On Thompson-like groups for Julia sets of quadratic maps

Shogo Matsuba

February 16, 2018

Abstract

We describe Thompson-like groups corresponding to some Julia sets and reveal their generators and abelianizations by following the construction of the basilica Thompson group by Belk and Forrest in [BF15b]. We also construct Thompson-like groups for Julia sets obtained by tuning two Julia sets that corresponding to Thompson-like groups which are already known.

Contents

1 Introduction

The Thompson group T was defined by Richard Thompson in the 1960's. It is a group of orientation preserving piecewise linear homeomorphisms on the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$ with their break points and slopes of linear intervals are dyadic rationals. The Thompson group T has many interesting properties, for example, T is one of the infinite but finitely presented simple groups.

Until now, many generalizations of T have been studied. Belk and Forrest introduced the Thompson-like group T_B for the basilica Julia set in [BF15b] and they also defined Thompson-like groups for other fractals as rearrangement groups, for instance the "rabbits" and the "airplane" Julia sets [BF15a]. Each Julia set of a quadratic map corresponds to a point in the Mandelbrot set $\mathcal{M}$.

In this paper, we study some properties of Thompson-like groups for other points in $\mathcal{M}$. We confirm that Thompson-like groups for the rabbits have some expected properties in Section 2. Next we construct a Thompson-like group $T\left(\frac{3}{15}\right)$ for the Julia set $J\left(\frac{3}{15}\right)$ using orbit portraits which show us combinatorial structures of Julia sets. The basilica and the rabbits are living in "satellite" components, on the other hand $J\left(\frac{3}{15}\right)$ lives in the "primitive" component of $\mathcal{M}$, and properties of $T\left(\frac{3}{15}\right)$ look different. Finally we define Thompson-like groups for more complicated Julia sets, "tuned" Julia sets.

Acknowledgement. I would like to thank my supervisor Takuya Sakasai for his support throughout the process of writing this paper. I am particularly grateful to Tomoki Kawahira for conversations and advice. Also, I would like to thank everyone in the laboratory for helpful suggestions and advice.

2 Preliminaries

Set $\mathbb{N} = \{0, 1, 2, \dots\}$.

2.1 Standard definitions of quadratic dynamics

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and $f = f_c \colon \mathbb{C} \to \mathbb{C}$ be a quadratic map $f(z) = z^2 + c$ where $c \in \mathbb{C}$.

Definition 2.1.1.

- The set $K = K_c = \{z \in \mathbb{C} \mid \text{The orbit } \{f^n(z)\}_{n=1}^{\infty} \text{ is bounded}\}$ of the union of all orbits for f is called the **filled Julia set**.

- The boundary $J = J_c$ of K_c is called the **Julia set**.
- A connected component of $\widehat{\mathbb{C}} - J_c$ is called a **Fatou component**.
- The set $\mathcal{M} = \{c \in \mathbb{C} \mid K_c \text{ is connected}\}$ is called the **Mandelbrot set** which is compact subset of the **parameter plane** $\mathbb{C}$.

By Riemann's mapping theorem, if K_c is connected, $\hat{\mathbb{C}} - K_c$ is biholomorphic to $\hat{\mathbb{C}} - \overline{\Delta}$ where Δ is the unit disc. In paticular, there is a unique biholomorphic map $\Phi_c\colon \hat{\mathbb{C}} - K_c \to \hat{\mathbb{C}} - \overline{\Delta}$ such that the following diagram commutes (cf. [Mil06]).

$$\begin{tikzcd}
\hat{\mathbb{C}} - K_c \arrow[r, "f"] \arrow[d, "\Phi_c"'] & \hat{\mathbb{C}} - K_c \arrow[d, "\Phi_c"] \\
\hat{\mathbb{C}} - \overline{\Delta} \arrow[r, "z \mapsto z^2"] & \hat{\mathbb{C}} - \overline{\Delta}
\end{tikzcd}$$

This map is called the **Böttcher map**. The pullback of a radial segment $\left\{re^{2\pi i t} \;\middle|\; 1 < r < \infty\right\}$ by the Böttcher map is called the **dynamical ray** for f at an angle $t \in \mathbb{R}/\mathbb{Z}$ and we denote it by $\mathcal{R}_t^{K_c} = \mathcal{R}_t^c = \mathcal{R}_t$.

In the same manner, there is a biholomorphic map which is also called the **Böttcher map** $\Phi_{\mathcal{M}}\colon \hat{\mathbb{C}} - \mathcal{M} \to \hat{\mathbb{C}} - \overline{\Delta}$ and similarly we can consider rays $\mathcal{R}_t^{\mathcal{M}}$ which is the pullback of a radial segment by $\Phi_{\mathcal{M}}$, called **parameter rays**.

Definition 2.1.2. We say that a quadratic map $f(z) = z^2 + c$ is **hyperbolic** if

$$J_c \cap \overline{C^+(f)} = \emptyset,$$

where $C^+(f) = \bigcup_{n=1}^{\infty} f^n(C_f)$ is the **postcritical set** of f and C_f is the set of all critical point of f. The set $\{c \in \mathcal{M} \mid f_c \text{ is hyperbolic}\}$ is open in $\mathcal{M}$ and each connected component H is called a **hyperbolic component** in $\mathcal{M}$.

According to Caratéodory's work, if J_c is locally connected, then we can extend $\Psi_c = \Phi_c^{-1}$ continuously on S^1. The induced map $\Psi_c\colon S^1 \to J_c$ is surjective and satisfies $\Psi_c(z^2) = (\Psi_c(z))^2$. For a point $w \in J_c$ with rays $\mathcal{R}_{a_1}^c, \ldots, \mathcal{R}_{a_n}^c$ landing on it, we write $w = (a_1; a_2; \cdots; a_n)$.

The next theorem gives a sufficient condition for local connectivity of Julia sets.

Theorem 2.1.3 (cf. [Mil06])**.** If the Julia set of a hyperbolic (quadratic) map is connected, then it is locally connected.

As above, we also parametrize each Fatou component which does not contain ∞.

Proposition 2.1.4 (cf. [Hub16])**.** Let $f = f_c$ be a quadratic map and assume c is a periodic point and forms the superattracting cycle $x_0 = c, x_1 = f(x_0), \ldots, f^k(x_{k-1}) = x_0$. Let V_i be the Fatou component containing x_i.

(1) There is a unique homeomorphism $\Psi_c^{V_0} \colon \overline{\Delta} \to \overline{V_0}$, analytic in Δ such that $\Psi_c^{V_0}(z^2) = f^k(\Psi_c^{V_0}(z))$.

(2) Let V be a Fatou component which does not contain ∞. Then there exists minimal m such that $f^m \colon V \to V_0$ is an analytic isomorphism so that the map $\Psi_c^V \colon \overline{\Delta} \to \overline{V}$ given by $\Psi_c^V = (f^m|_V)^{-1} \circ \Psi_c^{V_0}$ is a homeomorphism analytic in Δ.

Definition 2.1.5. Let $f = f_c$ be a quadratic map and assume c is the periodic point of period k. In each Fatou component V, the arc $\left\{\Psi_c^V(re^{2\pi it}) \;\middle|\; 0 \leq r < 1\right\}$ is called the **internal ray** of V at angle $t \in \mathbb{R}/\mathbb{Z}$. The points $\Psi_c^V(0)$ and $\Psi_c^V(1)$ are called the **center** and the **root** of V respectively.

Definition 2.1.6. A **regulated path** in K_c is an embedded arc that intersects each component of the interior only in internal rays. The regulated path connecting two points $z, w \in K_c$ is written by $[z, w]_K$, and $(z, w)_K$ denotes the regulated path without ends.

2.2 Orbit portraits

Definition 2.2.1. Let $\mathcal{O} = \{z_1, \ldots, z_p\}$ be a periodic orbit for quadratic map $f = f_c$ of period p. Suppose that there is some angle $t \in \mathbb{Q}/\mathbb{Z}$ so that the dynamic ray $\mathcal{R}_t^c$ lands at a point of $\mathcal{O}$. Then for each z_i let A_i be the set of all angles of dynamical rays which land at z_i. The set $\mathcal{P}(\mathcal{O}) = \mathcal{P} = \{A_1, \ldots, A_p\}$ is called the **orbit portrait**.

From now on, we only consider the case $c \in \mathcal{M}$ and let $\mathcal{O} = \{z_1, \ldots, z_p\}$ be a periodic orbit of $f = f_c$ with orbit points numbered so that $f(z_i) = z_{i+1}$. Furthermore we suppose that there is at least one rational angle $t \in \mathbb{Q}/\mathbb{Z}$ so that the dynamical ray $\mathcal{R}_t^c$ associated with f lands at some point of this orbit $\mathcal{O}$. The above p is called the **orbit period** of $\mathcal{P}$.

Proposition 2.2.2 ([Mil00b])**.** Under the above hypotheses we have:

(1) Each A_i is a finite subset of $\mathbb{Q}/\mathbb{Z}$.

(2) For each j modulo p, the doubling map $z \mapsto z^2$ carries A_j bijectively onto A_{j+1} preserving cyclic order around the circle.

(3) All of the angles $A_1 \cup \cdots \cup A_p$ are periodic under doubling, with a common period rp.

(4) The sets $A_1, \ldots, A_p$ are **pairwise unlinked**: that is, for each $i \neq j$ the sets A_i and A_j are contained in disjoint sub-intervals of $\mathbb{R}/\mathbb{Z}$.

The period for angles rp is called the **ray period** and the number of elements of A_i is called the **valence** v. By Proposition 2.2.2, the valence of A_i constant; independent of a choice of $i \in \{1, \ldots, p\}$, and thus it is well-defined. Now we assume $v \geq 2$. Then v tuples of rays cut the plane up into v open regions which are called the **sectors** based at $z \in \mathcal{O}$. The **angular width** of a sector S is the length of the open arc $I_S = \{t \in \mathbb{R}/\mathbb{Z} \mid \mathcal{R}_t^c \subset S\}$.

Definition 2.2.3. There is one exceptional orbit portrait $\{\{0\}\}$, called the **zero portrait**. A portrait $\mathcal{P}$ is said to be **non-trivial** if $\mathcal{P}$ has valence $v \geq 2$ or equals the zero portrait.

Theorem 2.2.4 ([Mil00b])**.** Let $\mathcal{O}$ be an orbit of period $p \geq 1$ for $f = f_c$ and assume its portrait $\mathcal{P}$ has valence $v \geq 2$. Then there is one and only one sector S_1 based at some point $z_1 \in \mathcal{O}$ which contains $c = f(0)$. This sector S_1 can be characterized as the unique sector of the smallest angular length. The interval $I_{\mathcal{P}} = I_{S_1}$ is called the **characteristic arc** and the angle corresponding to the ends of $I_{\mathcal{P}}$ is called the **characteristic angles**.

Definition 2.2.5. A set $\mathcal{P} = \{A_1, \dots A_p\}$ of subsets of $\mathbb{R}/\mathbb{Z}$ is called the **formal orbit portrait** if it satisfies four conditions of Proposition 2.2.2.

Theorem 2.2.6. For a formal orbit portrait $\mathcal{P}$, there exists a quadratic map $f = f_c$ and its orbit $\mathcal{O}$ realizing $\mathcal{P}$.

Proposition 2.2.7 ([Mil00b])**.** Any orbit portrait of valence $v > r$ must have $v = 2$ and $r = 1$. It follows that there are just two possibilities:

(1) If $r = 1$ then at most two rays land on each orbit point, namely $v = 2$. We say that this orbit portrait is **primitive**.

(2) If $r > 1$ then $v = r$ and all rays belong to a single cyclic orbit under angle doubling. We say that this orbit portrait is **satellite**.

Proposition 2.2.8 ([Mil00b])**.** Let $\mathcal{P}$ be an orbit portrait of valence $v \geq 2$, and let $I_{\mathcal{P}} = (t_-, t_+)$ be its characteristic arc. Then a quadratic map f_c has an orbit portrait with portrait $\mathcal{P}$ if and only if the two dynamical rays $\mathcal{R}_{t_-}$ and $\mathcal{R}_{t_+}$ landing at a common point in the Julia set J_c.

For parameter rays $\mathcal{R}_t^{\mathcal{M}}$ and $\mathcal{R}_{t'}^{\mathcal{M}}$ landing at a common point w, we call $\mathcal{R}_t^{\mathcal{M}} \cup \mathcal{R}_{t'}^{\mathcal{M}} \cup \{w\}$ a **parameter ray pair** and denote it by $P(t, t')$. If there exist a minimal integer $m \geq 1$ such that $P(t, t') = P(2^m t, 2^m t')$, $P(t, t')$ is said to be of **period** m.

Let $0 < t_- < t_+ < 1$ be the angles of two dynamical rays $\mathcal{R}^c_{t_\pm}$ bounding S_1.

