A COEFFICIENT INEQUALITY FOR BLOCH FUNCTIONS WITH APPLICATIONS TO UNIFORMLY LOCALLY UNIVALENT FUNCTIONS

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Abstract. We give a Fekete-Szegö type inequality for a Bloch function with Bloch seminorm ≤ 1. As an application of it, we derive a sharp coefficient inequality for a_3 for a uniformly locally univalent function $f(z) = z + a_2z^2 + a_3z^3 + \dots$ on the unit disk with pre-Schwarzian norm $\leq \lambda$ for a given $\lambda > 0$.

1. Introduction

Let $\mathscr{S}$ be the class of univalent (analytic) functions f on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 1$ and $f'(0) = 1$. Thus a function f in $\mathscr{S}$ can be expanded in the form

$$f(z) = z + a_2z^2 + a_3z^3 + \dots, \quad |z| < 1.$$

Bieberbach proved the inequality $|a_2| \leq 2$ and conjectured that $|a_n| \leq n$ holds for every n in 1916. After the proof of $|a_3| \leq 3$ by Löwner in 1923, Fekete and Szegö [3] surprised mathematicians by showing that the complicated inequality

$$|a_3 - \mu a_2^2| \leq 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right)$$

holds and is best possible for each $0 \leq \mu \leq 1$. We remark that $a_3 - a_2^2$ equals $S_f(0)/6$, where S_f is the Schwarzian derivative of f : $S_f = (f''/f')' - (f''/f')^2/2$. The above inequality suggests that the shape of the coefficient region $\{(a_2, a_3) \in \mathbb{C}^2 : \exists f \in \mathscr{S}$ such that $f(z) = z + a_2z^2 + a_3z^3 + \dots\}$ is quite complicated. Note that this coefficient region was thoroughly investigated by Schaeffer and Spencer [6].

In general, given a class $\mathscr{F}$ of normalized analytic functions on the unit disk $\mathbb{D}$ and a real (or, more generally, a complex) number μ, the Fekete-Szegö problem asks to find the best possible constant $C(\mu)$ so that $|a_3 - \mu a_2^2| \leq C(\mu)$ for every function $f(z) = z + a_2z^2 + a_3z^3 + \dots$ in $\mathscr{F}$. Many papers have been devoted to this problem (see, for instance, [2] and references therein).

A function F on $\mathbb{D}$ is called a *Bloch function* if the Bloch seminorm

$$\|F\|_{\mathscr{B}} = \sup_{z\in\mathbb{D}} (1 - |z|^2)|F'(z)|$$

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is finite. We denote by $\mathscr{B}$ the complex Banach space consisting of Bloch functions F on $\mathbb{D}$ normalized by $F(0)=0$ and set $\mathscr{B}_1=\{F\in\mathscr{B}:\|F\|_{\mathscr{B}}\le 1\}$. Our first principal result is stated as follows.

Theorem 1. *Let $\mu\in\mathbb{C}$. Then the sharp inequality*

$$|b_2+\mu b_1^2|\le\begin{cases}\dfrac{1+3\sqrt{3}|\mu|^3+(1+3|\mu|^2)^{3/2}}{6\sqrt{3}|\mu|^2} & (|\mu|>\frac{4}{3\sqrt{3}})\\[2ex] \dfrac{3\sqrt{3}}{4} & (|\mu|\le\frac{4}{3\sqrt{3}})\end{cases}$$

holds for every function $F(z)=b_1z+b_2z^2+\dots$ in $\mathscr{B}_1$.

The inequality in Theorem 1 can be regarded as a variant of the Fekete-Szegö inequality for $\mathscr{B}_1$.

An analytic function f on $\mathbb{D}$ is called *uniformly locally univalent* if there is a constant $\rho=\rho(f)$ such that f is univalent in each hyperbolic disk of radius ρ. It is known that f is uniformly locally univalent if and only if the norm

$$\|T_f\|_{\mathbb{D}}=\sup_{z\in\mathbb{D}}(1-|z|^2)|T_f(z)|$$

is finite, where $T_f=f''/f'$ is the pre-Schwarzian derivative of f. It is also known that f is (globally) univalent if $\|T_f\|_{\mathbb{D}}\le 1$ and, conversely, $\|T_f\|_{\mathbb{D}}\le 6$ holds if f is univalent. We denote by $\mathscr{U}$ the class of uniformly locally univalent functions f on $\mathbb{D}$ normalized by $f(0)=0$ and $f'(0)=1$. Let $\mathscr{U}(\lambda)$ be the subclass of $\mathscr{U}$ consisting of those functions f satisfying $\|T_f\|_{\mathbb{D}}\le\lambda$.

