On Teichmüller spaces of complex dynamics by entire functions

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Abstract

In this paper, we give a very brief exposition of the general Teichmüller theory for complex dynamics and illustrate it with examples in the case of entire functions.

1991 Mathematics Subject Classification. Primary 30D05, 30F60
Keywords and phrases. Entire functions, Teichmüller spaces, complex dynamics

1 A Teichmüller theory for entire functions

Recently, McMullen and Sullivan have constructed a general Teichmüller theory for complex dynamics [24], which gives a lot of new viewpoints to the research of complex dynamics. In this paper, we give a very brief exposition of the theory and illustrate it with examples in the case of entire functions. In the seqeul, we discuss the case of transcendental entire functions only.

The authors would like to express hearty thanks to the referee and our colleague, T. Sugawa for their helpful comments.

Definition A transcendental entire function f induces an equivalence relation on the whole plane $\mathbf{C}$: z and w are *equivalent* if there are positive integers n and m such that $f^n(z) = f^m(w)$. We call this equivalence relation the *grand orbit equivalence relation* for f, and the equivalence class *the grand orbit.*

A point of $\mathbf{C}$ belongs to either the Fatou set or the Julia set of a given entire function f, according as the iterations of f form a normal family in a neighborhood of the point or do not. The grand orbit of any component of the Fatou set contains one of the following domains; an attractive basin, a super-attractive basin, a parabolic basin, a Siegel disk, a Baker domain, and a wandering domain. (For the details, see for instance [4], [25] and references of them.)

Recall that entire functions have no Herman rings and that every Baker domain is simply connected.

Definition An *invariant Beltrami differential* μ for f on a completely invariant set E is a measurable function on E that satisfies

$$f^*\mu(z) \left(= \mu(f(z))\overline{f'(z)}/f'(z)\right) = \mu(z). \tag{1}$$

Now considering a transcendental entire function f as a (branched and incomplete) covering of $\mathbf{C}$ by itself, we can define its deformation space.

Definition We say that two quasiconformal automorphisms ϕ_1 and ϕ_2 of $\mathbf{C}$ are *conformally equivalent* if $\phi_1 = \psi \circ \phi_2$ with a conformal map ψ.

The *deformation space* $\mathrm{Def}(\mathbf{C}, f)$ of the covering $f : \mathbf{C} \to \mathbf{C}$ is the totality of the conformal equivalence classes of quasiconformal automorphisms ϕ of $\mathbf{C}$ satisfying

$$\phi \circ f = g \circ \phi \tag{2}$$

with some entire functions $g = g_\phi$.

Quasiconformal automorphisms ϕ satisfying (2) with some entire g send the grand orbit equivalence relation for f to that for g.

Remark By the Ahlfors-Bers measurable mapping theorem (see [20] Theorem 4.30), $\mathrm{Def}(\mathbf{C}, f)$ can be identified with the open unit ball $M_1(\mathbf{C}, f)$ of $M(\mathbf{C}, f)$ consisting of all invariant Beltrami differentials for f, whose elements are called *invariant Beltrami coefficients*.

Definition Let $\mathrm{QC}_0(\mathbf{C}, f)$ be the set of all quasiconformal automorphisms of $\mathbf{C}$ isotopic to the identity by uniformly quasiconformal maps which are commutative with f.

The *Teichmüller space* $\mathrm{Teich}(\mathbf{C}, f)$ of the covering $f : \mathbf{C} \to \mathbf{C}$ is the quotient space of $\mathrm{Def}(\mathbf{C}, f)$ by $\mathrm{QC}_0(\mathbf{C}, f)$.

Remark We also consider the set $\mathrm{Rep}(f)$ of all entire functions g_ϕ as in (2) with $\phi \in \mathrm{Def}(\mathbf{C}, f)$. This way was taken in the celebrated paper [15] of

Eremenko and Lyubich, and leads to explicit families of entire functions as are discussed later.

To state the fundamental structure theorem of $\mathrm{Teich}(\mathbf{C}, f)$, we further need several definitions.

