UNIFORM PERFECTNESS OF THE LIMIT SETS OF KLEINIAN GROUPS

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Abstract. A compact set C in the Riemann sphere is said to be uniformly perfect if bounded are the moduli of those annuli in the complement which separate C. The limit set of an analytically finite non-elementary Kleinian group is known to be uniformly perfect.

In this note, we shall show that the limit set of a non-elementary Kleinian group is uniformly perfect if the quotient orbifold is of Lehner type, i.e., satisfies that the space of integrable holomorphic quadratic differentials on it is continuously contained in the space of (hyperbolically) bounded ones. Indeed, we shall state the result in more precise and quantitative form. As applications, we present estimates of the Hausdorff dimension of the limit set and the translation length in the region of discontinuity.

1. Introduction

In this note, we shall consider uniform perfectness of the limit sets of Kleinian groups. Once one know the limit set is uniformly perfect, the estimation of various quantities involving Kleinian groups becomes easier (see Section 5). Bishop and Jones effectively used this fact in their paper [4].

As soon as a proof of uniform perfectness of the limit sets of Schottky groups appeared in [2], this result was generalized to the case of finitely generated non-elementary Kleinian groups by several specialists (cf. [13] and [14]). Afterward, Canary remarked in [5] that the same result holds for analytically finite Kleinian groups. As for Schottky groups, we should mention the pioneer work of Tsuji [18] (see the comment on Theorem C below).

Recently, Järvi and Vuorinen [6] proved the same result for finitely generated Kleinian groups in higher dimensional case. This is a generalization of Tukia's result [20] (geometrically finite case). It is noteworthy that their proof does not rely on Ahlfors' Finiteness Theorem.

In this note, we will present a more general condition for the limit sets of Kleinian groups to be uniformly perfect (Corollary 3.3). Our method also provides a bound for uniform perfectness by some geometric quantity. In practice, it is important to know an explicit bound because the uniform perfectness connects with various quantities involving geometry of the quotient surface (cf. [16]). Indeed, we shall give some applications in Sections 5 and 6. In Section 5, we state results relating to regularity in the sense of Dirichlet and Hausdorff dimension of the limit sets. Section 6 is devoted to an estimate of translation length in terms of the multiplier of a loxodromic element of a Kleinian group.

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We shall conclude this note by giving several examples of Kleinian groups, one of which has the limit set which is not uniformly perfect.

2. Riemann orbifolds

A 1-dimensional complex orbifold is a Hausdorff topological space locally modeled on the quotient of an open subset of the complex plane $\mathbb{C}$ under an action of finite group whose members consist of biholomorphic maps. Note that such a finite group is necessarily a cyclic group, thus determined by its order. By a *Riemann orbifold*, we will mean a connected 1-dimensional complex orbifold. Thus X can be regarded as a pair of a Riemann surface R and a multiplicity map $\nu : R \to \mathbb{N}$, where the value of ν corresponds to the order of the finite group, therefore the set of singularities (or, branch points) $b(X) = \{x \in R; \nu(x) > 1\}$ should be discrete. We set $\overset{\circ}{X} = R \setminus b(R)$. Then $\overset{\circ}{X}$ can be regarded as a subdomain of X without singularities. If ν is the constant 1, then X can be naturally indentified with the underlying Riemann surface R. For a precise definition and fundamental properties of Riemann orbifolds, we refer the reader to McMullen's book [10].

A Riemann orbifold $X = (R, \nu)$ is called *hyperbolic* if R has a holomorphic branched Galois covering map $p : \mathbb{H} \to R$ from the upper half plane $\mathbb{H}$ onto R such that $\mathrm{ord}_\zeta p = \nu(p(\zeta))$ for any $\zeta \in \mathbb{H}$, where $\mathrm{ord}_\zeta p$ denotes the local degree of p at ζ. The map p is called a universal covering map of X.

The covering transformation group $\Gamma = \Gamma_X = \{\gamma \in Aut(\mathbb{H}); p \circ \gamma = p\}$ is called a *Fuchsian model* of X. Through the universal covering map, the Riemann orbifold X inherits the hyperbolic metric ρ_X from $\mathbb{H}$, i.e., $\rho_{\mathbb{H}} = \frac{|dz|}{2\mathrm{Im} z} = p^*(\rho_X)$. We denote by d_X the hyperbolic distance on X canonically defined by the hyperbolic metric.

Let $\mathcal{C}_X$ denote the set of free homotopy classes $[\alpha] = [\alpha]_X$ of non-trivial closed curves α in $\overset{\circ}{X}$, where a curve α is said to be non-trivial if it is covered by an element $\gamma \in \Gamma$ of infinite order, more precisely, there exists a lift $\tilde{\alpha}$ via p such that the initial point is translated to the terminal one by γ, and said to be freely homotopic to another α' if both are covered by the same element of Γ. (The free homotopy class of a non-trivial curve precisely corresponds to the conjugacy class of an element of Γ which covers the curve and has infinite order.) And let $\mathcal{C}_X^*$ be the subset of $\mathcal{C}_X$ consisting of free homotopy classes of curves which are covered by hyperbolic elements of Γ.

Remark. It may be more natural to consider all curves in X, but a curve passing a branch point of X cannot be lifted via p in a unique way even if the initial point is assigned, so being freely homotopic fails to be an equivalence relation. To avoid this difficulty, we may assign a sort of "multiplicity data" to the curve. Precisely speaking, we may consider a (regular) curve as the pair (γ, n_γ) of a C^1 map $\gamma : \mathrm{S}^1 \to R$ with $\gamma'(\zeta) \neq 0$ and $n_\gamma : \mathrm{S}^1 \to \mathbb{Z}$ such that $0 \leq n_\gamma(\zeta) \leq \nu(\gamma(\zeta)) - 1$, where $n_\gamma(\zeta)$ indicates the winding number of γ around $\gamma(\zeta)$. But, we will not adopt this formulation for the sake of simplicity.

We write $\ell_X(\alpha) = \int_\alpha \rho_X$ and $\ell_X[\alpha] = \inf_{\alpha' \in [\alpha]} \ell_X(\alpha')$. By definition, if $\gamma \in \Gamma$ covers α, then

$$\ell_X[\alpha] = l_\gamma = \cosh^{-1}\left(\frac{|\mathrm{tr}\gamma|}{2}\right), \tag{2.1}$$

where l_γ denotes the translation length of γ and $\mathrm{tr}\gamma$ is the trace of a representative of γ in $\mathrm{SL}(2,\mathbb{R})$.