In [4] Y. C. Kim and the first author observed various properties of uniformly locally univalent functions. They obtained, among others, the asymptotic estimate $a_n=O(n^\alpha)$ for every function $f(z)=z+a_2z^2+a_3z^3+\dots$ in $\mathscr{U}(\lambda)$ and every number α with $\alpha<(\sqrt{1+\lambda^2}-3)/2$. However, they did not have a sharp coefficient inequality except for the trivial one: $|a_2|\le\lambda/2$. We apply Theorem 1 to obtain the following result.

Theorem 2. *Let $\lambda>0$. Then the sharp inequality*

$$|a_3|\le\begin{cases}\dfrac{8+3\sqrt{3}\lambda^3+(4+3\lambda^2)^{3/2}}{36\sqrt{3}\lambda} & (\lambda>\frac{8}{3\sqrt{3}})\\[2ex] \dfrac{\sqrt{3}}{4}\lambda & (\lambda\le\frac{8}{3\sqrt{3}})\end{cases}$$

holds for every function $f(z)=z+a_2z^2+a_3z^3+\dots$ in $\mathscr{U}(\lambda)$.

2. Proof of Theorems 1 and 2

For a positive integer n, we consider the set

$$\mathbf{B}_n=\{(b_1,\dots,b_n)\in\mathbb{C}^n:\exists F\in\mathscr{B}_1\text{ such that }F(z)=b_1z+\dots+b_nz^n+\dots\},$$

which is sometimes called the *coefficient region* of $\mathscr{B}_1$ with order n. Bonk studied in his dissertation [1] the coefficient regions $\mathbf{B}_n$ and observed that they are closed convex sets with non-empty interior. It is an easy exercise to show that $\mathbf{B}_1=\{|b_1|\le 1\}$. One of

Bonk's main contributions was to give a description of $\mathbf{B}_2$. To state his result, we need to introduce auxiliary functions. Let

$$P(x) = \frac{3\sqrt{3}}{2} x(1-x^2).$$

Then the function $P(x)$ increases from 0 to 1 when x moves from 0 to $1/\sqrt{3}$. Therefore, we can take a branch Q of P^{-1} on the interval $[0,1]$ so that $Q : [0,1] \to [0, 1/\sqrt{3}]$ is homeomorphic. Note that the relation

$$P(Q(t)) = \frac{3\sqrt{3}}{2} Q(t)\big(1 - Q(t)^2\big) = t \tag{2.1}$$

holds for $t \in [0,1]$. We are now ready to state Bonk's theorem.

Theorem A (Bonk [1, Satz 3.2.1])**.**

$$\mathbf{B}_2 = \left\{ (b_1, b_2) \in \mathbb{C}^2 : |b_1| \le 1 \ \mathit{and} \ |b_2| \le \tfrac{3\sqrt{3}}{4}\big(1 - 3Q(|b_1|)^2\big)\big(1 - Q(|b_1|)^2\big) \right\}.$$

In particular, we have the sharp bound $|b_2| \le 3\sqrt{3}/4$ for functions $F(z) = b_1 z + b_2 z^2 + \dots$ in $\mathscr{B}_1$. With this information about $\mathbf{B}_2$, we prove Theorem 1.

Proof of Theorem 1. Let $C(\mu)$ be the best possible constant C such that $|b_2 + \mu b_1^2| \le C$ holds for every function $F(z) = b_1 z + b_2 z^2 + \dots$ in $\mathscr{B}_1$, where μ is a fixed complex number. Then, by definition of the coefficient region, we have

$$C(\mu) = \sup_{(b_1,b_2)\in \mathbf{B}_2} |b_2 + \mu b_1|.$$

For $(b_1, b_2) \in \mathbf{B}_2$, by Theorem A,

$$|b_2 + \mu b_1^2| \le |b_2| + |\mu||b_1|^2 \tag{2.2}$$

$$\le \frac{3\sqrt{3}}{4}\big(1 - 3Q(|b_1|)^2\big)\big(1 - Q(|b_1|)^2\big) + |\mu||b_1|^2 = M(|b_1|), \tag{2.3}$$

where

$$M(t) = \frac{3\sqrt{3}}{4}\big(1 - 3Q(t)^2\big)\big(1 - Q(t)^2\big) + |\mu| t^2.$$