Definition Let $\hat{J}$ be the closure of grand orbits of periodic points and those intersecting $\mathrm{sing}(f^{-1})$. We write the complement of $\hat{J}$ as $\hat{F}$. Let F^{dis} be the subset of $\hat{F}$ consisting of all grand orbits discrete in $\hat{F}$. Finally, we set $F^{fol} = \hat{F} - F^{dis}$.

A theorem of Fatou implies that periodic points are dense in the Julia set J. Hence $\hat{J}$ contains the Julia set J of f. Also note that a deep theorem of Baker states that repelling ones only are dense in the Julia set.

A fundamental structure theorem of $\mathrm{Teich}(\mathbf{C}, f)$ is Theorem 1 below ([19]), which is essentially due to McMullen and Sullivan.

Theorem 1 *Suppose that the set* $\mathrm{sing}(f^{-1})$ *of all singular values is a countable set, then*

$$\mathrm{Teich}(\mathbf{C}, f) = M_1(\hat{J}, f) \times \mathrm{Teich}(F^{fol}, f) \times \mathrm{Teich}(F^{dis}, f)$$

Remark The same assertion follows under a weaker condition that the subset $\mathrm{sing}(f^{-1}) \cap F$ is countable. Indeed, the set Per consisting of periodic points in J is dense in J and we can apply the argument in the proof below to the grand orbits of $Per \cup (\mathrm{sing}(f^{-1}) \cap F)$, which is still countable.

Proof. By the invariance of Beltrami coefficients, the topological covering structures are unchanged under the isotopy in $\mathrm{QC}_0(\mathbf{C}, f)$. Since $\mathrm{sing}(f^{-1})$ is countable by the assumption, it is fixed pointwise under the isotopy in $\mathrm{QC}_0(\mathbf{C}, f)$, and hence so is $\hat{J}$. Thus

$$\mathrm{QC}_0(\mathbf{C}, f) = \mathrm{QC}_0(\hat{F}, f),$$

and we conclude

$$\mathrm{Teich}(\mathbf{C}, f) = M_1(\hat{J}, f) \times \mathrm{Teich}(\hat{F}, f).$$

■

Since f is a smooth covering of $\hat{F}$ onto itself, the general theory of McMullen and Sullivan ([24]) implies the following.

Proposition 2 *The factors* $M_1(\hat{J}, f)$ *and* $\mathrm{Teich}(F^{fol}, f)$ *are the unit balls of the* L^∞*-space and of the* ℓ^∞*-space, respectively.*

The discrete part $\mathrm{Teich}(F^{dis}, f)$ *is identified with the Teichmüller space* $T(F^{dis}/f)$ *of a union of Riemann surfaces.*

Fatou components producing non-trivial factors of $\mathrm{Teich}(F^{fol}, f)$ *are super-attractive basins, Siegel disks, and wandering domains.*

Those for $\mathrm{Teich}(F^{dis}, f)$ *are attractive basins, parabolic basins, wandering domains, and Baker domains.*

The discrete part $\mathrm{Teich}(F^{dis}, f)$ represents the visible action of f, and is important in itself.

Remark Eremenko and Lyubich constructed (in [14]) an example where the space $M(J, f)$ is of infinite dimension. And we will give various conditions which implies that $M(J, f)$ is trivial. Wandering domains and Baker domains contribute nothing in some cases, and infinite dimensions in other cases, to the Teichmüller space. See Examples 4 and 5. On the other hand, the authors do not give an explicit example of a wandering domain which gives a non-trivial factor of $\mathrm{Teich}(F^{fol}, f)$.

