Conformal mapping and universal Teichmüller space

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- Models of universal Teichmüller space
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Introduction

- Bieberbach conjecture
- Krushkal’s paper
  (i) Main theorem
  (ii) Main idea
1916. Bieberbach: $|a_n| \leq n$ for $\forall f \in S$

$z = \sum_{n=0}^{\infty} z^n \neq 0 \quad \Theta \in (0, 2\pi)$

1923. Loewner: $\forall f \in S, |a_3| \leq 3$

1984. de Branges: proved

Zalcman Conjecture: $\forall f \in S, |a_n^2 - a_{2n-1}| \leq (n-1)^2$

(Hayman)$\downarrow$

Bieberbach Conjecture.

Hayman theorem: $\forall f \in S$

$$\lim_{n \to \infty} \frac{|a_n|}{n} = \alpha \leq 1, \quad \text{only for } k_0$$

$$\alpha = \lim \max_{r \in \mathbb{D}} (1-r^2)^2 \frac{|f(rz)|}{|z|=1}$$
Krashkal’s main result.

\[ J_n(f) := \alpha_n^p - \alpha_{p(n-1)+1} + \sum_{k=2}^{p-1} \alpha_k, \quad \text{for all } f \in \mathcal{S}, p \geq 2. \]

\( p \) is a polynomial of indicated coefficients of \( f \) which is homogeneous of degree \( p(n-1) \) with respect to the stretching \( f(x) \rightarrow f_t(x) := \frac{1}{t} f(tx), t \in [0,1] \).

\text{Thm: } \forall J_n(f) \text{ with } n \geq 3, f \in \mathcal{S}.

\[ |J_n(f)| \leq \max\{ |J_n(K_0)|, |J_n(K_0,0)| \} \]

If \( p \equiv 0 \Rightarrow \text{only for } K_0. \)

\[ K_{2,0} := \sqrt[2]{K_0(2^2)} = \frac{2}{1 - e^{i\theta}} = \frac{2}{e^{i\theta}} \]

\[ p \equiv 0 \Rightarrow p = 2, \text{ Krashkal's theorem } \Rightarrow \text{Zalcman conjecture } \Rightarrow \text{Bieberbach conjecture.} \]
\[ S := \{ f \mid f: \mathbb{D} \xrightarrow{\text{cont}} \mathbb{C}, f(0) = 0, f'(0) = 1 \} \]

\[ S^0 := \{ f \in S \mid f \text{ has q.c. extension to } \mathbb{C} \} \]

\[ \Sigma := \{ F \mid F = \frac{1}{f(z)} \cdot f \in S \} \]

\[ \Sigma^0 := \{ F \in \Sigma \mid F \text{ has q.c. extension to } \mathbb{C} \} \]
\( F \in \Sigma^0 \rightarrow S_F(2) \xrightarrow{\phi_{1,n}} S_F^m \rightarrow \bar{F}(\bar{P}) \subset T \times \Delta(0,2) \)

\( \bar{f} = \bar{f}(\bar{P}) \)

\[ J^n(f) = J^n(\bar{f}) = J^n(S_F, a_2) \]

\[ J^n(S_F, a_2) \]

\[ \max | J^n(f)^2 | \leq S \]

\[ J^n^*(S_F, f^m(2m)) = J^n, m(S_F^m) \{ m=1, 2, \ldots \} \]

\[ J^n(S_F) = \sup | J^n, m(S_F^m) | ^{1/n} \]

\[ g_m(S) = J^n, m(S)^{2/m} = (J^n, m(S)|_{S^2}) \]

\[ \lambda_g(S) = g_m^* \lambda_0(S) = \frac{18^m |S|}{1 - 18^m |S|} (S \in \Omega) \]

\[ \lambda_f(S) = \sup \lambda g_m(S) \]

\[ \log \lambda_f(0) = g_f(0, 4) \]
$D = h(D)$ distinguished disk. $h'(s) \neq 0$ on $D \setminus \{0\}$

