Structurally finite meromorphic functions

Masahiko Taniguchi

1 Definitions and main theorems

In general, for a possibly incomplete and branched holomorphic covering $\pi : R \to \hat{\mathbb{C}}$ of $\hat{\mathbb{C}}$ by a simply connected Riemann surface R, we say that a point α in $\hat{\mathbb{C}}$ is a *singular value* of the projection π if, for every neighborhood U of α, there exists a connected component V of $\pi^{-1}(U)$ such that $\pi : V \to U$ is *not* a biholomorphic surjection, which is called a *singular component* of $\pi^{-1}(U)$. In other words, a point α is not a singular value of π if and only if α is *evenly covered* by π, i.e., we can find a neighborhood U of α such that π maps every connected component of $\pi^{-1}(U)$ biholomorphically onto U. The projection π is called a *Speiser function* if it has only a finite number of singular values.

Next, we say that a possibly incomplete and branched holomorphic covering $\pi : R \to \hat{\mathbb{C}}$ covers a domain D in $\hat{\mathbb{C}}$ *almost evenly* if there are only a finite number of points in D which are not evenly covered, and at every such point α, there is a finite number of singular components of $\pi^{-1}(B)$ for every sufficiently small disk B with center α.

If such a holomorphic covering $\pi : R \to \hat{\mathbb{C}}$, which covers $\hat{\mathbb{C}}$ almost evenly, has no algebraic singularities, i.e. branch points, then the projection π is a meromorphic function with a polynomial Schwarzian derivative ([3], and also see [1]). And similarly, we can see that, when the projection π has a finite number of branch points, π has a rational Schwarzian derivative (cf. [2]). In the case of a holomorphic covering of $\mathbb{C}$ which covers $\mathbb{C}$ almost evenly, the author showed in [4] that the projection π is a *structurally finite* entire function. And we know that f is a sturcturally finite entire function, possibly postcomposed by a Möbius transformation, if and only if the Schwarzian derivative $S(f)$ has the form

$$\left(\frac{P'}{P} + Q\right)' - \frac{1}{2}\left(\frac{P'}{P} + Q\right)^2$$

with suitable polynomials P and Q.

Here recall that, for an entire function f, we can also consider the *non-linearity* or the *pre-Schwarzian derivative* $N(f) = f''/f'$ of f, which determines the affine structure induced by f.

Example 1 *$g(z) = e^z$ and $f(z) = \tan z$ has the same Schwarzian derivative $S(g) = S(f) = -1/2$, but $N(g) = 1$ and $N(f) = 2f$.*

Definition 1.1 We say that a possibly incomplete and branched holomorphic covering $\pi : R \to \hat{\mathbb{C}}$ is *structurally finite* if it covers $\mathbb{C}$ almost evenly and has ∞ as a Picard's exceptional value.

Theorem 1.1 (Structure) *A possibly incomplete and branched holomorphic covering $\pi : R \to \hat{\mathbb{C}}$ is structurally finite if and only if it is obtained from a structurally finite entire function $\mathbb{C} \to \mathbb{C}$ by applying the surgeries attaching $\hat{\mathbb{C}}$ finitely many times.*

Here the *surgery attaching $\hat{\mathbb{C}}$ to the covering $\pi_0 : R_0 \to \hat{\mathbb{C}}$* (*in the sense of Schiffer*) is the one constructing a new possibly incomplete and branched holomorphic covering $\pi : R \to \hat{\mathbb{C}}$, from $\pi_0 : R_0 \to \hat{\mathbb{C}}$ by taking a simple closed arc ℓ in $\hat{\mathbb{C}}$ whose interior is in $\mathbb{C}$ and disjoint from the set of all singular values of π_0, and a lift ℓ' of ℓ with respect to π_0, by connecting $\mathbb{C} - \ell'$ and $\hat{\mathbb{C}} - \ell$ crosswise, and by filling end points if necessary, so that π_0 can be extended to a holomorphic projection π from a newly obtained simply connected Riemann surface R to $\hat{\mathbb{C}}$.

Proof. First we suppose that the convering $\pi : R \to \hat{\mathbb{C}}$ is structurally finite and show the "only-if" part.

Deforming $\pi : R \to \hat{\mathbb{C}}$ by the surgeries relaxing the relation between the singurality data as in [4] §5, we may assume that the singularities of $\pi : R \to \hat{\mathbb{C}}$ are *in general position*, i.e. every singular value is in $\mathbb{C}$ and corresponds to either a single branch point or a logarithmic singurality of π.