2 Dimension estimates

To estimate the dimension of the Teichmüller space, it is natural to consider some representation space containing it. In the case of rational functions (or polynomials), the coefficients of functions give parameters of such a space. In the case of entire functions, Eremenko and others used the loci of singular values possibly with two accessary parameters [15], [17]. Suitable choice of such a representation space provides a universe of various (possibly some quotients of) Teichmüller spaces, as the c-plane in the case of the quadratic family $\{z^2 + c\}$. In particular, we know the following

Proposition 3

$$\dim \mathrm{Rep}(f) \leq \#\mathrm{sing}(f^{-1}) + 2,$$

or equivalently

$$\dim \mathrm{Teich}(\mathbf{C}, f) \leq \#\mathrm{sing}(f^{-1}).$$

On the other hand, the McMullen-Sullivan theory gives a lower bound of the dimensions. As far as entire functions concern, the hard obstruction to get the exact value is the existence of wandering and Baker domains. Actually, we have known little about wandering domains and Baker domains. So, no wandering domains theorems such as follows are useful.

Proposition 4 (No wandering domains theorem of primitive type) *If the discrete part* $\mathrm{Teich}(F^{dis}, f)$ *is of finite dimension, then there are no eventually singular-value free, simply connected wandering domains.*

Remark Absence of multiply connected wandering domains is equivalent to the condition that $J \cup \{\infty\}$ is connected in $\hat{\mathbf{C}}$ (cf.[22]). On the other hand, sufficient conditions for connectedness of the Julia set have been given in [8], [26], which imply absence of multiply connected wandering domains.

Now we turn to famous tame families of entire functions.

Definition

1. The class B consists of *critically bounded* entire functions, namely those with bounded singular values.

2. The *Speiser class* S consists of those with a finite number of singular values.

3. The class C consists of entire functions f such that the closure of the forward orbits of $\mathrm{sing}(f^{-1})$ is compact and has a positive distance from the Julia set.

Recall that any composition of two elements of S again belongs to S ([3], [17]) and that every function in C has no Siegel disks. In the case of rational functions, every function in the class C is called *hyperbolic*, and its Julia set has vanishing area (cf. [23]). But in the case of entire functions, the Julia set of a function in $S \cap C$ may have a positive area. The sine family gives such an example ([23]). See Example 2.

The McMullen-Sullivan theory gives the precise dimensions of the Teichmüller spaces in some cases.

Theorem 5 *Let $f \in B \cap C$, and suppose that the number N_{AC} of the foliated equivalence classes of acyclic (i.e. neither periodic nor strictly preperiodic) singular values in the Fatou set is finite. Then*

$$\dim \mathrm{Teich}(\mathbf{C}, f) = N_{AC} + \dim M(J, f).$$

For every $f \in S$,

$$\dim \mathrm{Teich}(\mathbf{C}, f) = N_{AC} + \dim M(J, f) - N_P,$$

where N_P is the number of cycles of parabolic periodic points.

Here we say that two points are in the same *foliated equivalence class* if the closures of their grand orbits coincide.

Proof. First, since N_{AC} is finite, the closure of the grand orbits of acyclic singular values in the Fatou set is a countable union of points and analytic curves. Hence $M(\hat{J}, f) = M(J, f)$.

It is known [15] that every $f \in B$ has no Baker domains. Further, if $f \in C$, then f has no wandering domains ([7]). This is true also when $f \in S$ by [15], [17]. Thus by the same argument as in [24], we conclude the assertion. ∎

Remark One of Bergweiler's conjectures states that every $f \in B$ has no wandering domains. Clearly, entire functions not belonging to B may have no wandering domains. $e^{e^z} - e^z$ is such an example ([3]). Also cf. [4].

For other conditions on absence of Baker domains, see [27].

We can consider some larger class EL of all entire functions such that the Julia set is coincident with the closure of the set of *escape points* (i.e. points whose forward orbits tend to the infinity). Eremenko and Lyubich showed that the class EL contains the class B, and we have another kind of no wandering domains theorem.

Theorem 6 (No wandering domains theorem) *Suppose that $f \in EL$ and that $\mathrm{sing}(f^{-1}) \cap F$ is finite. Then f has no wandering domains if and only if $\dim \mathrm{Teich}(F^{dis}, f)$ is finite.*

Moreover, if so,

$$\dim \mathrm{Teich}(F^{dis}, f) = N_{AC} - N_P$$

Proof. The first assumption implies that there are no Baker domains and no multiply connected wandering domains. If there exists a simply connected wandering domain D, then the assumption and Propostion 4 imply that $\dim \mathrm{Teich}(F^{dis}, f)$ should be infinite. Hence the if-part of the first assertion follows.

