n \frac{\log q_{n+1}}{q_n} < +\infty, \tag{$\mathcal{B}$}$$

then any analytic germ f at the origin with $f(0) = 0$ and $f'(0) = \lambda$ is linearizable at the origin. We call α satisfying $\mathcal{B}$ a Brjuno number. Conversely, Yoccoz showed in [9] that the quadratic polynomial $Q(z) = \exp(2\pi i\alpha)z + z^2$ is non-linearizable at the origin if α is not a Brjuno number. Therefore Q is linearizable at the origin if and only if α is Brjuno number. However it remains open whether there exists a non-linear rational map linearizable at an irrationally indifferent fixed point with a non-Brjuno multiplier.

On the other hand, there is a well-known sufficient condition to obtain the non-linearizability of any element of $\mathcal{R}_{\lambda,d}$. Under the Cremer condition

$$\sup_n \frac{\log q_{n+1}}{d^{q_n}} = +\infty, \tag{$\mathbf{Cr}_d$}$$

any rational map $f \in \mathcal{R}_{\lambda,d}$ is non-linearizable at the origin (see [3]). Furthermore, Tortrat showed in [8] strictly weaker condition than $\mathbf{Cr}_d$

$$\limsup_{n\to+\infty} \frac{\log q_{n+1}}{d^{q_n}} > 0 \tag{$\mathbf{T}_d$}$$

implies that any polynomial $f \in \mathcal{P}_{\lambda,d}$ is non-linearizable at the origin. However these conditions depend on $d \geq 2$. In fact, $\mathbf{Cr}_{d+1}$ (resp. $\mathbf{T}_{d+1}$) is strictly stronger than $\mathbf{Cr}_d$ (resp. $\mathbf{T}_d$). Can we obtain a weaker non-linearizability condition independent of d?

The main theorem in this paper is the following.

Main Theorem. *Let* $\lambda = \exp(2\pi i\alpha)$ *for* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *and* f *be an analytic germ with* $f(0) = 0$ *and* $f'(0) = \lambda$*. There exists an absolute constant* $c_1 > 0$ *such that if* α *satisfies* $\mathbf{T}_3$*, then a cubic-perturbation* $f(z) + bz^3$ *of* f *is non-linearizable at the origin for any complex number* b *with* $|b| > c_1/(r_f^2)$*. Here* $r_f > 0$ *is the radius of univalency of* f *at the origin, i.e.*

$$r_f := \sup\{r > 0; f \textit{ is injective on } |z| < r\}.$$

In Section 2, we shall prove Main Theorem.

2 Proof of Main Theorem

Let S be the set of holomorphic functions univalent on $\mathbb{D}$ with $f(0) = 0$ and $|f'(0)| = 1$ and let S_λ be the set of elements f of S which satisfies $f'(0) = \lambda$. We set $\mathbb{D}_r := \{z; |z| < r\}$ for $r > 0$.

For $f \in S$, we define a *cubic-perturbation* of f:

$$f_{a,b}(z) := a^{-1}f(az) + bz^3$$

where $0 < |a| \le 1$ and $b \in \mathbb{C}$. Furthermore we set $W := \mathbb{D}_{15/2}$, $\widetilde{W} := \mathbb{D}_{1/3} \cap f_{a,b}^{-1}(W)$ and a constant $c_0 := 225$. Recall that a triplet $(\widetilde{U}, U, f)$ is called a *cubic-like* map if $\widetilde{U}$ and U are simply connected proper subdomains of $\mathbb{C}$, $\widetilde{U}$ is relatively compact in U and $f : \widetilde{U} \to U$ is a proper holomorphic map of degree three.

The following is a special case of Lemma 2.1 in [6].

Lemma 2.1. *If* $|b| \ge c_0$*, the triplet* $(\widetilde{W}, W, f_{a,b})$ *is a cubic-like map.*

We take a smooth function $\eta : \mathbb{R} \to [0, 1]$ which is identically 1 on $(-\infty, 1/3]$ and 0 on $[15/2, +\infty)$ and define

$$\tilde{f}_{a,b}(z) := \eta(|z|)f_{a,b}(z) + (1 - \eta(|z|))(\lambda z + bz^3)$$

for $f \in S_\lambda$, $0 < |a| < 2/15$ and $|b| \ge c_0$. Then $\tilde{f}_{a,b} : \mathbb{C} \to \mathbb{C}$ is in C^∞, coincides with $f_{a,b}$ on $\overline{\mathbb{D}_{1/3}}$ and with $\lambda z + bz^3$ on $\mathbb{C} \setminus \mathbb{D}_{15/2}$. Moreover it converges to $\lambda z + bz^3$ in C^∞-topology on $\mathbb{C}$ if a tends to 0, and this convergence is uniform in $f \in S_\lambda$ and $|b| \ge c_0$.

From now on we assume $|b| = c_0$. We can conclude the following.

