AN EXPLICIT BOUND FOR UNIFORM PERFECTNESS OF THE JULIA SETS OF RATIONAL MAPS

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Abstract. A compact set C in the Riemann sphere is called uniformly perfect if the moduli of annuli separating C are bounded. Mañé-da Rocha and Hinkkanen showed independently uniform perfectness of the Julia sets of rational maps of degree ≥ 2, but they presented no explicit bounds for uniform perfectness. In this note, we shall provide such an explicit bound and, as a result, we give another proof of uniform perfectness of the Julia sets. As an application, we refer to a lower estimate of the Hausdorff dimension of the Julia sets. We also give a concrete bound for the family of quadratic polynomials $f_c(z) = z^2 + c$ in terms of the parameter c.

1. Introduction

Let C be a closed set in the Riemann sphere $\widehat{\mathbb{C}}$ with $\#C \geq 3$ and Ω its complement. We say C is *uniformly perfect* if there exists a constant $0 < c < 1$ such that $C \cap \{z \in \mathbb{C}; cr < |z-a| < r\} \neq \emptyset$ for any $a \in C$ and $0 < r < \operatorname{diam}(C)$, where diam denotes the Euclidean diameter.

The notion of uniform perfectness first appeared in Beardon-Pommerenke [3], and was investigated more deeply by Pommerenke [12] and [13], and afterwards by many authors (see [15] and its references). By definition, the sets with some kind of self-similarities are expected to have uniform perfectness. In fact, the limit sets are known to be uniformly perfect for a wide class of Kleinian groups (cf. Sugawa [16]). On the other hand, Pommerenke [13] first showed the uniform perfectness of the Julia sets of hyperbolic rational maps. Later, Mañé-da Rocha [11] and Hinkkanen [7] proved in the case of general rational maps of degree ≥ 2, independently. For a simpler proof, see the textbook [4] by Carleson and Gamelin. But, their proofs are done by contradiction, thus no explicit bounds for uniform perfectness are given. In this note, we shall present such an explicit bound and also exhibit some applications of this result. Our proof employs the hyperbolic geometry, and hence is different from ones of the above authors. As Hinkkanen remarked in [7], we should note that there exists an entire function whose Julia set is not uniformly perfect. We also note that Hinkkanen and Martin [8] recently proved uniform perfectness for the Julia sets of finitely generated rational semigroups.

We will state the main result in the next section as well as fundamental definitions and notation. In Section 3, we shall discuss the connection between branched coverings and uniform perfectness, which will be a key to prove our main theorem. Section 4 is devoted to prove our main theorem, and we make an essential use of Sullivan's No Wandering Domains Theorem. We shall give applications of the main result to estimations of Hausdorff

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dimension, the (logarithmic) capacity density of the Julia set and the Poincaré metric of the Fatou set in Section 5. In the last section, as a special case, we investigate the quadratic family of polynomials $f_c(z) = z^2 + c$ with $c \in \mathbb{C}$ outside the Mandelbrot set. We have rather satisfactory result in the case $c < -2$. For a general c, we shall explain a method using a holomorphic motion of the Julia set.

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2. Main result

Let C be a closed set in the Riemann sphere containing at least three points and Ω its complement. We denote by $\mathcal{A}_\Omega$ and $\mathcal{A}^\circ_\Omega$ the sets of annuli and round annuli, respectively, in Ω separating C, where annuli mean doubly connected domains and round annuli do special annuli of the form $\{z \in \mathbb{C}; r_1 < |z-a| < r_2\}$ for some $a \in \mathbb{C}$ and $0 \le r_1 < r_2 \le \infty$, and we say that an annulus A separates C if $A \cap C = \emptyset$ and if each component of $\widehat{\mathbb{C}} \setminus A$ intersects C. The modulus $m(A)$ of an annulus A is the number m such that A is conformally equivalent to the round annulus $\{z; 1 < |z| < e^m\}$. We set

$$M_\Omega = \sup_{A \in \mathcal{A}_\Omega} m(A), \quad M^\circ_\Omega = \sup_{A \in \mathcal{A}^\circ_\Omega} m(A),$$

and call them the modulus and the round modulus of Ω. (If $\mathcal{A}_\Omega$ and/or $\mathcal{A}^\circ_\Omega$ is empty, then we define $M_\Omega = 0$ and/or $M^\circ_\Omega = 0$, respectively.) For these constants, it is known that $\frac{1}{2}M_\Omega - 1.7332\cdots \le M^\circ_\Omega \le M_\Omega$, and that if $\Omega \subset \mathbb{C}$, $M_\Omega - 2.8911\cdots \le M^\circ_\Omega \le M_\Omega$ (cf. [15]).

It is easily verified that C is uniformly perfect if and only if $M^\circ_\Omega < \infty$, equivalently $M_\Omega < \infty$.

Next, for the later use, we consider quantities determined by the hyperbolic geometry of Ω. Let C be a closed set in the Riemann sphere with $\#C \ge 3$ and $\Omega = \widehat{\mathbb{C}} \setminus C$. Then each component D of Ω is hyperbolic, i.e., there exists a holomorphic universal covering map $p : \mathbb{H} \to D$ from the upper half plane onto D. Thus D can be regarded as the quotient space $\mathbb{H}/\Gamma$ of $\mathbb{H}$ over the covering transformation group $\Gamma = \{\gamma \in \mathrm{PSL}(2,\mathbb{R}); p \circ \gamma = p\}$. Since the hyperbolic (or Poincaré) metric $\rho_{\mathbb{H}}(z)|dz| = \frac{|dz|}{2\mathrm{Im} z}$ is invariant under the action of $\mathrm{PSL}(2,\mathbb{R})$, D inherits the hyperbolic metric $\rho_D(z)|dz|$ so that $p : \mathbb{H} \to D$ is a local isometry with respect to the hyperbolic metric, i.e., $\rho_{\mathbb{H}} = p^*\rho_D$. Therefore, we can define the hyperbolic metric ρ_Ω of Ω componentwise.

The hyperbolic distance $d_\Omega(z_0, z_1)$ of a pair of points z_0, z_1 in the same component of Ω can be defined by $\inf_\alpha \int_\alpha \rho_\Omega(z)|dz|$, where the infimum is taken over all paths joining z_0 to z_1 in Ω. We also difine $d_\Omega(z_0, z_1) = +\infty$ if z_0 and z_1 do not belong to the same component of Ω. For $z \in \Omega$ we denote by $\iota_\Omega(z)$ the injectivity radius of Ω at z, that is, $\iota_\Omega(z)$ is the maximal radius r so that the hyperbolic disk $\{w \in \Omega; d_\Omega(z, w) < r\}$ is simply connected.