Now we set

$$L(X) = \inf_{[\alpha]\in\mathcal{C}_X} \ell_X[\alpha] = \inf_{\gamma\in\Gamma_X:\text{ of infinite order}} l_\gamma \quad \text{and}$$
$$L^*(X) = \inf_{[\alpha]\in\mathcal{C}_X^*} \ell_X[\alpha] = \inf_{\gamma\in\Gamma_X:\text{ hyperbolic}} l_\gamma.$$

We call X is *modulated* if $L(X) > 0$. We note here that if X is a Riemann surface R, the constant $L(R)$ is nothing but $2I_R$, where I_R is the injectivity radius of R, so we may say that R is *of bounded geometry* if $L(R) > 0$ (see [16]).

When the Fuchsian model Γ_X of X is finitely generated, the quantity $L^*(X)$ represents the hyperbolic length of the shortest closed geodesic in X. A closed geodesic shortest in the curves in X other than boundary curves is called *systole* of X. Hence, $L^*(X)$ can be thought as the length of systole of X if Γ is finitely generated and of the first kind. It has been recognized that the length of systole is quite an important function on the moduli space of X (cf. [15]).

Let $A_2(X)$ and $B_2(X)$ be the complex Banach spaces consisting of holomorphic quadratic differentials $\varphi = \varphi(z)dz^2$ on X with norms

$$\|\varphi\|_1 = \iint_X |\varphi| = \iint_X |\varphi(z)|dxdy,$$
$$\|\varphi\|_\infty = \sup \rho_X^{-2}|\varphi| = \sup \rho_X^{-2}(z)|\varphi(z)|,$$

respectively. The spaces $A_2(X)$ and $B_2(X)$ are canonically isomorphic to the spaces $A_2(\mathbb{H},\Gamma)$ and $B_2(\mathbb{H},\Gamma)$ of integrable and bounded holomorphic automorphic forms on $\mathbb{H}$ of weight -4 for Γ, respectively. And we set

$$\kappa(X) = \sup\{\|\varphi\|_\infty; \varphi \in A_2(X) \text{ with } \|\varphi\|_1 \le 1\}.$$

For these spaces, the inclusion problem was first settled by Niebur and Sheingorn [12]. The following strong form is due to Matsuzaki [8].

Theorem 2.1. *The space $A_2(X)$ is (continuously) included by $B_2(X)$ if and only if $L^*(X) > 0$. Furthermore, there exist universal constants r_0 and r_1 such that*

$$\frac{1}{2\pi L^*(\mathring{X})} \le \kappa(X) \le \max\{\frac{r_0}{L^*(X)}, r_1\}.$$

Note that the inclusion map $\mathring{X} \hookrightarrow X$ induces the restriction maps $A_2(X) \to A_2(\mathring{X})$ and $B_2(X) \to B_2(\mathring{X})$, which are, respectively, an isometric isomorphism and a bounded linear operator with $\|\varphi\|_{B_2(\mathring{X})} \le \|\varphi\|_{B_2(X)}$ by virtue of the monotonicity of hyperbolic metrics: $\rho_X \le \rho_{\mathring{X}}$. Therefore, we have $\kappa(\mathring{X}) \le \kappa(X)$. In fact, $\kappa(\mathring{X}) \le \kappa(X) \le 3\kappa(\mathring{X})$ holds (see [17]).

In this article, we will say that X is *of Lehner type* if $L^*(X) > 0$. Generally, for a (possibly disconnected) 1-dimensional complex orbifold X, we define $L(X) = \inf L(X_0)$ and $L^*(X) = \inf L^*(X_0)$, where the infima are taken over all connected components X_0 of X, and we say that X is modulated or of Lehner type if $L(X) > 0$ or $L^*(X) > 0$, respectively.

A closed set C in the Riemann sphere $\widehat{\mathbb{C}}$ with $\#C \geq 3$ is called *uniformly perfect* if there exists a constant $c > 0$ such that $C \cap \{z; cr < |z - a| < r\} \neq \emptyset$ for all $a \in C$ and $0 < r < \operatorname{diam} C$. This is equivalent to the condition that the complement $R = \widehat{\mathbb{C}} \setminus C$ is modulated (see Section 5).

3. Kleinian groups and main results

Let G be a Kleinian group acting on the Riemann sphere $\widehat{\mathbb{C}}$, i.e., G is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$ whose region of discontinuity $\Omega(G)$ in $\widehat{\mathbb{C}}$ is not empty. We denote by $\Lambda(G)$ the limit set of G, i.e., $\Lambda(G) = \widehat{\mathbb{C}} \setminus \Omega(G)$.

In the following, we always assume that G is non-elementary, in ohter words, $\#\Lambda(G) \geq 3$. Then $\Lambda(G)$ is known to be a perfect set. The quotient space $X(G) = \Omega(G)/G$ has a natural hyperbolic 1-dimensional complex orbifold structure with which the canonical projection $\pi : \Omega(G) \to X(G)$ is a holomorphic covering map.

Let X be a connected component of $X(G)$ and Ω be a connected component of $\pi^{-1}(X)$. If $q : \mathbb{H} \to \Omega$ is a holomorphic universal covering map, then clearly $p = \pi \circ q$ is a holomorphic universal covering map of X. Let $H = H_\Omega$ be the component subgroup of G corresponding to Ω, i.e., $H = \mathrm{Stab}_G(\Omega) = \{g \in G; g(\Omega) = \Omega\}$. And denote by Γ and $\widetilde{\Gamma}$ the covering transformation groups of q and p, respectively. Then, we have a natural exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \widetilde{\Gamma} \xrightarrow{\ \chi\ } H \longrightarrow 1$$

of group homomorphisms, here χ can be described as $q \circ \gamma = \chi(\gamma) \circ q$ for all $\gamma \in \widetilde{\Gamma}$.