Theorem 2.2.9 ([Mil00b])**.** Two parameter rays $\mathcal{R}^c_{t_\pm}$ land at a **root point** $\mathbf{r}_{\mathcal{P}} \in \mathcal{M}$. The ray pair $P(t_-, t_+)$ cuts the parameter plane up into open subsets $W_{\mathcal{P}}$ and $\mathbb{C} - \overline{W_{\mathcal{P}}}$. $W_{\mathcal{P}}$ is called the ($\mathcal{P}$-)**wake** rooted at $\mathbf{r}_{\mathcal{P}}$. A quadratic map f_c has a repelling orbit with portrait $\mathcal{P}$ if and only if $c \in W_{\mathcal{P}}$ and has a parabolic orbit if and only if $c = \mathbf{r}_{\mathcal{P}}$.

Let n be a period of an attracting orbit of $f = f_c$, and let $\lambda_n = \lambda_n(f_c)$ be its multiplier, in other words, $\lambda_n = (f^n)'(p)$.

Theorem 2.2.10 ([DH84], [DH85a], [Mil00b], [Sch04])**.** (1) For any two parameters c and c' in a hyperbolic component H, f_c and $f_{c'}$ have attracting orbits of the same period n. The period n is called the **period of** H.

(2) Each hyperbolic component H is conformally isomorphic to the unit disk Δ under the map

$$\lambda_n \colon H \to \Delta; \;\; c \mapsto \lambda_n(f_c).$$

In particular, each H has a unique **center** c_H which maps to $\lambda_n(c_H) = 0$. This map extends uniquely to a homeomorphism between $\overline{H}$ and $\overline{\Delta}$.

(3) The point $\boldsymbol{r}_H$ in the boundary of H which satisfies $\lambda_n(r_h) = 1$ is a root point, and it is called the **root point for** H. The ray pair containing the root point for H also has the period n.

(4) If f_c has a parabolic periodic orbit of period n then c is a root point for one and only one hyperbolic component H. If $\lambda_n(f_c) = e^{\frac{2\pi i m}{n'}}$ then the period of H is nn' where $m \in \mathbb{Z},\ n' \in \mathbb{N}_{>0}$ and they are relatively prime.

Theorem 2.2.11 ([Mil00b], [Lav86])**.** If $\mathcal{P}$ and $\mathcal{Q}$ are two distinct non-trivial orbit portraits, then the closure of the wakes $\overline{W_{\mathcal{P}}}$ and $\overline{W_{\mathcal{Q}}}$ are either disjoint or strictly nested. In particular, if $I_{\mathcal{P}} \subset I_{\mathcal{Q}}$ with $\mathcal{P} \neq \mathcal{Q}$, then it follows that $\overline{W_{\mathcal{Q}}} \subset W_{\mathcal{P}}$, and the ray period of $\mathcal{P}$ is strictly grater than that of $\mathcal{Q}$.

Definition 2.2.12. Let H be a hyperbolic component of $\mathcal{M}$ whose root point $\boldsymbol{r}_{\mathcal{P}}$ has two rays $\mathcal{R}_{t_-}$ and $\mathcal{R}_{t_+}$ ($t_- < t_+$) and let $\mathcal{P}$ be the orbit portrait whose characteristic arc is (t_-, t_+). Then H is said to be **primitive** (resp. **satellite**) if $\mathcal{P}$ is primitive (resp. satellite).

2.3 Internal addresses of the Mandelbrot set

D.Schleicher introduced an internal address of the Mandelbrot set $\mathcal{M}$, which describes the combinatorial structure of $\mathcal{M}$ well (see [Sch17]).

Definition 2.3.1. For a parameter $c \in \mathcal{M}$, the **internal address** $S_0 \to S_1 \to S_2 \to \cdots$ of c is a strictly increasing finite or infinite sequence of integers defined as follows:

(1) The internal address starts with $S_0 = 1$ associated with the ray pair $P(0, 1)$.

(2) If $S_0 \to \cdots \to S_k$ is an initial segment of the internal address of c, where S_k associated with a ray pair $P(t_k, t'_k)$ of period S_k, then let $P(t_{k+1}, t'_{k+1})$ be the ray pair of least period which separates $P(t_k, t'_k)$ from c or for which $c \in P(t_{k+1}, t'_{k+1})$. Let S_{k+1} be the period of $P(t_{k+1}, t'_{k+1})$.

The case (2) continues for every $k \geq 1$ unless there is a finite k so that $P(t_k, t'_k)$ is not separated from c by any periodic ray pair.

For a parameter $c \in \mathcal{M}$, the internal address of c is unique by Theorem 2.2.11.

Definition 2.3.2. For a parameter $c \in \mathcal{M}$, the **angled internal address** for c is the sequence

$$(S_0)_{p_0/q_0} \to (S_1)_{p_1/q_1} \to (S_2)_{p_2/q_2} \to \cdots$$

where $S_0 \to S_1 \to S_2 \to \cdots$ is the internal address of c, and the angles p_k/q_k are defined as follows: for $k \geq 0$, let $P(t_k, t'_k)$ be the parameter ray pair associated with S_k. The landing point of $P(t_k, t'_k)$ is the root of a hyperbolic component H_k of period S_k. The angle p_k/q_k is defined so that c is contained in the wake W_k rooted at the point of ∂H_k at internal angle p_k/q_k.

If the internal address of c terminates with S_k, then the angled internal address of c is also finite and terminates with S_k:

$$(S_0)_{p_0/q_0} \to (S_1)_{p_1/q_1} \to \cdots \to (S_{k-1})_{p_{k-1}/q_{k-1}} \to S_k.$$

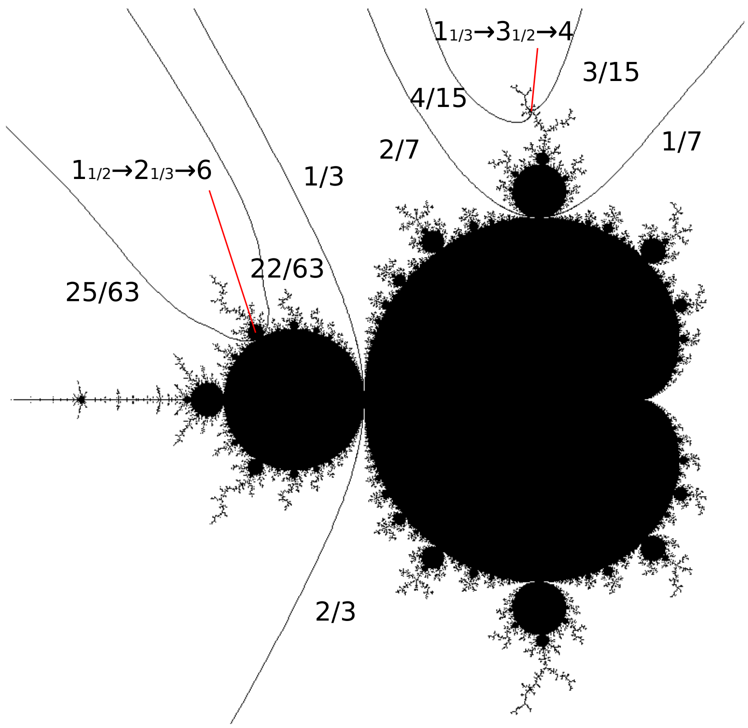


Figure 1: The Mandelbrot set with some parameter rays and angled internal addresses

2.4 Tuning

Definition 2.4.1. A **polynomial-like map** is a triple (g, U, V) of bounded simply connected domains U and V such that $\overline{U} \subset V$ and a holomorphic proper map $g\colon U \to V$. The **degree** of the polynomial-like map (g, U, V) is the degree of g. A polynomial-like map of degree 2 is called a **quadratic-like map**.

Definition 2.4.2. For a polynomial-like map (g, U, V), we define the **filled Julia set**

$$K(g) = \{z \in U \mid g^n(z) \in U \text{ for every } n \in \mathbb{N}\}.$$

Definition 2.4.3. Polynomial like mappings (g, U, V) and (g', U', V') are **hybrid equivalent** if there exists a quasiconformal map h of a neighborhood W of the filled Julia set$K(g) \subset U$ onto a neighborhood W' of the filled Julia set $K(g') \subset U'$ which satisfies

(1) $h(K(g)) = K(g')$,

(2) the dilatation of h, $\mu(h) = 0$ a.e. on $K(g)$, and

(3) $h \circ g = g' \circ h$ on $W \cap f^{-1}(W)$.

Definition 2.4.4. Let f be a quadratic map and let m be a positive integer. Then f^m is said to be (***c***-)**renormalizable** if there are simply connected domains U and V such that $c \in U$, (g, U, V) is a quadratic-like map where $g = f^m|_U$, and the filled Jula set $K_m = K(g)$ is connected. The quadratic-like map is called the (***c***-)**renormalization** of f^m.

Let $\mathcal{P}$ be an orbit portrait of ray period $n \geq 2$ and valence $v \geq 2$. Set $c \in W_{\mathcal{P}} \cup \{\boldsymbol{r}_{\mathcal{P}}\}$ such that $f = f_c$ has a periodic orbit $\mathcal{O}$ with the orbit portrait $\mathcal{P}$, and let S be the sector containing the critical value of f. The **Green function** or the **canonical potential function** for f is the function $G\colon \mathbb{C} \to [0, \infty)$ such that $G_c(z) = \log |\Phi_c(z)|$ for $z \in \mathbb{C} - K_c$ and it vanishes on K_c where Φ_c is the Böttcher map for f.

According to [DH85b] and [Mil00b], there are neighborhoods U and V of $S \cap \{G_c(z) < 1/2^n\}$ such that f^n has a c-renormalization $(g = f^n|_U, U, V)$ which is hybrid equivalent to uniquely defined quadratic map $f_{c'}$, with $c' \in \mathcal{M}$. We write $c = \mathcal{P} * c'$ or say that c equals $\mathcal{P}$ **tuned by** c'.

The correspondence

$$\mathcal{M} - \{1/4\} \to \mathcal{M} - \{1/4\}; \quad c' \mapsto \mathcal{P} * c'$$

is a continuous embedding onto a proper subset of $\mathcal{M} - \{1/4\}$.

For special cases, we define

$$\mathcal{P} * \frac{1}{4} := \boldsymbol{r}_{\mathcal{P}},$$

$$\{\{0\}\} * c' := c', \quad \text{for all } c' \in \mathcal{M}.$$

For details, see [DH85a], [DH85b] and [Hs00].

Theorem 2.4.5 ([Hs00], [Mil00b])**.** For each non-trivial orbit portrait $\mathcal{P}$, the correspondence $c' \mapsto \mathcal{P} * c'$ defines a continuous embedding of $\mathcal{M}$ into itself. Furthermore, there is a unique composition operation $(\mathcal{P}, \mathcal{Q}) \mapsto \mathcal{P} * \mathcal{Q}$ for a pair of non-trivial orbit portraits so that

$$(\mathcal{P} * \mathcal{Q}) * c = \mathcal{P} * (\mathcal{Q} * c)$$

for all $\mathcal{P}, \mathcal{Q}$ and c.

For example, let $\mathcal{B} = \{\{1/3, 2/3\}\}$ be an orbit portrait and $c_R \in \mathcal{M}$ be the center of the hyperbolic component rooted at the landing point of parameter rays of angle 1/7 and 2/7. The portrait $\mathcal{B}$ and the point c_R correspond to the Julia sets "basilica" and "(Douady) rabbit" respectively (see Figures 2 and 3). The filled Julia set $K_{\mathcal{B} * c_R}$ by tuning the basilica by the rabbit is shown in Figure 4.

There is a one-to-one correspondence between a non-trivial orbit portrait $\mathcal{P}$ and the center c_0 of a hyperbolic component rooted at $\boldsymbol{r}_{\mathcal{P}}$. Then for each $c' \in \mathcal{M}$, we sometimes write $\mathcal{P}$ tuned by c' as $c_0 * c'$ in place of $\mathcal{P} * c'$.

The next theorem gives an algorithm for computing angles of rays for tuned Julia sets.