Actually, first we can approximate the given π by a structurally finite holomorphic covering $\pi_1 : R_1 \to \hat{\mathbb{C}}$ whose singularities except for poles are in general position with respect to the synthetic Teichmüller topology (which is defined in the next section). Next, by applying Whitehead surgeries as in

[4], we can deform $\pi_1 : R_1 \to \hat{\mathbb{C}}$ to another structurally finite holomorphic covering $\pi_2 : R_2 \to \hat{\mathbb{C}}$ whose singularities are in general position. Here it is clear that, if we show the assertion for $\pi_2 : R_2 \to \hat{\mathbb{C}}$, then we can conclude that the assertion also holds for the given $\pi : R \to \hat{\mathbb{C}}$.

Now fix a spider at ∞ (cf. [4] §4), and consider the *plates*, i.e. lifts of the complement of all legs of this spider at ∞ in $\mathbb{C}$ (cf. [6]). On every plate Π, a leg ℓ corresponds to two borders of Π. If they are the same arc on R, we say that ℓ has *trivial lifts* on Π. And we say that a leg ℓ is *polar* if there is a non-trivial lift of ℓ ending at a pole, which in turn, is called a *polar lift* of ℓ.

To show the assertion by induction, first suppose that π has a single simple pole. Fix a plate Π_0 having a polar lift ℓ_0 as a border. Then Π_0 is connected with another plate Π_1 along ℓ_0. Furthermore since the single pole of f is simple, there are a finite number of legs $\ell_1, \cdots, \ell_N$ and plates $\Pi_2, \cdots, \Pi_{N+1}$ such that $\ell_0, \ell_1, \cdots, \ell_{j+1}$ are located in this order, the plates $\Pi_0, \cdots, \Pi_N$ are mutually different, $\Pi_0 = \Pi_{N+1}$, and Π_j and Π_{j+1} are connected along ℓ_j. Then by applying the positive permutation of spider legs $\{\ell_1, \cdots, \ell_N\}$ just $N-1$ times (cf. [4] Definition 7.11), we have a new leg ℓ^*_{N+1} neighboring to ℓ_0 which also connected Π_0 and Π_1. Deforming $\ell_0 \cup \ell^*_{N+1}$ continuously, we have a compact simple arc C in $\mathbb{C}$, and R is obtained from a simply connected Riemann surface R_1 by the surgery attaching $\hat{\mathbb{C}}$ to the holomorphic covering $\pi_1 : R_1 \to \hat{\mathbb{C}}$ along C in the sense of Schiffer, where π_1 is the projection of a structurally finite holomorphic covering of $\mathbb{C}$ by R_1 induced from π.

Next suppose that the assertion holds if the number of poles is not greater than $n\,(\geq 1)$. And consider such a π with $n+1$ simple poles. Again, fix a plate Π_0 having a polar lift ℓ_0 as a border. Then similarly as in the previous paragraph, we can have a compact simple arc C in $\mathbb{C}$ along which suitably deformed Π_0 and another such one are connected crosswise. Hence we can decompose $\pi : R \to \hat{\mathbb{C}}$ into two structurally finite holomorphic covering $\pi_j : R_j \to \hat{\mathbb{C}}$ with n_j simple poles ($j = 1, 2$) where n_j are non-negative integers satisfying $n_1 + n_2 = n + 1$. Thus if both of n_j are positive, then by the induction we have the assertion for π. If one of n_j, say n_1, is 0, then the number of the singularities of π_2 is less than that of π. Hence repeating the above decomposition finitely many times, we can decompose π into a finite number of structurally finite holomorphic coverings, each of which has at most n simple poles, and we can complete the proof of the "only-if" part by induction.

Since the "if" part is clear, we have finished the proof. ■

Theorem 1.2 (Representation) *The projection of every structurally finite holomorphic covering $\pi : R \to \hat{\mathbb{C}}$ can be reperesnted as*

$$\int^z \frac{P(t)}{\prod_{j=1}^k (t-b_j)^{m_j+1}} e^{Q(t)} dt$$

with suitable polynomials P and Q satisfying the residue condition*:*

$$\left(\frac{P(z)}{\prod_{j\neq\ell} (z-b_j)^{m_j+1}} e^{Q(z)} \right)^{(m_\ell)} (b_\ell) = 0$$

for every ℓ. Here m_j is the order of the pole b_j.