For each $h \in H$, we denote by $l_{h,\Omega}$ the translation length of h in Ω :

$$l_{h,\Omega} := \inf_{z \in \Omega} d_\Omega(z, h(z)). \tag{3.1}$$

Set

$$H_e = \{h \in H; h \text{ has a fixed point in } \Omega\} \quad \text{and}$$
$$H_p = \{h \in H; h \text{ corresponds to a puncture of } X = \Omega/H\}$$

Precisely speaking, an element h of H belongs to H_p if and only if there exist an element $h_0 \in H$ of infinite order and a subdomain ω of Ω with the following properties:

1. $h = h_0^n$ for a non-zero integer n,
2. $h_0(\omega) = \omega$ and $g(\omega) \cap \omega = \emptyset$ for all $g \in H \setminus \langle h_0 \rangle$, and
3. $\omega/\langle h_0 \rangle$ is conformally equivalent to the punctured disk.

We remark that there may exist an elliptic or parabolic element h of H with $h \notin H_e$ or $h \notin H_p$, respectively (see Example 7.2 below). (Of course, an element of $H_e \setminus \{1\}$ or H_p is necessarily elliptic or parabolic, respectively.) Now we define $\lambda_{H,\Omega}, \lambda^*_{H,\Omega}$ as follows:

$$\lambda_{H,\Omega} = \inf_{h \in H \setminus H_e} l_{h,\Omega}, \quad \text{and} \quad \lambda^*_{H,\Omega} = \inf_{h \in H \setminus (H_e \cup H_p)} l_{h,\Omega}.$$

Noting that an elliptic or parabolic element in a Fuchsian group always represents a branch point or a puncture of the quotient surface, similarly we set

$$\lambda_{\widetilde{\Gamma}} = \inf_{\gamma \in \widetilde{\Gamma} \setminus \widetilde{\Gamma}_e} l_\gamma = \inf_{\widetilde{\Gamma} \ni \gamma:\ \text{of infinite order}} l_\gamma \quad \text{and}$$

$$\lambda^*_{\widetilde{\Gamma}} = \inf_{\gamma \in \widetilde{\Gamma} \setminus (\widetilde{\Gamma}_e \cup \widetilde{\Gamma}_p)} l_\gamma = \inf_{\widetilde{\Gamma} \ni \gamma:\ \text{hyperbolic}} l_\gamma,$$

where l_γ is the translation length of γ in $\mathbb{H}$, i.e., $l_\gamma = \cosh^{-1}(|\mathrm{tr}\gamma|/2)$. We can define $\lambda_\Gamma, \lambda^*_\Gamma$ in the same manner, however $\lambda_\Gamma = \lambda^*_\Gamma$ holds since $\Lambda(G)$ has no isolated points. Let $N_{H,\Omega}$ be the number defined by $N_{H,\Omega} = \sup_{h \in H_e} \mathrm{ord} h$. And we define the number $N^*_{H,\Omega}$ as follows: If H_p is non-empty, then $N^*_{H,\Omega} = +\infty$, otherwise we set $N^*_{H,\Omega} = N_{H,\Omega}$.

For these constants, the next result is fundamental for our present aim.

Lemma 3.1.

$$\min\{\frac{1}{N_{H,\Omega}}\lambda_\Gamma, \lambda_{H,\Omega}\} \le \lambda_{\widetilde{\Gamma}} \le \min\{\lambda_\Gamma, \lambda_{H,\Omega}\}, \qquad \textit{and} \tag{3.2}$$

$$\min\{\frac{1}{N^*_{H,\Omega}}\lambda_\Gamma, \lambda^*_{H,\Omega}\} \le \lambda^*_{\widetilde{\Gamma}} \le \min\{\lambda_\Gamma, \lambda^*_{H,\Omega}\}. \tag{3.3}$$

By the relation (2.1), we obtain that $\lambda_\Gamma = L(\Omega)$ and $\lambda_{\widetilde{\Gamma}} = L(X)$ and the similar relations for λ^* and L^* hold, hence the above theorem is equivalent to the following:

$$\min\{\frac{1}{N_{H,\Omega}}L(\Omega), \lambda_{H,\Omega}\} \le L(X) \le \min\{L(\Omega), \lambda_{H,\Omega}\}, \qquad \text{and} \tag{3.4}$$

$$\min\{\frac{1}{N^*_{H,\Omega}}L(\Omega), \lambda^*_{H,\Omega}\} \le L^*(X) \le \min\{L(\Omega), \lambda^*_{H,\Omega}\}. \tag{3.5}$$

Now we define $\lambda(G), \lambda^*(G), N(G)$ and $N^*(G)$ by $\inf_\Omega \lambda_{H,\Omega}, \inf_\Omega \lambda^*_{H,\Omega}, \sup_\Omega N_{H,\Omega}$ and $\sup_\Omega N^*_{H,\Omega}$, respectively, where Ω runs over all components of $\Omega(G)$ and $H = H_\Omega$. Remark here that the constants $\lambda_{H,\Omega}, \lambda^*_{H,\Omega}, N_{H,\Omega}$ and $N^*_{H,\Omega}$ depend only on the conjugacy class of Ω under the action of G, i.e., on the component $X = \Omega/H$ of $X(G)$. Then, we immediately obtain the following

Theorem 3.2. *For a non-elementary Kleinian group G it follows that*

$$\min\{\frac{1}{N(G)}L(\Omega(G)), \lambda(G)\} \le L(X(G)) \le \min\{L(\Omega(G)), \lambda(G)\}, \qquad \textit{and}$$

$$\min\{\frac{1}{N^*(G)}L(\Omega(G)), \lambda^*(G)\} \le L^*(X(G)) \le \min\{L(\Omega(G)), \lambda^*(G)\}.$$

Corollary 3.3. *Thus we have $L(\Omega(G)) \ge L^*(X(G))$. In particular, if $X(G)$ is of Lehner type then $\Lambda(G)$ is uniformly perfect.*

Corollary 3.4. *For a torsion-free non-elementary Kleinian group G, the complex orbifold $X(G) = \Omega(G)/G$ satisfies*

$$L(X(G)) = \min\{L(\Omega(G)), \lambda(G)\}.$$

The counterpart $L^*(X(G)) = \min\{L(\Omega(G)), \lambda^*(G)\}$ does not hold in general (see Example 7.3 below).