Let $\mathcal{C}_\Omega$ denote the set of free homotopy classes of non-trivial loops in Ω, where a loop (=closed curve) is called *non-trivial* if this is not null-homotopic (=contractible) in Ω. For a loop α in Ω, we define the length of it by

$$\ell_\Omega(\alpha) = \int_\alpha \rho_\Omega(z)|dz|,$$

and for the free homotopy class $[\alpha]$ represented by α, we define

$$\ell_\Omega[\alpha] = \inf_{\alpha' \in [\alpha]} \ell_\Omega(\alpha').$$

Finally, we set

$$L_\Omega = \inf_{[\alpha]\in \mathcal{C}_\Omega} \ell_\Omega[\alpha].$$

(If $\mathcal{C}_\Omega$ is an empty set, we set $L_\Omega = +\infty$.) We remark that the injectivity radius $\iota_\Omega(z)$ is equal to half the infimum of lengths of non-trivial loops in Ω passing through z. In particular, L_Ω is nothing other than twice the (global) injectivity radius $\inf_{z\in\Omega}\iota_\Omega(z)$ of Ω.

Concerning the constant L_Ω, the following estimate is fundamental.

Proposition 2.1 ([15]).

$$L_\Omega \le \frac{\pi^2}{M_\Omega} \le \min\{L_\Omega e^{L_\Omega}, \frac{L_\Omega^2}{2}\coth^2(L_\Omega/2)\}.$$

In particular, $M_\Omega < \infty$ *if and only if* $L_\Omega > 0$.

In order to estimate M_Ω from above, by this proposition, we have only to do L_Ω from below. Now let us state the main theorem. For basic definitions and results about the complex dyamics of the rational maps, we refer to the textbook [2] by Beardon as a general reference.

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \ge 2$. We denote by $J = J_f$ and $\Omega = \Omega_f$ the Julia set and the Fatou set of f, respectively. (In other words, Ω_f is the domain of normality of the iteration family $\{f^n\}_{n=1,2,\ldots}$ of f and $J_f = \widehat{\mathbb{C}} \setminus \Omega_f$.) Note that Ω_f is completely invariant under f, precisely, $f(\Omega_f) = \Omega_f = f^{-1}(\Omega_f)$.

We denote by $\mathrm{Crit}(f)$ the set of critical points of f in the Fatou set Ω_f and let $U_1, \cdots, U_s$ be the complete list of the components of Ω_f which contains at least one critical point of f and is not simply connected. And we set $W_j = f(U_j)$ and $C_j = \mathrm{Crit}(f) \cap U_j$ for $j = 1, \cdots, s$. Note here that $\#\mathrm{Crit}(f) \le 2d-2$, so $s \le 2d-2$. Now we introduce two kinds of curve family: $\mathcal{S}(v_1, v_2)$ and $\mathcal{T}(v)$, for $v_1, v_2, v \in f(C_j)$ with $v_1 \ne v_2$. Let $\mathcal{S}(v_1, v_2)$ and $\mathcal{T}(v)$ consist of the loops $\beta : \mathrm{S}^1 \to W_j$, where S^1 denotes the unit circle $\{z \in \mathbb{C}; |z| = 1\}$, satisfying the conditions (a), (b), (c) and (a), (b'), (c), respectively, in the following:

(a) β is contractible in W_j.
(b) β passes through v_1 and v_2.
(b') β passes through v essentially two times, at least.
(c) There exists a non-trivial loop α in U_j such that $f_*(\alpha) = \beta$.

More precisely, the condition (b') says that there exist distinct points ζ_0 and ζ_1 in S^1 with $\beta(\zeta_0) = \beta(\zeta_1) = v$ such that the restrictions $\beta|_{\overline{I_1}}$ and $\beta|_{\overline{I_2}}$ of the loop β are both non-trivial closed curves in W_j, where I_1 and I_2 are the connected component of $\mathrm{S}^1 \setminus \{\zeta_0, \zeta_1\}$. In particular, $\mathcal{T}(v)$ is empty if W_j is simply connected. And we set

$$a_j(v_1, v_2) = \inf_{\beta\in\mathcal{S}(v_1,v_2)} \ell_\Omega(\beta), \quad b_j(v) = \inf_{\beta\in\mathcal{T}(v)} \ell_\Omega(\beta) \text{ and}$$

$$a_j = \min_{v_1,v_2\in f(C_j), v_1\ne v_2} a_j(v_1, v_2), \quad b_j = \min_{v\in f(C_j)} b_j(v),$$

where we set $a_j = +\infty$ if $\#f(C_j) = 1$.

Finally, let $A_1, \cdots, A_t$ be all of the cycles of Herman rings of f. We note here that, by Shishikura's theorem, $0 \leq t \leq d-2$, in particular, if $d = 2$ there are no Herman rings. And, since the Julia set has no isolated points, the Herman rings have finite moduli, so $L_{A_k} > 0$ for all k.

Now we are ready to state our main theorem.

Theorem 2.2 (Main Thoerem). *For an arbitrary rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$, the following holds.*

$$L_{\Omega_f} \geq \min\{a_1, \cdots, a_s, b_1, \cdots, b_s, L_{A_1}, \cdots, L_{A_t}\}.$$

The proof of this theorem will be given in Section 4.

For any $\beta \in \mathcal{S}(v_1, v_2)$, it is clear by definition that $\ell_\Omega(\beta) \geq 2d_\Omega(v_1, v_2)$. Similarly, for $\beta \in \mathcal{T}(v)$, we have $\ell_\Omega(\beta) \geq 4\iota_\Omega(v)$. Thus, we conclude that $a_j(v_1, v_2) \geq 2d_\Omega(v_1, v_2)$ and $b_j(v) \geq 4\iota_\Omega(v)$ and hence have the following

Corollary 2.3. *Under the same situation as the Main Theorem, it follows that*

$$L_{\Omega_f} \geq \min\{C_1, C_2, C_3\}(> 0),$$

where

$$\begin{aligned} C_1 &= \min_{v_1 \neq v_2 \in f(\mathrm{Crit}(f))} 2d_{\Omega_f}(v_1, v_2), \\ C_2 &= \min_{v \in f(\mathrm{Crit}(f))} 4\iota_{\Omega_f}(v), \quad \textit{and} \\ C_3 &= \min_{k=1,\cdots,t} L_{A_k}. \end{aligned}$$

In particular, the Julia set J_f is uniformly perfect.

