GEOMETRIC PROPERTIES OF FUNCTIONS WITH SMALL SCHWARZIAN DERIVATIVE

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ABSTRACT. The main aim in the present article is to give sufficient conditions for a locally univalent meromorphic function in the unit disk to have specific geometric properties such as starlikeness and convexity in terms of the Schwarzian derivative. To this end, we establish estimates of fundamental solutions to an ODE of the form $2y''+\varphi y=0$ in the unit disk, where φ is an analytic function satisfying a given growth condition. As by-products, growth and distortion estimates are derived for a locally univalent strongly normalized analytic function f in the unit disk with a prescribed growth of the Schwarzian derivative.

1. Preliminaries and main results

We denote by $\mathcal{M}$ the set of meromorphic functions f in the unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ with $f(0)=0, f'(0)=1$. For a complex number c, we set $\mathcal{M}(c)=\{f\in\mathcal{M}:f''(0)=2c\}$. For $c\in\mathbb{C}$ and for an analytic function φ in the unit disk, there exists the unique function $f\in\mathcal{M}(c)$ such that $S_f=\varphi$, where S_f stands for the Schwarzian derivative of f :

$$S_f=\left(\frac{f''}{f'}\right)'-\frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

Indeed, the existence of such a function f is explained by the well-known relationship between the Schwarzian derivative and the linear second-order differential equation which is sometimes called complex oscillation:

$$2y''+\varphi y=0 \quad \text{in } \mathbb{D}. \tag{1.1}$$

Let y_0 and y_1 be analytic solutions to (1.1) with the initial conditions $y_0(0)=1, y_0'(0)=0, y_1(0)=0$ and $y_1'(0)=1$. These are called the fundamental solutions to the equation (1.1). It is easy to see that the Wronskian satisfies the identity

$$y_0y_1'-y_0'y_1=1. \tag{1.2}$$

The function $f=y_1/(y_0-cy_1)$ thus satisfies $f(0)=0, f'(0)=1, f''(0)=2c$ and $S_f=\varphi$. Uniqueness of such a function follows from the fact that $g=L\circ f$ for some Möbius

Date: January 28, 2004, `FILE: osc2003a.tex`.

1991 *Mathematics Subject Classification.* Primary 30C45; Secondary 34M10.

Key words and phrases. Schwarzian derivative, starlike function, convex function.

The first named author's research was supported in part by the Grant KRF-2002-070-C 00005 of the Korea Research Foundation. The second named author was partially supported by the Ministry of Education, Grant-in-Aid for Encouragement of Young Scientists, 14740100.

transformation L if two functions f and g in $\mathcal{M}$ satisfy $S_f = S_g$. In what follows, we sometimes write $f_{\varphi,c}$ for this function f. Note, in particular, that the relation

$$f_{\varphi,c} = \frac{f_{\varphi,0}}{1 - cf_{\varphi,0}}$$

holds. Hence, $f_{\varphi,c}$ is univalent in $\mathbb{D}$ if and only if so is $f_{\varphi,0}$. Notice that the function $f = f_{\varphi,0}$ of the simple form y_1/y_0 satisfies the unexpected condition $f''(0) = 0$, which has been missed by some authors (cf. [7]). Also, $f_{\varphi,c}$ is analytic, namely, it has no poles in $\mathbb{D}$, precisely when $c \in K(\varphi)$, where $K(\varphi)$ is the set of inversion of omitted values of $f_{\varphi,0}$, namely,

$$K(\varphi) = \{c \in \mathbb{C} : 1/c \notin f_{\varphi,0}(\mathbb{D})\}.$$

Although it is known as the Koebe one-quarter theorem that omitted values w of a univalent analytic function in the unit disk satisfy $|w| \geq 1/4$, a stronger assertion can be said for $K(\varphi)$. If $f_{\varphi,0}$ is a univalent meromorphic function in $\mathbb{D}$, then $|c| \leq 2$ for $c \in K(\varphi)$, in other words, omitted values w of the univalent function $f_{\varphi,0}$ satisfy $|w| \geq 1/2$. Indeed, in view of the expansion $f_{\varphi,c}(z) = z + cz^2 + \cdots$, the above inequality immediately follows from Bieberbach's theorem on the second coefficient of a normalized univalent function in the unit disk.

Let $A(x)$, $0 \leq x < 1$, be a locally Lipschitz, non-decreasing, positive function. We call such $A(x)$ as above a *weight function.* An important example is given by

$$A(x) = C(1 - x^2)^{-\mu}, \tag{1.3}$$

where C and μ are non-negative constants, or a linear combination of such functions.

We now consider the linear second-order ordinary differential equations $2y'' = \pm Ay$ on the interval $[0, 1)$, where $y'' = d^2y/dx^2$. It is well known that the equations have unique solutions for any initial data at $x = 0$. (For basic knowledge of the theory of ordinary differential equations, we refer the reader to Walter's book [13].) Let U_0, U_1, V_0 and V_1 be the functions on $[0, 1)$ determined by

$$\begin{aligned} 2U_0'' &= AU_0, \quad U_0(0) = 1, \quad U_0'(0) = 0, \\ 2U_1'' &= AU_1, \quad U_1(0) = 0, \quad U_1'(0) = 1, \\ 2V_0'' &= -AV_0, \quad V_0(0) = 1, \quad V_0'(0) = 0, \\ 2V_1'' &= -AV_1, \quad V_1(0) = 0, \quad V_1'(0) = 1. \end{aligned} \tag{1.4}$$

When we need to indicate the weight function A, we write, for example, $U_0(x, A) = U_0(x)$. As we see later, the inequalities $U_0 > 0$ and $U_0' > 0$ hold on the interval $[0, 1)$ for any weight function A. In the sequel, we will write, for example, $U_0(1) = \lim_{x\to 1-} U_0(x)$ and $U_0'(1) = \lim_{x\to 1-} U_0'(x)$ whenever these limits exist.