Remark. To guarantee that $\lambda(G) > 0$ it is sufficient to assume that $\sup_{z\in\Omega(G)} \iota(z) < \infty$, where $\iota(z)$ denotes the injectivity radius of $\Omega(G)$ at z (oral communication with Prof. Matsuzaki). But, this condition seems to be hard to check.

In the case that G is analytically finite, i.e., $X(G)$ consists of a finite number of Riemann orbifolds of finite type, it is easily verified that $L^*(X(G)) > 0$ thus we have the following collorary.

Corollary 3.5 (Canary [5]). *For an analytically finite non-elementary Kleinian group G, the limit set $\Lambda(G)$ is uniformly perfect.*

We should remark that by Ahlfors' Finiteness Theorem this result produces the finitely generated case.

4. Proof of Lemma 3.1

Under the situation of the lemma, first we prove the next elementary

Lemma 4.1. *For an element $h \in H$, we have*

$$\inf_{\gamma\in\chi^{-1}(h)} l_\gamma = l_{h,\Omega}. \tag{4.1}$$

Proof. Let $\gamma \in \chi^{-1}(h)$, then $q\circ\gamma = h\circ q$ by definition. For a $\zeta \in \mathbb{H}$ we put $z = q(\zeta)$. Then, by the definition of metrics and the Schwarz-Pick lemma, we have

$$d_{\mathbb{H}}(\zeta,\gamma(\zeta)) \geq d_\Omega(q(\zeta), q(\gamma(\zeta))) = d_\Omega(z, h(z)) \geq l_{h,\Omega}.$$

Since ζ is arbitrary, it follows that $l_\gamma \geq l_{h,\Omega}$.

Now we prove the reverse inequality. Take a point z in Ω, and let α be a geodesic arc joining z and $h(z)$ in Ω such that $d_\Omega(z, h(z)) = \int_\alpha \rho_\Omega$. Choose a point $\zeta \in \mathbb{H}$ with $q(\zeta) = z$, and let β be a lift of α via q with initial point ζ, then the terminal point of β can be written by $\gamma(\zeta)$ for some $\gamma \in \tilde{\Gamma}$. We note here that $\chi(\gamma) = h$ by definition. Therefore, we have

$$d_\Omega(z, h(z)) = \int_\alpha \rho_\Omega = \int_\beta \rho_{\mathbb{H}} \geq d_{\mathbb{H}}(\zeta, \gamma(\zeta)),$$

thus $d_\Omega(z, h(z)) \geq l_\gamma \geq \inf_{\gamma\in\chi^{-1}(h)} l_\gamma$. Since z is arbitrary, we have $l_{h,\Omega} \geq \inf_{\gamma\in\chi^{-1}(h)} l_\gamma$. Now the proof is completed. □

Now we prove Lemma 3.1. First, we note that if $\gamma \in \tilde{\Gamma}$ is elliptic, then $h = \chi(\gamma) \in H_e \setminus \{1\}$. Note also that $\chi^{-1}(1) = \Gamma$. Therefore we conclude that

$$(\Gamma \setminus \{1\}) \cup \chi^{-1}(H \setminus H_e) \subset \tilde{\Gamma} \setminus \tilde{\Gamma}_e,$$

and this and the above lemma immediately yield the right-hand side inequality in (3.2).

In order to get the left-hand side inequality, we consider an element γ of the residual part $\chi^{-1}(H_e \setminus \{1\}) \setminus \tilde{\Gamma}_e$. Since $h = \chi(\gamma)$ is of finite order, say n, we have $\gamma^n \in \chi^{-1}(1) = \Gamma$, thus $nl_\gamma = l_{\gamma^n} \geq \lambda_\Gamma$. Hence $l_\gamma \geq \lambda_\Gamma/n \geq \lambda_\Gamma/N_{H,\Omega}$. By this observation, we are convinced the validity of the left-hand side of (3.2). Noting that any parabolic element of $\tilde{\Gamma}$ is mapped to H_p by the homomorphism χ, we can show the inequality (3.3) in the same way as above.

5. Some consequences

In this section, we shall exhibit several applications of our theorems. We denote by $M_{\Omega(G)}$ $(M^\circ_{\Omega(G)})$ the supremum of the moduli of annuli (round annuli, respectively) separating $\Lambda(G)$, where the modulus of an annulus is defined here as the number m when this annulus is conformally equivalent to the round annulus $\{z \in \mathbb{C}, 1 < |z| < e^m\}$ and the round annulus means a bounded annulus with boundary consisting of concentric circles, and we say that an annulus A separates $\Lambda(G)$ if $A \cap \Lambda(G) = \emptyset$ and if both components of $\widehat{\mathbb{C}} \setminus A$ intersects $\Lambda(G)$. If $\infty \in \Lambda(G)$, then we can define another constant $C_{\Omega(G)}$ by $\inf_{z\in\Omega(G)} \delta(z)\rho_{\Omega(G)}(z)$, where $\delta(z)$ denotes the Euclidean distance from z to $\Lambda(G)$. Here it should be noted that $\delta(z)\rho_{\Omega(G)}(z) \le 1$ is always true. For these constants, we know the following estimates.

Theorem A (cf. [16]). *For a non-elementary Kleinian group G, we obtain*

$$L \le \frac{\pi^2}{M_{\Omega(G)}} \le \min\{Le^L, \frac{1}{2}L^2 \coth^2(L/2)\}, \quad \textit{and}$$

$$\frac{1}{2}M_{\Omega(G)} - K_0 \le M^\circ_{\Omega(G)} \le M_{\Omega(G)},$$

where $L = L(\Omega(G))$ and K_0 is an absolute constant $\le 1.7332\ldots$. Moreover if $\infty \in \Lambda(G)$, we also have

$$M_{\Omega(G)} - K_1 \le M^\circ_{\Omega(G)} \le M_{\Omega(G)}, \quad \textit{and}$$

$$\frac{\tanh L/2}{4} \le C_{\Omega(G)} \le \frac{\sqrt{3}L}{\sqrt{\pi^2 + 4L^2}}, \tag{5.1}$$

where K_1 is an absolute constant $\le 2.8911\ldots$.

In particular, $L(\Omega(G)) > 0$ if and only if $M_{\Omega(G)} < \infty$. The first inequality in the above partly follows from the next result, which is an improvement of Maskit's one [7] and will be used later.