- A holomorphic disk in $\mathbb{D}$ is called distinguished if it touch the zero-set

$$\mathcal{Z}_f := \bigcup_{m \in \mathbb{N}} \{ s \in \mathbb{D} : J_m(s) = 0 \}$$

only at the origin.
Preliminary

- Quasiconformal mapping
- Complex dilatation
- Schwarzian derivative
Quasiconformal mapping

- A sense preserving homeomorphism with a finite maximal dilatation is quasiconformal. If the maximal dilatations is bounded by a number $K$, the mapping is said to be $K$-quasiconformal.
quadrilateral:

\[ Q(z_1, z_2, z_3, z_4) \] Jordan domain
\( \{z_1, z_2, z_3, z_4\} \subset \partial Q \) following each other determine a positive orientation of \( \partial Q \) with respect to \( Q \)

\( R \): canonical rectangle of \( Q(z_1, z_2, z_3, z_4) \)
\( \{i.e. \exists f: Q \xrightarrow{\text{cont}} R \text{ s.t. } z_1, z_2, z_3, z_4 \ \text{correspond to the vertices of } R \} \)
If $R = \{x + iy \mid 0 < x < a, 0 < y < b\}$

$(z_1, z_2)$ corresponds to $0 \leq x \leq a$

module of $Q(z_1, z_2, z_3, z_4)$:

$\text{N}(Q(z_1, z_2, z_3, z_4)) = 9b$
maximal dilatation:

\[ k := \sup_Q \frac{M(f(Q)(1+z_1, 1+z_2, 1+z_3, 1+z_4))}{M(Q(z_1, z_2, z_3, z_4))} \]

\[ f: A \rightarrow A': \text{ sense-preserving homeomorphism} \]

\[ \bar{Q} \subset A \]

- \( K \) invariant under conformal
- \( K \geq 1 \). \( K = 1 \) if \( f \) conformal.
Complex dilatation:

- \( f: A \rightarrow A', \ k.q.c. \)

Suppose \( J_f(z) > 0 \) then \( \text{det}(f) \neq 0 \)

Define: \( \mu(z) = \frac{\bar{\partial} f(z)}{\partial f(z)} \) : complex dilatation of \( f \).

\[ |\mu(z)| \leq \frac{k-1}{k+1} < 1. \]

\( \mu(z) = 0 \) iff \( f \) conformal.
\[ f, g : \text{q. c. of } A \text{ with } \mu_f, \mu_g. \]

\[\mu_{f \circ g^{-1}}(E) = \frac{\mu_f(E) - \mu_g(E)}{1 - \frac{\mu_f(E)}{\mu_g(E)}} \left( \frac{dg(E)}{dg(E)} \right)^2 \quad E \in \mathcal{G}_2\]

\[\Rightarrow \mu_f = \mu_g \text{ a.e. in } A \text{ iff } f \circ g^{-1} \text{ is conformal}\]

(Existence theorem): \( \mu \) measurable in \( A \) with \( \|\mu\|_\infty < 1 \), then \( \exists \) q.c. \( f \) of \( A \) s.t.

\[\mu_f = \mu \text{ a.e. in } A\].
Schwarzian derivative

- Definition
- Existence and uniqueness
- Norm of the Schwarzian derivative
- Convergence of Schwarzian derivatives
**Definition:** \( f : A \to \mathbb{C} \), mero, locally injective.

\[ f'(z) \neq 0 \]

\[ S_f(z) := (f^{(n)})' - \frac{1}{2}(f^{(n)})^2 = (f^{(n+1)}) - \frac{3}{2}(f^{(n)})^2 \quad z \in A \]

If \( \infty \in A \) let \( \phi_\infty(f) = f(\frac{1}{z}) \), \( S_f(\infty) = \lim_{z \to 0} z^4 S_f(z) \)