In connection with univalent functions, Nehari [12] proved the following result.

Theorem A. *Suppose that a weight function A has the properties:*

(i) $A(x)(1 - x^2)^2$ *is non-increasing in* $0 \leq x < 1$, *and*
(ii) *the solution* $V_0(x, A)$ *is positive for* $0 \leq x < 1$.

Then, the condition $|S_f(z)| \leq A(|z|)$ for a function $f \in \mathcal{M}$ implies univalence of f in $\mathbb{D}$.

On the other hand, the Kraus-Nehari theorem states that $|S_f(z)| \le 6(1-|z|^2)^{-2}$ holds for any univalent meromorphic function f in $\mathbb{D}$. Therefore, it may be natural to impose the condition (i) on the weight function $A(x)$. We, however, do not need it in the sequel.

The following are typical cases where (i) and (ii) hold and sharp multipliers are obtained (see also [2]):

(0) For $A(x) = \pi^2/2$, one has $V_0(x) = \cos(\pi x/2)$.
(1) For $A(x) = 4(1-x^2)^{-1}$, one has $V_0(x) = 1 - x^2$.
(2) For $A(x) = 2(1-x^2)^{-2}$, one has $V_0(x) = \sqrt{1-x^2}$.

Here, all the multipliers $\pi^2/2, 4$ and 2 cannot be replaced by larger ones, respectively.

A function $f \in \mathcal{M}$ is called *starlike* (respectively, *convex*) if f is univalent analytic and the image $f(\mathbb{D})$ is starlike with respect to the origin (respectively, convex). Since starlikeness and convexity are not preserved by Möbius transformations, unlike univalence, starlikeness (convexity) of $f_{\varphi,0}$ does not necessarily imply that of $f_{\varphi,c}$ for $c \in K(\varphi)$. It is well known that $f \in \mathcal{M}$ is starlike if and only if $\operatorname{Re}(zf'(z)/f(z)) > 0$ and that $f \in \mathcal{M}$ is convex if and only if $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$. Furthermore, for a constant $\alpha \in [0,1)$, $f \in \mathcal{M}$ is called *starlike of order* α if $\operatorname{Re}(zf'(z)/f(z)) > \alpha$. It is known as the Strohhäcker theorem that a convex function is starlike of order $1/2$, where the number $1/2$ is sharp (see [5, p. 251]).

In the present article, we give sufficient conditions for a function in $\mathcal{M}(c)$ to be starlike of order $1/2$ or convex in a similar way to Theorem A.

Theorem 1.1. *Let A be a weight function and k be a non-negative number. Suppose that the functions U_0 and U_1 defined by* (1.4) *satisfy the inequality*

$$2\int_0^1 {U_0}'(x)U_1(x)dx + kU_1(1)^2 \le 1. \tag{1.5}$$

If a function f belongs to $\mathcal{M}(c)$ for some c with $|c| \le k$ and satisfies the inequality $|S_f(z)| \le A(|z|)$ in $|z| < 1$, then f is starlike of order $1/2$. Moreover, if A extends to an analytic function in the unit disk in such a way that $|A(z)| \le A(|z|)$ holds in $|z| < 1$ and if equality holds in (1.5)*, then this condition is sharp, namely, for each number $\varepsilon > 0$, there exists a function $f \in \mathcal{M}(-k)$ which is not starlike of order $1/2$ but satisfies the condition $|S_f(z)| \le (1+\varepsilon)A(|z|)$ in $|z| < 1$.*

As the special case when A is a positive constant and $k = 0$, we obtain the following result.

Corollary 1.2. *Let $C_0 = 2\beta_0^2 \approx 2.37036$, where β_0 is the unique positive root of the equation $\sinh(2\beta) = 4\beta$. If a function $f \in \mathcal{M}(0)$ satisfies the inequality $|S_f(z)| \le C_0$ in $|z| < 1$, then f is a starlike function of order $1/2$. The constant C_0 is sharp.*

The sharp constant for starlikeness is unknown. At least, Gabriel [7] proved that the inequality $|S_f(z)| \le C_0'$ in $|z| < 1$ implies starlikeness of $f \in \mathcal{M}(0)$, where $C_0' = 2{\beta_0'}^2 \approx 2.71707$ and β_0' is the unique root of the equation $2\beta = \tan\beta$ in $0 < \beta < \pi/2$. On the other hand, Chiang [3, Proposition 4] showed that C_0' cannot be replaced by a larger number than $C_0'' = (\xi^2 + \eta^2)/2 \approx 4.6351$, where ξ and η are the smallest positive roots of the equations $\xi \tan\xi = -1$ and $\eta \tanh\eta = 1$. By some experiments, it is likely that C_0'' is the best possible constant.