Next, if there are no wandering domains, then by considering invariant Beltrami coefficients supported on the Fatou set, we can show the other assertions as in [24]. ■

Corollary 1 *Suppose that $f \in EL$ and that every singular value is either an escape point, or strictly preperiodic, or repelling periodic. Further if there are no Siegel disks and no simply connected wandering domains, the Julia set of f is* $\mathbf{C}$.

There may be a set of infinitely many singular values whose closure contains the boundary of either a Siegel disk or a simply connected wandering domain without singular values.

Proof. The first assumptions gives that there are no Baker domains and no multiply connected wandering domains. The second assumption means that every singular value in the Fatou set should be strictly preperiodic. Then the other assumptions and Theorem 6 gives that $\dim \mathrm{Teich}(F^{dis}, f) = N_{AC} = N_P = 0$. Hence there are no attractive basins, no parabolic ones, and no super-attractive basins. ■

Next, a condition for absence of invariant line fields (which represent non-trivial invariant Beltrami differentials) can be restated as structural instability as follows: Suppose that the Julia set of an entire function f_0 is $\mathbf{C}$, and that, for every holomorphic slice $\{f_t\}$ through f_0 in a representation space natural and faithful in a sense, there is a sequence $\{t_n\}$ tending to 0 such that the Julia set of every f_{t_n} is a proper subset of $\mathbf{C}$. Then

$$\dim M(\mathbf{C}, f_0) = 0,$$

namely there are no invariant line fields for f_0 on $\mathbf{C}$.

Also, Proposition 3 and Theorem 5 give the following

Corollary 2 (No invariant line fields theorem) *Let $f \in S$ and have no parabolic periodic points. Suppose that all singular values are acyclic and* *belong to mutually different foliated equivalence classes in the Fatou set. Then there are no invariant line fields on the Julia set J, or equivalently*

$$\dim M(J,f)=0.$$

Note that absence of invariant line fields on the Julia set J does not imply ergodicity of the action of f on J. See Example 2.

3 Case studies

We will discuss some examples. First, we consider the exponential family.

Example 1 Since e^z has the single singular value 0 and hence by Proposition 3, we need to prepare the 3-dimensional family

$$\{e^{az+b}+c \mid a\in \mathbf{C}^*,\ b,c\in \mathbf{C}\}.$$

By taking conformal conjugacy equivalence classes, we may use the 1-dimensional exponential family

$$\mathcal{E}=\{f_\lambda(z)=\lambda e^z \mid \lambda\in \mathbf{C}^*\}.$$

Fix an element $f=f_\lambda\in\mathcal{E}$. If $\mathrm{Teich}(\mathbf{C},f)$ (which is at most one-dimensional, for $\mathrm{sing}(f^{-1})$ consists of only one asymptotic value) is not trivial, then the family $\mathcal{E}$ gives a local chart of $\mathrm{Teich}(\mathbf{C},f)$ near f. (More explicitly, there are a domain D containing λ and a holomorphic covering map of $\mathrm{Teich}(\mathbf{C},f)$ onto D.) Also recall that the exponential family $\mathcal{E}$ is *topologically complete* in the sense that any entire function topologically conjugate to an element of the family is actually conformally conjugate to an element.

In particular, if $f\in\mathcal{E}$ has an attracting periodic point, then the Julia set admits no invariant line fields (and is area 0 in this case. See [23]). This is still true for every element accumulated by such ones.

Recall the similarity of the exponential family to the quadratic family $\{z^2+c\}$; both are controled by the forward orbit of a single point. Cf. [2], [10], [11], [13], [16], [30], [31]. And we can formulate the hyperbolic-dense conjecture.

Conjecture ([4] Question 19) In the exponential family $\mathcal{E}$, the subset of functions belonging to the class C is dense.

When this conjecture is true, every function in the family admits no invariant line fields on the Julia set.