- \( S_f \) holo in \( A \)
- \( S_g = 0 \) if \( g \in \text{Möb} \)
- \( S_{f \circ g} = S_g \) if \( f \in \text{Möb} \)
- \( S_{f \circ g} = (S_f \circ g)(g')^2 + S_g \)
Thm: A: simply connected

\( \varphi: \text{ holo in } A \)

\( \Rightarrow \exists f: \text{ mero in } A \text{ s.t. } \varphi = f \)

The solution is unique up to an arbitrary Möbius transformation.
norm: A is simply connected, conformally equivalent to A

\[ \eta_A : \text{Poincare density of } A \]

\[ \eta_A = \frac{f'(z)}{|1 - f'(z)|^2} \cdot f: A \xrightarrow{\text{conf}} D. \]

hyperbolic sup-norm:

\[ ||f||_A = \sup_{z \in A} \eta(z)^2 |s_f(z)| \]

\[ |s_f| \eta^{-2} \text{ is a function on Riemann Surface} \]
Convergence:

\[ f_n, f: \text{mero, locally injective in } A \]

If \( f_n \to f \), locally uniformly in \( A \)

\[ \Rightarrow S_{f_n} \to S_f, \text{ locally uniformly in } A \]

\[ \Rightarrow \lim_{n \to \infty} \| S_{f_n} - S_f \|_A = 0 \]
Models of the universal Teichmüller space $T$

1. $T$ is the set of the equivalence classes of For B.
2. $T$ is the set of all normalized quasisymmetric functions.
3. $T$ is the normalized conformal mappings.
4. $T$ is the collection of all normalized quasidiscs.
\[ F_i = \{ f \mid f : H \to H \text{ fixed } 0, 1, \infty \} \]

\[ f_1 \sim f_2 \iff f_1 \|R = f_2 \|R, \quad f_1, f_2 \in F_i. \]

\[ B = \{ \mu(2) \mid \mu(2) \text{ measurable, } \| \mu(2) \| < 1, 2 \in H \} \]

\[ M_{f_1} \sim M_{f_2} : \iff f_1 \sim f_2. \]

(a) \[ T : = \{ [f] \mid f \in F_i \} = \{ [\mu] \mid \mu \in B \} \]

(b) \[ \mathcal{T} : = \{ h \mid h : \mathbb{R} \to \mathbb{R} \text{ quasisymmetric, } h(x, 0, 1) \] bijective \[ \overset{\text{onto}}{\longrightarrow} X \]

\[ [f] \mapsto f|_R. \]

(c) \[ \mathcal{T} : = \{ h \mid h \in X \} \]
\[ B^* = \{ \mu^* | \mu^*(z) = \begin{cases} \mu(z) & z \in \mathcal{H} \\ 0 & z \notin \mathcal{H} \end{cases} \} \]

\[ F^* = \{ f_\mu | \mu \in B^*, f_\mu : \hat{\mathcal{E}} \rightarrow \hat{\mathcal{E}} \text{ fix } 0, 1, \infty \}
\quad f_\mu \vert \mathcal{H}' \text{ cont } \}

\[ f_\mu f_\nu \in F^* \quad f_\mu \sim f_\nu \iff f_\mu \vert \mathcal{H}' = f_\nu \vert \mathcal{H}' \]

\[ \mathcal{P} = \{ [f_\mu] | f_\mu \in F^* \} = \{ [f^m] | f^m \in F \} \]

\[ \Delta = \{ A | A \text{ is a normalized quasidisc } \} \]

\[ f \in F^* \quad \text{bijec} \quad \mathcal{P} \overset{\text{bijec}}{\longrightarrow} \mathcal{H'} \]

\[ \mathcal{P} = \{ f(\mathcal{H}') | f(\mathcal{H}') \in \Delta \} \]
Normalized quasidiscs