Theorem 1.3. *Let A be a weight function and k be a non-negative number. Suppose that the functions V_0 and V_1 given by* (1.4) *satisfy the inequalities $V_0(x) - kV_1(x) > 0,\ 0 \le x < 1$, and*

$$-\lim_{x\to 1-} \frac{{V_0}'(x) - k{V_1}'(x)}{V_0(x) - kV_1(x)} \le 1. \tag{1.6}$$

If a function f belongs to $\mathcal{M}(c)$ for some c with $|c| \le k$ and satisfies the inequality $|S_f(z)| \le A(|z|)$ in $|z| < 1$, then $|zf''(z)/f'(z)| < 1$ in $|z| < 1$ and, in particular, f is convex. Moreover, if A extends to an analytic function in the unit disk in such a way that $|A(z)| \le A(|z|)$ holds in $|z| < 1$ and if equality holds in (1.6)*, then this condition is sharp, namely, for each number $\varepsilon > 0$, there exists a function $f \in \mathcal{M}(k)$ for which the inequality $|zf''(z)/f'(z)| < 1$ fails to hold in $|z| < 1$ but $|S_f(z)| \le (1+\varepsilon)A(|z|)$ in $|z| < 1$.*

As we will see in the proof of Lemma 2.4 below, the limit in (1.6) always exists in $[-\infty, -k]$. See also Proposition 4.3 for additional information.

As a corollary of the theorem, we obtain an improvement of a result of Chiang [3].

Corollary 1.4. *Let $C_1 = 2\beta_1^2 \approx 0.853526$, where β_1 is the unique root of the equation $2\beta \tan\beta = 1$ in $0 < \beta < \pi/2$. If a function $f \in \mathcal{M}(0)$ satisfies the inequality $|S_f(z)| \le C_1$ in $|z| < 1$, then $|zf''(z)/f'(z)| < 1$ in $|z| < 1$ and, in particular, f is convex. The constant C_1 is sharp.*

The above C_1 is smaller than the sharp constant $\tilde{C}_1$ for convexity. More precisely, $C_1 < \tilde{C}_1$ and $\tilde{C}_1$ is the possible largest constant C such that the condition $|S_f(z)| \le C$ in $|z| < 1$ implies convexity of f for functions $f \in \mathcal{M}(0)$. Chiang showed in [3, Propositoin 4] the inequality $\tilde{C}_1 \le C_1' = 2{\beta_1'}^2 \approx 1.19105$, where β_1' is the unique positive root of the equation $\beta \tanh \beta = 1/2$.

2. Growth theorems

For the proof of our main results, we need some growth theorems for solutions to the associated complex oscillation, or related functions to them. We relied heavily upon comparison theorems in the theory of ordinary differential equations with real coefficients, while Chiang [3] relied on Gronwall's inequality.

For convenience, we define the new function $\tilde{V}$ for a given continuous function V in $[0,1)$ by $\tilde{V}(x) = V(x)$ for $0 \le x < x_0$ and $\tilde{V}(x) = 0$ for $x \ge x_0$, where x_0 is the smallest positive zero of $V(x)$ (if there is no such zero, set $x_0 = 1$). By using this notation, we state comparison theorems in the following form. As will be indicated in the proof, some of those inequalities are essentially known.

Lemma 2.1. *Let A be a weight function and let an analytic function φ in $\mathbb{D}$ be majorized by A, namely, $|\varphi(z)| \le A(|z|)$ in $|z| < 1$. The solutions y_0 and y_1 to the differential equation $2y'' + \varphi y = 0$ in $\mathbb{D}$ with the initial conditions $y_0(0) = 1, y_0'(0) = 0, y_1(0) =$*

$0, y_1'(0) = 1$ *then satisfy the inequalities*

$$\begin{aligned}\tilde{V_0}(|z|) \le & |y_0(z)| \le U_0(|z|),\\ & |y_0'(z)| \le U_0{}'(|z|),\\ \tilde{V_1}(|z|) \le & |y_1(z)| \le U_1(|z|),\\ & |y_1'(z)| \le U_1{}'(|z|)\end{aligned}$$

for $z \in \mathbb{D}$, *where* U_0, U_1, V_0, V_1 *are functions given in* (1.4). *Nontrivial equality holds at* z_0 *in any of these inequalities if and only if* $A(t) = |\varphi(tz_0/|z_0|)|$ *holds in* $0 < t \le |z_0|$.

Proof. The inequalities $|y_j(z)| \le U_j(|z|),\ j = 0,1$, follow from the same argument as in the proof of Lemma 8 in [6]. On the other hand, the inequalities $\tilde{V_j}(|z|) \le |y_j(z)|,\ j = 0,1$, can be deduced by Lemmas 1 and 2 in [4] (see also [4, (2.4)]).