Proof. First by Structure Theorem, we can find a compact set K in $\mathbb{C}$, a conformal map ϕ of $R-K$ into $\mathbb{C}$ such that $\mathbb{C} - \phi(R-K)$ is compact, and a structurally finite entire function g such that $\pi = g \circ \phi$ on $R-K$. Then it is well-know that ϕ can be extended to a quasiconformal map of R onto $\mathbb{C}$. In particular, π is a meromorphic function on $\mathbb{C}$. Furthermore, since ϕ has a simple pole at ∞, it is easy to see that π and hence also π' has a finite order. The assumption implies that π' has only a finite number of zeros and poles with residue 0, we conclude that π has the form as asserted. ∎

Corollary 1 *The projection of a structurally finite holomorphic covering of $\hat{\mathbb{C}}$ is a meromorphic function on $\mathbb{C}$ having a rational non-linearity.*

Definition 1.2 We call the projection of a strucutrally finite holomorphic covering $f : \mathbb{C} \to \hat{\mathbb{C}}$ a *structurally finite meromorphic function.*

If such an f is obtained from a structurally finite entire function of type (p,q) by the surgeries attaching $\hat{\mathbb{C}}$ n times, then we say that f is *of type* (p,q,n).

Example 2 *The function* $(z^2+1)/z$ *is of type* $(0,0,1)$, $(z+1)e^z/z$ *and* e^z/z *are of type* $(0,1,1)$, *and* $\int^z e^{t^2} dt/t^2$ *is of type* $(0,2,1)$.

Actually, Representation Theorem implies the following

Corollary 2 *A meromorphic function f is structurally finite if and only if the non-linearity $N(f)$ of f has the form*

$$\sum_{i=1}^{m} \frac{1}{z-a_i} - \sum_{j=1}^{k} \frac{m_j+1}{z-b_j} + Q'(z)$$

with a suitable polynomial Q, points $\{a_i\}_{i=1}^k$ and $\{b_j\}_{j=1}^k$, and positive integers $\{m_j\}_{j=1}^k$, such that

$$\frac{\prod_{i=1}^m (z-a_i)}{\prod_{j=1}^k (z-b_j)^{m_j+1}} e^{Q(z)}$$

satisfies the residue condition.

Finally, as in [5], we can show the following

Corollary 3 *The Hausdorff dimension of the Julia set of a transcendental structurally finite meromorphic function is two.*

Proof. Let f be a transcendental structurally finite meromorphic function. Then exactly as in [5], we can construct a compact subset E_∞ such that the Hausdorff dimension of E_∞ is two, and the orbit of every point in E_∞ is contained in the set where $|f'| \geq 2$ (cf. (4-d) in [5]). Hence E_∞ is contained in the Julia set of f, which implies the assertion. ∎

2 Several deformation spaces

Let f be a meromorphic function. Then we can consider several deformation spaces of f.

Definition 2.1 Let f_1 and f_2 be meromorphic functions. We say that f_1 and f_2 determine the same *covering structure* if there are a similarity g and a Möbius transformation h such that $f_2 = h \circ f_1 \circ g^{-1}$. We denote by $\mathcal{C}_f$ the covering structure determined by f, and call the set of all covering structures determined by meromorpihc functions g which are quasiconformally equivalent to f the *prime Hurwitz space* of f, which is denoted by $H^{\#}(f)$. Here we say that f and g are *quasiconformally equivalent* if there are quasiconformal self-maps φ and ψ of $\mathbb{C}$ and $\hat{\mathbb{C}}$, respectively such that $g = \psi \circ f \circ \varphi^{-1}$.

The *prime Hurwitz distance* $d_{H^\#}$ on $H^\#(f)$ is defined by setting

$$d_{H^\#}(\mathcal{C}_{f_1}, \mathcal{C}_{f_2}) = \inf_{\varphi} K(\varphi),$$

where the infimum is taken over all quasiconformal maps φ of $\mathbb{C}$ which satisfy $\psi \circ f_2 \circ \varphi^{-1} = f_1$ with suitable quasiconformal maps ψ of $\hat{\mathbb{C}}$. $d_{H^\#}$ is actually a distance and complete on $H^\#(f)$.

Definition 2.2 We say that meromorphic functions f_1 and f_2 determine the same *isomorphism class* if there is a similarity φ such that $f_2 = f_1 \circ \varphi$. We call the set of all isomorphism classes $\mathcal{I}_g$ of meromorphic functions g which are quasiconformally equivalent to f the *Hurwitz space* of f, and denote it by $H(f)$. Here, we distinguish the value ∞ and assume that $\psi(\infty) = \infty$ for every quasiconformal map ψ appeared in quasiconformal equivalence relation $g = \psi \circ f \circ \varphi^{-1}$

If f is a Speiser function, then we can define the *normalized Hurwitz distance* d_H on $H(f)$ by setting

$$d_H(\mathcal{I}_{f_1}, \mathcal{I}_{f_2}) = \inf_{\varphi} K(\varphi),$$

where the infimum is taken over all quasiconformal maps φ of $\hat{\mathbb{C}}$ satisfying $f_1 = \psi \circ f_2 \circ \varphi^{-1}$ with quasiconformal maps ψ of $\hat{\mathbb{C}}$ normalized as follows: We prescribe $N+2$ points in $\mathbb{C}$ and assume that each ψ fixes two of them, where N is the number of finite singular values of f. d_H is actually a distance and complete on $H(f)$.

