The Hurwitz space of a Bell representation

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1 The coefficient body

We say that a bounded domain D in $\mathbb{C}$ is an *n-ply connected non-degenerate planar domain* if the boundary of D in $\mathbb{C}$ consists of n simple closed curves. Bell proposed in [1] a family of simple such domains, and we have shown in [4] that every non-degenerate n-ply connected planar domain with $n > 1$ is mapped biholomorphically onto a domain $W_{\mathbf{a},\mathbf{b}}$ proposed by Bell, which is defined by

$$W_{\mathbf{a},\mathbf{b}} = \left\{ z \in \mathbb{C} : \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} \right| < 1 \right\}$$

with suitable complex vectors $\mathbf{a} = (a_1, a_2, \cdots, a_{n-1})$ and $\mathbf{b} = (b_1, b_2, \cdots, b_{n-1})$. We call such a domain $W_{\mathbf{a},\mathbf{b}}$ a *Bell representation* of W. (Also see [5].)

Definition 1.1 Let $\mathbf{B}_n$ be the locus in $\mathbb{C}^{2n-2}$ consisting of all points $(\mathbf{a}, \mathbf{b})$ such that the corresponding domains $W_{\mathbf{a},\mathbf{b}}$ are non-degenerate n-ply connected planar domains. We call this locus $\mathbf{B}_n$ the *coefficient body* of degree n.

Also we set

$$\mathbf{B}_n^* = \{(a_1, \cdots, a_{n-1}, \mathbf{b}) \mid (a_1^2, \cdots, a_{n-1}^2, \mathbf{b}) \in \mathbf{B}_n\},$$

and call it the *modified coefficient body* (of degree n).

It is obvious that $\mathbf{B}_n$ and $\mathbf{B}_n^*$ are contained in the product space $(\mathbb{C}^*)^{n-1} \times F_{0,n-1}\mathbb{C}$, where

$$F_{0,n-1}\mathbb{C} = \{(z_1, \cdots, z_{n-1}) \in \mathbb{C}^{n-1} \mid z_j \neq z_k \quad \text{if} \quad j \neq k\}$$

is the configuration space of $n-1$ points in $\mathbb{C}$. And $\mathbf{B}_n^*$ is a 2^{n-1}-sheeted smooth holomorphic covering space of $\mathbf{B}_n$.

In the sequel, we assume that $n > 2$, since $\mathbf{B}_2$ and $\mathbf{B}_2^*$ are explicitly known. Note that $\mathbf{B}_n^*$ is *circular*, i.e. for every point $(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n^*$ and every $\theta \in \mathbb{R}$, $e^{i\theta}(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n^*$. Also for every point $(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n^*$ and every $0 < r \leq 1$, $r(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n^*$.

Theorem 1.1 ([6]) *The modified coefficient body $\mathbf{B}_n^*$ is a circular domain homeomorphic to $\mathbf{B}_n$, and $\mathbf{B}_n$ has the same homotopy type as that of $(\mathbb{C}^*)^{n-1} \times F_{0,n-1}\mathbb{C}$.*

Remark The fundamental group of $F_{0,n-1}\mathbb{C}$ is called the *pure braid group*, and its structure is well-known. See for instance [2].

2 The Hurwitz spaces

Now, rational functions can be parametrized also by the set of critical values.

Definition 2.1 Let Γ be the set of all points $(\mathbf{a}, \mathbf{b})$ of $\mathbf{B}_n$ such that the corresponding rational function

$$f_{\mathbf{a},\mathbf{b}} = z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k}$$

has either a non-simple critical point or has a pair of critical points whose images by $f_{\mathbf{a},\mathbf{b}}$ are the same. We call Γ the *collision locus* of $\mathbf{B}_n$.