Remark 1. As is well-known, any polynomial has no Herman rings. In general, if there exsits a Herman ring A, it is known that the boundary of A is contained in the closure of forward orbits of the critical points of f. Therefore, if each critical point of f is (pre)periodic or contained in a (super)attracting or parabolic basin, then we can conclude that f has no Herman rings. We also note that a cycle of (super)attracting or parabolic components always contains a critical point, thus a component of it appears as a member of the list $U_1, \cdots, U_s$.

Remark 2. Let B_1 and B_2 be connected components of a cycle of Herman rings A_j. Then $B_2 = f^l(B_1)$ for some $l \in \mathbb{N}$. Since $f^m : B_1 \to B_1$ is known to be analytically conjugate to an irrational rotation of a round annulus, where m is the period of A_j, we can see that $f^l : B_1 \to B_2$ is biholomorphic. Hence, L_{A_j} is equal to the hyperbolic length of the core curve of any component of A_j.

Remark 3. A pair of critical values v_1, v_2 can accidentally be very close to each other in Ω, i.e., $d_\Omega(v_1, v_2)$ is very small, while $a_j(v_1, v_2)$ is not so small. (The phenomenon $\mathcal{T}(v) \neq \emptyset$ can be considered as a limiting case of the above situation.) So, the formulation in the above corollary does not always provide a good estimate for uniform perfectness.

3. Branched coverings and uniform perfectness

In this section, we shall investigate the connection between branched coverings and uniform perfectness. Let $f : U \to W$ be a holomorphic (possibly branched) covering map from a (connected) hyperbolic Riemann surface U onto another W. Precisely speaking, for each point $w \in W$ there exists an open neighborhood V of w satisfying the condtion: For each component $\tilde{V}$ of $f^{-1}(V)$ there exist a natural number $n \geq 1$ and conformal homeomorphisms $\varphi : \tilde{V} \to \varDelta_r$ and $\psi : V \to \varDelta_{r^n}$ with $\psi(w) = 0$ such that $\psi \circ f \circ \varphi^{-1}(\zeta) = \zeta^n$, where $\varDelta_r$ denotes the disk $\{|\zeta| < r\}$.

When a loop α is freely homotopic to another α' in U, $f_*\alpha := f \circ \alpha$ is freely homotopic to $f_*\alpha'$. Therefore, the natural homomorphism $f_* : \mathcal{C}_U \to \mathcal{C}_W$ can be defined by $f_*[\alpha] = [f_*\alpha]$.

First suppose that f is unbranched, then by the homotopy lifting property we can see that the induced map f_* is injective. And moreover $\ell_U[\alpha] = \ell_W[f_*\alpha]$ because f is a local isometry, therefore we have the next

Proposition 3.1. *If $f : U \to W$ is an unbranched holomorphic covering map, then $L_U \geq L_W$.*

In the case when f is branched, we need more efforts to estimate L_U from below. In fact, for any finitely connected planar Jordan domain U, it is known that there exists a branched holomorphic covering map from U onto the unit disk (so-called the Ahlfors map), thus L_U cannot be estimated from below by only the data of W (in this case, $L_W = +\infty$).

Let $\mathrm{Crit}(f)$ be the set of critical points of f and for $v_1, v_2, v \in f(\mathrm{Crit}(f))$ with $v_1 \neq v_2$ define the curve families $\mathcal{S}(v_1, v_2)$ and $\mathcal{T}(v)$ by the same way as in the previous section. And we set

$$a(v_1, v_2) = \inf_{\beta \in \mathcal{S}(v_1, v_2)} \ell_W(\beta), \quad b(v) = \inf_{\beta \in \mathcal{T}(v)} \ell_W(\beta), \text{ and}$$
$$a = \inf_{v_1 \neq v_2 \in f(\mathrm{Crit}(f))} a(v_1, v_2), \quad b = \inf_{v \in f(\mathrm{Crit}(f))} b(v).$$

Then, the following lemma is a key step to our proof of the Main Theorem.

Lemma 3.2. *For a non-trivial loop α in U such that $\beta = f_*\alpha$ is contractible in W, it follows that*

$$\ell_U[\alpha] \geq \min\{a, b\}.$$

Proof. First we show that $\ell_U[\alpha] > 0$. In fact, if $\ell_U[\alpha] = 0$ then α surrounds a puncture of U, in other words, there exists a holomorphic injection $g : \varDelta^* = \varDelta \setminus \{0\} \to U$ such that α is freely homotopic to ε^n for some integer $n \neq 0$, where $\varepsilon = g(\{|\zeta| = 1/2\})$. As is easily seen, $f(g(\varDelta^*))$ is a neighborhood of a puncture of W and thus β is freely homotopic to non-zero multiple of a simple loop around the puncture in W. On the other hand, β is contractible in W, therefore W must be conformally equivalent to the complex plane $\mathbb{C}$, but this is impossible because W is hyperbolic. Hence we have $\ell_U[\alpha] > 0$. In particulur, we see that $\ell_U[\alpha] = \ell_U(\alpha_0)$ for the closed geodesic α_0 freely homotopic to α in U. So, for the proof, it suffices to that $\ell_U(\alpha) \leq \min\{a, b\}$ in the case α is a smooth curve. Approximating α by another smooth curve if necessary, we may further assume that α does not pass any critical point. Here, we should observe $\ell_U(\alpha) \geq \ell_W(\beta)$ by the Schwarz-Pick lemma: $f^*\rho_W \leq \rho_U$.

Let $p : \Delta \to W$ be a holomorphic universal covering map of W from the unit disk Δ and set $C = p^{-1}(f(\mathrm{Crit}(f)))$. Since β is contractible, a lift $\tilde{\beta} : \mathrm{S}^1 \to \Delta$ of β via p is closed. Let K be the holomorphically convex hull of $\tilde{\beta}(\mathrm{S}^1)$ in Δ. In other words, $K = \Delta \setminus D_0$, where D_0 is the relatively non-compact component of $\Delta \setminus \tilde{\beta}(\mathrm{S}^1)$ in Δ.