We next show the inequality $|y_j'(z)| \le U_j{}'(|z|)$ for $j = 0,1$. For a moment, we assume the inequality $|\varphi(0)| < A(0)$ to hold. Set $w(t) = y_j(t\zeta),\ 0 \le t < 1$, for a point $\zeta \in \partial\mathbb{D}$. Put $\Phi(t) = \zeta^2 \int_0^t \varphi(s\zeta)ds/2$ and $Q(t) = \int_0^t A(s)ds/2$. Then $|\Phi(t) - \Phi(s)| \le Q(t) - Q(s)$ holds for $0 < s < t$. Since $2w''(t) + \zeta^2\varphi(t\zeta)w(t) = 0$, by integration by parts, we obtain

$$\begin{aligned}w'(t) &= \int_0^t w''(s)ds + w'(0)\\ &= -\int_0^t \Phi'(s)w(s)ds + w'(0)\\ &= \int_0^t (\Phi(s) - \Phi(t))w'(s)ds - \Phi(t)w(0) + w'(0).\end{aligned}$$

Therefore,

$$|w'(t)| \le \int_0^t (Q(t) - Q(s))|w'(s)|ds + Q(t)w(0) + w'(0).$$

On the other hand, by the same computation, we obtain

$$U_j{}'(t) = \int_0^t (Q(t) - Q(s))U_j{}'(s)ds + Q(t)U_j(0) + U_j{}'(0).$$

If we set $u(t) = |w'(t)| - U_j{}'(t)$, then we obtain the integral inequality

$$u(t) \le \int_0^t (Q(t) - Q(s))u(s)ds \tag{2.1}$$

because $U_j(0) = w(0)$ and $U_j{}'(0) = w'(0)$. Since $||w'|'(t)| \le |w''(t)|$ (see [4, Lemma 2]), we see that $u'(0) \le |w''(0)| - U_j{}''(0) = (|\varphi(0)| - A(0))w(0)$. In particular, $u'(0) < 0$ when $j = 0$ by the assumption $|\varphi(0)| < A(0)$. When $j = 1$, $|w'|'(0) = 0$ and thus $u'(0) = 0$. In this case, we have $|w'(t)| = 1 - \mathrm{Re}\,(\zeta^2\varphi(0))t^2/4 + O(t^3)$ and $U_j{}'(t) = 1 + A(0)t^2/4 + O(t^3)$ as $t \to 0$. Therefore, $u''(0) = -(\mathrm{Re}\,(\zeta^2\varphi(0)) + A(0))/2 < 0$. In this way, at any event, we observe that $u(t) < 0$ for sufficiently small $t > 0$. We suppose now that $u(t) < 0$ in $(0, t_0)$ but $u(t_0) = 0$ for some $t_0 \in (0,1)$. By (2.1), we obtain

$$0 = u(t_0) \le \int_0^{t_0} (Q(t_0) - Q(s))u(s)ds < 0,$$

which is impossible. Therefore, the function $u(t)$ must be negative throughout $0 < t < 1$, and hence, $|y_j'(t\zeta)| = |w'(t)| < {U_j}'(t)$ holds for any $0 < t < 1$ and $\zeta \in \partial\mathbb{D}$ under the assumption $|\varphi(0)| < A(0)$.

The general case follows from an approximation argument. Indeed, for $\varepsilon > 0$, we consider the solution $y_{j,\varepsilon}$ to the equation $2y'' + (1-\varepsilon)\varphi y = 0$ with the same initial conditions as y_j. Then the previous assertion implies $|y_{j,\varepsilon}'(z)| < {U_j}'(|z|)$. On the other hand, as is well known, $y_{j,\varepsilon}'(z)$ tends to $y_j'(z)$ as $\varepsilon \to 0$. Therefore, by taking the limit, we obtain the inequality $|y_j'(z)| \le {U_j}'(|z|)$ for each $z \in \mathbb{D}$.

Equality conditions can be deduced easily from the above proof. For instance, if $|y_j'(z_0)| = U_j(|z_0|)$ holds at some point $z_0 \neq 0$, then the integral inequality (2.1) leads to the required conclusion.

□

As an immediate consequence, we obtain the following result, which may be of independent interest. Note that the special cases when $A(x) = 2t/(1-x^2)^2$ and $A(x) = \pi^2 t/2$ for $0 \le t \le 1$ were given in [4].

Corollary 2.2. *Let A be a weight function. If a function $f \in \mathcal{M}(0)$ satisfies the inequality $|S_f(z)| \le A(|z|)$ in $|z| < 1$, then*

$$U_0(|z|)^{-2} \le |f'(z)| \le \tilde{V}_0(|z|)^{-2},$$
$$|f(z)| \le \frac{V_1(|z|)}{\tilde{V}_0(|z|)},$$

in $|z| < 1$ and the image of the unit disk under f contains the disk $\{|w| < U_1(1)/U_0(1)\}$. Moreover, if f is univalent in $|z| < r$, then $U_1(|z|)/U_0(|z|) \le |f(z)|$ holds in $|z| < r$. These inequalities are sharp if A extends to an analytic function in the unit disk so that $|A(z)| \le A(|z|)$.

Proof. Let $\varphi = S_f$ and let y_0 and y_1 be fundamental solutions to the equation $2y'' + \varphi y = 0$. Then, as was seen in Section 1, $f = y_1/y_0$ and $f' = 1/y_0^2$. Lemma 2.1 now implies the first inequality. By integration, we obtain the second one.