Then for every point $(\mathbf{a}, \mathbf{b})$ in $\mathbf{B}_n - \Gamma$, the rational function $f_{\mathbf{a},\mathbf{b}}$ has $2n-2$ critical values, which we denote by

$$S_{\mathbf{a},\mathbf{b}} = \{\alpha_1, \cdots, \alpha_{2n-2}\}.$$

This set can be considered as a point in the *unordered configuration space* $B_{0,2n-2}\mathbb{C}$, the quotient space of $F_{0,2n-2}\mathbb{C}$ by the symmetric group. Actually, we know that $S_{\mathbf{a},\mathbf{b}}$ is a point of the unordered configuration space $B_{0,2n-2}U$ of $2n-2$ points on the unit disc U. Hence we can define the projection

$$\pi_S : \mathbf{B}_n - \Gamma \to B_{0,2n-2}U$$

by setting $\pi_S(\mathbf{a}, \mathbf{b}) = S_{\mathbf{a},\mathbf{b}}$. We have the following theorem about the projection π_S.

Theorem 2.1 ([6]) *The projection π_S gives a $(2n-2)!\, n^{n-3}$-sheeted proper holomorphic covering of $B_{0,2n-2}U$ by $\mathbf{B}_n - \Gamma$ for every $n > 2$.*

We can see this theorem by regarding $\mathbf{B}_n - \Gamma$ as a Hurwitz space. First we intruduce a conceptually natural kind of such spaces.

Let f_1 and f_2 be rational functions. We say that f_1 and f_2 determine the same *covering structure* if there are Möbius transformations γ and δ of $\hat{\mathbb{C}}$ such that $f_2 = h \circ f_1 \circ g^{-1}$. We denote by $\mathcal{C}_f$ the covering structure determined by f, and call the set of all covering structures determined by rational functions g which are quasiconformally equivalent to f the *prime Hurwitz space* of f, which is denoted by $H^{\#}(f)$. Here we say that f and g are *quasiconformally equivalent* if there are quasiconformal self-maps φ and ψ of $\hat{\mathbb{C}}$ such that $g = \psi \circ f \circ \varphi^{-1}$.

We introduce the *prime Hurwitz distance* $d_{H^\#}$ on $H^{\#}(f)$ by setting

$$d_{H^\#}(\mathcal{C}_{f_1}, \mathcal{C}_{f_2}) = \inf_{\varphi} \log K(\varphi),$$

where the infimum is taken over all quasiconformal maps φ of $\hat{\mathbb{C}}$ which satisfy $\psi \circ f_2 \circ \varphi^{-1} = f_1$ with suitable quasiconformal maps ψ of $\hat{\mathbb{C}}$.

Proposition 2.1 *$d_{H^\#}$ is actually a distance and complete on $H^{\#}(f)$.*

Proof. If $d_{H^\#}(\mathcal{C}_{f_1}, \mathcal{C}_{f_2}) = 0$, then there are sequences $\{\psi_n\}$ and $\{\varphi_n\}$ of quasiconformal maps of $\hat{\mathbb{C}}$ fixing $0, 1, \infty$ such that $\psi_n \circ f_2 \circ \varphi_n^{-1} \in \mathcal{C}_{f_1}$ for every n, and $K(\varphi_n) = K(\psi_n)$ tend to 1 as n tend to ∞. Then ψ_n and φ_n converge to Möbius transformations γ and δ, respectively, which shows that $d_{H^\#}$ is actually a distance.

Similarly, let $\{\mathcal{C}_{f_n}\}$ be a Cauchy sequence with respect to $d_{H^\#}$. Then by a standard argument and taking subsequences if necessary, we can find sequences $\{\psi_n\}$ and $\{\varphi_n\}$ of quasiconformal maps of $\hat{\mathbb{C}}$ fixing $0, 1, \infty$ such that $\psi_n \circ f_1 \circ \varphi_n^{-1} \in \mathcal{C}_{f_n}$ for every n, and quasiconformal maps ψ and φ of $\hat{\mathbb{C}}$, such that $f_\infty = \psi \circ f_1 \circ \varphi^{-1}$ is a rational function and $K(\varphi_n \circ \varphi^{-1})$ and $K(\psi_n \circ \psi^{-1})$ tend to 1 as n tend to ∞. This implies that $\mathcal{C}_{f_n}$ converges to $\mathcal{C}_{f_\infty}$ with respect to $d_{H^\#}$, which shows completeness. ∎

But a finer equivalence relation is often considered.