We note here that we can choose $(b_1, b_2) \in \mathbf{B}_2$ so that equality holds at both (2.2) and (2.3). Since $|b_1|$ can take any value in $[0,1]$, we obtain

$$C(\mu) = \max_{0 \le t \le 1} M(t). \tag{2.4}$$

We have thus to compute the value of the maximum of $M(t)$ over $0 \le t \le 1$. Since $P'(Q(t))Q'(t) = 1$, we obtain the relation

$$Q'(t) = \frac{2}{3\sqrt{3}\big(1 - 3Q(t)^2\big)}.$$

Therefore, by substituting the last relation and (2.1), we get

$$\begin{aligned}M'(t) &= -3\sqrt{3}\big(2-3Q(t)^2\big)Q(t)Q'(t) + 2|\mu|t \\ &= -\frac{2Q(t)\big(2-3Q(t)^2\big)}{1-3Q(t)^2} + 3\sqrt{3}|\mu|Q(t)\big(1-Q(t)^2\big) \\ &= \frac{Q(t)}{1-3Q(t)^2}\Big\{2\big(3Q(t)^2-2\big) + 3\sqrt{3}|\mu|\big(Q(t)^2-1\big)\big(3Q(t)^2-1\big)\Big\}.\end{aligned}$$

Solving the quadratic equation $2(3x-2)+3\sqrt{3}|\mu|(x-1)(3x-1)=0$, we have the solutions $x=(2\sqrt{3}|\mu|-1\pm\sqrt{1+3|\mu|^2})/(3\sqrt{3}|\mu|)$. Because $(2\sqrt{3}|\mu|-1+\sqrt{1+3|\mu|^2})/(3\sqrt{3}|\mu|)\ge 2/3>1/3$, if the derivative $M'(t)$ has a zero t_0 in the interval $(0,1)$ it must satisfy the relation

$$Q(t_0)^2 = \frac{2\sqrt{3}|\mu|-1-\sqrt{1+3|\mu|^2}}{3\sqrt{3}|\mu|}.$$

We now set

$$R(s) = \frac{2\sqrt{3}s-1-\sqrt{1+3s^2}}{3\sqrt{3}s}, \quad s>0.$$

Since

$$R'(s) = \frac{1+\sqrt{1+3s^2}}{3s\sqrt{3(1+3s^2)}} > 0,$$

the function $R(s)$ is increasing in $s>0$. Note that $R(\frac{4}{3\sqrt{3}})=0$ and $\lim_{s\to+\infty}R(s)=\frac{1}{3}$. Therefore, the equation $Q(t)^2=R(|\mu|)$ has a solution $t=t_0$ in the interval $(0,1)$ precisely when $\frac{4}{3\sqrt{3}}<|\mu|$.

First we consider the case when $|\mu|\le\frac{4}{3\sqrt{3}}$. In this case, $M'(t)<0$ in $0<t<1$ and hence $M(|b_1|)$ takes its maximum as $|b_1|=0$. Therefore, we obtain $C(\mu)=M(0)=3\sqrt{3}/4$ by (2.4).

Secondly, we assume that $|\mu|>\frac{4}{3\sqrt{3}}$. Then, as was seen above, there is a unique point $t_0\in(0,1)$ such that $Q(t_0)^2=R(|\mu|)$. Since $M'(t)>0$ for $0<t<t_0$ and $M'(t)<0$ for $t_0<t<1$, the function $M(t)$ takes its maximum at $t=t_0$. Thus, $C(\mu)=M(t_0)$ by (2.4). Let us now compute the value of $M(t_0)$. In view of the relation $t_0=P(Q(t_0))=P(\sqrt{R(|\mu|)})$, we have the expression

$$\begin{aligned}M(t_0) &= \frac{3\sqrt{3}}{4}\big(1-3Q(t_0)^2\big)\big(1-Q(t_0)^2\big) + |\mu|t_0^2 \\ &= \frac{3\sqrt{3}}{4}\big(1-3R(|\mu|)\big)\big(1-R(|\mu|)\big) + |\mu|P(\sqrt{R(|\mu|)})^2 \\ &= \frac{1+3\sqrt{3}|\mu|^3+(1+3|\mu|^2)^{3/2}}{6\sqrt{3}|\mu|^2}.\end{aligned}$$