Theorem B (cf. [16]). *Let R be a hyperbolic Riemann surface. For the free homotopy class $[\alpha]$ of a non-trivial loop α in R, we have the following estimate.*

$$\ell_R[\alpha] \le \frac{\pi}{2}E_R[\alpha] \le \ell_R[\alpha]e^{\ell_R[\alpha]}.$$

In the above, $E_R[\alpha]$ denotes the extremal length of the curve family $[\alpha]$, more precisely, $E_R[\alpha] = \sup_\sigma(\iint_R \sigma(z)^2 dxdy)^{-1}$, where the supremum is taken over all Borel measurable conformal metrics σ satisfying that $\int_{\alpha'} \sigma(z)|dz| \ge 1$ for any $\alpha' \in [\alpha]$ (such a metric σ is called *admissible* for $[\alpha]$).

Furthermore, Pommerenke has given a remarkable characterization of uniform perfectness in terms of capacity density.

Theorem C (Pommerenke [13]). *$\Lambda(G)$ is uniformly perfect if and only if there exists a constant $c \in (0,1]$ such that* $\mathrm{Cap}(\Lambda(G) \cap B(a,r)) \ge cr$ *for any $a \in \Lambda(G)$ and $0 < r < \mathrm{diam}(\Lambda(G))$, where* Cap *denotes the logarithmic capacity and $B(a,r)$ the closed disk centered at a with radius r.*

Here we mention the work of Tsuji. He proved in [18] that any point of the limit set of a non-elementary (finitely generated) Schottky group has a positive capacity density (with a uniform bound). In view of the above theorem, this is a substantial proof of uniform perfectness of it. Earlier than this, Myrberg [11] showed that any non-elementary Kleinian group has the limit set of positive capacity.

We also note that $\mathrm{Cap}(B(a,r)) = r$ and $2^{-7}e^{-M^\circ_{\Omega(G)}}$ can be taken as the constant c in the above statement (see [16]). In particular, by virtue of Wiener's criterion, we have the following

Corollary 5.1. *If $L(\Omega(G)) > 0$ the limit set $\Lambda(G)$ is regular in the sense of Dirichlet.*

For general Kleinian groups, at least we can state the following

Corollary 5.2. *Let G be a non-elementary Kleinian group. Any loxodromic or parabolic fixed point of G is a regular point of $\Lambda(G)$ in the sense of Dirichlet.*

In fact, if z_0 is a fixed point of a loxodromic or parabolic element γ of G, then γ is contained in a finitely generated non-elementary subgroup G_0 of G. Since $\Lambda(G_0)$ is uniformly perfect, by Theorem C, we see that $\overline{\lim}_{r\to 0}\mathrm{Cap}(\Lambda(G)\cap B(z_0,r))/r \geq \overline{\lim}_{r\to 0}\mathrm{Cap}(\Lambda(G_0)\cap B(z_0,r))/r > 0$, which implies the regularity of $\Lambda(G)$ at z_0 (see, for example, Tsuji [19] p.104). Further, we should note that the set of loxodromic fixed points of a non-elementary Kleinian group is dense in the limit set.

Another application of unifom perfectness is concerned with the Hausdorff dimension. This sort of result is essentially due to Järvi-Vuorinen [6]. The following quantitative form follows from a result in [16].

Theorem D. *The Hausdorff dimension* $\text{H-dim}(\Lambda(G))$ *of* $\Lambda(G)$ *can be estimated from below as follows.*

$$\text{H-dim}(\Lambda(G)) \geq \frac{\log 2}{\log(2e^{M^\circ_{\Omega(G)}}+1)} \left(\geq \frac{\log 2}{M^\circ_{\Omega(G)} + \log 3} \right).$$

As an immediate consequence of this, we can see that any non-elementary Kleinian group has the limit set of positive Hausdorff dimemsion.

6. Estimate of translation length

As an application of (5.1), we present here an estimate of the translation length of a loxodromic element of Kleinian group in the region of discontinuity in terms of the trace or the multiplier. Before stating our result, we refer to a general result on translation length, which is suggested to the author by K. Matsuzaki. The original idea is due to Bers [3]. See also the proof of Proposition 6.4 in [9].

Let G be a non-elementary Kleinian group and H a component subgroup of G corresponding to a component Ω of the region of discontinuity $\Omega(G)$. Let h be a loxodromic element of H, i.e., $\eta = \mathrm{tr}^2(h) \in \mathbb{C}\setminus[0,4]$. By the Möbius invariance of $l_{h,\Omega}$ and $\mathrm{tr}^2(h)$, we may assume that h has the form $h(z) = \lambda z$ with $|\lambda| > 1$ where $\eta = (\sqrt{\lambda}+\sqrt{\lambda}^{-1})^2 = \lambda+\lambda^{-1}+2$. We note that $0, \infty \in \Lambda(G)$ under this assumption. For an arbitrary $z_0 \in \Omega$, let α be a geodesic arc joining z_0 and $h(z_0)$ in Ω such that $d_\Omega(z_0, h(z_0)) = \ell_\Omega(\alpha)$. Without loss of generality, we may further assume that $z_0 = 1$. Now we consider the quotient map

$p : \mathbb{C}^* \to \mathbb{C}^*/\langle h \rangle =: T$, and set $R = p(\Omega) = \Omega/\langle h \rangle \subset T$. Then $\beta = p_*\alpha = p(\alpha)$ is a closed geodesic in R, where R is endowed with the hyperbolic metric. We note here that $\ell_\Omega(\alpha) = \ell_R(\beta)$. In view of Theorem B, we have $\frac{\pi}{2} E_R[\beta] \le \ell_R[\beta] e^{\ell_R[\beta]}$. Since $[\beta] = [\beta]_R \subset [\beta]_T$, by the monotonicity of extremal length (cf. [1]), it turns out that $E_R[\beta] \ge E_T[\beta]_T$. Thus it is sufficient to compute $E_T[\beta]_T$.