Also since every f in the class C can not have $\mathbf{C}$ as the Julia set, we can consider another kind of

Conjecture

The set of all f whose Julia sets have no interior points are dense in $\mathcal{E}$.

Every f having $\mathbf{C}$ as the Julia set is structurally unstable.

Such conjectures can be formulated for other examples below. On the other hand, each family has clearly an individual character. For instance, in the case of the exponential family, there exist no super-attracting periodic points. Hence the center of each stable component should disappears, except for the main one.

Example 2 Next, we consider the sine family. Since $\sin z$ has two critical values $\{\pm 1\}$, we consider the 4-dimensional family

$$\{c \sin(az+b)+d\},$$

or by taking conformal conjugacy equivalence classes, the 2-dimensional family

$$\mathcal{S} = \{f_{a,b}(z) = \sin(az+b) \mid a \in \mathbf{C}^*,\ b \in \mathbf{C}\}.$$

Let $f = f_{a,b} \in \mathcal{S}$. Then $\mathrm{Teich}(\mathbf{C}, f)$ is at most two-dimensional, and this family gives a locally faithful representation space of $\mathrm{Teich}(\mathbf{C}, f)$. Recall that every element of this family has the Julia set of positive measure ([23]).

Actually, the sine family $\mathcal{S}$ is contained in an infinite dimensional family, which distinguishes every critical points. But $\mathcal{S}$ is topologically complete, and hence of great importance.

Now apply Proposition 3 and Theorem 5, and we have the following

Proposition 7 *Let $f \in C \cap \mathcal{S}$, and suppose that two singular values belong to different foliated equivalence classes. Then*

$$\mathrm{Teich}(\mathbf{C}, f) = \mathrm{Teich}(F^{dis}, f),$$

and is two-dimensional. In particular, there are no invariant line fields on the Julia set J.

Proof. Teich(F^{dis}, f) is the Teichmüller space of a twice punctured torus or of two once punctured tori, and hence of two dimension. Hence the assertion follows. ∎

Remark For instance, if $b = 0$ and $0 < |a| < 1$, we can apply Proposition 7.

McMullen further showed in [23] that, for every element in the sine family, the action on the Julia set is not ergodic, and that, under the assumption of Proposition 7, the set of escape points has full measure in the Julia set.

Example 3 A family

$$\{-(e^{2a}/2a)z^2e^z \mid 0 < |a-1| < 1/2\}$$

has an attracting fixed point $-2a$ near, but not equal to -2 (which is super-attracting if $a = 1$). It has two critical points at $0, -2$ and asymptotic value 0. So we may consider that this family is a complex submanifold of a more general representation space.

Such a 3-dimensional representation space is

$$\mathcal{F}_2 = \{f_{a,b,c}(z) = \lambda(z+b)(z+c)e^{az} \mid \lambda(b-2)(c-2)e^{-2a} = -2\}.$$

The normalization conditions of this family is that 0 is the asymptotic value and -2 is one of fixed points. Take an f such that 0 and -2 are attracting fixed points, and that two critical values are near, but not equal to, 0 and -2, respectively: $abc + b + c$ and $a(2-b)(2-c) + (b+c-4)$ are sufficiently small, but nonzero. Also note that $\mathcal{F}_2$ is topologically complete (cf. [29]).

And the general theory in §2 means the following

Proposition 8 *For such an f as above,*

$$\mathrm{Teich}(\mathbf{C}, f) = \mathrm{Teich}(F^{dis}, f),$$

and is three-dimensional. In particular, there are no invariant line fields on the Julia set J.

Proof. Teich(F^{dis}, f) contains the Teichmüller spaces of a once punctured torus and a twice punctured torus. Hence it is at least three-dimensional. ∎

As for the first family in Example 3, we have the following

Proposition 9 *Each element of a family*

$$\{(e^{2a}/2a)z^2e^{-z} \mid 0 < |a-1| < 1/2\},$$

admits no invaliant line fields on the Julia set and has the one-dimensional Teichmüller space (coming from the attractive basin for $-2a$).