Thus, the assertion of Theorem 1 has been confirmed. □

Proof of Theorem 2. For a function $f(z)=z+a_2z^2+a_3z^3+\dots$ in $\mathscr{U}(\lambda)$, we set $F=\lambda^{-1}\log f'$. Then, $\|F\|_{\mathscr{B}}=\lambda^{-1}\|T_f\|_{\mathbb{D}}\le 1$ and thus $F\in\mathscr{B}_1$. We expand F in a power series: $F(z) = b_1 z + b_2 z^2 + \dots$. A comparison of the Taylor coefficients of the both sides of $f' = \mathrm{e}^{\lambda F}$ yields the relations

$$2a_2 = \lambda b_1 \quad \text{and} \quad 3a_3 = \lambda\left(b_2 + \frac{\lambda}{2} b_1^2\right).$$

Thus, the maximum of $|a_3|$ for $f \in \mathscr{U}(\lambda)$ is given as $\lambda C(\lambda/2)/3$. Theorem 1 now yields the required assertion. □

Under the same circumstances as in the above proof, we further obtain the expression

$$a_3 - \mu a_2^2 = \frac{\lambda}{3}\left[b_2 + \frac{\lambda}{4}(2 - 3\mu) b_1^2\right].$$

Hence, as an immediate consequence of Theorem 1, we also have the Fekete-Szegö inequality for the class $\mathscr{U}(\lambda)$.

Theorem 3. *Let a fimctopm $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ belong to $\mathscr{U}(\lambda)$ for a $\lambda > 0$. Then the sharp inequality*

$$\begin{aligned} |a_3 - \mu a_2^2| &\le \frac{\lambda}{3} C\left(\tfrac{(2-3\mu)\lambda}{4}\right) \\ &= \begin{cases} \dfrac{64 + 3\sqrt{3}\lambda^3 |2-3\mu|^3 + \left(16 + 3\lambda^2|2-3\mu|^2\right)^{3/2}}{72\sqrt{3}\lambda|2-3\mu|^2} & (\lambda|2-3\mu| > \frac{16}{3\sqrt{3}}) \\ \dfrac{\sqrt{3}}{4}\lambda & (\lambda|2-3\mu| \le \frac{16}{3\sqrt{3}}) \end{cases} \end{aligned}$$

holds for each $\mu \in \mathbb{C}$.

Since $S_f(0) = 6(a_3 - a_2^2)$, we obtain the following corollary.

Corollary 4. *For $f \in \mathscr{U}(\lambda)$, $\lambda > 0$, the sharp inequality*

$$|S_f(0)| \le 2\lambda C(-\lambda/4) = \begin{cases} \dfrac{64 + 3\sqrt{3}\lambda^3 + \left(16 + 3\lambda^2\right)^{3/2}}{12\sqrt{3}\lambda} & (\lambda > \frac{16}{3\sqrt{3}}) \\ \dfrac{3\sqrt{3}}{2}\lambda & (\lambda \le \frac{16}{3\sqrt{3}}) \end{cases}$$

holds.

3. Extremal functions

We end the paper with a remark on functions extremal in $\mathscr{U}(\lambda)$. First we observe extremal functions for the coefficient functional $|b_2 + \mu b_1^2|$ in $\mathscr{B}_1$. It is clear that such an extremal function $F(z) = b_1 z + b_2 z^2 + \dots$ has to satisfy the condition $(b_1, b_2) \in \partial \mathbf{B}_2$, in other words, either

(i) $|b_1| = 1$ and $b_2 = 0$, or
(ii) $|b_1| < 1$ and $|b_2| = \frac{3\sqrt{3}}{4}(1 - 3Q(|b_1|)^2)(1 - Q(|b_1|)^2)$.

In case (i), an extremal function is given by $F(z) = b_1 z$. In case (ii), setting $t_0 = P(|b_1|)$, we define F by

$$F(z) = \frac{3\sqrt{3}\varepsilon}{4}\left\{\left(\frac{z+z_0}{1+\overline{z_0}z}\right)^2 - z_0^2\right\},$$

where $\varepsilon \in \partial\mathbb{D}$ and $z_0 \in \mathbb{D}$ are chosen so that $\arg\varepsilon = \arg b_2$, $|z_0| = Q(|b_1|)$, and $\arg z_0 = \arg b_1 - \arg b_2$. Then, it is checked that $\|F\|_{\mathscr{B}} = 1$, $F'(0) = \varepsilon(z_0/|z_0|)P(|z_0|) = b_1$ and $F''(0)/2 = \varepsilon\frac{3\sqrt{3}}{4}(1-3|z_0|^2)(1-|z_0|^2) = b_2$. Therefore, $F(z) = b_1 z + b_2 z^2 + \dots$.

As for uniqueness of extremal functions, at least, we have the following.