- We call a quasidisc normalized if its boundary passes through the points 0, 1, infinity, and is so oriented that the direction from 0 to 1 to infinity is negative with respect to the domain.
A quasicircle in the extended plane is the image of a circle under a quasiconformal mapping of the plane. A domain bounded by a quasicircle is called a quasidisc.
Metric of T

- T has a natural metric, we obtain this metric by measuring the distance between quasiconform mappings in terms of their maximal dilatations.
- Some properties
  (i) Teichmüller distance and complex dilatation
  (ii) geodesics, contractibility, incompatibility
metric of $\mathcal{P}$:

$\tau(p, q) = \frac{1}{2} \inf \{ \log K_{g_{o \to f}} \mid f \in \mathcal{P}, g \in q \}$

$= \frac{1}{2} \inf \{ \log K_{g_{o \to f}} \mid f \circ p, g \in q \}$

$= \frac{1}{2} \min \{ \log K_h \mid h = g_{o \to f}^{-1} \} \mid \mathbb{R}$

It makes $\mathcal{P}$ into a complete metric space.
Teichmüller distance and complex dilatation

\[ T(p, q) = \frac{1}{2} \min \left\{ \log \frac{1 + \| \frac{\mu - \nu}{1 - \mu \nu} \|_{\infty}}{1 - \| \frac{\mu - \nu}{1 - \mu \nu} \|_{\infty}} \mid \mu \in p, \nu \in q \right\} \]

\[ \beta(p, q) = \min \| \frac{\mu - \nu}{1 - \mu \nu} \|_{\infty} \mid \mu \in p, \nu \in q \right\} \]

- \( \beta \) makes \( T \) into a metric space
- \( \beta = \tanh T \Rightarrow (T, \beta) \) topologically equivalent
- \( (T, \beta) \) arcwise connected, complete
Geodesics for the Teichmüller metric

- The length of an arc
- An arc is a geodesic if the length of every subarc is equal to the distance between the endpoints.
- Geodesic of $T$ can be described explicitly with the help of extremal complex dilatation
- Theorem:
\[ Y : [0, 1] \rightarrow (\mathcal{P}, \tau) \]
\[ l(Y) = \sup \{ \sum \tau(Y(t_{j-1}), Y(t_j)) \mid 0 = t_0 < t_1 < \cdots < t_n = 1 \} \]
Thm: \( u \) is extremal for \( p \in p \). Then:

\[
M_t = \frac{(1 + t \mu)^t - (1 - t \mu)^t}{(1 + t \mu)^t + (1 - t \mu)^t} \frac{u}{1/t} \quad t \in [0, 1]
\]

is extremal for \( p_t = \frac{t}{u_t} \).

The arc \( t \to p_t \) is a geodesic from 0 to \( p \).

and \( \mathcal{I}(p_t, 0) = t \mathcal{I}(p, 0) \).
μερισμός είναι εξτρεμαλ. Εάν
||μ||₁₀₀₀ = \min \left\{ \frac{||ν||₁₀₀₀}{ν ∈ P} \right\}
Contractibility of $T$

- $T$ is contractible.

\[ \exists w : T \times [0,1] \xrightarrow{\text{conti}} T \]

- $\exists \omega : T$

- $\exists \pi : T \times [0,1] \xrightarrow{\text{conti}} T$

such that $\pi(p,0) = p$, $\pi(p,1) = \text{constant}$
Distance between quasisymmetric functions

\[ K_h^* := \sup \frac{\mathcal{M}(h(x_1), h(x_2), h(x_3), h(x_4))}{\mathcal{M}(h(x_1, x_2, x_3, x_4))} \]

\( x_1, x_2, x_3, x_4 \in \mathbb{R} \) determine the positive orientation with respect to \( H \)

\[ p(h_1, h_2) := \frac{1}{2} \log K_{h_2/h_1}^* \quad , \; h_1, h_2 \in X \]