Now we show that $\#(K \cap C) \geq 2$. If $K \cap C$ is an empty set, then it is clear that β is homotopic to a point with a homotopy in $W \setminus f(\mathrm{Crit}(f))$. Since $f : U \setminus f^{-1}(f(\mathrm{Crit}(f))) \to W \setminus f(\mathrm{Crit}(f))$ is an unbranched covering map, this homotopy can be lifted via f to a homotopy from α to a point, but this contradicts the assumption that α is non-trivial. Next, suppose that $K \cap C$ consists of one point ζ_0. By assumption, we note that $\zeta_0 \in K \setminus \tilde{\beta}(\mathrm{S}^1)$. Then it is not difficult to see that the loop $\tilde{\beta}$ is freely homotopic to ε^n in $\Delta \setminus C$, where ε is a sufficiently small simple loop around ζ_0 in $\Delta \setminus C$ and n is the winding number of $\tilde{\beta}$ around ζ_0. This implies that β is freely homotopic to $p_*(\varepsilon^n)$ in $W \setminus f(\mathrm{Crit}(f))$, therefore α is freely homotopic to a loop δ with $f_*\delta = p_*(\varepsilon^n)$. In particular, $\ell_U[\alpha] \leq \ell_U(\delta)$ and the length of δ can be arbitrarily small, therefore $\ell_U[\alpha] = 0$, this is not the case. Now we have proved that $\#(K \cap C) \geq 2$.

Here we recall that $\tilde{\beta}$ is parametrized by $\mathrm{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. For each $\theta \in \mathbb{R}$, we denote by S_θ the hyperbolic segment joining $\tilde{\beta}(1)$ and $\tilde{\beta}(e^{i\theta})$ in Δ. Now we define positive numbers θ_+ and θ_- by

$$\theta_\pm = \max\{\theta \geq 0; S_{\pm u} \cap C = \emptyset \quad \text{for all } u \in [0, \theta)\}.$$

Then we see that $\theta_+ + \theta_- \leq 2\pi$ and if the equality occurs we have $\#(S_{\theta_+} \cap C) \geq 2$ since $K \subset \bigcup_{t \in \mathbb{R}} S_t$ and $\#(K \cap C) \geq 2$. In any case, there exist distinct two points $\tilde{v}_+$ and $\tilde{v}_-$ such that $\tilde{v}_\pm \in S_{\pm\theta_\pm} \cap C$. We put $v_\pm = p(\tilde{v}_\pm)$.

Let $u_n^\pm \quad (n = 1, 2, \cdots)$ be an increasing sequence of positive numbers which converges to $\theta_\pm$ for each signature, and $\tilde{\beta}_n$ the curve obtained from $\tilde{\beta}$ by replacing its subarcs $\tilde{\beta}|_{I_n^+}, \tilde{\beta}|_{I_n^-}$ by the hyperboic segments $S_{u_n^+}, S_{-u_n^-}$, respectively, where $I_n^\pm$ denotes the subinterval $\{e^{i\theta}; \pm\theta \in [0, u_n^\pm]\}$ of S^1. And set $\beta_n = p_*\tilde{\beta}_n$ for each $n = 1, 2, \cdots$. By construction, β_n is freely homotopic to β in $W \setminus f(\mathrm{Crit}(f))$, and it holds that $\ell_W(\beta_n) = \ell_\Delta(\tilde{\beta}_n) \leq \ell_\Delta(\tilde{\beta}) = \ell_W(\beta)$. Let α_n be the lift of β_n via f determined by $\alpha_n(1) = \alpha(1)$, then α_n is closed and homotopic to α.

Let $\alpha' = \lim \alpha_n$ and $\beta' = f_*\alpha'$. Then, we note that $\ell_W(\beta') = \lim \ell_W(\beta_n) \leq \ell_W(\beta)$. Further, we can see that $\beta' \in \mathcal{S}(v_+, v_-)$ or $\beta' \in \mathcal{T}(v_+)$ according to that $v_+ \neq v_-$ or not. Therefore, we can compute as follows.

$$\ell_U(\alpha) \geq \ell_W(\beta) \geq \ell_W(\beta') \geq \min\{a(v_+, v_-), b(v_+)\} \geq \min\{a, b\}.$$

If $\beta = f_*\alpha$ is not contractible in W, then $\ell_U(\alpha) \geq \ell_W(\beta) \geq L_W$. Whence we have the following

Corollary 3.3. *Let $f : U \to W$ be a holomorphic branched covering between hyperbolic Riemann surfaces U and W. Then it follows that*

$$L_U \geq \min\{L_W, a, b\},$$

where the constants a and b are as in the above.

4. Proof of the Main Theorem

Let α be a non-trivial closed curve in $\Omega = \Omega_f$. In order to prove our main theorem, we should show that $\ell_\Omega(\alpha) \geq C$, where $C = \min\{a_1, \cdots, a_s, b_1, \cdots, b_s, L_{A_1}, \cdots, L_{A_t}\}$. We denote by α_n the image $f^n \circ \alpha = (f^n)_*(\alpha)$ of α under the n-th iterate of f. We note here that $\ell_\Omega(\alpha) \geq \ell_\Omega(\alpha_1) \geq \ell_\Omega(\alpha_2) \cdots$ by the Schwarz-Pick lemma. Let U be the component of Ω containing α. Then, by Sullivan's No Wandering Domains Theorem, U is eventually periodic, i.e., $D = f^k(U)$ is a periodic component for some integer k. As is well-known, a periodic component D is one of the following:

1. a (super)attracting immediate basin. In this case, the sequence of curves α_n is attracted to a (super)attracting cycle (in Ω), in particular, α_n is contractible in Ω for sufficiently large n.
2. a parabolic immediate basin. In this case, a subsequence of α_n is absorbed by a simply connected attracting petal (in Ω), therefore α_n is contractible in Ω, too, for sufficiently large n.
3. a Siegel disk. In this time, D is simply connected itself, thus α_k is of course contractible in D.
4. a Herman ring.

Hence, we can conclude that if α_n is non-trivial for any n, then α_n is contained in a cycle of Herman rings A_j for sufficiently large n. In this case, α_n is freely homotopic to a non-zero multiple of the core curve β of a component of A_j, thus $\ell_\Omega(\alpha_n) \geq \ell_{A_j}(\beta) = L_{A_j}$. In particular, we have $\ell_\Omega(\alpha) \geq L_{A_j} \geq C$.