Definition 2.2 We say that rational functions f_1 and f_2 determine the same *isomorphism class* if there is a Möbius transformation γ such that $f_2 = f_1 \circ \gamma$. We call the set of all isomorphism classes $\mathcal{I}_g$ of rational functions g which are quasiconformally equivalent to f the *Hurwitz space* of f, and denote it by $H(f)$.

Here, we distinguish the value ∞ and assume that $\psi(\infty) = \infty$ for every quasiconformal map ψ appeared in quasiconformal equivalence relation $g = \psi \circ f \circ \varphi^{-1}$

Example 1 The set $H_{0,n}[n]$ of genus 0 and degree n with type (n), consisting of all isomorphism classes of polynomials of degree n in general position, is the Hurwitz space $H(f)$ for any such an f.

The set $H_{0,n}[1^n]$ of genus 0 and degree n with type $1^n = (1, \cdots, 1)$, consisting of all isomorphism classes of rational functions of degree n in general position with n simple poles, is the Hurwitz space $H(f)$ for any such an f.

Remark The coefficient representation for $H(f)$ is not faithful in general. Also see [10].

We can define the *normalized Hurwitz distance* d_H on $H(f)$ by setting

$$d_H(\mathcal{I}_{f_1}, \mathcal{I}_{f_2}) = \inf_{\varphi} \log K(\varphi),$$

where the infimum is taken over all quasiconformal maps φ of $\hat{\mathbb{C}}$ satisfying $f_1 = \psi \circ f_2 \circ \varphi^{-1}$ with quasiconformal maps ψ of $\hat{\mathbb{C}}$ normalized as follows: Let n be the degree of f. Prescribe $2n+1$ distinct points in $\hat{\mathbb{C}}$ including ∞ and assume that each ψ fixes three of them including ∞.

Then similarly as in the proof of the previous proposition, we have

Proposition 2.2 *d_H is actually a distance and complete on $H(f)$.*

Remark Historically, the Hurwitz space of a rational function f is defined algebraically. Natanzon showed that such Hurwitz spaces are the same as the set $\mathrm{Top}(f)$ of all isomorphism classes of rational functions g which are *topologically equivalent* to f, i.e. there are self-homeomorphisms φ and ψ of $\hat{\mathbb{C}}$ such that $\psi(\infty) = \infty$ and $g = \psi \circ f \circ \varphi^{-1}$ (cf. [7]).

Finally we consider the *marked Hurwitz space* $MH_{0,n}[1^n]$ of all isomorphism classes of rational functions of degree n in general position with n *ordered* simple poles. Then another Hurwitz space MH_nU in $MH_{0,n}[1^n]$, consisting of all isomorphism classes of those with critical values, all of which are contained in U, is a complete metric space with another normalized Hurwitz metric

$$d_{MH,U}(\mathcal{I}_{f_1}, \mathcal{I}_{f_2}) = \inf_{\varphi} \log K(\varphi),$$

where the infimum is taken over all quasiconformal maps φ of $\hat{\mathbb{C}}$ preserving the order of poles and satisfying $f_1 = \psi \circ f_2 \circ \varphi^{-1}$ with quasiconformal self-maps ψ of $\hat{\mathbb{C}}$ which satisfy $\psi(U) = U$ and fix points ± 1 and ∞.

Theorem 2.2 ([6]) $\mathbf{B}_n - \Gamma$ *can be identified with* MH_nU.

3 The synthetic Teichmüller spaces

To recover $\mathbf{B}_n$ from $\mathbf{B}_n - \Gamma$, we need, for instance, to compactify the marked Hurwitz spaces $MH_{0,n}[1^n]$.

Remark The Hurwitz spaces of a rational function can be compactified naturally. See [3] and [8]. We call these compactifications as the *DENT compactification*. But they have the singularities in general. And the boundary part of $\mathbf{B}_n - \Gamma$ corresponding to Γ in the DENT compactification of the marked Hurwitz space $MH_{0,n}[1^n]$ may be different from Γ.