Remark We say that f and g are *topologically equivalent* if there are self-homeomorphisms φ and ψ of $\mathbb{C}$ and $\hat{\mathbb{C}}$, respectively, such that $\psi(\infty) = \infty$ and $g = \psi \circ f \circ \varphi^{-1}$. Let $\mathrm{Top}(f)$ be the set of all isomorphism classes of meromorphic functions topologically equivalent to f. If f is a Speiser function, we have

$$\mathrm{Top}(f) = H(f).$$

Definition 2.3 The *full deformation set* $FD(f)$ of f is the set of all meromorphic functions g on $\mathbb{C}$ such that there are quasiconformal maps φ of $\mathbb{C}$ which fix 0 and 1, and satisfy the *qc-L^∞ condition*:

$$D_f(g;\varphi) = \|f - g \circ \varphi\|_\infty \left(= \sup_{\mathbb{C}} |f - g \circ \varphi|\right) < \infty.$$

For every pair of functions f_1 and f_2 in $FD(f)$, we set

$$d(f_1, f_2) = \inf \left(\log K(\varphi_1 \circ \varphi_2^{-1}) + \|f_1 \circ \varphi_1 - f_2 \circ \varphi_2\|_\infty \right),$$

where the infimum is taken over all quasiconformal maps φ_1 and φ_2 of $\mathbb{C}$ which fix 0 and 1, and satisfy the qc-L^∞ conditions $D_f(f_j; \varphi_j) < \infty$. d is actually a distance and complete on $FD(f)$. We call the distance d defined above the *synthetic Teichmüller distance* on $FD(f)$. The space $FD(f)$ equipped with this synthetic Teichmüller distance is called the *full synthetic deformation space* of f and is denoted as $FSD(f)$. (Cf. [4].)

Theorem 2.1 *For every structurally finite meromorphic functions f of type (p, q, n) and with the polar type $[\mathbf{n}]$, the space $SF(f)$ consists of all strucutrally finite meromorphic functions of type (p', q, n) with the polar type $[\mathbf{n}]$, where $p' = p$ when $q = 0$ and $p' \leq p$ when $q > 0$, is complete, in the sense that if $f_n \in SF(f)$ converge to $g \in FSD(f)$ then $g \in SF(f)$, with respect to the synthetic Teichmüller topology.*

Here, for a meromorphic function f with n poles counting multiplicities, we say that $[\mathbf{n}] = [n_1, \cdots, n_k]$ is the *polar type* of f if f has k poles which have orders $n_1, \cdots, n_k$, where $\sum_{j=1}^{k} n_j = n$.

Proof. First, it is easy to see that every structurally finite meromorphic functions of type (p', q, n) with the polar type $[\mathbf{n}]$, where $p' = p$ when $q = 0$ and $p' \leq p$ when $q > 0$, in $SF(f)$.

On the other hand, let g be a meromorphic function belonging to $FSD(f)$. Take a quasiconformal map φ of $\mathbb{C}$ such that $D_f(g; \varphi) < +\infty$. Let $\{b_j\}_{j=1}^{k}$ be the poles of f. And for a sufficiently small $\epsilon > 0$, take a disc $D_j = \{|z - b_j| < \epsilon\}$ for every j. (Here, when $q = 0$, then we regard ∞ also as a pole, and take one more disc $D_0 = \{|z| > 1/\epsilon\} \cap \{\infty\}$.)

Now we may assume that $f(D_j)$ is disjoint from $\{|z| \leq 2D_f(g; \varphi)\}$, and that the winding number of the image $f(C_j)$ around 0 is $-n_j$, for every $j = 1, \cdots, k$. Then the assumption implies that $g(\varphi(D_j))$ is disjoint from $\{|z| \leq D_f(g; \varphi)\}$ and that the winding number of $g(\varphi(C_j))$ arround 0 is $-n_j$, for every $k = 1, \cdots, k$. Thus $\varphi(D_k)$ contains no zeros and a pole with order n_j for every j, which implies that the polar type of g is $[\mathbf{n}]$.

Thus we have shown the assertion when $q = 0$. When $q > 0$, by the same argument as in the proof of [4] Theorem 2.18, we can show the assertion. ■

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Masahiko TANIGUCHI
Department of Mathematics
Kyoto University
Kyoto 606, JAPAN