Otherwise, there exists an integer $n \geq 0$ such that α_n is non-trivial while α_{n+1} is trivial in Ω. Since $\ell_\Omega(\alpha) \geq \ell_\Omega(\alpha_n)$, we may assume that $n = 0$, in other words, α is non-trivial but $\beta = f_*(\alpha)$ is contractible in Ω. If $f : U \to W := f(U)$ is unbranched covering, a homotopy connecting β with a constant curve in W can be lifted to a homotopy connecting α with a constant curve in U via f, thus α is contractible in U, this is a contradiction. Therefore, $f : U \to W$ must be branched, i.e., $U = U_j$ for some $j = 1, \cdots, s$. Now we can apply the key lemma in the previous section! By Lemma 3.2, we have $\ell_\Omega(\alpha) \geq \min\{a_j, b_j\} \geq C$, thus the proof is now completed.

5. Applications

We now present some applications of the main result. For a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$, we have seen that $L_{\Omega_f} \geq C$, where Ω_f denotes the Fatou set $\widehat{\mathbb{C}} \setminus J_f$ and $C > 0$ is the constant which appears in Theorem 2.2 or Corollary 2.3.

First of all, we state a result concerning Hausdorff dimension. The following theorem is essentially due to Järvi-Vuorinen [9], while a quantitative version as in the following can be found in [15].

Theorem 5.1. *The Hausdorff dimension of the Julia set J_f of a rational map f can be estimated as*

$$\text{H-dim}(J_f) \geq \frac{\log 2}{\log(2e^{M^\circ_{\Omega_f}} + 1)} \geq \frac{\log 2}{M^\circ_{\Omega_f} + \log 3} \geq \frac{\log 2}{\pi^2/L_{\Omega_f} + \log 3}.$$

In particular, any rational map of degree≥ 2 has always the Julia set of positive Hausdorff dimension. This is a well-known fact and is also shown in [4] by uniform perfectness of the Julia set in another context.

The next theorem ensures the regularity of the Julia set in the sense of Dirichlet (cf. [17]) by Wiener's criterion.

Theorem 5.2 (Pommerenke [12]. See also [15]). *Let f be a rational map of degree at least two. Then, for each point $a \in J_f$ and $0 < r < \operatorname{diam} J_f$, it holds that* $\operatorname{Cap}(J_f \cap B(a,r)) \geq cr$, *where $c \leq 1$ is a constant satisfying* $\log 1/c \leq M^\circ_{\Omega_f} + 7\log 2$, diam *stands for the Euclidean diameter,* Cap *the logarithmic capacity and $B(a,r)$ is the closed disk centered at a with radius r.*

In fact, the above property characterizes uniform perfectness of J_f (see [12]). Similarly, we can state a characterization of uniform perfectness of the closed set in terms of Hausdorff contents [9] (see also [15]).

Finally, we mention the estimate of the hyperbolic (or Poincaré) metric $\rho(z)|dz| = \rho_\Omega(z)|dz|$ of $\Omega = \Omega_f$ in terms of the distance function $\delta(z) = \delta_\Omega(z) = \operatorname{dist}(z, J_f) = \inf_{a\in J_f} |z-a|$, provided that $\infty \in J_f$. It is always true that $\rho(z) \leq 1/\delta(z)$. On the other hand, if Ω is simply connected, it is well-known that $\rho(z) \geq 1/4\delta(z)$, while this kind of inequality need not hold in general, even in the case $\partial\Omega$ is a perfect set. But this is true in our situation, indeed the validity of this inequality characterizes uniform perfectness.

Theorem 5.3 (cf. [15]). *For a rational map f of degree at least two, we set $L = L_{\Omega_f}$. If $\infty \in J_f$ then we have*

$$\frac{1}{4}\tanh L/2 \leq \inf_{z\in\Omega_f} \rho_{\Omega_f}(z)\delta_{\Omega_f}(z) \leq \frac{\sqrt{3}L}{\sqrt{\pi^2+4L^2}}.$$

For other applications and characterizations of uniform perfectness, see [15] and its references.

6. Quadratic polynomials

In this section, as the simplest example, we shall consider the quadratic polynomials $f(z) = f_c(z) = z^2 + c$ and attempt to give concrete lower and upper bounds for the uniform perfectness constant L_{Ω_f} (abbreviated by L_c) of the Jula set J_c of f_c in terms of the parameter c. (For a general rational map f, we may estimate L_{Ω_f} in the similar way as below, in principle.) For general results of the dynamics of quadratic polynomials, the reader will find a good account in the book [4] by Carleson and Gamelin.

Since f_c is a polynomial, the point at infinity is a superattracting fixed point of f_c. And 0 is a unique finite critical point of f_c and c is the corresponding critical value. Let $\mathcal{M}$ denote the Mandelbrot set $\{c \in \mathbb{C}; (f_c^n(0))_{n=1,2,\cdots}$ is a bounded sequence$\}$. As is well-known, $c \in \mathcal{M}$ if and only if the Julia set J_c is connected, in which case $L_c = +\infty$ since $\Omega = \Omega_c := \widehat{\mathbb{C}} \setminus J_c$ is simply connected, thus we have nothing to do. So we assume that $c \notin \mathcal{M}$ in the sequel. In this case, the Julia set J_c is a Cantor set, therefore the Fatou set Ω_c is connected. In order to estimate L_c from below, by Corollary 2.3, it is sufficient to estimate $d_\Omega(c,\infty), \iota_\Omega(c)$ and $\iota_\Omega(\infty)$ from below. To accomplish it, we may utilize the monotonicity property of the hyperbolic metrics. If we find a hyperbolic domain $\widetilde{\Omega}$

containing Ω which is easier to estimate its hyperbolic metric, then $\rho_\Omega \geq \rho_{\widetilde{\Omega}}$ by the Schwarz-Pick lemma. Therefore, it holds that $d_\Omega(a,b) \geq d_{\widetilde{\Omega}}(a,b)$ for any $a,b \in \Omega$. On the other hand, it is not always true that $\iota_\Omega(a) \geq \iota_{\widetilde{\Omega}}(a)$, but we can avoid this difficulty as follows. Fix $a \in \Omega$. Let D be an arbitrary simply connected subdomain of Ω containing a, then we have

$$\iota_\Omega(a) \geq \inf_{w \in \partial D} d_\Omega(a,w) \geq \inf_{w \in \partial D} d_{\widetilde{\Omega}}(a,w).$$

The most useful (but not necessarily sufficient) domain D is thought to be a thrice punctured sphere, since it has been studied for a long time and its hyperbolic metric can be expressed almost explicitly (see, for example, [1] and [3]). Any thrice punctured sphere is conformally (indeed, Möbius) equivalent to the canonical one: $D_0 = \widehat{\mathbb{C}} \setminus \{0,1,\infty\}$. The following is the precise version of Landau's theorem due to Hempel [6]: The hyperbolic metric $\rho_0(z)|dz|$ of D_0 satisfies

$$\rho_0(z) \geq \frac{1}{2|z|(|\log|z|| + K)}, \tag{6.1}$$

where $K = \Gamma(\frac{1}{4})^4/4\pi^2 = 4.3768796\cdots$, and the equality occurs if $z=-1$. Note that this estimate is efficient only on the half plane $\mathrm{Re} z \leq \frac{1}{2}$, otherwise we have only to use the functional equation $\rho_0(1-z) = \rho_0(z)$.