Theorem 2.4.6 ([Dou86])**.** Let $a^0 < a^1$ be characteristic angles for an orbit portrait $\mathcal{P}$ and suppose they have periodic binary expansions of the form $.\overline{a_1^0 \cdots a_k^0}$ and $.\overline{a_1^1 \cdots a_k^1}$ of period exactly k. If the

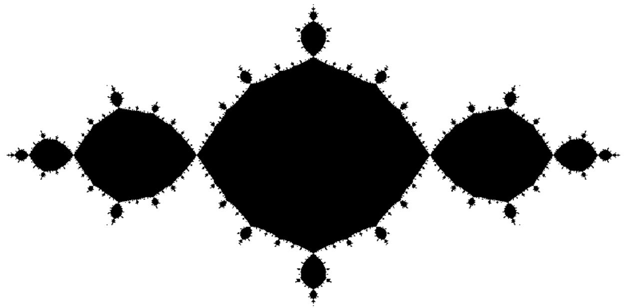


Figure 2: The basilica filled Julia set

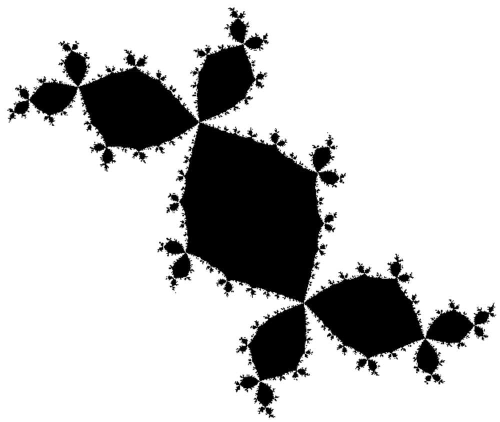


Figure 3: The rabbit filled Julia set

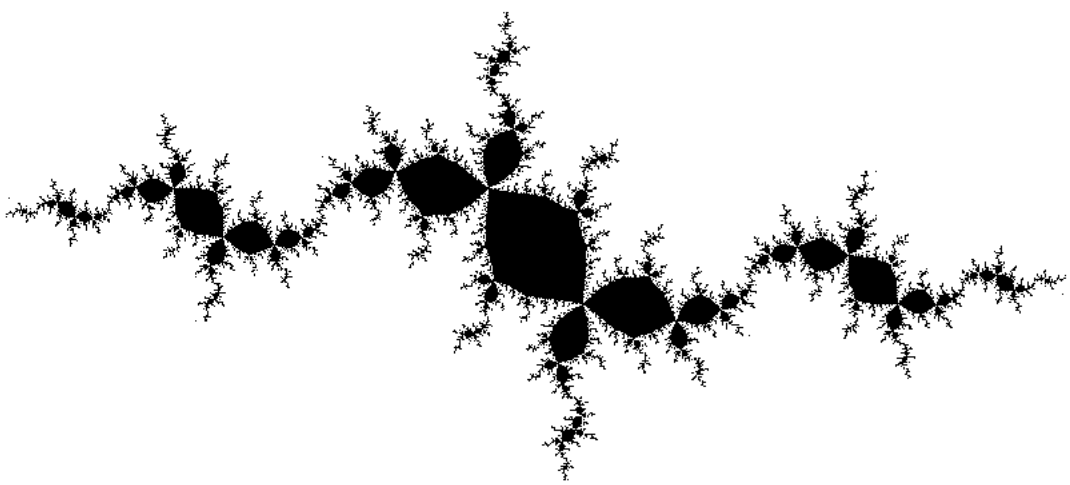


Figure 4: The filled Julia set for the basilica tuned by the rabbit

point $c' \in \mathcal{M}$ has a landing parameter ray of angle t with binary expansion $.\overline{t_1 \cdots t_n}$, then the image $\mathcal{P} * c'$ is the landing point of a parameter ray of angle t' whose binary expansion is obtained by

$$.\overline{a_1^{t_1} \cdots a_k^{t_1} a_1^{t_2} \cdots a_k^{t_2} \cdots a_1^{t_n} \cdots a_k^{t_n}}.$$

We write $t' = \mathcal{P} * t = (a^0, a^1) * t$ or simply $a^0 * t$.

For example, the smaller characteristic angle of the orbit portrait corresponding to the basilica Julia set tuned by the rabbit is $\mathcal{B} * \frac{1}{7} = (\frac{1}{3}, \frac{2}{3}) * \frac{1}{7} = \frac{1}{3} * \frac{1}{7} = \frac{22}{63}$.

In the filled Julia set for the basilica tuned by the rabbit, there are small "copies" of the rabbit (see Figure 4). In fact, the tuned filled Julia set is obtained by replacing each Fatou component of the basilica Julia set for the filled Julia set by the rabbit. Douady and Hubbard claimed the above property in [DH85b]. However for a complete proof we have to refer Haïssinsky's paper [Hs00].

Assume $K = K_c$ is locally connected and connected. Let $\{U_i\}_{i \in \mathbb{N}}$ be a family of Fatou components such that $K = \left(\bigcup_{n \in \mathbb{N}} U_i\right) \cup J_c$. We define an equivalence relation $\sim$ on K as follows:

$$x \sim y \quad \Longleftrightarrow \quad x = y \text{ or there exists a Fatou component } U_i \text{ such that } x, y \in U_i.$$

Since for every $\epsilon > 0$ the number of Fatou components whose diameters are larger than ϵ is finite ([Mil06], §19), the quotient space $\widehat{K} = K/\sim$ is compact and metrizable, and the quotient map $q\colon K \to \widehat{K}$ is continuous and proper (see [Hs00], §5).

Let $\infty \notin U$ be a Fatou component of K and x_U be its center. We define a continuous surjection $\pi_U\colon K \to \overline{U}$ so that:

(1) if $x \in \overline{U}$ then $\pi_U(x) = x$,

(2) otherwise there is an unique point $y \in \overline{U}$ such that $y \in [x, x_U]_K$ and $(x, y)_K \cap U = \emptyset$, and then we set $\pi_U(x) = y$.

Also we define a continuous map

$$\begin{array}{cccc} \varphi\colon & K & \longrightarrow & \widehat{K} \times \prod_{i \in \mathbb{N}} U_i \\ & \Cup & & \Cup \\ & x & \longmapsto & (q(x), (\pi_{U_i})_{i \in \mathbb{N}}). \end{array}$$

We can easily see that this map is injective. Let $L = K_{c'}$ be the filled Julia set for a parameter $c' \in \mathcal{M}$ with the extended Böttcher map $\Psi_{c'}\colon S^1 \to J_{c'}$.

For each $i \in \mathbb{N}$, let L_i be a copy of L.

Definition 2.4.7. The set

$$K_L = \left\{ (\xi, (\xi_i))_{i\in\mathbb{N}} \in \widehat{K} \times \prod_{i\in\mathbb{N}} L_i \;\middle|\; \begin{array}{l} \xi_i = \psi_{U_i}\pi_{U_i}(q^{-1}(\xi)) \text{ if } q(U_i) \neq \{\xi\}, \\ \text{otherwise, if there exist } k \in N \text{ such that } q(U_k) = \{\xi\} \text{ then} \\ \xi_i = \psi_{U_i}\pi_{U_i}(x_{U_k}) \end{array} \right\}$$

is called the filled Julia set K **tuned by** L.

Theorem 2.4.8 ([Hs00]). Let $c' \in \mathcal{M}$ and c_0 be the center of a hyperbolic component of $\mathcal{M}$ of period k, and set $c = c_0 * c$. Then K_c is homeomorphic to $(K_{c_0})_{K_{c'}}$.

2.5 Thompson groups F and T

The Thompson groups F and T were defined by Richard Thompson in 1965. Let us recall their definitions and properties without proofs. For details, see [CFP96].

Definition 2.5.1.

- The **Thompson group** T is the group of orientation preserving piecewise linear homeomorphisms on $S^1 = \mathbb{R}/\mathbb{Z}$ that map dyadic rational numbers to themselves, and that are differentiable except at finitely many dyadic rational numbers, and the derivatives on intervals of differentiability are powers of 2.
- The **Thompson group** F is the subgroup of T consisting of elements fixing $0 \in S^1$. This group can be regarded as a subgroup of the group of homeomorphisms of the unit interval $[0,1]$.

For example, the functions A, B and C defined below are elements of T. In particular, A and B are contained in F.

$$A(x) = \begin{cases} \frac{x}{2} & x \in \left[0, \frac{1}{2}\right] \\ x - \frac{1}{4} & x \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ 2x - 1 & x \in \left[\frac{3}{4}, 1\right] \end{cases}, \quad B(x) = \begin{cases} x & x \in \left[0, \frac{1}{2}\right] \\ \frac{x}{2} + \frac{1}{4} & x \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ x - \frac{1}{8} & x \in \left[\frac{3}{4}, \frac{7}{8}\right] \\ 2x - 1 & x \in \left[\frac{7}{8}, 1\right] \end{cases}, \quad C(x) = \begin{cases} \frac{x}{2} + \frac{3}{4} & x \in \left[0, \frac{1}{2}\right] \\ 2x - 1 & x \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ x - \frac{1}{4} & x \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

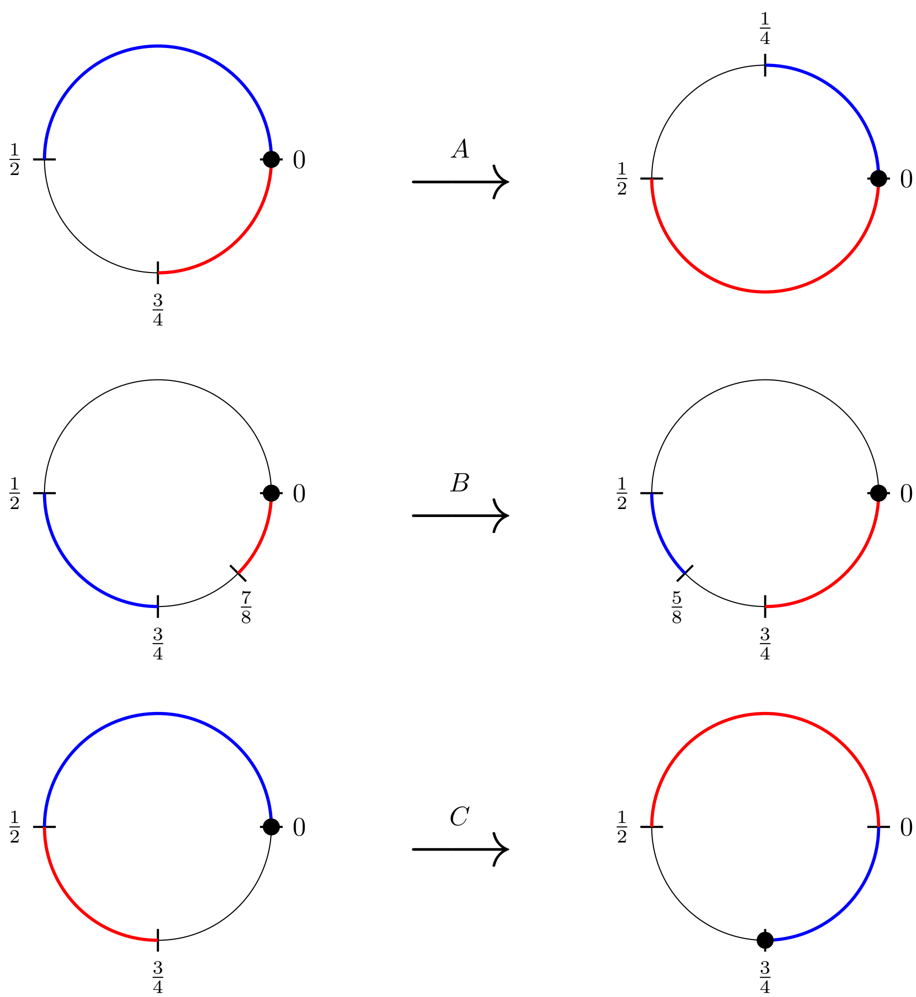
These elements are presented as diagrams in Figure 5.

Proposition 2.5.2. Let $0 = x_0 < x_1 < \cdots < x_m = 1$ and $0 = y_0 < y_1 < \cdots < y_m = 1$ be partitions of S^1 consisting of dyadic rational numbers.

(1) There exists $f \in F$ such that $f(x_i) = y_i$ for all $i = 0, \ldots, n-1$. Furthermore, if $x_{i-1} = y_{i-1}$ and $x_i = y_i$ for some $i \in \{1, \ldots, N\}$, then f can be taken to be trivial on the interval $[x_{i-1}, x_i]$.

(2) For each $j \in \{0, \ldots, n-1\}$ there exists $f \in T$ such that $f(x_i) = y_{i+j}$ for $i = 0, \ldots, n-1$.

Theorem 2.5.3 (R. Thompson (1965), cf. [CFP96]). (1) The Thompson group T is generated by A, B, and C, and is finitely presented.



$\frac{1}{2}$
0
$\frac{3}{4}$
A
$\frac{1}{4}$
$\frac{1}{2}$
0
$\frac{1}{2}$
0
$\frac{7}{8}$
$\frac{3}{4}$
B
$\frac{1}{2}$
0
$\frac{5}{8}$
$\frac{3}{4}$
$\frac{1}{2}$
0
$\frac{3}{4}$
C
$\frac{1}{2}$
0
$\frac{3}{4}$

Figure 5:

(2) The Thompson group F is generated by A and B, and is finitely presented.