The function $q(z) = \exp(2\pi i z)$ is a universal covering map of $\mathbb{C}^*$ from the complex plane $\mathbb{C}$. Let $\tilde{\alpha}$ be a lift of α via q with initial point 0 and τ denote the terminal point of $\tilde{\alpha}$. In particular, $\exp(2\pi i \tau) = \lambda$, so we may write $\tau = \frac{1}{2\pi i} \log \lambda$. Further note that $p \circ q : \mathbb{C} \to T$ is a quotient map of $\mathbb{C}$ by the lattice generated by 1 and τ over $\mathbb{Z}$. And it follows that any curve in $[\beta]_T$ is lifted to an arc with initial point a in $[0, 1]$ and terminal point $a + \tau$, and vice versa. Thus, by the standard length-area method (cf. [1]), one can easily see that the extremal admissible metric for $[\beta]_T$ is given by the projection σ of $|\tau|^{-1}|dz|$ under $p \circ q$, hence $E_T[\beta]_T = (\iint_T \sigma^2)^{-1} = (|\tau|^{-2} \cdot |\mathrm{Im}\tau|)^{-1} = |\tau|^2/|\mathrm{Im}\tau| = |\mathrm{Im}(1/\tau)|^{-1}$. Thus we have

$$\ell_R[\beta] e^{\ell_R[\beta]} \ge \frac{\pi |\tau|^2}{2|\mathrm{Im}\tau|} = \frac{|\log \lambda|^2}{4 \log |\lambda|}.$$

The quantity $\log \lambda$ is sometimes called the *complex length* of h (cf. [9]). Here, denote by $\mathrm{Log} z$ the principal branch of $\log z$, i.e., $\mathrm{Log} z$ is the branch of the logarithm determined by $-\pi < \mathrm{Im}\mathrm{Log} z \le \pi$. Since $|\log \lambda| \ge |\mathrm{Log}\lambda|$ and z_0 is arbitrary, we have the following

Proposition 6.1. *Let G be a non-elementary Kleinian group and H its component subgroup which corresponds to a component Ω of $\Omega(G)$. For any loxodromic element h of H with multiplier λ we have an estimate of the translation length of h in Ω as in the following:*

$$l_{h,\Omega} e^{l_{h,\Omega}} \ge \frac{|\mathrm{Log}\lambda|^2}{4 \log |\lambda|} \,(\ge \log |\lambda|/\,4).$$

The left-hand side of this inequality is of exponential order, but in general it seems to be difficult to improve this order. But, if the limit set $\Lambda(G)$ is uniformly perfect, we have an estimate of linear order.

Theorem 6.2. *Let H be a component subgroup of a non-elementary Kleinian group G corresponding to a component Ω of $\Omega(G)$, and suppose that $C = \frac{1}{4} \tanh(L(\Omega)/2) > 0$. Then, for a loxodromic element $h \in H$, the translation length $l_{h,\Omega}$ can be estimated as*

$$l_{h,\Omega} \ge C |\mathrm{Log}\lambda|,$$

where λ is the multiplier of h, i.e., $\mathrm{tr}^2(h) = \lambda + \lambda^{-1} + 2$.

Remarks. Noting that $C \ge \frac{1}{4} \tanh(L(\Omega(G))/2)$, we can see $C > 0$ if $\Lambda(G)$ is uniformly perfect.

In case Ω is simply connected, $L(\Omega) = \infty$ by definition, so we have $l_{h,\Omega} \ge |\mathrm{Log}\lambda|/4$.

We further remark that, in general, we cannot estimate the translation length from above by the trace or multiplier. This can be understood by the existence of accidental parabolic transformations.

Proof. We shall prove the theorem under the exactly same normalizations and notation in the previous sentences. If we denote by $\delta(z)$ the distance from z to $\Lambda(G)$, by (5.1), we have $\delta(z)\rho_\Omega(z) \geq C$ for $z \in \Omega$ and $\delta(z) \leq |z|$ because $0 \in \Lambda(G)$. Then, we compute

$$\begin{aligned} d_\Omega(z_0, h(z_0)) &= \int_\alpha \rho_\Omega(z)|dz| \geq C \int_\alpha \frac{|dz|}{\delta(z)} \geq C \int_\alpha \frac{|dz|}{|z|} \\ &= C \int_{\tilde{\alpha}} |d\zeta| \geq C \left| \int_{\tilde{\alpha}} d\zeta \right| = C|\log\lambda| \geq C|\mathrm{Log}\lambda|. \end{aligned}$$

Now the theorem is proved. □ □

7. Examples

In this section, we present simple examples of Kleinian groups of Schottky type. The first construction provides an example of an infinitely generated Kleinian group whose boundary is not uniformly perfect. Looking at Theorem 3.2, one may guess that it is sufficient to construct a Kleinian group whose quotient orbifold has arbitrarily short geodesics which are lifted to closed curves in the region of discontinuity. In fact, such an example can be given by infinitely generated Schottky groups as Pommerenke indicated in [13].

Example 7.1. Let $a_j, b_j \in \mathbb{C}$ be sequences tending to ∞, and $r_j > 0$ and $\alpha_j \in \mathbb{C}$ with $|\alpha_j| = 1$ be given so that all closed disks $A_j = B(a_j, r_j), B_j = B(b_j, r_j)$ are disjoint $(j = 1, 2, \dots)$. We set $g_j(z) = b_j - \frac{\alpha_j r_j^2}{z - a_j}$, then A_j and B_j are the isometric circles for g_j and g_j^{-1}, respectively, thus $G = \langle g_1, g_2, \dots \rangle$ is an infinitely generated Schottky group with a fundamental domain $\mathbb{C} \setminus \cup_j (A_j \cup B_j) \subset \Omega(G)$.

Now we set $\tilde{r}_j = \mathrm{dist}(a_j, (\cup_{k \neq j} A_k) \cup (\cup_k B_k)) > r_j$. Then, we can directly see that $M^\circ_{\Omega(G)} \geq \sup_j \tilde{r}_j / r_j$, hence $\Lambda(G)$ is not uniformly perfect if $\sup_j \tilde{r}_j / r_j = \infty$.

The second construction serves an infinitely generated Kleinian group G which contains a parabolic element h which does not represent any puncture of $\Omega(G)/G$.