In order to find out a thrice punctured sphere containing Ω, we have only to specify three points a_1, a_2, a_3 in the Julia set, for example, repelling periodic points or their inverse images. In our present case, any periodic point is repelling since $c \notin \mathcal{M}$. For example, fixed points of f_c are solutions of the equation: $z^2 + c = z$, thus $(1 \pm \sqrt{1-4c})/2$. We note that if α is in J_c, so is $-\alpha$. If we selected the three points a_1, a_2, a_3 in the Julia set, let T be the Möbius transformation mapping a_1, a_2 and a_3 to $0, 1$ and ∞, respectively. Then, $d_{\Omega_c}(c,\infty) \geq d_{\widehat{\mathbb{C}}\setminus\{a_1,a_2,a_3\}}(c,\infty) = d_0(T(c), T(\infty))$, where d_0 denotes the hyperbolic distance in D_0, however it seems impossible to estimate $\iota_{\Omega_c}(c)$ and $\iota_{\Omega_c}(\infty)$ by only the data a_j.

For simplicity, we further assume that $c < -2$ for a moment. Set $\alpha = (1 + \sqrt{1-4c})/2$ and $\beta = 1 - \alpha$, then these are fixed points of $f = f_c$. Then we see that $S = \cup_{n=1}^\infty f^{-n}(\alpha) \subset [-\alpha, \alpha]$, hence $J_c \subset [-\alpha,\alpha]$ since $\overline{S} = J_c$.

For the later convenience, we set $t = \sqrt{1-4c} - 3 > 0$. Let $T(z) = \frac{(\beta-\alpha)(z+\alpha)}{(\beta+\alpha)(z-\alpha)} = (3+t)\frac{\alpha+z}{\alpha-z}$. Then $T(\infty) = -(3+t)$ and $T(c) = -\frac{t(3+t)}{4+t}$. We also note that $T(J_c) \subset [0,\infty]$. Using (6.1), we can calculate as

$$d_{\Omega_c}(c,\infty) \geq d_0(T(c), T(\infty)) = \int_{T(\infty)}^{T(c)} \rho_0(x)dx \geq \frac{1}{2}Q(t),$$

where

$$Q(t) = \begin{cases} \log \dfrac{\log(3+t) + K}{K} + \log \dfrac{\log \frac{4+t}{t(3+t)} + K}{K} & \text{if } t \leq t_0 \\ \log \dfrac{\log(3+t) + K}{\log \frac{4+t}{t(3+t)} + K} & \text{if } t > t_0 \end{cases}$$

and $t_0 = 0.38297\cdots$ is the positive root of the equation $3 + t = (4+t)/t(3+t)$.

Next, we shall estimate the injectivity radii of Ω_c at c and ∞. As a preparation, we consider the quantity $h(a) = \iota_{D_0}(-a)$ for $a > 0$. First we assume that $0 < a \leq 1$. Let

Δ be the domain defined by $\{\tau \in \mathbb{H}; 0 < \mathrm{Re}\tau < 1, |\tau - \frac{1}{2}| > \frac{1}{2}\}$, and $\lambda : \Delta \to \mathbb{H}$ the conformal homeomorphism from Δ onto the upper half plane $\mathbb{H}$ which maps $0, 1, \infty$ to $1, \infty, 0$, respectively. We denote by $g : \mathbb{H} \to \Delta$ the inverse map of λ. Then, as is well-known, λ is analytically continued to the universal covering map of D_0 from $\mathbb{H}$ by the reflection principle, in particular, $1/2\mathrm{Im}\tau = \rho_0(\lambda(\tau))|\lambda'(\tau)|$. The map λ is nothing but the classical elliptic modular function. For the point $\tau_0 = (e^{i\theta}+1)/2 = g(1+a) \in g((1,2])$ $\quad(\pi/2 \le \theta < \pi)$ we can see that $d_{\mathbb{H}}(\tau_0, g((0,1))) \le d_{\mathbb{H}}(\tau_0, g((-\infty, 0)))$ and that the shortest hyperbolic segment γ connecting τ_0 and $g((0,1)) = \{ti; t > 0\}$ is contained in $\{\tau \in \Delta; \mathrm{Re}\tau \le \frac{1}{2}\}$. Noting that $\lambda(\{\tau \in \Delta; \mathrm{Re}\tau = \frac{1}{2}\}) = \{z \in \mathbb{H}; |z-1| = 1\}$, we have $h(a) = \iota_{D_0}(-a) = \iota_{D_0}(1+a) = \int_{\lambda_*\gamma} \rho_0(z)|dz|$ and $\lambda_*\gamma$ is contained in $\{z \in \overline{\mathbb{H}}; |z-1| \le 1\}$. Denote by β the closed curve obtained as the union of $1 - \lambda_*\gamma$ and its complex conjugate, then $|\beta| \le 1$ and $2h(a) = \int_\beta \rho_0(z)|dz|$. Note that $|dz| \ge (|dr| + r|d\theta|)/\sqrt{2}$, where $z = re^{i\theta}$. Put $a_0 = \min|\beta|$, then by (6.1) we have

$$\begin{aligned}
2h(a) &\ge \int_\beta \frac{|dz|}{2|z|(-\log|z| + K)} \ge \int_\beta \frac{|dr| + r|d\theta|}{2r\sqrt{2}(-\log r + K)} \\
&\ge \frac{2}{2\sqrt{2}} \log\left(\frac{-\log a_0 + K}{-\log a + K}\right) + \frac{1}{2\sqrt{2}} \frac{2\pi}{-\log a_0 + K} \\
&\ge \frac{\pi/\sqrt{2}}{-\log a + K},
\end{aligned}$$

because $K > \pi$.