To deduce the covering estimate, we follow the argument used in the proof of Corollary 2.4 in [10]. Consider the set W of omitted values of f in the disk $|z| < r$, namely, $\mathbb{C} \setminus W = \{f(z) : |z| < r\}$ for $0 < r < 1$. Take a point w_0 in W with the minimum modulus and denote by γ the connected component of the inverse image of the line segment $[0, w_0)$ under f which contains the origin. Since γ does not end at an interior point of the circle $|z| = r$, we obtain the estimate

$$\begin{aligned} |w_0| &\ge \int_{f(\gamma)} |dw| = \int_\gamma |f'(z)||dz| \\ &\ge \int_\gamma \frac{|dz|}{U_0(|z|)^2} \\ &\ge \int_0^r \frac{dx}{U_0(x)^2} = \frac{U_1(r)}{U_0(r)}. \end{aligned}$$

Letting r tend to 1, we obtain the covering theorem. The lower estimate $|f(z)| \geq U_1(|z|)/U_0(|z|)$ also follows from this observation when f is univalent in $|z| < r$. The sharpness assertion is obvious. □

We need also a variant of comparison theorem in the following specialized form to prove our second main lemma.

Lemma 2.3 (cf. [13, p. 96])**.** *Let A be a non-negative continuous function on $J = [0, 1)$ and set $Pw = w' - A/2 - w^2$. If absolutely continuous real-valued functions u, v on J satisfy the inequalities*
(a) *$Pu \leq Pv$ a.e. in J and*
(b) *$u(0) \leq v(0)$,*
then $u \leq v$ holds in J.

Before stating the lemma, we draw the reader's attention to the following fact. Let V be a solution to the differential equation $2V'' = -AV$. If we assume that $V > 0$ on the interval $(0, a)$, then $V'(x)$ is decreasing in $0 < x < a$ because $V'' = -AV < 0$ there. In particular, if $V'(0) \leq 0$, then $V' < V'(0) \leq 0$ on $(0, a)$. We are now in a position to state our second main lemma.

Lemma 2.4. *Under the same hypothesis as in Lemma 2.1, suppose that the function $V_2 = V_0 - kV_1$ is positive on $(0, 1)$ for a non-negative constant k. For a complex number c with $|c| \leq k$, set $y_2 = y_0 - cy_1$. Then the inequality*

$$\left| \frac{y_2'(z)}{y_2(z)} \right| \leq -\frac{V_2{}'(|z|)}{V_2(|z|)}$$

holds for every $z \in \mathbb{D}$.

Proof. For a fixed $\zeta \in \partial\mathbb{D}$, we set $w(t) = y_2'(t\zeta)/y_2(t\zeta)$ and $v(t) = -V_2'(t)/V_2(t)$. Then, the function w satisfies the Riccati equation

$$w' = -\frac{\varphi}{2} - w^2.$$

Hence, the function $u(t) = |w(t)|$ satisfies the differential inequality

$$u' \leq |w'| \leq \frac{A}{2} + u^2.$$

Similarly, the function v satisfies $v' = A/2 + v^2$. Note also that $u(0) = |c| \leq k = v(0)$. We now apply Lemma 2.3 to deduce the desired inequality $u \leq v$ on $(0, 1)$. □

By the above proof, we also see that $v = -V_2'/V_2$ is increasing on $(0, 1)$. Integrating the above inequality, we get the following result as a corollary.

Corollary 2.5. *Under the same circumstances as in Lemma 2.4, the inequality $|\log y_2(z)| \leq -\log V_2(|z|)$ holds in $|z| < 1$ and, in particular,*

$$V_0(|z|) - kV_1(|z|) \leq |y_0(z) - cy_1(z)| \leq \frac{1}{V_0(|z|) - kV_1(|z|)}, \quad |z| < 1.$$

3. Proof of main theorems

Proof of Theorem 1.1. Let y_0 and y_1 be the solutions to the differential equation $2y'' + S_f y = 0$ in $\mathbb{D}$ with the initial conditions $y_0(0) = 1, y_0'(0) = 0, y_1(0) = 0$ and $y_1'(0) = 1$. We now set $y_2 = y_0 - cy_1$ and $U_2 = U_0 + kU_1$ for the given numbers c and k with $|c| \le k$. Then, by (1.2), the identity $y_2 y_1' - y_2' y_1 = 1$ holds. Therefore $f \in \mathcal{M}(c)$ can be written in the form y_1/y_2, where $y_2 = y_0 - cy_1$. Also, the quantity $p(z) = zf'(z)/f(z)$ satisfies the relation

$$\begin{aligned} \frac{1}{p(z)} &= \frac{y_1(z)y_2(z)}{z} \\ &= \int_0^1 \big(y_1y_2\big)'(tz)dt \\ &= 1 + 2\int_0^1 y_1(tz)y_2'(tz)dt. \end{aligned}$$

We now use Lemma 2.1 to get the estimate

$$\begin{aligned} \left|\frac{1}{p(z)} - 1\right| &\le 2\int_0^1 U_1(t|z|){U_2}'(t|z|)dt \\ &\le 2\int_0^1 U_1(t){U_2}'(t)dt \\ &= 2\int_0^1 U_1(t){U_0}'(t)dt + kU_1(1)^2. \end{aligned}$$

By (1.5), we conclude that $|1/p(z) - 1| < 1$, which is equivalent to $\operatorname{Re} p(z) > 1/2$. We have shown that f is starlike of order $1/2$.