Now, take a logarithmic lift (cf. [15]), we have a family

$$\mathcal{L} = \{2a - \log 2a + 2z - e^z \mid 0 < |a-1| < 1/2\}.$$

The element with $a = 1$ is an example of Bergweiler in [6], which is also interesting in our viewpoint.

Example 4 Wandering domains may contribute nothing to $\mathrm{Teich}(\mathbf{C}, f)$. A famous example in [1] and [18] (also cf. [21])

$$f(z) = z - e^z + 1 + 2\pi i$$

gives a wandering domain D contained in F^{fol} and $\mathrm{Teich}([D]/f)$ is trivial. Here $[D]$ is the grand orbit of D. Bergweiler's example

$$2 - \log 2 + 2z - e^z$$

stated above gives another such wandering domain.

In fact, since D is simply connected and f maps singular values to themselves, taking copies $\Delta_n = \{|z_n| < 1\}$ of the unit disk, we may assume that f on $f^n(D)$ are $P(z_n) = z_{n+1}^2$. Suppose that there is an invariant Beltrami differential μ on $[D]$. Let μ_n be μ restricted on Δ_n. Then

$$\mu_n(P^n(z_0))\,\overline{(P^n)'(z_0)}/(P^n)'(z_0) = \mu_0(z_0).$$

Set $\omega_n = e^{2^{1-n}\pi i}$, and we see that all $\omega_n^m z_0$ belong to the grand orbit of z_0, and that

$$\mu_0(\omega_n z_0)\,\overline{\omega_n}/\omega_n = \mu_n(P^n(z_0))\,\overline{(P^n)'(z_0)}/(P^n)'(z_0) = \mu_0(z_0).$$

Thus by a direct construction or by the same argument as in [24], we conclude that μ_0, and hence μ itself, corresponds to an element of $\mathrm{QC}_0(\mathbf{C}, f)$.

Here also note that, elements of the family $\mathcal{L}$ with $a \neq 1$ have wandering domains D in F^{dis} such that $\mathrm{Teich}([D]/f)$ is that of $\mathbf{C} - \mathbf{Z}$, and hence of infinite dimension.

In fact, we can take as a fundamental set all copies of a suitable ring domain in D with the single critical value on the boundary, and patching up all of them canonically, we have a Riemannn surface equivalent to $\mathbf{C} - \mathbf{Z}$.

Remark Recall that Eremenko and Lyubich [14] showed existence of a wandering domain D where f^n are eventually univalent, which means that $D \subset F^{dis}$ and $\mathrm{Teich}(F^{dis}, f)$ is of infinite dimension. As an explicit known example, we cite Herman's one [18]

$$f(z) = z + (\lambda - 1)(e^z - 1) + 2\pi i$$

with a linearizable λ such that $|\lambda| = 1$.

Example 5 Baker domains may contribute nothing to $\mathrm{Teich}(\mathbf{C}, f)$. An example is

$$f(z) = z + e^z.$$

This f has infinitely many Baker domains, each D of which is contained in F^{dis} and $\mathrm{Teich}([D], f)$ is the Teichmüller space of a thrice punctured sphere and hence trivial.

On the other hand, Bergweiler's f in Example 4 again gives an example of a Baker domain D without singular values, and contained in F^{dis}. $\mathrm{Teich}([D]/f)$ is that of an annulus, and hence $\mathrm{Teich}([D]/f)$ is of infinite dimension.

In fact, on a half plane $\{\mathrm{Re}\ z \leq -R\}$ with sufficiently large R, the hyperbolic metric on D is nearly that on the left half plane, and $f(z)$ is nearly $z \mapsto 2z$. Thus on the quotient hyperbolic surface $[D]/f$, the hyperbolic length of any non-trivial loop passing through the point corresponding to z tends to infinity as $|\mathrm{Im}\ z|$ tends to ∞. Hence $[D]/f$ is an annulus.

A classical example of Fatou

$$f(z) = z + e^{-z} + 1$$

also gives a Baker domain D such that $\mathrm{Teich}([D]/f)$ is that of $\mathbf{C} - \mathbf{Z}$, and hence again of infinite dimension.

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