Next, we consider the case $a > 1$. Since the Möbius transformation $z \mapsto 1/z$ preserves D_0, we have $h(a) = h(1/a)$, thus $h(a) \ge \pi/2\sqrt{2}(\log a + K)$. Therefore, for any $a > 0$, we have

$$\iota_{D_0}(-a) \ge \frac{\pi}{2\sqrt{2}(|\log a| + K)}.$$

Letting $D = \mathbb{C} \setminus [0, \infty)$, we can estimate the injectivity radius of Ω_c at $T^{-1}(-a)$ as follows:

$$\iota_{\Omega_c}(T^{-1}(-a)) = \iota_{T(\Omega_c)}(-a) \ge \inf_{w \in \partial D} d_0(-a, w) = \iota_{D_0}(-a) \ge \frac{\pi}{2\sqrt{2}(|\log a| + K)}.$$

Hence,

$$\begin{aligned}
&\min\{2d_{\Omega_c}(c, \infty), 4\iota_{\Omega_c}(c), 4\iota_{\Omega_c}(\infty)\} \\
&\ge R(t) := \min\left\{Q(t), \frac{\sqrt{2}\pi}{\log(3+t) + K}, \frac{\sqrt{2}\pi}{\log\frac{4+t}{t(3+t)} + K}\right\} \qquad (6.2) \\
&= \begin{cases}
\dfrac{\sqrt{2}\pi}{\log\frac{4+t}{t(3+t)} + K} & \text{if } 0 < t < 0.12626\cdots, \\[2ex]
\log\dfrac{\log(3+t) + K}{K} + \log\dfrac{\log\frac{4+t}{t(3+t)} + K}{K} & \text{if } 0.12626\cdots < t \le t_0, \\[2ex]
\log\dfrac{\log(3+t) + K}{\log\frac{4+t}{t(3+t)} + K} & \text{if } t_0 = 0.38297\cdots < t,
\end{cases}
\end{aligned}$$

where $K = \Gamma(\frac{1}{4})^4/4\pi^2 = 4.3768796\cdots$.

In contrast, the estimation of L_c from above is rather easy. In the same assumption as the above, we set $\gamma = \sqrt{-c-\alpha} > 0$. Note that $\gamma \in J_c \subset [-\alpha, \alpha]$ since $f_c(\gamma) = -\alpha \in J_c$. Then, for $x \in (-\gamma, \gamma)$, we see that $f_c(x) = x^2 + c < \gamma^2 + c = -\alpha$, hence $(-\gamma, \gamma) \subset \Omega_c$. This implies that the annulus $A = \widehat{\mathbb{C}} \setminus ([-\alpha, -\gamma] \cup [\gamma, \alpha])$ separates the Julia set J_c. The Möbius transformation $T(z) = \frac{\gamma+z}{\alpha-z}$ maps A onto Teichmüller's extremal domain $\widehat{\mathbb{C}} \setminus ([-r_1, 0] \cup [r_2, +\infty])$, where $r_1 = (\alpha-\gamma)/2\alpha$ and $r_2 = 2\gamma/(\alpha-\gamma)$. Thus we have $m(A) = 2\mu(\sqrt{r_1/(r_1+r_2)})$, where $\mu(r)$ denotes the modulus of Grötzsch's extremal domain $\mathbb{D} \setminus [0, r]$ for $0 < r < 1$, where $\mathbb{D}$ denotes the unit disk (see [10]). The behaviour of the function $\mu(r)$ is well understood. Amongst them, it will be useful to record the following (cf. [10]):

$$\log \frac{(1+\sqrt{1-r^2})^2}{r} < \mu(r) < \log \frac{2(1+\sqrt{1-r^2})}{r} < \log \frac{4}{r} \quad \text{and}$$

$$\mu(r)\mu\left(\frac{1-r}{1+r}\right) = \frac{\pi^2}{2}.$$

In particular, we can see that

$$m(A) = 2\mu\left(\frac{\alpha-\gamma}{\alpha+\gamma}\right) = \frac{\pi^2}{\mu(\gamma/\alpha)}.$$

Therefore, we have $L_c \le \pi^2/M_{\Omega_c} \le \pi^2/2\mu((\alpha-\gamma)/(\alpha+\gamma)) = \mu(\gamma/\alpha)$. Noting Corollary 2.3, we summarize the results obtained above.

Theorem 6.1. *For $c < -2$, the Fatou set Ω_c of $f_c(z) = z^2 + c$ satisfies*

$$R(t) \le L_{\Omega_c} \le \frac{\pi^2}{2\mu(\frac{\alpha-\gamma}{\alpha+\gamma})} = \mu\left(\frac{\gamma}{\alpha}\right), \tag{6.3}$$

where $t = \sqrt{1-4c} - 3 > 0, \alpha = (1+\sqrt{1-4c})/2 > 2, \gamma = \sqrt{-c-\alpha} > 0, R(t)$ is the function defined by (6.2) *and $\mu(r)$ denotes the modulus of Grötzsch's extremal domain $\mathbb{D} \setminus [0, r]$.*

Remark . Since $(\alpha-\gamma)/(\alpha+\gamma) = 2\alpha/(\alpha+\gamma)^2 \sim 1/t$, the upper bound in (6.3) behaves like $\pi^2/2 \log t$ as $t \to \infty$, while $R(t) \sim 4/t \log t$.

When $t \to +0$, the upper bound in (6.3) is $\frac{1}{2} \log 1/t + O(1)$ since $\gamma/\alpha = \sqrt{t/8}(1+O(t))$, however $R(t) \sim \sqrt{2}\pi/\log 1/t$.

The badness of the lower bound $R(t)$ is mainly caused by having replaced the Julia set with only three points in the estimation when the critical values are very close to the Julia set.

In order to get a result for any $c \in \mathbb{C} \setminus \mathcal{M}$, we can use the following fundamental property of the uniform perfectness constants (cf. [15]).