We next show the sharpness. Assume that equality holds in (1.5) and that A can be extended to an analytic function in the unit disk $\mathbb{D}$ so that $|A(z)| \le A(|z|)$ in $|z| < 1$. For a given number $\varepsilon > 0$, consider the fundamental solutions y_0 and y_1 to the equation $2y'' + (1+\varepsilon)Ay = 0$ in $\mathbb{D}$. Then the function $f = y_1/(y_0 + ky_1) \in \mathcal{M}(-k)$ satisfies $|S_f(z)| = (1+\varepsilon)|A(z)| \le (1+\varepsilon)A(|z|)$ in $|z| < 1$. On the other hand, by Lemma 2.1, $U_j(t) < y_j(t)$ holds for each $j = 1, 2$ and $0 < t < 1$. Therefore, by the above computation, we see that the function $p(z) = zf'(z)/f(z)$ satisfies

$$\lim_{x\to 1-} \frac{1}{p(x)} - 1 = 2\int_0^1 y_1(t)(y_0'(t) + ky_1'(t))dt > 2\int_0^1 U_1(t)(U_0'(t) + kU_1'(t))dt = 1.$$

Therefore, the function f is not starlike of order $1/2$. Thus the proof is now complete. □

Proof of Theorem 1.3. We use the same notation as in the proof of Theorem 1.1. Further we set $V_2 = V_0 - kV_1$. Then, since $f' = y_2^{-2}$, we have the expression

$$1 + \frac{zf''(z)}{f'(z)} = 1 - 2z\,\frac{y_2'(z)}{y_2(z)}.$$

By Lemma 2.4, we estimate

$$\left|2z\,\frac{y_2'(z)}{y_2(z)}\right| \le -2|z|\,\frac{{V_2}'(|z|)}{V_2(|z|)} < -2\,\frac{{V_2}'(1)}{V_2(1)}$$

The last term is certainly not greater than 1 by (1.6). Therefore, we obtain the inequality $\mathrm{Re}\,(1+zf''(z)/f'(z)) > 0$. The sharpness assertion can be obtained in the same way as in the proof of Theorem 1.1. □

4. Examples and concluding remarks

For a given weight function $A(x)$, it is generally difficult to give explicit expressions to the functions $U_j(x,tA)$ and $V_j(x,tA)$ for each constant $t>0$.

The simplest case is when A is a positive constant. If we write $A=2\beta^2$, where β is a positive number, then $U_0(x)=\cosh(\beta x), U_1(x)=\sinh(\beta x)/\beta, V_0(x)=\cos(\beta x)$ and $V_1(x)=\sin(\beta x)/\beta$. By using this knowledge, we can deduce Corollary 1.2 from Theorem 1.1.

On the other hand, the most important case is when $A(x)=C(1-x^2)^{-2}$, where the constant C is allowed to be negative for convenience. If we write $C=2(4\alpha^2-1)=-2(4\beta^2+1)$, then it is classically known (cf. [9, 2·369]) that the functions U_0 and U_1 are given in terms of α by

$$U_0(x)=\frac{\sqrt{1-x^2}}{2}\left\{\left(\frac{1+x}{1-x}\right)^{\alpha}+\left(\frac{1+x}{1-x}\right)^{-\alpha}\right\}=\sqrt{1-x^2}\cosh\left[\alpha\log\left(\frac{1+x}{1-x}\right)\right],$$

$$U_1(x)=\frac{\sqrt{1-x^2}}{4\alpha}\left\{\left(\frac{1+x}{1-x}\right)^{\alpha}-\left(\frac{1+x}{1-x}\right)^{-\alpha}\right\}=\frac{\sqrt{1-x^2}}{2\alpha}\sinh\left[\alpha\log\left(\frac{1+x}{1-x}\right)\right]$$

and in terms of β by

$$U_0(x)=\sqrt{1-x^2}\cos\left[\beta\log\left(\frac{1+x}{1-x}\right)\right],$$

$$U_1(x)=\frac{\sqrt{1-x^2}}{2\beta}\sin\left[\beta\log\left(\frac{1+x}{1-x}\right)\right].$$

In the above, when $C=2$, the function $U_1(x)$ should take the form

$$U_1(x)=\frac{\sqrt{1-x^2}}{2}\log\left(\frac{1+x}{1-x}\right).$$

This can be seen by taking limit when $\alpha\to 0$. Note also that the function V_j can be understood by $V_j(x,A)=U_j(x,-A)$. There is, however, no hope that the inequality (1.5) in Theorem 1.1 would hold for $A(x)=C(1-x^2)^{-2},\ C>0$, even in the case when $c=0$. This can be seen by considering the example $f(z)=(1/2\alpha)\tanh(\alpha\log(1+z)/(1-z))$, where $\alpha=\varepsilon+1/2$ and ε is a complex number with small modulus. Then

$$\frac{zf'(z)}{f(z)}=\frac{\alpha z}{(1-z^2)\sinh[(1+2\varepsilon)\log(1+z)/(1-z)]}.$$

Noting that $\tan[\arg(\sinh(a+ib))]=\tan b/\tanh a$ for real numbers a and b, we observe that the argument of $xf'(x)/f(x)$ can assume any given number when $x\to 1-$ and thus

f is not starlike as long as $\operatorname{Im}\varepsilon \neq 0$. On the other hand, $S_f(z) = 2(4\alpha^2-1)/(1-z^2)^2 = 8(1+\varepsilon)\varepsilon/(1-z^2)^2$, and therefore, $\sup_{z\in\mathbb{D}}(1-|z|^2)^2|S_f(z)| \le 8(1+|\varepsilon|)|\varepsilon|$ can be small as much as we wish.