Example 7.2. Let h be the transformation $z \mapsto z + 2i$. For $j \in \mathbb{Z}$, we take $a_j, b_j \in \mathbb{R}$ and $0 < r_j < 1$ in such a way that $b_j - a_j > 2r_j$ and $a_{j+1} - b_j > r_j + r_{j+1}$ and $\lim_{j \to \pm\infty} a_j = \pm\infty$. Taking a sequence α_j in the unit circle, we set $g_j(z) = b_j - \frac{\alpha_j r_j^2}{z - a_j}$. Then the domain $\omega = \{z \in \mathbb{C}; |\mathrm{Im} z| < 1\} \setminus \cup_j (A_j \cup B_j)$ is a fundamental domain of the Kleinian group G with free generators $h, g_j (j \in \mathbb{Z})$, where $A_j = B(a_j, r_j)$ and $B_j = B(b_j, r_j)$. In particular, one can observe that the Riemann surface $\Omega(G)/G$ has no punctures. Let $\beta_j : [-1, 1] \to \Omega(G)$ be the curve given by $\beta_j(t) = ti + (a_j + b_j)/2$. Since $\delta(\beta_j(t)) \geq (b_j - a_j)/2 - r_j$, we have

$$\ell_{\Omega(G)}(\beta_j) \leq \int_{\beta_j} \frac{|dz|}{\delta(z)} \leq \frac{4}{b_j - a_j - 2r_j},$$

where $\delta(z)$ denotes the Euclidean distance from z to $\Lambda(G)$. Since $h(\beta_j(-1)) = \beta_j(1)$, we see that $l_{h,\Omega(G)} \leq \inf_{j \in \mathbb{Z}} \ell_{\Omega(G)}(\beta_j)$. Therefore, if $\sup_j (b_j - a_j) = \infty$, then we obtain $l_{h,\Omega(G)} = 0$. On the other hand, if $r_j = r_0, \alpha_j = \alpha_0, a_j = a_0 + 2j(b_0 - a_0)$ and $b_j = b_0 + 2j(b_0 - a_0)$ for all $j \in \mathbb{Z}$, then G is a normal subgroup of the Kleinian group $\widetilde{G} = \langle h, h^*, g_0 \rangle$, where $h^*(z) = z + 2(b_0 - a_0)$. The property $G \lhd \widetilde{G}$ implies $\Omega(\widetilde{G}) = \Omega(G)$. Since $Y = \Omega(G)/\widetilde{G}$ is a compact Riemann surface of genus 2, thus $L(Y) > 0$, we have $\lambda(G) \geq L(X(G)) \geq L(Y) > 0$. In particular, $l_{h,\Omega} > 0$ in this case.

Finally we give a family of finitely generated torsion-free Kleinian groups which shows that the counterpart for L^* in Corollary 3.4 does not hold.

Example 7.3. Let $0 < r < 1$ and $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ be fixed. Set $h(z) = z + 2i$ and $g_t(z) = 2t - \alpha r^2/z$, where $t > 1$ is a real parameter. Then $G_t = \langle h, g_t \rangle$ is a free Kleinian group with fundamental domain $\omega = \hat{\omega} \setminus (B \cup B')$, where $\hat{\omega} = \{z \in \mathbb{C}; |\mathrm{Im} z| < 1\}, B = \{|z| \leq r\}$ and $B' = \{|z - 2t| \leq r\}$. Since h represents a pair of punctures of the quotient surface, the element h is contained in G_p. We now prove the following claim: *There exist constants $K < +\infty$ and $c > 0$ depending only on r such that*

$$c \leq \min\{L(\Omega(G_t)), \lambda^*(G_t)\} \quad \textit{and} \tag{7.1}$$

$$L^*(X(G_t)) \leq \frac{K}{t}. \tag{7.2}$$

Let us write $g = g_t, G = G_t$ and $\Omega = \Omega(G_t)$. We begin with a few preliminary observations. We set $B_n = h^n(B)$ and $B'_n = h^n(B')$ for $n \in \mathbb{Z}$. Then we know that $\Lambda(G) \cap \mathbb{C} \subset \tilde{\omega} := \mathbb{C} \setminus \cup_{n \in \mathbb{Z}} (B_n \cup B'_n)$. Since $\infty \in \Lambda(G)$, it follows that $0 = g^{-1}(\infty)$ and $2t = g(\infty)$ belong to $\Lambda(G)$. For later use, we estimate the quantity $d = \sup\{|z|; z \in B \cap \Lambda(G)\}$. First we note

$$d = \sup_{w \in \Lambda(G) \setminus B'} |g^{-1}(w)| = \sup_{w \in \Lambda(G) \setminus B'} \frac{r^2}{|w - 2t|} = r^2/\mathrm{dist}(2t, \Lambda(G) \setminus B').$$

Since $2 - r \leq \mathrm{dist}(2t, \Lambda(G) \setminus B') \leq 2$, we obtain $r^2/2 \leq d \leq r^2/(2 - r) < r^2$. In particular, we see

$$\mathrm{dist}(\partial B, B \cap \Lambda(G)) \geq 1 - r^2. \tag{7.3}$$

Noting $g^{-1}(2t \pm 2i) = \pm \alpha r^2/2 \in \Lambda(G)$, we also have $\mathrm{diam} \Lambda_0 \geq r^2$.

For simplicity, we further assume that $0 < r \leq 1/4$ in the sequel. First we prove $L(\Omega) \geq c_1 > 0$. To see this, it suffices to show that M°_Ω is uniformly bounded by Theorem A. Let A be an arbitrary round annulus in Ω which separates $\Lambda(G)$. Set $\rho = \exp(-m(A))$, where $m(A)$ denotes the modulus of A. We note here the following elementary lemma.

Lemma 7.1. *For the round annuli $A = \{\rho < |z| < 1\}$, we set $A_0 = \{\rho \frac{3+\rho}{1+3\rho} < |z| < \frac{1+3\rho}{3+\rho}\}$. Then, if a Möbius transformation T maps A into $\mathbb{C}$, we can take a round annulus A' in $T(A)$ in such a way that $T(A_0) \subset A'$. In particular, $m(A') \geq \log 1/\rho + 2 \log \frac{1+3\rho}{3+\rho} \geq m(A) - 2 \log 3$.*

Now we take the annulus A_0 in A with the property similar to the above. And choose an element $f \in G$ such that $f(A_0) \cap \omega \neq \emptyset$. Then $f(A)$ contains a round annulus $A' = \{z; r_0 < |z - a| < r_1\}$ with $f(A_0) \subset A'$. We note here that $\log r_1/r_0 = m(A') \geq m(A_0) \geq m(A) - 2 \log 3$. Now we estimate r_1/r_0 from above. Since A' separates $\Lambda(G)$, we have $r_1 - r_0 \leq 2$, hence $r_1/r_0 \leq 1 + 2/r_0$. By construction, the closed disk $E = \{z; |z - a| \leq r_0\}$ intersects $\Lambda_n = B_n \cap \Lambda(G)$ or $\Lambda'_n = B'_n \cap \Lambda(G)$ for some $n \in \mathbb{Z}$. Conjugating by h^n, we may assume that E intersects Λ_0 or Λ'_0. Assume that $E \cap \Lambda_0 \neq \emptyset$. (The other case can be treated similarly, so we omit it.) We further divide the case into three parts.