Proposition 6.2. *The constants M_Ω and L_Ω are quasi-invariant, in other words, if $f : \Omega \to \Omega'$ is a K-quasiconformal homeomorphism ($K \ge 1$) then*

$$M_\Omega/K \le M_{\Omega'} \le KM_\Omega \quad \text{and} \quad L_\Omega/K \le L_{\Omega'} \le KL_\Omega.$$

Next we consider a holomorphic motion of the Julia set, which is an important tool introduced by Mañé-Sad-Sullivan. For a subset E of $\widehat{\mathbb{C}}$ and a pointed hyperbolic Riemann surface (or, more generally, complex hyperbolic manifold) (X, x_0) with hyperbolic distance d_X, a map $F : X \times E \to \widehat{\mathbb{C}}$ is called a *holomorphic motion* of E parametrized by (X, x_0) if the following holds:

1. $F(\cdot, a) : X \to \widehat{\mathbb{C}}$ is holomorphic for each $a \in E$,
2. $F_x := F(x, \cdot) : E \to \widehat{\mathbb{C}}$ is injective for each $x \in X$, and
3. F_{x_0} is the identity map of E.

We can state the optimal λ-lemma proved by Słodkowsky [14] (see also [5]) as in the following form.

Theorem 6.3 (Optimal λ-lemma). *Let F be a holomorphic motion of a subset E of the Riemann sphere parametrized by a simply connected hyperbolic Riemann surface X with basepoint x_0. Then, F can be extended to a holomorphic motion $\widetilde{F}$ of the whole sphere $\widehat{\mathbb{C}}$ parametrized by (X, x_0) with the following properties.*

1. *$\widetilde{F} : X \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is (jointly) continuous,*
2. *for each $x \in X$, the map $\widetilde{F}_x : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasiconformal homeomorphism with Beltrami coefficient $\mu_x = \partial_{\bar{z}}\widetilde{F}_x / \partial_z \widetilde{F}_x$ satisfying $d_T(\mu_x, 0) \le d_X(x, x_0)$, where d_T denotes the Teichmüller distance*

$$d_T(\mu, \nu) = \operatorname*{ess.sup}_{z \in \mathbb{C}} d_{\mathbb{D}}(\mu(z), \nu(z)) = \operatorname{arctanh}\left(\left\| \frac{\mu - \nu}{1 - \bar{\nu}\mu} \right\|_\infty \right).$$

Now we construct a holomorphic motion of the Julia set by the standard method (see, for instance, [5]). As is well-known (cf. [4]), the functions $\Phi_n(c) := (f_c^n(c))^{2^{-n}}$ converges locally uniformly in $\mathbb{C} \setminus \mathcal{M}$ to a holomorphic function $\Phi(c)$, which is, in turn, a conformal mapping of $\mathbb{C} \setminus \mathcal{M}$ onto $D := \{z \in \mathbb{C}; |z| > 1\}$, where we take the branch Φ_n so as to $\Phi_n(c) = c + O(1)$ as $c \to \infty$. By the symmetry of the Mandelbrot set $\mathcal{M}$, we note that $\overline{\Phi(\bar{z})} = \Phi(z)$, in particular, $\Phi((-\infty, -2)) = (-\infty, -1)$.

Now we define the function $p : H = \{\zeta \in \mathbb{C}; \mathrm{Re}\zeta > 0\} \to D$ by $p(\zeta) = -e^\zeta$, then $q := \Phi^{-1} \circ p$ is a universal covering map of $\mathbb{C} \setminus \mathcal{M}$ from the right half plane H. Fix an arbitrary point c_1 in $\mathbb{C} \setminus \mathcal{M}$. Then there exists a ζ_1 in H such that $q(\zeta_1) = c_1$ and $-\pi < \mathrm{Im}\zeta_1 \le \pi$. Set $\zeta_0 = |\zeta_1|$, $c_0 = q(\zeta_0)$ and let E_0 be the set of repelling periodic points of f_{c_0}. Then, considering the roots of the equation $f_c^n(z) = z$, where $c = q(\zeta)$ and n is taken over all positive integers, we can obtain a holomorphic motion of the set E_0 parametrizedby the right half plane H with basepoint ζ_0 because f_c has no parabolic periodic points for $c \in \mathbb{C} \setminus \mathcal{M}$, thus the roots do not collide. By the optimal λ-lemma, we have a holomorphic motion $F : H \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $F_{\zeta_0} = \mathrm{id}$ and that $F_\zeta(E_0)$ is the set of repelling periodic points of $f_{q(\zeta)}$. (We can take an F compatible with the dynamics, thus F_ζ is a quasiconformal conjugate of f_{c_0} to $f_{q(\zeta)}$, but we do not this property here.) Since the set of repelling periodic points is dense in the Julia set, $F_\zeta(J_{c_0}) = J_{q(\zeta)}$ holds. Hence, by Proposition 6.2 and the second property in Theorem 6.3, we have

$$1/K \le L_{\Omega_{c_1}} / L_{\Omega_{c_0}}, M_{\Omega_{c_1}} / M_{\Omega_{c_0}} \le K,$$

where $\frac{1}{2}\log K = d_H(\zeta_1, \zeta_0)$.

Now we compute K. We write $\zeta_1 = |\zeta_1|e^{i\theta} = z_0 e^{i\theta}$ with $\theta \in (-\pi/2, \pi/2)$. Then we can calculate as

$$d_H(\zeta_1, \zeta_0) = \operatorname{arctanh}\left(\left|\frac{\zeta_1 - \zeta_0}{\zeta_1 + \zeta_0}\right|\right) = \operatorname{arctanh}\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\log\left(\frac{1+\sin\theta}{\cos\theta}\right).$$

Now we have shown the following.

Theorem 6.4. *For an arbitray $c \in \mathbb{C} \setminus \mathcal{M}$, take a point $\zeta = re^{i\theta}$ with $\theta \in (-\pi/2, \pi/2)$ such that $-e^{\zeta} = \Phi(c)$ and that $-\pi < \mathrm{Im}\zeta = r\sin\theta \le \pi$. Then, we have the following estimates:*

$$L_{\Omega_{c_0}}/K \le L_{\Omega_c} \le K L_{\Omega_{c_0}} \quad \text{and} \quad M_{\Omega_{c_0}}/K \le M_{\Omega_c} \le K M_{\Omega_{c_0}},$$

where $c_0 < -2$ is the number determined by $\Phi(c_0) = -e^r$, and

$$K = \frac{1+\sin\theta}{\cos\theta}.$$

We remark that $\log|\Phi(c)| = r\cos\theta$ is Green's function of the domain $\widehat{\mathbb{C}} \setminus \mathcal{M}$ with pole at the infinity.

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