Theorem 2.5.4 (cf. [CFP96])**.** The commutator subgroup $[F, F]$ of F consists of all elements in F which are trivial in a neighborhood of $0 \in S^1$. Furthermore, $F/[F, F] \cong \mathbb{Z} \oplus \mathbb{Z}$.

Theorem 2.5.5 (R. Thompson (1965), cf. [CFP96])**.** The Thompson group T and the commutator subgroup $[F, F]$ of F are simple.

3 Thompson-like groups for satellite components

The Thompson group for the basilica T_B was defined in [BF15b], where the basilica is the Julia set of the quadratic dynamical system $f_{-1}(z) = z^2 - 1$. In this section we introduce T_B and some generalizations in parallel.

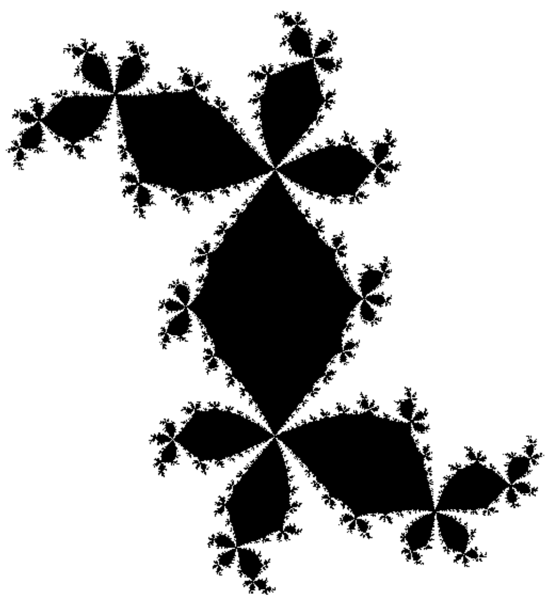


Figure 6: The filled Julia set $J\left(\frac{1}{2^4-1}\right) = J\left(\frac{1}{15}\right)$

3.1 Basic definitions

Let c be the center of a hyperbolic component, with (finite) angled internal address

$$(S_0)_{p_0/q_0} \to (S_1)_{p_1/q_1} \to \cdots \to (S_{k-1})_{p_{k-1}/q_{k-1}} \to S_k$$

associated with the ray pairs $P(t_l, t_l')$ with $t_l \leq t_l'$ for $0 \leq l \leq k$. We denote the Julia set and filled Julia set of f_c by $J_c = J(t_l)$ and $K_c = K(t_l)$ respectively. Let $\Psi \colon S^1 \to J(t_k)$ be the extended Böttcher map, and let $\mathcal{P}^l = \{A_1^l, \ldots, A_{S_l}^l\}$ be an orbit portrait whose characteristic arc is $I_{\mathcal{P}^l} = (t_l, t_l')$ for $0 \leq l \leq k$.

Definition 3.1.1. Set $A_1^l := \{a_1, \ldots, a_q\}$ with $1 \le l \le k$. For $m \in \mathbb{N}$, let $B^l(m) = \{b_1(m), \ldots, b_q(m)\}$ be a subset of $S^1 = \mathbb{R}/\mathbb{Z}$ such that $\{2^m b_1(m), \ldots, 2^m b_q(m)\} = \{a_1, \ldots, a_q\} = A_1^l$. If $\Psi(b_1(m)) = \cdots = \Psi(b_q(m)) =: w \in J(t_k)$, we write $w = (b_1(m); \cdots ; b_q(m))$.

For a point $w = (b_1(m); \cdots ; b_q(m))$, the convex hull of the set $\{b_1(m), \ldots, b_q(m)\} \subset S^1$ in the closed unit disc with respect to the Poincaré metric is called the **pinching locus** for $J(t_k)$, and we also denote it by $(b_1(m); \cdots ; b_q(m))$ identifying with $w \in J(t_k)$. The closed unit disc with all pinching loci for $J(t_k)$ for all $l \in \{1, \ldots, k\}$ is called the **pinching lamination** for $J(t_k)$ and is denoted by $\mathcal{L}(t_k)$.

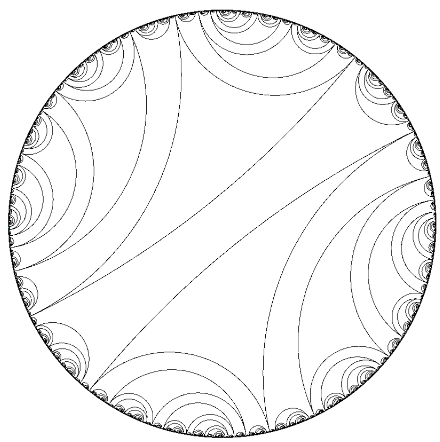


Figure 7: The pinching lamination $\mathcal{L}\left(\frac{1}{7}\right)$

In this section, we mainly regard the simplest case $t_k = \frac{1}{2^n-1}$, in other words, the angled internal address of c and the corresponding orbit portrait are

$$1_{1/n} \to n \quad \text{and} \quad \mathcal{P}^1 = \{\{2^0/(2^n-1), 2^1/(2^n-1), \ldots, 2^{n-1}/(2^n-1)\}\}.$$

In order to lighten the notations, we set $J^{(n)} = J\left(\frac{1}{2^n-1}\right)$, $K^{(n)} = K\left(\frac{1}{2^n-1}\right)$ and $\mathcal{L}^{(n)} = \mathcal{L}\left(\frac{1}{2^n-1}\right)$.

Definition 3.1.2. A **finite locus diagram** for $J^{(n)}$ is the closed unit disc $\overline{\Delta}$ with:

(1) the **primary loci** : $\left(\dfrac{2^0}{2^n-1}; \cdots ; \dfrac{2^{n-1}}{2^n-1}\right)$, $\left(\dfrac{1}{2} + \dfrac{2^0}{2^n-1}; \cdots ; \dfrac{1}{2} + \dfrac{2^{n-1}}{2^n-1}\right)$, and

(2) a finite number of pinching loci; each of them is added one by one so that it subdivides an interval with ratios $1 : 2 : 4 : \cdots : 2^{n-1} : 1$.

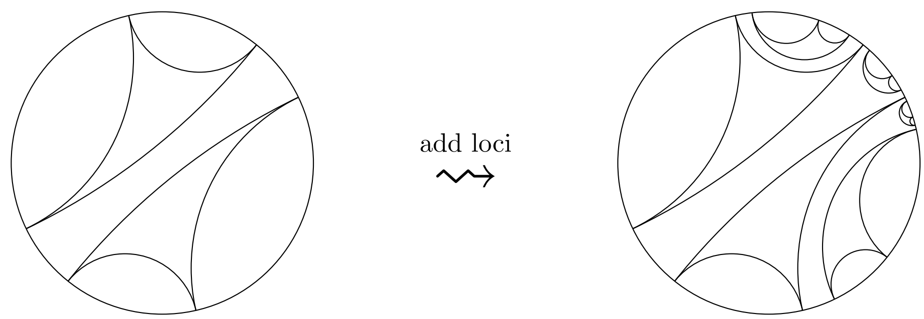


Figure 8: An example of a finite locus diagram for $J^{(3)}$

Definition 3.1.3. We consider each finite locus diagram as a 2-complex. Two finite locus diagrams G and H are **isomorphic** if there exists an orientation preserving isomorphism $f\colon G' \to H'$ where G' and H' are 2-complexes corresponding to G and H respectively. G is called the **domain diagram** and H is called the **range diagram**. A pair (G, H) of a domain and range diagram is called a **locus pair diagram** for $J^{(n)}$.

Definition 3.1.4. An **expansion** of a locus pair diagram (G, H) consists of adding a locus to G subdividing an interval of G, and adding the image of the locus to H. A **reduction** is the inverse operation. (G, H) is said to be **reduced** if no reductions are possible.

Proposition 3.1.5. Every locus pair diagram for $J^{(n)}$ has a unique reduced locus pair diagram.

Proof. Let $f \in T^{(n)}$. A standard interval $I \subset S^1$ of a locus L in $\mathcal{L}^{(n)}$ is said to be **regular** (with respect to f) if f is linear on I and $f(I)$ is also a standard interval of $f(L)$. Each standard interval of the domain diagram D_f of f must be regular. An locus pair diagram for f is reduced if and only if each regular interval in D_f is maximal under inclusion. Since any two maximal regular intervals have disjoint interiors, there can only one subdivision of the circle into regular intervals. $\square$

A locus pair diagram induces an orientation preserving piecewise linear homeomorphism on S^1 whose breakpoints are vertices of loci lying on the domain diagram. This homeomorphism induces an orientation preserving homeomorphism again on $J^{(n)}$ and we call this homeomorphism a **rearrangement** for $J^{(n)}$.

Theorem 3.1.6. Let f be an orientation preserving piecewise linear homeomorphism of the unit circle. The map f induces a rearrangement for $J^{(n)}$ if and only if:

(1) the pinching lamination for $J^{(n)}$ is invariant under f, and

(2) every breakpoint of f is the vertex of a pinching locus.

Proof. If a map f induces a rearrangement for $J^{(n)}$, then the conditions (1) and (2) are clearly satisfied.

For the converse direction, suppose each locus in $\mathcal{L}^{(n)}$ has the form

$$\left(\frac{k+1}{(2^n-1)2^m}; \frac{k+2}{(2^n-1)2^m}; \cdots; \frac{k+2^{n-1}}{(2^n-1)2^m}\right), \quad m, k \in \mathbb{N}.$$

Therefore each linear segment of f must preserve this set of ends of loci. Then f must have the form

$$f(x) = 2^p \left(x + \frac{q}{2^r} \right), \quad p, q, r \in \mathbb{Z}.$$

Let L be a locus. The shortest closed interval which contains all endpoints of L is called the **standard interval** for L. Let D be a locus diagram. Endpoints of loci of D subdivide the unit circle into intervals. Assume D contains enough loci so that f is linear on each interval obtained as above. Since f sends standard intervals to standard intervals, the image of D by f forms a locus diagram R, then f is a rearrangement. $\square$

Definition 3.1.7. The above theorem shows that

$$T^{(n)} = T\left(\tfrac{1}{2^n-1}\right) := \left\{ f \colon J^{(n)} \to J^{(n)} \mid f \text{ is a rearrangement for } J^{(n)} \right\}$$

has a group structure under composition. We call this group the **rearrangement group** for the Julia set $J^{(n)}$.

The Thompson group for the basilica in [BF15b] coincides with the rearrangement group $T\left(\frac{1}{3}\right) = T^{(2)}$.

Proposition 3.1.8. The rearrangement group $T\left(\frac{1}{2^n-1}\right)$ can be embedded into T as a subgroup.

Proof. We consider a piecewise linear homeomorphism on S^1;

$$h(x) = \begin{cases} \frac{2^n-1}{2}x - \frac{1}{8} & x \in \left[\frac{1}{2(2^n-1)}, \frac{1}{2^n-1}\right] \\ \frac{2^n-1}{4(2^{n-1}-1)}x - \frac{3\cdot 2^{n-1}-5}{8(2^{n-1}-1)} & x \in \left[\frac{1}{2^n-1}, \frac{2^{n-1}}{2^n-1}\right] \\ \frac{2^n-1}{2}x + \frac{5}{8} - 2^{n-2} & x \in \left[\frac{2^{n-1}}{2^n-1}, \frac{2^n+1}{2(2^n-1)}\right] \\ \frac{2^n-1}{4(2^{n-1}-1)}x + \frac{7}{8} - \frac{2^n-1}{8(2^{n-1}-1)} & x \in \left[\frac{2^n+1}{2(2^n-1)}, \frac{1}{2(2^n-1)} + 1\right]. \end{cases}$$

Since this map h sends the ends of each locus in $\mathcal{L}^{(n)}$ to dyadic points, $hT^{(n)}h^{-1}$ is a subgroup of T. $\square$

3.2 Generators of $T^{(n)}$

Let us introduce some fundamental elements $\alpha_1, \ldots, \alpha_{n-1}, \beta, \gamma, \delta \in T^{(n)}$ (see Figure 9).