Finally, we consider the case $A(x) = C(1-x^2)^{-1}$ with positive constant C. It seems that explicit forms of V_j and U_j for A are less known except for the elementary case when $C=4$ (see Section 1). We, however, can still give some expressions for the general case. Indeed, letting $\alpha = \sqrt{1-2C}$ for a given $C \in \mathbb{R}$, we obtain the representations

$$U_0(x) = F(-\tfrac{1+\alpha}{4}, -\tfrac{1-\alpha}{4}; \tfrac{1}{2}; x^2)$$
$$U_1(x) = x\,F(\tfrac{1+\alpha}{4}, \tfrac{1-\alpha}{4}; \tfrac{3}{2}; x^2),$$

where $F(a,b;c;x)$ stands for the hypergeometric function. (When $C<0$, we interpret as $U_j(z,-A) = V_j(z,A)$.) The above formulae are confirmed by the fact that the function $y = F(a,b;c;x)$ satisfies the hypergeometric equation

$$x(1-x)y'' + \big[c-(a+b+1)x\big]y' - aby = 0.$$

Note that the above representations are still valid when $C > 1/2$ and thus when α is pure imaginary.

As a corollary of Theorem 1.1, we obtain the following result.

Theorem 4.1. *Let $A(x) = C/(1-x^2)$ for a positive constant C and let c be a complex number. Suppose that the inequality*

$$C\int_0^1 x^2 F(\tfrac{3+\alpha}{4}, \tfrac{3-\alpha}{4}; \tfrac{3}{2}; x^2)F(\tfrac{1+\alpha}{4}, \tfrac{1-\alpha}{4}; \tfrac{3}{2}; x^2)dx + \frac{16\pi|c|}{C^2\,\Gamma(\frac{1+\alpha}{4})^2\Gamma(\frac{1-\alpha}{4})^2} \le 1$$

holds, where $\alpha = \sqrt{1-2C}$. If a function $f \in \mathcal{M}(c)$ satisfies $|S_f(z)| \le A(|z|)$ in $|z|<1$, then f is starlike of order $1/2$.

Proof. Noting the relations

$$U_0{}'(x) = (C/2)xF(\tfrac{3+\alpha}{4}, \tfrac{3-\alpha}{4}; \tfrac{3}{2}; x^2)$$

and

$$U_1(1) = F(\tfrac{1+\alpha}{4}, \tfrac{1-\alpha}{4}; \tfrac{3}{2}; 1) = \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{5+\alpha}{4})\Gamma(\frac{5-\alpha}{4})} = \frac{4\sqrt{\pi}}{C\,\Gamma(\frac{1+\alpha}{4})\Gamma(\frac{1-\alpha}{4})}$$

(see, for example, [1, 15.1.20]), the assertion follows from Theorem 1.1. □

Corollary 4.2. *Let $C_2 = (1+\beta_2^2)/2 \approx 1.52444$, where β_2 is the unique positive root of the equation*

$$\frac{1+\beta^2}{2}\int_0^1 x^2 F(\tfrac{3+i\beta}{4}, \tfrac{3-i\beta}{4}; \tfrac{3}{2}; x^2)F(\tfrac{1+i\beta}{4}, \tfrac{1-i\beta}{4}; \tfrac{3}{2}; x^2)dx = 1.$$

If a function $f \in \mathcal{M}(0)$ satisfies the inequality $|S_f(z)| \le C_2/(1-|z|^2)$ in $|z|<1$, then f is a starlike function of order $1/2$. The constant C_2 is sharp.

On the other hand, Theorem 1.3 does not yield any meaningful result in the case when $A(x) = C(1-x^2)^{-1}$. Indeed, by the asymptotic behavior of the hypergeometric functions (cf. [1, 15.3.10]), we obtain

$$V_0'(x) - kV_1'(x) = \left[\frac{\sqrt{\pi}C}{3\Gamma(\frac{3+\alpha}{4})\Gamma(\frac{3-\alpha}{4})} + k\frac{4\sqrt{\pi}}{\Gamma(\frac{1+\alpha}{4})\Gamma(\frac{1-\alpha}{4})}\right]\log\frac{1}{1-x} + O(1)$$

as $x \to 1-$, where $\alpha = \sqrt{1+2C}$. In particular, the left-hand side in (1.6) is $+\infty$ unless $C = 0$. This is, however, a critical case as we see below.

Proposition 4.3. *Let $A(x)$ be a weight function. Suppose that the functions V_0 and V_1 given by* (1.4) *satisfy the inequality $V_0(x) - kV_1(x) > 0,\ 0 \le x < 1$. Then*

$$-\lim_{x\to 1-}(V_0'(x) - kV_1'(x)) \le \frac{1}{2}\int_0^1 A(x)dx + k. \tag{4.1}$$

Proof. Set $V_2 = V_0 - kV_1$ as before. By assumption $V_2 > 0$ in the interval $J = [0,1)$. Therefore, $V_2'' = -AV_2/2 < 0$ and thus V_2' is decreasing in J. In particular, $V_2' \le V_2'(0) = -k \le 0$, and hence, V_2 is decreasing in J. In particular, $0 < V_2 \le V_2(0) = 1$. We now obtain

$$-V_2'(x_0) - k = -\int_0^{x_0} V_2''(x)dx = \frac{1}{2}\int_0^{x_0} A(x)V_2(x)dx \le \frac{1}{2}\int_0^{x_0} A(x)dx,$$

which implies the required inequality. □

As an immediate consequence, we have the estimate for the left-hand side in (1.6):

$$-\lim_{x\to 1-}\frac{V_0'(x) - kV_1'(x)}{V_0(x) - kV_1(x)} \le \frac{(1/2)\int_0^1 A(x)dx + k}{V_0(1) - kV_1(1)}.$$

Note also that, for $A(x) = C(1-x^2)^{-\mu}$, the integral in (4.1) is convergent if and only if $\mu < 1$.