1. In the case $E \cap \Lambda(G) \setminus \Lambda_0 \neq \emptyset$, we have $2r_0 \geq 2 - 2r^2$, thus $r_1/r_0 \leq 1 + 2/(1 - r^2) < 5$.
2. In the case $E \cap \Lambda(G) = \Lambda_0$, we have $2r_0 \geq \mathrm{diam} \Lambda_0 \geq r^2$, thus $r_1/r_0 \leq 1 + 2/r^2$.

3. In the case $E \cap \Lambda(G) \subsetneq \Lambda_0$, the annulus A' separates Λ_0, therefore $r_1 - r_0 \leq \mathrm{diam}\Lambda_0 \leq 2r^2$. On the other hand, by assumption, we can take points $z \in \omega \cap A'$ and $w \in E \cap \Lambda(G)$. Then $r_1 + r_0 \geq |z-w| > \mathrm{dist}(\partial B, B \cap \Lambda(G)) \geq 1 - r^2$. By these inequalities, it follows that $r_1/r_0 \leq (1+r^2)/(1-3r^2) \leq 5$.

In any case, $e^{m(A)} - 2\log 3 \leq r_1/r_0 \leq 1 + 2/r^2$ holds provided that $r \leq 1/4$. Hence we have $M_\Omega^\circ \leq \log(1+2/r^2) + 2\log 3$, which implies $L(\Omega) \geq c_1 = c_1(r) > 0$.

In particular, by Theorem A, $\delta(z)\rho_\Omega(z) \geq C_1$ holds for $z \in \Omega$, where $\delta(z) = \mathrm{dist}(z, \partial\Omega)$ and C_1 is a positive constant depending only on r. Using this fact, we next show $\lambda^*(G) \geq c_2 > 0$. First we note the following estimate:

$$\lambda^*(G) = \inf_{f \in G \setminus (G_e \cup G_p)} \inf_{z \in \omega} d_\Omega(z, f(z)) \geq \inf_{f \in G \setminus \langle h \rangle} \inf_{z \in \omega} d_\Omega(z, f(z)).$$

For $f = g^{\pm 1}$, we can use Theorem 6.2 to show

$$d_\Omega(z, f(z)) \geq l_{g,\Omega} \geq C_1 \log|\lambda|,$$

where λ is the multiplier of g with $|\lambda| \geq 1$. By the relation $\lambda + \lambda^{-1} = 4t^2/\alpha r^2 - 2$, we have $|\lambda| \geq 4t^2/r^2 - 3 > 1$.

Now suppose that $f \in G \setminus \langle h \rangle$ is not $g^{\pm 1}$. Let z_0 be a point in ω. If $f(z_0) \in B$ then $f(\hat{\omega}) \subset B$, equivalently, $f^{-1}(\widehat{\mathbb{C}} \setminus B) \subset \widehat{\mathbb{C}} \setminus \hat{\omega}$, which implies $f^{-1}(z_0) \in \mathbb{C} \setminus \hat{\omega}$. When $f(z_0) \in B'$, the same result holds. Since $f \notin \langle h \rangle$, we actually see $f^{-1}(z_0) \in \cup_{n \in \mathbb{Z} \setminus \{0\}} (B_n \cup B_n')$ in these cases. Thus, without loss of generality, we may assume that z_0 belongs to B_n or B_n' for some non-zero integer n because $d_\Omega(z_0, f(z_0)) = d_\Omega(z_0, f^{-1}(z_0))$. Let α be a geodesic arc joining z_0 and $f(z_0)$ in Ω such that $\int_\alpha \rho_\Omega = d_\Omega(z_0, f(z_0))$. Since $\omega_n := h^n(\omega)$ separates $f(z_0)$ from z_0, a component α_0 of $\alpha \cap \omega_n$ connects a bounded boundary component of ω_n with an unbounded one. We now estimate the hyperbolic length of α_0 in Ω from below. By translation invariance under h and symmetry of Ω, we may assume that α_0 connects ∂B with $\{z; \mathrm{Im} z = 1\}$.

Letting ζ_0 and ζ_1 be endpoints of α_0 with $|\zeta_0| = r$ and $\mathrm{Im}\zeta_1 = 1$, we can estimate as follows:

$$\int_\alpha \rho_\Omega(z)|dz| \geq \int_{\alpha_0} \rho_\Omega(z)|dz| \geq C_1 \int_{\alpha_0} \frac{|dz|}{\delta(z)} \geq C_1 \int_{\alpha_0} \frac{|dz|}{|z|} \geq C_1 \log\frac{1}{r}.$$

Here, we used the fact $\delta(z) \leq |z|$. Consequently, we obtain $\lambda^*(G) \geq c_2$, where $c_2 = C_1 \min\{\log(4t^2/r^2 - 3), \log 1/r\}$. Whence, the inequality (7.1) is now shown.

Finally, we show the inequality (7.2). Let $\beta : [-1, 1] \to \Omega$ be the curve given by $\beta(s) = 1 + si$. Then the image of β in $X(G)$ is a non-trivial closed curve which is not homotopic to any puncture. Therefore, by the method same as in Example 7.2, we have

$$L^*(X(G)) \leq \ell_\Omega(\beta) \leq \frac{2}{t-r} \leq \frac{K}{t},$$

where $K = 2/(1-r)$.

Remark. Further, in the same fashion as in Example 7.2, we can construct an infinitely generated free Kleinian group G with the property that $\min\{L(\Omega(G)), \lambda^*(G)\} > 0$ while $L^*(X(G)) = 0$.

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