$$\alpha_i(x) = \begin{cases} \frac{2^n-2^{n-i}-2}{2^i-2}x + \frac{-2^n+2^{n-i}-2^i}{2(2^n-1)(2^i-2)} & x \in \left[\frac{1}{2^n-1}, \frac{2^{i-1}}{2^n-1}\right] \\ 2^{n-i}x - \frac{1}{2} & x \in \left[\frac{2^{i-1}}{2^n-1}, \frac{2^i}{2^n-1}\right] \\ \frac{2^{n-i-1}-1}{2(2^n-2^i)}x + \frac{2^{2n}-2^{n+i}-2^{n-1}+2^i}{2(2^n-1)(2^n-2^i)} & x \in \left[\frac{2^i}{2^n-1}, \frac{2^{n-1}}{2^n-1}\right] \\ 2^{i-n}x + \frac{(2^{n-1}-1)(2^i+1)}{2(2^n-1)} & x \in \left[\frac{2^{n-1}}{2^n-1}, \frac{1}{2^n-1} + 1\right] \end{cases},$$

$$\beta(x) = \begin{cases} \frac{x}{2^n} + \frac{1}{2^{n+1}} & x \in \left[\frac{1}{2(2^n-1)}, \frac{2^{n-1}}{2^n-1}\right] \\ x - \frac{2^{2n}-2^{n+1}+1}{2^{n+1}(2^n-1)} & x \in \left[\frac{2^{n-1}}{2^n-1}, \frac{2^{n-1}}{2^n-1} + \frac{1}{2^{n+1}(2^n-1)}\right] \\ 2^n x - \frac{2^n+1}{2} & x \in \left[\frac{2^{n-1}}{2^n-1} + \frac{1}{2^{n+1}(2^n-1)}, \frac{2^n+1}{2(2^n-1)}\right] \\ x & x \in \left[\frac{2^n+1}{2(2^n-1)}, \frac{1}{2(2^n-1)} + 1\right] \end{cases},$$

$$\gamma(x) = \begin{cases} \frac{x}{2^n} + \frac{1}{2} & x \in \left[\frac{1}{2(2^n-1)}, \frac{1}{2^n-1} - \frac{1}{2^{n+1}(2^n-1)}\right] \\ x - \frac{2^{2n}-2^{n+1}+1}{2^{2n+1}(2^n-1)} & x \in \left[\frac{1}{2^n-1} - \frac{1}{2^{n+1}(2^n-1)}, \frac{1}{2^n-1} - \frac{1}{2^{n+1}(2^n-1)} + \frac{1}{2^{2n+1}(2^n-1)}\right] \\ \frac{x}{2^n} + \frac{1}{2} & x \in \left[\frac{1}{2^n-1} - \frac{1}{2^{n+1}(2^n-1)} + \frac{1}{2^{2n+1}(2^n-1)}, \frac{1}{2^n-1}\right] \\ x & x \in \left[\frac{1}{2^n-1}, \frac{1}{2(2^n-1)} + 1\right] \end{cases},$$

$$\delta(x) = x + \frac{1}{2}.$$

A domain in $\mathcal{L}^{(n)}$ surrounded by (infinitely many) loci is called a **gap** and the gap corresponding to the Fatou component which contains 0 is called the **critical gap** C. A locus surrounding the critical gap is called a **critical locus** and let $\mathcal{L}_C^{(n)}$ be the set of all critical loci in $\mathcal{L}^{(n)}$.

Definition 3.2.1.

- The **stabilizer** $\mathrm{stab}(C) = \left\{f \in T^{(n)} \mid f(C) = C\right\}$ is the group of elements of $T^{(n)}$ which send C to itself.
- The **rigid stabilizer** $\mathrm{rist}(C) = \{f \in \mathrm{stab}(C) \mid$ the reduced locus pair diagram for f has only critical loci $\}$.

We prove the next proposition by the same way as in [BF15b].

Proposition 3.2.2.

(1) Each element of $\mathrm{stab}(C)$ acts on $\mathcal{L}_C^{(n)}$ as an element of the Thompson group T.

(2) The rigid stabilizer $\mathrm{rist}(C)$ acts on $\mathcal{L}_C^{(n)}$ as an isomorphic copy of the Thompson group T.

Proof. (1) For an element $f \in \mathrm{stab}(C)$, let D_f and R_f be the reduced domain and range diagram respectively. We will define a bijection $\tau\colon \mathcal{L}_C^{(n)} \to \{b/2^a \mid a, b \in \mathbb{N}\}$.

Set $\tau(L_-) = 1/2$, $\tau(L_+) = 0$ where $L_- = \left(\frac{2^0}{2^n-1}; \cdots; \frac{2^{n-1}}{2^n-1}\right), L_+ = \left(\frac{1}{2} + \frac{2^0}{2^n-1}; \cdots; \frac{1}{2} + \frac{2^{n-1}}{2^n-1}\right)$. Note that L_- corresponds to the root of the Fatou component containing 0. Let $L (\neq L_+, L_-)$ be a critical locus and let L' and L'' be loci surrounding L which have longer standard intervals than that of L. Assume that there are no loci whose standard intervals are longer than that of L between L and L' or L and L''. Then we define $\tau(L) = \frac{1}{2}\left(\frac{a'}{2^{b'}} + \frac{a''}{2^{b''}}\right)$ where $\tau(L') = \frac{a'}{2^{b'}}$ and $\tau(L'') = \frac{a''}{2^{b''}}$. We obtain a dyadic subdivision $\tau(D_f)$ of S^1 for D_f, and since $f \in \mathrm{stab}(C)$ also we may obtain that for R_f. The pair of these two dyadic subdivisions yields an element of T and we denote it by $\tau(f)$.

(2) The map $\tau\colon \mathrm{stab}(C) \to T$ clearly induces an isomorphism from $\mathrm{rist}(C)$ to T. $\square$

Corollary 3.2.3. The rigid stabilizer $\mathrm{rist}(C)$ is genarated by β, γ, and δ.

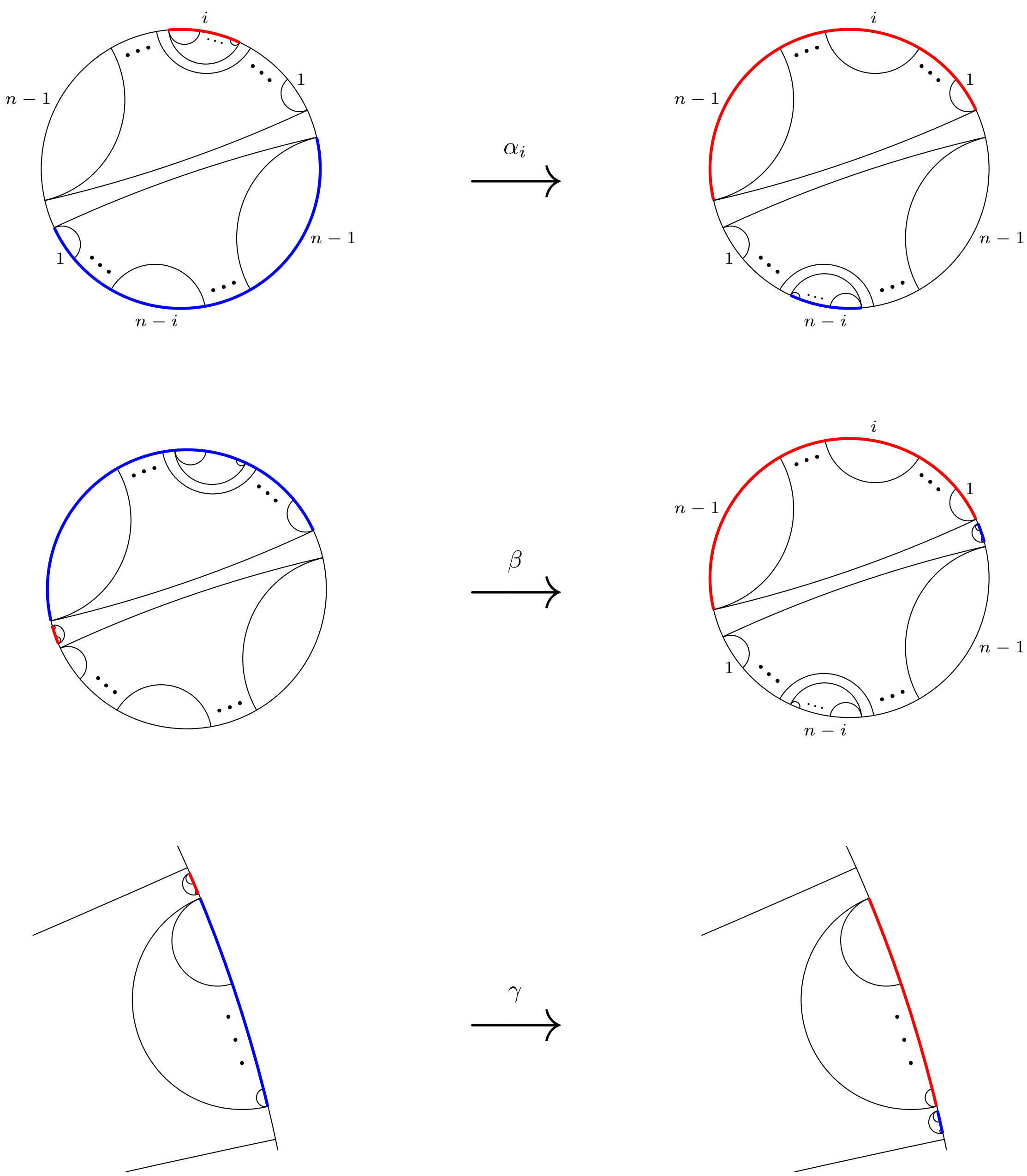


Figure 9:

Proof. We can easily see that $\tau(\beta) = A, \tau(\gamma^\delta) = B$ and $\tau(\beta^{-1}\delta) = C$. Since T is generated by A, B and C, the claim is proved. $\square$

By calculation, we can show that each α_i is in the group $\langle \alpha_1, \delta \rangle$.

Lemma 3.2.4. We consider the subscriptions in modulo n. Then

$$\delta = \begin{cases} \alpha_i \delta \alpha_j & (i + j = n) \\ \alpha_j \alpha_{i+j}^{-1} \alpha_i & (i + j \neq n) \end{cases}.$$

Lemma 3.2.5. The group $\langle \alpha_1, \beta, \gamma, \delta \rangle$ acts transitively on the gaps of $\mathcal{L}^{(n)}$.

Proof. The **depth** of a gap L of $\mathcal{L}^{(n)}$ is the number of loci separating L from the critical gap C. Let G_m be a gap of depth m. It is enough to show that G_m is mapped to C by an element of the group $\langle \alpha_1, \beta, \gamma, \delta \rangle$. We use induction on m. If $m = 0$ it is trivial. Suppose $m = 1$. By Proposition 2.5.2, there exists $f \in \langle \beta, \gamma, \delta \rangle$ and $i \in \{1, \ldots, n-1\}$ such that $f(G_1) = C_i$ where C_i is the gap which has the arc $\left(\frac{2^{i-1}}{2^n-1}; \frac{2^i}{2^n-1}\right)$ as a part of its boundary. Then we find $\alpha_i f(G_1) = C$.

Finally we consider the case $m \geq 2$. Let $G_1, G_2, \ldots, G_m$ be a sequence of gaps such that G_i and G_{i+1} are adjacent to the same locus. Then the sequence $\alpha_{i_1} f(G_2), \ldots, \alpha_{i_1} f(G_n)$ also satisfies the above condition and each $\alpha_{i_1} f(G_k)$ is of depth $k-1$, and $\alpha_{i_1} f(G_1) = C$, where $\alpha_{i_1}(G_1) = C$. $\square$

Theorem 3.2.6. The rearrangement group $T^{(n)}$ is generated by α_1, β, γ and δ.

Proof. Let f be an element of $T^{(n)}$. By Lemma 3.2.5, we may assume that $f \in \mathrm{stab}(C)$. Let m be the number of the loci of the reduced domain diagram D_f of f. We use induction on m. If $m = 2$, then $f = \mathrm{id}$ or δ. Suppose $m \geq 3$. There exists $g \in \mathrm{rist}(C)$ such that $h = g \circ f$ fixes each critical locus. By Lemma 3.2.4 it is enough to show $h \in \langle \alpha_1, \ldots, \alpha_{n-1}, \beta, \gamma, \delta \rangle$. Assume that the reduced domain diagram D_h of h contains critical loci $L_1, \ldots, L_k$.

<u>Case1</u> Assume D_h has loci in more than one standard interval for L_i. Then we can write

$$h = h_1 \circ h_2 \circ \cdots \circ h_k$$

where each $h_i \in T^{(n)}$ is an rearrangement which has the same critical loci as h but has non-critical loci only in the standard interval for L_i. Each h_i has fewer than m loci, then by induction, $h_i \in \langle \alpha_1, \ldots, \alpha_{n-1}, \beta, \gamma, \delta \rangle$.