We end this article with the remark that the Schwarzian radius of convexity (cf. [3]) must be zero unless we impose some restriction on the second coefficient $a_2 = f''(0)/2$. More strongly, we can show the following.

Proposition 4.4. *Let φ be analytic in the unit disk. Suppose that $f_{\varphi,0}$ is univalent and that $f_{\varphi,c}$ is convex for every $c \in K(\varphi)$. Then, $\varphi = 0$.*

As we shall see soon, this follows from the next more geometric assertion, which gives a new characterization of a (round) disk in the complex plane. A nice survey on characterizations of a disk was given by K. Hag [8].

Proposition 4.5. *Let D be a proper subdomain of the complex plane $\mathbb{C}$. Suppose that $L(D)$ is convex for each Möbius transformation L such that $L^{-1}(\infty) \notin D$. Then D is a disk or a half-plane.*

Proof. First we observe that the set $C \cap D$ is connected for every line or circle C in $\widehat{\mathbb{C}}$. Indeed, otherwise, there are at least two connected components I_1 and I_2 of $C \cap D$. Let a and b be the end points of the interval I_1 and choose $z_1 \in I_1$ and $z_2 \in I_2$. Then the Möbius transformation $L(z) = (z-a)/(z-b)$ satisfies $b = L^{-1}(\infty) \notin D$, and thus, $L(D)$

is convex by assumption. In particular, the segment $[L(z_1), L(z_2)]$ must be contained in $L(D)$ entirely. This is, however, impossible because the boundary point $L(a)$ of $L(D)$ lies in the segment. The first claim has now been shown.

Next, by assumption, the domain D is Möbius equivalent to a bounded convex domain. In particular, D is a Jordan domain. We may assume that $\infty \in \partial D$. Let a and b be distinct two points in $\Gamma = \partial D \setminus \{\infty\}$. Suppose now that D is not a half-plane. Then, there exists a point c in Γ which does not lie on the line passing through the points a and b. In other words, the three points a, b, c form a non-degenerate triangle. We relabel these points, if necessary, so that b separates a from c in Γ. We can now choose a circle C which separates a and c from b and ∞. By the first claim, $I = C \cap D$ must be connected. In particular, the relative boundary ∂I of I in C consists of at most two points. On the other hand, the set $\Gamma \setminus \{a, b, c\}$ is divided into four connected components, say, $\Gamma_j,\ j = 1, 2, 3, 4$. Since the two end points of each Γ_j are separated by the circle C, any neighbourhood of Γ_j intersects $C \cap D$. Therefore, $\Gamma_j \cap \partial I \neq 0$ for $j = 1, 2, 3, 4$. This contradicts the fact $\#\partial I \leq 2$. Therefore, D must be a half-plane under the assumption $\infty \in \partial D$. □

Proof of Proposition 4.4. Set $D = f_{\varphi,0}(\mathbb{D})$. By assumption, $L_c(D)$ is convex for each $c \in K(\varphi) = \{c \in \mathbb{C} : 1/c \notin D\}$, where $L_c(w) = w/(1 - cw)$. Let $L(w) = (pw + q)/(rw + s)$, $ps - qr = 1$, be an arbitrary Möbius transformation with $L^{-1}(\infty) \notin D$. Note that $s \neq 0$ since $L(0) \in L(D) \subset \mathbb{C}$. If we put $c = -r/s$, the Möbius transformation L can be written in the form

$$L(w) = \frac{q}{s} + \frac{w}{s(rw + s)} = s^{-2} L_c(w) + \frac{q}{s}.$$

Since $c = 1/L^{-1}(\infty) \in K(\varphi)$, the set $L(D)$ is the image of a convex domain under a complex Affine map, and thus, $L(D)$ itself is convex. Proposition 4.5 now guarantees that D is a disk or a half-plane. Therefore, $f_{\varphi,0}$ must be a Möbius transformation and, in particular, $\varphi = S_{f_{\varphi,0}} = 0$. □

Remark. Näkki and Väisälä [11] have given the theorem similar in nature to Proposition 4.5: *A simply connected proper subdomain D of $\mathbb{C}$ is a quasidisk if and only if $L(D)$ is a John disk for every Möbius transformation L with $L^{-1}(\infty) \notin D$.*

Acknowledgements. The authors would like to express sincere thanks to Professor Kang-Tae Kim for his supports and valuable comments. They are also grateful to Katsuhiko Matsuzaki for his help to the first named author's trip to Japan, which initiated the present research. The second named author had nice conversations with Kari Hag and Raimo Näkki about Proposition 4.5 when he was staying at Jyväskylä, Finland, in June, 2003. He thanks also them.

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