Lemma 5. *Let $(b_1, b_2) \in \partial\mathbf{B}_2$. If $|b_1| = 1$, then there are infinitely many functions $F \in \mathscr{B}_1$ such that $F(z) = b_1 z + O(z^3)$. If $b_1 = 0$ then a function $F \in \mathscr{B}_1$ with $F(z) = b_1 z + b_2 z^2 + \dots$ necessarily has the form $F(z) = b_2 z^2$.*

Proof. We may first assume that $b_1 = 1$. Let ω be an analytic map of $\mathbb{D}$ into itself with $\omega(0) = \omega'(0) = 0$. Then, consider the function

$$F(z) = \int_0^z \frac{\mathrm{d}\zeta}{1-\omega(\zeta)} = \int_0^1 \frac{z\mathrm{d}t}{1-\omega(tz)}.$$

Then F is analytic on $\mathbb{D}$ and satisfies $F(0) = 0$ and $F'(0) = 1$. On the other hand, since $|\omega(z)| \le |z|^2$, we have

$$(1-|z|^2)|F'(z)| = \frac{1-|z|^2}{|1-\omega(z)|} \le \frac{1-|z|^2}{1-|\omega(z)|} \le 1$$

for $|z| < 1$ with equality for $z = 0$. Thus, we see that $\|F\|_{\mathscr{B}} = 1$. In this way, we can construct a plenty of such functions.

Next we assume that $b_1 = 0$ and $|b_2| = 3\sqrt{3}/4$. Let F be a function in $\mathscr{B}_1$ such that $F(z) = b_2 z^2 + c_3 z^3 + c_4 z^4 + \dots$. Then,

$$F'(z) = 2b_2 z + 3c_3 z^2 + \dots = F_0'(z)\big(1 + h(z)\big),$$

where $F_0(z) = b_2 z^2$ and h is analytic on $\mathbb{D}$ with $h(0) = 0$. In particular, we have

$$(1-|z|^2)|F'(z)| = \frac{3\sqrt{3}}{2}|z|(1-|z|^2)|1+h(z)| = |1+h(z)| \le 1$$

for $|z| = 1/\sqrt{3}$, by the assumption $\|F\|_{\mathscr{B}} \le 1$. By the maximum modulus principle, this forces h to be identically 0. Thus, the proof is complete. □

As conseqences of the last lemma together with the proof of Theorem 1, we can deduce some information about extremal functions in $\mathscr{B}_1$ and $\mathscr{U}(\lambda)$.

Theorem 6. *Let $\mu \in \mathbb{C}$ satisfy $|\mu| \le \frac{4}{3\sqrt{3}}$. Then an extremal function F_0 for the coefficient functional $|b_2 + \mu b_1^2|$ for functions $F(z) = b_1 z + b_2 z^2 + \dots$ in $\mathscr{B}_1$ must have the form $F_0(z) = \varepsilon\frac{3\sqrt{3}}{4}z^2$ for a complex constant ε with $|\varepsilon| = 1$.*

We recall the definition of the error function:

$$\mathrm{Erf}(z) = \int_0^z \mathrm{e}^{-\zeta^2}\mathrm{d}\zeta.$$

Then extremal functions in $\mathscr{U}(\lambda)$ can be expressed in terms of the error function for a small λ.

Theorem 7. *Let $0 < \lambda \le \frac{8}{3\sqrt{3}}$. Suppose that a function $f \in \mathscr{U}(\lambda)$ maximizes the functional $|a_3|$ within $\mathscr{U}(\lambda)$. Then f has to be represented by*

$$f(z) = \frac{\mathrm{Erf}(\alpha z)}{\alpha}$$

for a complex constant α with $|\alpha|^2 = 3\sqrt{3}\lambda/4$.

Kreyszig and Todd [5] obtained the radius ρ of univalence of the error function up to 7 decimal places by using large-scale computers. According to their observations, the radius ρ is given by $\rho = \sqrt{(\theta + \pi/2)/\sin 2\theta}$, where $\theta \in (0, \pi/2)$ is determined by the equation

$$\mathrm{Im}\ \mathrm{Erf}\left(\sqrt{\frac{\theta + \pi/2}{\sin 2\theta}}\mathrm{e}^{\mathrm{i}\theta}\right) = 0.$$

Using Mathematica 5.2, we obtained numerically

$$\rho = 1.57483758917543224805\ldots.$$

Especially, we admit that their computation was correct. Since α in Theorem 7 satisfies $|\alpha| \le \sqrt{2} = 1.414\ldots$, the extremal function $f(z) = \mathrm{Erf}(\alpha z)/\alpha$ is univalent in the unit disk for such an α.

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