Lemma 2.2. *There exist an* $a_0 \in (0, 2/15]$ *and a continuous function* $k : [0, a_0] \to [0, 1)$ *with* $k(0) = 0$ *such that for* $f \in S_\lambda$, $|b| = c_0$ *and* $0 < |a| < a_0$*, the map* $\tilde{f}_{a,b}$ *is a branched covering map of* $\mathbb{C}$ *of degree three and it satisfies*

$$\left| \frac{\bar{\partial} \tilde{f}_{a,b}(z)}{\partial \tilde{f}_{a,b}(z)} \right| \le k(|a|) \quad (1/3 \le |z| \le 15/2).$$

We identify a Beltrami coefficient on an open set U with a function $\mu \in L^\infty(U)$ such that $\|\mu\|_\infty < 1$. For a C^1-function $f : U \to V$ and a Beltrami coefficient μ on V, we define the pullback $f^*\mu$ of μ on U by

$$(f^*\mu)(z) = \frac{\overline{\partial f(z)}\mu(f(z)) + \bar\partial f(z)}{\overline{\bar\partial f(z)}\mu(f(z)) + \partial f(z)}.$$

For $f \in S_\lambda$, $|b| = c_0$ and $0 < |a| < a_0$, there exists a unique Beltrami coefficient $\mu = \mu_{f,a,b}$ on $\mathbb{C}$ which is invariant under the pullback by $\tilde f_{a,b}$ and agrees with $\frac{\bar\partial \tilde f_{a,b}}{\partial \tilde f_{a,b}}$ on $1/3 \le |z| \le 15/2$ and is 0 on $(\mathbb{C} - W) \cup \bigcap_{n\ge 0} f_{a,b}^{-n}(\widetilde{W})$. Since $\operatorname{supp}\mu \subset W$ and $\|\mu\|_\infty \le k(a) < 1$, by the Ahlfors-Bers theorem [1], there exists a unique quasiconformal homeomorphism $\phi = \phi_{f,a,b}$ of $\mathbb{C}$ onto itself which satisfies the following:

(i) for a.e.$z \in \mathbb{C}$, $\bar\partial\phi(z) = \mu(z)\partial\phi(z)$,

(ii) $\phi(0) = 0$ and

(iii) $\phi(z) - z$ is bounded on $\mathbb{C}$.

Lemma 2.3 (cf. [4]). *There exists an $A \in \mathbb{C}$ such that $\phi \circ \tilde f_{a,b} \circ \phi^{-1}(z) = \lambda z + Az^2 + bz^3$.*

Proof. Since $\mu(\phi \circ f_{a,b}) = \mu(\phi)$, the map $\phi \circ \tilde f_{a,b} \circ \phi^{-1} : \mathbb{C} \to \mathbb{C}$ is holomorphic and fixes the origin. Thus it is a branched holomorphic covering map of $\mathbb{C}$ of degree three fixing the origin so we can write $\phi \circ \tilde f_{a,b} \circ \phi^{-1}(z) = \lambda' z + Az^2 + b'z^3$. The multiplier of a fixed point of a holomorphic germ is topologically invariant if its modulus is equal to 1, so we have $\lambda' = \lambda$ (cf. [5], see also [7]). On a neighborhood of the point at infinity, $\phi_{f,a,b}(z) = z +$ (lower terms) and $\tilde f_{a,b}(z) = \lambda z + bz^3$. On the other hand, $\phi(\tilde f_{a,b}(z)) = \lambda\phi(z) + A(\phi(z))^2 + b'(\phi(z))^3$. Thus $\phi(\lambda z + bz^3) - (\lambda z + bz^3) = (b' - b)z^3 +$ (lower terms) when $|z|$ is sufficiently large. Since this quantity remains bounded as $|z| \to +\infty$ by (iii), it is necessary that $b' - b = 0$. □

We set $c_1 := c_0/(a_0^2)$. The following is equivalent to Main Theorem.

Proposition. *Let $\lambda = \exp(2\pi i\alpha)$ ($\alpha \in \mathbb{R} \setminus \mathbb{Q}$) and $f \in S_\lambda$. If $\alpha \in \mathbf{T}_3$, then $f_{1,b}$ is non-linearizable at the origin for any $|b| > c_1$.*

Proof. Noting that $\tilde f_{a,b} \equiv f_{a,b}$ on $\mathbb{D}_{1/3}$, we can see $f_{a,b}$ is non-linearizable at the origin if and only if the cubic polynomial $\phi \circ \tilde f_{a,b} \circ \phi^{-1} \in \mathcal{P}_{\lambda,d}$ is so. If $\alpha \in \mathbf{T}_3$, it is non-linearizable by the Tortrat Theorem. On the other hand, $af_{a,b}(z/a) = f_{1,b/(a^2)}$ and for any $|b_0| > c_1$, there exist $a \in \{0 < |a| < a_0\}$ and $b \in \{|b| = c_0\}$ such that $b/(a^2) = b_0$. □

Consequently the proof of Main Theorem is completed. □

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References

[1] Ahlfors L V and Bers L 1960 Riemann's mapping theorem for variable metrics *Ann. of Math.* **72** 385–404.

[2] Brjuno A D 1971 Analytical form of differential equations *Trans. Moscow Math. Soc.* **25** 199–239.

[3] Cremer H 1928 Zum Zentrumproblem *Math. Ann.* **98** 151–163.

[4] Douady A and Hubbard J H 1985 On the dynamics of polynomial like mappings *Ann. Sc. E.N.S.,4ème série* **18** 287–343.

[5] Naishul V I 1983 Topological invariants of analytic and area preserving mappings and their application to analytic differential equations in $\mathbb{C}^2$ and $\mathbb{CP}^2$ *Trans. Moscow Math. Soc.* **42** 239–250.

[6] Okuyama Y 1999 Non-Linearizability of Polynomials at Irrationally Indifferent Fixed Points *Kodai Math. J.* **22** 56–65.

[7] Pérez-Marco R 1997 Fixed points and circle maps *Acta Math.* **179** 243–294.

[8] Tortrat P 1987 Aspects potentialistes de l'itération des polynomials *Séminaire de Théorie du Potentiel, Paris, No. 8, Lecture Note in Math.* **1235** 195–209.

[9] Yoccoz J-C 1996 Thèorém de Siegel, nombres de Bruno et polynômes quadratiques *Astérisque* **231** 3–88.