<u>Case2</u> Assume D_h has non-critical loci only behind the critical locus $L = L_i$. By Proposition 2.5.2 and Lemma 3.2.2, we may assume $L = L_- = \left(\frac{2^0}{2^n-1}; \cdots; \frac{2^{n-1}}{2^n-1}\right)$. Then D_h must contain at least one locus $\alpha_j(L)$, $j \in \{1, \ldots, n-1\}$. The domain diagram $\alpha_j D_h \alpha_j^{-1}$ of $\alpha_j h \alpha_j^{-1}$ has m loci, however we can reduce it: the locus $\alpha_j(L_+)$ can be reduced where $L_+ = \left(\frac{1}{2} + \frac{2^0}{2^n-1}; \cdots; \frac{1}{2} + \frac{2^{n-1}}{2^n-1}\right)$. Since the reduced domain diagram for $\alpha_j h \alpha_j^{-1}$ has fewer than m loci, $\alpha_j h \alpha_j^{-1} \in \langle \alpha_1, \ldots, \alpha_{n-1}, \beta, \gamma, \delta \rangle$ by induction. $\square$

In particular, Theorem 3.2.6 says that $T^{(n)}$ is finitely generated, in other words, it is of type F_1. The Thompson group T is of type F_∞, nevertheless $T^{(2)}$ is not even finitely generated, in other words it is not of type F_2.

Theorem 3.2.7 ([WZ16])**.** The group $T^{(2)}$ is not finitely presentable.

Let $A = \left(\frac{2^0}{2^n-1}; \cdots ; \frac{2^{n-1}}{2^n-1}\right)$ be a locus in $\mathcal{L}^{(n)}$ (which was denoted by L_- before). The locus A is adjacent to the critical gap C. We give a color $a \in \mathbb{Z}_n$ to C and give colors $a+1, a+2, \dots, a+n-1 \in \mathbb{Z}_3$ to gaps surrounding A counterclockwise. In the same manner, we give colors to all gaps inductively. Now we define a homomorphism

$$\phi\colon T^{(n)} \to \mathbb{Z}_n$$

where $\phi(f) = k$ if $f \in T^{(n)}$ changes the color a of C to $a + k$.

Theorem 3.2.8. $\phi\colon T^{(n)} \big/ \left[T^{(n)}, T^{(n)}\right] \to \mathbb{Z}_n$ induces a group isomorphism.

We need Schreier's lemma to show Theorem 3.2.8.

Lemma 3.2.9 (Schreier's lemma, cf. [Ser03])**.** Let G be a finitely generated group with a generating set S and H be a subgroup of G, and let $\sigma\colon G/H \to G$ be a section of the quotient map $G \to G/H$ and denote $\sigma(G/H) = R$ and $\sigma(gH) = \overline{g}$ for $g \in G$. Then H is generated by the set $\left\{(\overline{sr})^{-1}sr \;\middle|\; s \in S, r \in R\right\}$.

Proof of Theorem 3.2.8. Set $G = T^{(n)}$, $H = \ker\phi$, $S = \{\alpha_1, \beta, \gamma, \delta\}$. Let $\sigma\colon G/H \to G$ be a section of the quotient map defined by $\overline{g} = \sigma(gH) = (\delta\alpha_1)^k$ if $\phi(g) = k$, and R denotes $\sigma(g/H) = \left\{(\delta\alpha_1)^k \;\middle|\; k \in \mathbb{Z}\right\}$. Since $(\delta\alpha_1)^n = \mathrm{id}$, σ is well-defined and it is easy to see that σ is group homomorphism. Set $U = \left\{(\overline{sr})^{-1}sr \;\middle|\; s \in S, r \in R\right\}$. We have to show that $H = \left[T^{(n)}, T^{(n)}\right]$. Since $\mathbb{Z}_3$ is abelian, $\left[T^{(n)}, T^{(n)}\right] < H$ is trivial.

For the converse direction, it is enough to show that $U \subset \left[T^{(n)}, T^{(n)}\right]$ since the generating set of H is U by Schreier's lemma. By calculation, we can see that $(\overline{\eta(\delta\alpha_1)^k})^{-1}\eta(\delta\alpha_1)^k = \eta^{(\delta\alpha_1)^k}$ where $\eta \in \{\mathrm{id}, \beta, \gamma, \delta\}$, and since $\delta^2 = \mathrm{id}$, $(\overline{\alpha_1(\delta\alpha_1)^k})^{-1}\alpha_1(\delta\alpha_1)^k = (\delta\alpha_1)^{-(k+1)}\delta\delta\alpha_1(\delta\alpha_1)^k = \delta^{(\delta\alpha_1)^{k+1}}$. Since $\langle\beta, \gamma, \delta\rangle \cong T = [T, T]$, β, γ and δ are elements of $\left[T^{(n)}, T^{(n)}\right]$. Then it follows that $\eta^{(\delta\alpha_1)^k}$, $\delta^{(\delta\alpha_1)^{k+1}} \in \left[T^{(n)}, T^{(n)}\right]$, and we find $U \subset \left[T^{(n)}, T^{(n)}\right]$. □

Theorem 3.2.8 shows that there is an exact sequence

$$1 \to \left[T^{(n)}, T^{(n)}\right] \to T^{(n)} \to \mathbb{Z}_n \to 1.$$

Furthermore the section $\sigma\colon G/H \to G$ yields the right splitting of the exact sequence, then it follows

$$T^{(n)} = \left[T^{(n)}, T^{(n)}\right] \rtimes \mathbb{Z}_n.$$

Since the Thompson group T is simple and the commutator subgroup $[T,T]$ is not trivial, $T=[T,T]$.

Corollary 3.2.10. (1) For every $n \geq 2$, $T \ncong T^{(n)}$.

(2) If $n \neq m$ then $T^{(m)} \ncong T^{(n)}$.

Differ from the Thompson group T, each $T^{(n)}$ is not simple but we will show the following.

Theorem 3.2.11. The commutator subgroup $\left[T^{(n)}, T^{(n)}\right]$ of $T^{(n)}$ is simple.

Lemma 3.2.12. The commutator subgroup $\left[T^{(n)}, T^{(n)}\right]$ acts transitively on the set of all loci in $\mathcal{L}^{(n)}$.

Proof. Let L be a locus in $\mathcal{L}^{(n)}$. It is enough to show that L can be sent to the locus $A = \left(\frac{2^0}{2^n-1}; \cdots ; \frac{2^{n-1}}{2^n-1}\right)$ by an element of $\left[T^{(n)}, T^{(n)}\right]$. We may assume that L lies in the closet subset of the closed unit disk whose boundary is the union of the interval $\left[\frac{1}{2^n-1}, \frac{2^{n-1}}{2^n-1}\right]$ and the arc $\left(\frac{1}{2^n-1}; \frac{2^{n-1}}{2^n-1}\right)$ since the group $\langle \beta, \gamma, \delta \rangle < \left[T^{(n)}, T^{(n)}\right]$ acts on $\mathcal{L}_C^{(n)}$.

The **depth** n of L is the number of gaps separating L from A. If $n=0$, it is clear.

Assume $n=1$. If L is adjacent to G_i, then $\delta^{\alpha_i}(L) = A$, where G_i is the gap defined in proof of Lemma 3.2.5. By Lemma 3.2.4, $\delta^{\alpha_i} = ((\delta\alpha_1)^{n-i}\delta)\delta((\delta\alpha_1)^{n-i}\delta)^{-1} = \delta^{(\delta\alpha_1)^{-(n-i)}}$, and then $\delta^{\alpha_i} \in \left[T^{(n)}, T^{(n)}\right]$.

In the case $n \geq 2$, we can show the claim using induction in the same manner as in proof of Lemma 3.2.5. □

Proof of Theorem 3.2.11. Let $N \neq \{\mathrm{id}\}$ be a normal subgroup of $\left[T^{(n)}, T^{(n)}\right]$. For an element $f \in N - \{\mathrm{id}\}$, there exists a dyadic rational $q \in S^1$ such that $f(q) \neq q$ and there exists a standard interval I containing q and satisfying $f(I) \cap I = \emptyset$. Let g and h be elements of $T^{(n)}$ with supports in I. Then $f \circ g \circ f^{-1}$ has support in $J = f(I)$, therefore $[g,f] = g \circ f \circ g^{-1} \circ f^{-1}$ has support in $I \cup J$ and $[g,f] = g$ on I. It follows that $[g,f] = [[g,f],h] \in N$.

By Lemma 3.2.12, there exists an element $w \in \left[T^{(n)}, T^{(n)}\right]$ such that $w(c) = \left(\frac{1}{2^n-1}; \frac{2^{n-1}}{2^n-1}\right)$ where c is the arc connecting the endpoints of I. Then we find $w(I) = \left[\frac{1}{2^n-1}, \frac{2^{n-1}}{2^n-1}\right]$ or $\left[\frac{2^{n-1}}{2^n-1}, \frac{1}{2^n-1}+1\right]$. Taking I smaller if necessary, we may assume $w(I) = \left[\frac{2^{n-1}}{2^n-1}, \frac{1}{2^n-1}+1\right]$.

Since $[g,h]$ is in N for elements $g, h \in \left[T^{(n)}, T^{(n)}\right]$ with supports in $w(I)$, the group $\mathrm{rist}(C)^\dagger = \{f \in \mathrm{rist}(C) \mid \text{the support of } f \text{ lies in } w(I)\}$ is a subgroup of N and is isomorphic to the subgroup $T^\dagger = \left\{f \in T \;\middle|\; f\left(\frac{1}{2}\right) = \frac{1}{2}\right\}$ of T under the isomorphism $\tau\colon \mathrm{rist}(C) \to T$. Since $T^\dagger$ is isomorphic to the Thompson group F and it is not abelian, there are elements $g, h \in \mathrm{rist}(C)^\dagger$ such that $[g,h]$ is not trivial. Therefore $N \cap \mathrm{rist}(C)$ is not trivial and is a normal subgroup of $\mathrm{rist}(C)$. Since $\mathrm{rist}(C) \cong T$ is simple, $N \cap \mathrm{rist}(C)$ must coincide with $\mathrm{rist}(C)$ and we have $\mathrm{rist}(C) < N$. By the same argument, we can show that $(\mathrm{rist}(C)^\dagger)^{(\delta\alpha_1)^k} < N$. Hence for every $k \in \{0, 1, \ldots, n-1\}$ and $\eta \in \{\beta, \gamma, \delta\}$ we have $\eta^{(\delta\alpha_1)^k} \in N$. Since the set $\left\{\eta^{(\delta\alpha_1)^k} \;\middle|\; \eta \in \{\beta, \gamma, \delta\}, k \in \mathbb{Z}_n\right\}$ generates $\left[T^{(n)}, T^{(n)}\right]$, $\left[T^{(n)}, T^{(n)}\right]$ is a subgroup of N and we conclude that $\left[T^{(n)}, T^{(n)}\right] = N$. □

Remark 3.2.13. In Section 3, we considered Thompson-like groups corresponding to some satellite components attaching to the main cardioid (the hyperbolic component containing $0 \in \mathbb{C}$) of the

Mandelbrot set $\mathcal{M}$. Now the orbit portrait for the main cardioid is $\mathcal{P}^0 = \{\{0\}\}$. Then we formally define the pinching lamination $\mathcal{L}(0)$ for $J(0)$ by

$$\mathcal{L}(0) = \left\{ a/2^b \in S^1 = \mathbb{R}/\mathbb{N} \;\middle|\; a, b \in \mathbb{N} \right\} = \{\text{dyadic rationals in } S^1 = \mathbb{R}/\mathbb{Z}\}$$

where $J(0)$ is the Julia set of $f_0(z) = z^2$. A finite locus diagram for $J(0)$ is a **dyadic subdivision** of S^1. A dyadic subdivision of S^1 is a subdivision of S^1 obtained by cutting the interval $(0, 1)$ in half repeatedly finitely many times. Each interval of a dyadic subdivision is called a **dyadic interval**. This shows that we may suppose the group corresponding to the main cardioid is the Thompson group T itself. Hence we formally define

$$T(0) = T\left(\tfrac{0}{1}\right) = T\left(\tfrac{1}{1}\right) = T(1) := T.$$

4 Thompson-like groups for some primitive components

Let $c \in \mathcal{M}$ be the center of the primitive hyperbolic component whose internal address and corresponding orbit portraits are

$$1_{1/3} \to 3_{1/2} \to 4, \text{ and}$$
$$\mathcal{P}^1 = \{\{1/7, 2/7, 4/7\}\} \text{ and } \mathcal{P}^2 = \{\{3/15, 4/15\}, \{6/15, 8/15\}, \{12/15, 1/15\}, \{2/15, 9/15\}\}.$$

Let $f(z) = f_c(z) = z^2 + c$ be the corresponding quadratic map. We will first consider this special case. We should note that all the Julia sets corresponding to the groups we considered in Section 2 come from satellite components. The definitions and results in this section are related to those in [Smi13] and we refer to some arguments in that unpublished paper.