The group isomorphism

\[ [f] \rightarrow f|_{\mathbb{R}} \quad \eta \]

\[ (\mathbb{T}, \mathbb{C}) \rightarrow (X, \rho) \text{ homeomorphism.} \]
Incompatibility of the **group structure** with the metric

- The topological structure and the group structure of $X$ are not compatible.
- $T$ is **not** a topological group.

$$
\exists [f] \in \mathcal{P}, [g_n] \in \mathcal{P} \text{ s.t. } [g_n] \to [g] \text{ but } [fog_n] \not\to [fog]
$$
\[ f \circ f, \Rightarrow f \circ f. \]
\[ f, g \circ f \Rightarrow f \circ g \circ f. \]
So, \( f \) can be regarded as a group.

\[ \mathcal{P} \] inherits this group structure.

The rule \([f] \circ [g] = [f \circ g]\) defines the group operation in \( \mathcal{P} \).

The point of \( \mathcal{P} \) determined by the identity mapping is called the origin of \( \mathcal{P} \) and denoted by \( 0 \).
counterexample:

\[
\begin{cases}
  f(x) = x & x \geq 0 \\
  \frac{x}{2} & -2 \leq x < 0 \\
  x+1 & x < -2
\end{cases}
\]

2-quasisymmetric of \( X \).

\[
ln(x) = \begin{cases} 
  x & x \geq 0 \\
  (1 + \frac{1}{n})x & x < 0 
\end{cases}
\]

\( n = 1, 2, ... \) \( \ln(x) \in X \).

\((1 + \frac{1}{n})\)-quasisymmetric. so \((1 + \frac{1}{n})^2 q. c.\)

\[
\log = x \ (x \in \mathbb{R})
\]

\[
p(\ln, l) \leq \log (1 + \frac{1}{n})
\]

let \( g_n = \ln o f^{-1} \) \( \Rightarrow p(g_n, f^{-1}) = p(\ln, l) \)

\[
\lim_{n \to \infty} p(g_n, f^{-1}) = 0
\]

but \( f o g_n = f o (\ln o f^{-1}) \rightarrow f (\ln (n g_n)) = l \) in the \( p \)-metric.
Mapping into the space of schwarzian derivatives

- Comparison of distance
- Imbedding of $T$

\[ T(U) := \{ S_f_{\mu} |_{H} : \mu \rightarrow S_f_{\mu}^{1/2H}, \mu \in B \} \]

\[ = \{ S_f | f \text{ is conf in } H \text{ has q.c. to } C \} \]

\[ \therefore [\mu] \rightarrow S_f^{1/2H} \text{ homeo} \]

(Bers imbedding of Teichmüller space.)
Comparison of distance

\[ f_m : \mathbb{C} \xrightarrow{g.c.} \mathbb{C} \quad f_m \restriction_{\mathbb{H}} : \text{conf} \]

\[ Q = \{ \varphi \mid \varphi \text{ hole in } \mathbb{H} \text{ with } \| \varphi \| = \sup_{y \in \mathbb{H}} |\varphi(y)\| < \infty \} \]

\[ \beta(\varphi_1, \varphi_2) = \| \varphi_1 - \varphi_2 \| \quad \varphi_1, \varphi_2 \in Q. \]

\[ S_m = S_{f_m} \restriction_{\mathbb{H}} \]

\[ \varepsilon(K) \beta(\mathbb{M}, [\nu]) \leq \beta(\mathbb{M}, [\nu]) \leq \varepsilon_0(\nu) \beta(\mathbb{M}, [\nu]) \]

\[ \varepsilon_0(\nu) = 6 + \sqrt{\delta(A)} \quad (\delta(A) = \frac{1}{2} S_m \mathbb{H}) \]

\[ \beta - \text{ and } 9 - \text{ metrics are topologically equivalent}. \]
\[ g(z) = i \frac{1 + z}{1 - z} : D \rightarrow H \]

(inverse of the Cayley transform)

\[ z \mapsto \frac{z - i}{z + i} \]
thank you very much!