Definition 3.1 Let $f = f_{\mathbf{a},\mathbf{b}}$ correspond to $\mathbf{a}, \mathbf{b} \in \mathbf{B}_n$. Then the *full deformation set* $FD(f)$ of f is the set of all meromorphic functions g on $\mathbb{C}$ such that there are quasiconformal self-maps φ of $\mathbb{C}$ which fix 0 and 1, and satisfy the *qc-L^∞ condition*:

$$D_f(g; \varphi) = \|f - g \circ \varphi\|_\infty \left(= \sup_{\mathbb{C}} |f - g \circ \varphi| \right) < \infty.$$

For every pair of functions f_1 and f_2 in $FD(f)$, we set

$$d(f_1, f_2) = \inf \left(\log K(\varphi_1 \circ \varphi_2^{-1}) + \|f_1 \circ \varphi_1 - f_2 \circ \varphi_2\|_\infty \right),$$

where the infimum is taken over all quasiconformal maps φ_1 and φ_2 of $\mathbb{C}$ which fix 0 and 1, and satisfy the qc-L^∞ conditions $D_f(f_j; \varphi_j) < \infty$. This d is actually a distance, and $FD(f)$ equipped with this distance is a complete metric space. We call the distance d defined above the *synthetic Teichmüller distance* on $FD(f)$. The space $FD(f)$ equipped with this synthetic Teichmüller distance is called the *full synthetic deformation space* of f and is denoted as $FSD(f)$.

Remark The synthetic deformation space is firstly defined for an entire functions. See [9]. The results for structurally finite entire functions such as in [9] can be formulated and proved also for structurally finite meromorphic functions.

Theorem 3.1 *For every $f = f_{\mathbf{a},\mathbf{b}}$ with $(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n$, $FSD(f)$ is the set $Rat_n[1^n]$ consisting of all rational functions of degree n with simple poles which include ∞.*

Proof. First, it is easy to see that every $g \in Rat_n[1^n]$ belongs to $FSD(f)$.

On the other hand, let g be a meromorphic function belonging to $FSD(f)$. Take a quasiconformal map φ of $\mathbb{C}$ such that $D_f(g; \varphi) < +\infty$. Then

$$\lim_{z\to\infty} g(z) = \infty,$$

i.e., g has a pole at ∞, and hence is a rational function.

Next for a sufficiently small $\epsilon > 0$, take discs

$$D_k = \{|z - b_k| < \epsilon\} \qquad (k = 1, \cdots, n-1)$$

$$D_n = \{|z| > 1/\epsilon\} \cup \{\infty\}.$$

Here we may assume that $f(D_k)$ is disjoint from

$$\{|z| \le 2D_f(g; \varphi)\},$$

and that the winding number of the image $f(C_k)$ around 0 is -1 for every $k = 1, \cdots, n$, where C_k is the boundary of D_k. Then the assumption implies that $g(\varphi(D_k))$ is disjoint from $\{|z| \le D_f(g; \varphi)\}$ and that the winding number of $g(\varphi(C_k))$ around 0 is -1 for every $k = 1, \cdots, n$. Thus $\varphi(D_k)$ contains no zeros and a single simple pole for every k, which implies that the degree of g is n. $\blacksquare$

Now for our purpose, we consider the *marked full synthetic deformation space* $MFSD(f)$ of a given $f = f_{\mathbf{a},\mathbf{b}}$ with $(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n$, by keeping the order of simple poles as before, which is again a complete metric space.

Corollary 1 *$MFSD(f)$ is the set $MRat_n[1^n]$ of all rational functions of degree n with ordered simple poles which include ∞, and hence is independent of the choice of f.*

The coefficient body $\mathbf{B}_n$ is a subset of $MFSD(f)$, and the relative topology on $\mathbf{B}_n$ is the same as the relative topology induced from that of $(\mathbb{C}^*)^{n-1} \times F_{0,n-1}\mathbb{C}$.

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