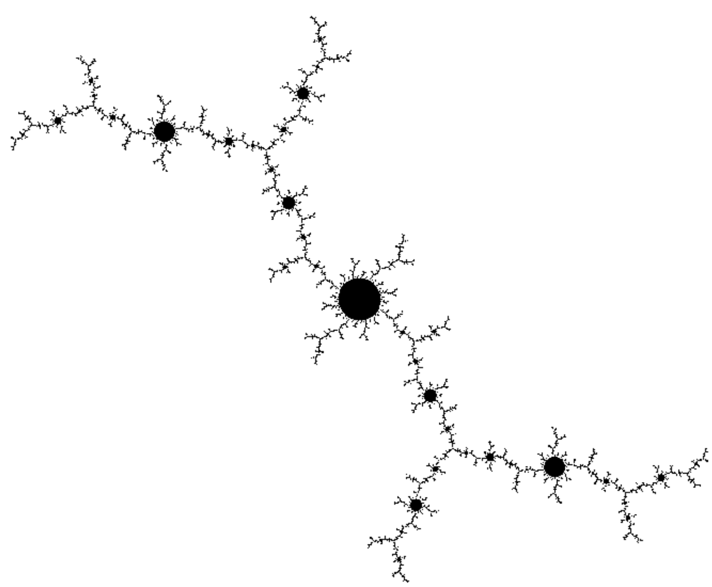


Figure 10: The filled Julia set $J\left(\frac{3}{15}\right)$

4.1 Basic definitions

Definition 4.1.1. A point $p \in J = J(f) = J\left(\frac{3}{15}\right)$ is called a **branch point** if there are more than two dynamical rays landing on p.

Definition 4.1.2. For a point $p \in K$ with $n(> 1)$ dynamical rays landing on it, let

$$[a_1, a_2], [a_2, a_3] \ldots, [a_{n-1}, a_n], [a_n, a_1] \subset \mathbb{R}/\mathbb{Z}$$

be intervals bounded by adjacent two rays. Then each $[a_j, a_{j+1}]$ (with subscripts modulo n) is called an **arm**.

Set $\hat{f}(z) := z^2$. Since $J\left(\frac{3}{15}\right)$ is locally connected, Φ_f extends to the boundary of $\hat{\mathbb{C}} - K(f)$, in other words, to $J\left(\frac{3}{15}\right)$. We denote this extension by Φ_f again and its restriction to $J\left(\frac{3}{15}\right)$ by $\Psi_f \colon J\left(\frac{3}{15}\right) \to S^1$. By definition, Ψ_f satisfies $\Psi(\hat{f}(z)) = \hat{f}(\Psi(z))$.

Definition 4.1.3.

- A boundary of a Fatou component of $J\left(\frac{3}{15}\right)$ is called a **pool**, and if the Fatou compontent contains the critical point $0 \in \mathbb{C}$ then it is called the **critical pool** and is denoted by P_C.
- A point of a pool on which two dynamical rays land is called a **pool point**.
- A locus corresponding to a pool point in a pool P is called a **pool arc** for P.

Definition 4.1.4. Assume that S is a closed and connected subset of $J\left(\frac{3}{15}\right)$ and is bounded by two branch points. Let N be the maximal subset of $J\left(\frac{3}{15}\right)$ bounded by $\left(\frac{1}{7}; \frac{2}{7}; \frac{4}{7}\right)$ and $\left(\frac{9}{14}; \frac{11}{14}; \frac{15}{14}\right)$. If there exists $n \in \mathbb{N}$ such that $f^n|_S \colon S \to N$ is bijective, S is called a **segment**. A pool lying on S is called the **dominant pool** for S if its image under f^m coincides with the critical pool P_C.

Definition 4.1.5. Assume that S is a closed and connected subset of $J\left(\frac{3}{15}\right)$ and is bounded by a branch point and a pool point. Let M be the maximal subset of $J\left(\frac{3}{15}\right)$ bounded by $\left(\frac{1}{7}; \frac{2}{7}; \frac{4}{7}\right)$ and $\left(\frac{12}{15}; \frac{1}{15}\right)$. If there exists $n \in \mathbb{N}$ with $n \geq 3$ such that $f^n|_S \colon S \to M$ is bijective, S is called a **half segment**.

Remark 4.1.6. Two dominant points of a pool P correspond to points with internal angles 0 and 1/2 respectively.

Definition 4.1.7. Let P be a pool and S be a segment bounded by two branch points b, b' whose dominant pool is P. A **dominant point** $q \in J\left(\frac{3}{15}\right)$ for P is a pool point lying on the intersection of P and the regulated path $[b, b']_{K\left(\frac{3}{15}\right)}$. By definition, each pool has two dominant points.

Definition 4.1.8. Let S be a closed and connected subset of $J\left(\frac{3}{15}\right)$ and suppose it is bounded by two pool points lying on a pool P. Let U be the maximal subset of $J\left(\frac{3}{15}\right)$ bounded by $\left(\frac{2}{15}; \frac{19}{30}\right)$ and $\left(\frac{12}{15}; \frac{1}{15}\right)$. If there exists $n \in \mathbb{N}$ with $n \geq 3$ such that $f^n|_S \colon S \to U$ is bijective, S is called a **pool segment** in P.

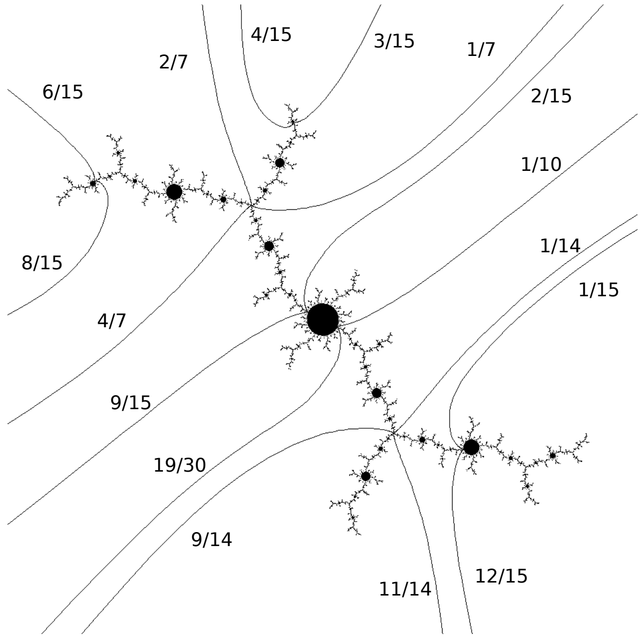


Figure 11: The filled Julia set $J\left(\frac{3}{15}\right)$ with dynamical rays

Definition 4.1.9. Let P be a pool.

- An arm $[a, b]$ is called a **branch arm** if there exists $n \in \mathbb{N}$ such that $\hat{f}^n|_{[a,b]}$ maps $[a, b]$ bijectively to $[\frac{4}{7}, \frac{8}{7}]$.
- An arm $[a, b]$ is called a **dominant arm** for P if a point $(a; b)$ is a dominant point of P and there exists $n \in \mathbb{N}$ such that $\hat{f}^n|_{[a,b]}$ maps $[a, b]$ bijectively to $[\frac{19}{30}, \frac{11}{10}]$.
- An arm $[a, b]$ is called a **pool arm** if a point $(a; b)$ is a pool point of P but is not a dominant point, and if there exists $n \in \mathbb{N}$ such that $\hat{f}^n|_{[a,b]}$ maps $[a, b]$ bijectively to $[\frac{19}{30}, \frac{11}{10}]$.

Definition 4.1.10. Let P, P' and Q be branch points, pools or pool points.
P (and P') is an (are) **ancestor(s)** of Q if either:

(1) Q lies in a branch arm of P.

(2) Q lies in a segment having a branch point P as an end point.

(3) Q lies in a half segment having a pool point P as an end point.

(4) Q lies in a pool segment having a pool point P as an end point.

(5) Q lies in a pool arm rooted at a pool point P in a pool P'.

(6) P, P' and Q are pool points of the same pool with internal angles $a, a' \in S^1$ and $b \in [a, a']$ respectively such that $[a, a']$ is a dyadic interval and b is a dyadic rational.

When P is a ancestor of Q, we write $Q \preceq P$.

Definition 4.1.11. Let P, P' and Q be branch points, pools or pool points.

- P is the **parent** of Q, and Q is a **child** of P if and only if: $Q \preceq P$ and if $Q \preceq P'$ then $P \preceq P'$.
- The branch point $\left(\frac{1}{7}; \frac{2}{7}; \frac{4}{7}\right)$ is said to be of **generation** 0. For other branch or pool points, each of them is said to be of **generation** n ($n > 0$) if its parent is of generation $n-1$. The generation of a pool P is defined to be the generation of dominant of P.

4.2 Finite locus diagrams

Definition 4.2.1. Let P and Q either be a branch point or a pool point and let P' and Q' be the corresponding points in $J\left(\frac{3}{15}\right)$. P' is an **ancestor** of Q' if P is an ancestor of Q and we write $Q' \preceq P'$.

Definition 4.2.2. Let G be a finite set of branch points and pool points. G is called a **finite locus diagram** for $J\left(\frac{3}{15}\right)$ if:

(1) For each $P \in G$, every ancestor of P is also contained in G, and

(2) if G has a dominant point of a pool P, then G also has the other dominant point of P.

The closed unit disc with pinching locus corresponding to each point in G is also called a finite locus diagram.

Definition 4.2.3. We regard each finite locus diagram as a 2-complex. Two finite locus diagrams G and H are **isomorphic** if there exists an orientation preserving isomorphism $f\colon G' \to H'$ where G' and H' are 2-complexes corresponding to G and H respectively. G is called the **domain diagram** and H is called the **range diagram**. A pair (G, H) of a domain and range diagram is called a **locus pair diagram** for $J\left(\frac{3}{15}\right)$.

A locus pair diagram for $J\left(\frac{3}{15}\right)$ induces an orientation preserving piecewise linear homeomorphism on S^1 whose breakpoints are vertices of loci lying on the domain diagram. This homeomorphism induces orientation preserving homeomorphism again on $J\left(\frac{3}{15}\right)$ and we call this homeomorphism a **rearrangement** for $J\left(\frac{3}{15}\right)$.

Theorem 4.2.4. Let f be an orientation preserving piecewise linear homeomorphism of the unit circle. The homeomorphism f induces a rearrangement for $J\left(\frac{3}{15}\right)$ if and only if

(1) The pinching lamination for $J\left(\frac{3}{15}\right)$ is invariant under f, and

(2) Every breakpoint of f is the vertex of a pinching locus.

The proof of Theorem 4.2.4 is essentially the same as that of Theorem 3.1.6.

Definition 4.2.5. The above theorem shows that

$$T\left(\tfrac{3}{15}\right) := \left\{ f\colon J\left(\tfrac{3}{15}\right) \to J\left(\tfrac{3}{15}\right) \;\middle|\; f \text{ is a rearrangement for } J\left(\tfrac{3}{15}\right) \right\}$$

has a group structure under composition. It is called the **rearrangement group** for $J\left(\frac{3}{15}\right)$.

Definition 4.2.6. An **expansion** of a locus pair diagram (G, H) consists of adding a locus to G, all of whose ancestors are already included, and adding the image of the locus to H. A **reduction** is the inverse operation. (G, H) is said to be **reduced** if no reductions are possible.

Proposition 4.2.7. Every locus pair diagram for $J\left(\frac{3}{15}\right)$ has a unique reduced locus pair diagram.

4.3 Generators of $T\left(\frac{3}{15}\right)$

Let us define some fundamental elements $\alpha_1, \alpha_2, \beta, \gamma, \delta, \epsilon \in T\left(\frac{3}{15}\right)$.

$$\alpha_1(x) = \begin{cases} \frac{1}{2}x & x \in \left[-\frac{5}{7}, \frac{1}{7}\right] \\ 4x - \frac{1}{2} & x \in \left[\frac{1}{7}, \frac{2}{7}\right] \end{cases}$$

$$\alpha_2(x) = \begin{cases} x & x \in \left[-\frac{6}{15}, \frac{2}{15}\right] \\ \frac{1}{16}x + \frac{1}{8} & x \in \left[\frac{2}{15}, \frac{1}{7}\right] \\ \frac{1}{2}x + \frac{1}{16} & x \in \left[\frac{1}{7}, \frac{9}{56}\right] \\ 4x - \frac{1}{2} & x \in \left[\frac{9}{56}, \frac{15}{56}\right] \\ \frac{1}{2}x + \frac{7}{16} & x \in \left[\frac{15}{56}, \frac{2}{7}\right] \\ \frac{1}{16}x + \frac{9}{16} & x \in \left[\frac{2}{7}, \frac{9}{15}\right] \end{cases}$$

$$\beta(x) = \begin{cases} x & x \in \left[-\frac{11}{30}, \frac{1}{10}\right] \\ \frac{1}{16}x + \frac{5}{32} & x \in \left[\frac{1}{10}, \frac{9}{15}\right] \\ x - \frac{15}{32} & x \in \left[\frac{9}{15}, \frac{329}{480}\right] \\ 16x - \frac{19}{2} & x \in \left[\frac{329}{480}, \frac{19}{30}\right] \end{cases}$$

$$\gamma(x) = \begin{cases} x & x \in \left[-\frac{2}{15}, \frac{11}{10}\right] \\ \frac{1}{16}x - \frac{3}{32} & x \in \left[\frac{1}{10}, \frac{63}{480}\right] \\ x - \frac{15}{512} & x \in \left[\frac{63}{480}, \frac{1009}{7680}\right] \\ 16x - 2 & x \in \left[\frac{1009}{7680}, \frac{13}{15}\right] \end{cases}$$

$$\delta(x) = x + \frac{1}{2}, \qquad \epsilon = \alpha_1\alpha_2^{-1}$$

Figure 12 shows the corresponding locus pair diagrams.

Definition 4.3.1. Let G be a finite locus diagram. A **finite representing graph** for G is a graph that one vertex for each locus in G and its edges are constructed as follows: Let v, w be vertices whose corresponding loci are V and W respectively. We add an edge connecting v and w if either:

(1) V and W are not separated by other loci in G and at least one of them is a triangle, or

(2) there are no ends of other loci of G in the interval $[\theta_1, \theta_2]$ surrounded by pool arcs V and W where θ_1 and θ_2 are ends of V and W respectively.

We also denote this graph by G. A graph corresponding to a domain (resp. range) diagram is called a **domain** (resp. **range**) **graph**.

A vertex corresponding to a branch point or a pool point is called a **branch vertex** or a **pool vertex** respectively. The terms arm, pool, segment, half segment, ancestor, parent, child, generation, etc... are used for corresponding subgraphs of finite representing graphs.

Definition 4.3.2. A **pair graph diagram** for $f \in T\left(\frac{3}{15}\right)$ is the pair of finite representing graph for locus pair diagrams for f. A pair graph diagram is said to be **reduced** if corresponding finite representing graph is reduced.

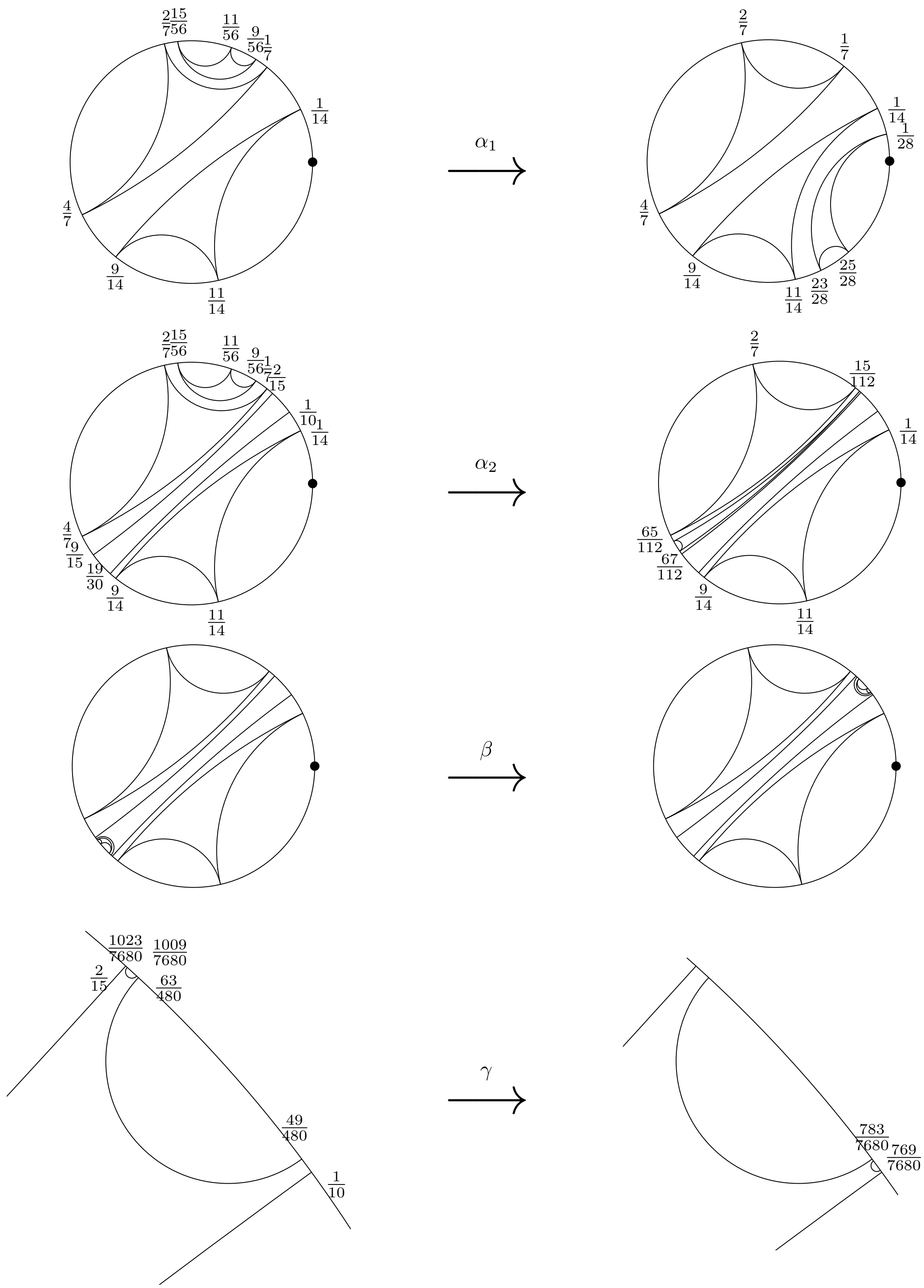


Figure 12: Locus pair diagrams

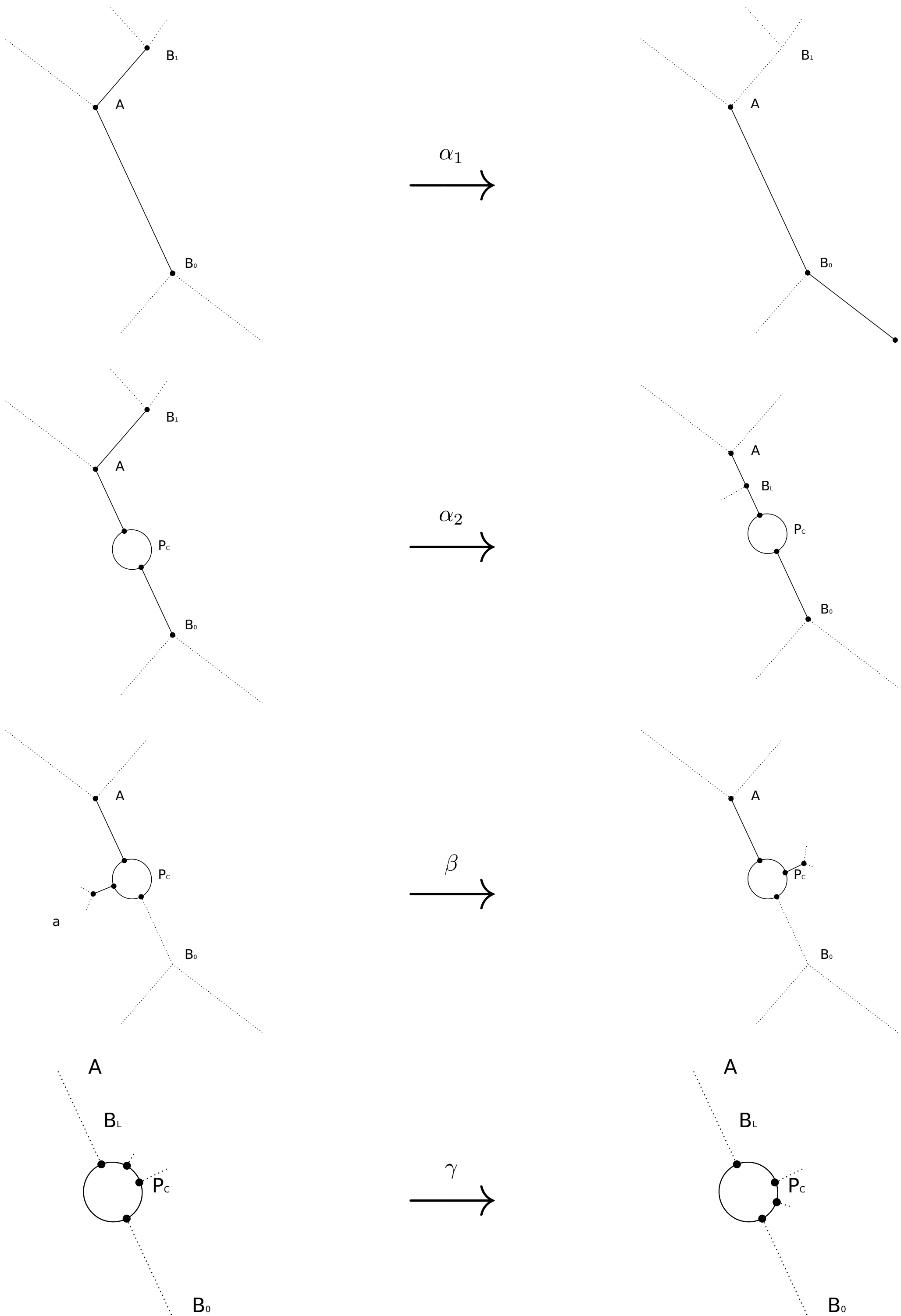


Figure 13: Graph pair diagrams

Definition 4.3.3. The **base tree** for $J\left(\frac{3}{15}\right)$ is the graph constructed as follows; first we begin with a graph consisting of a single vertex A corresponding to $\left(\frac{1}{7};\frac{2}{7};\frac{4}{7}\right)$ without edges. Then we add branch vertices of generation 1 and connect each of them to its parent by an edge. Next we add branch vertices of generation 2 and connect each of them to its parent by an edge. Iterating this operation, we obtain an infinite graph, and is called the base tree (see Figure 14).

The **base graph** for $J\left(\frac{3}{15}\right)$ is a graph obtained by adding two vertices and two edges which correspond to a dominant pool to each edge of the base tree (see Figure 15).

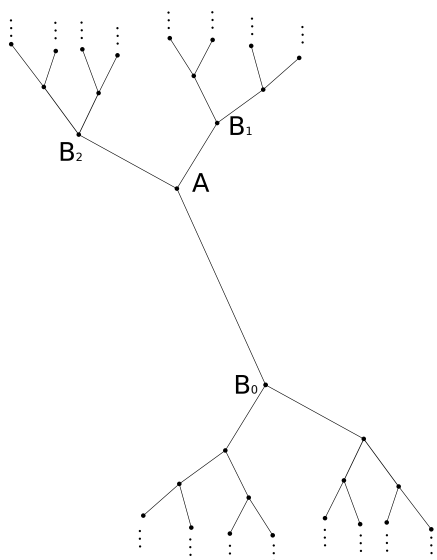


Figure 14: The base tree

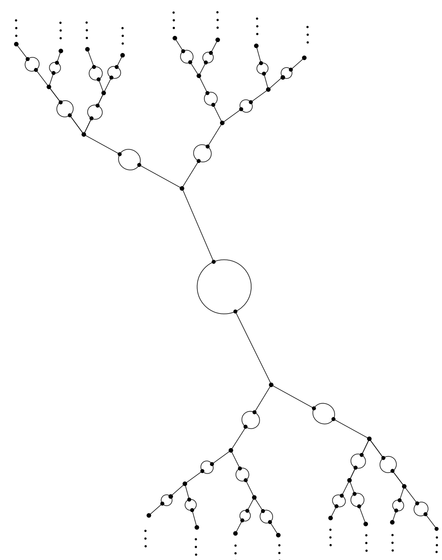


Figure 15: The base graph

The vertices corresponding to the points $\left(\frac{1}{7};\frac{2}{7};\frac{4}{7}\right), \left(\frac{9}{14};\frac{11}{14};\frac{15}{14}\right)$ are denoted by A and B_0. Remaining two vertices of generation 1 are written counterclockwise by B_1 and B_2. The edge connecting vertices v, w is denoted by $\overline{vw}$.

Lemma 4.3.4. Every edge on the base tree can be mapped to the edge $\overline{AB_0}$ by an element of the group $\langle \alpha_1, \delta\rangle$. Equivalently, every dominant pool lying on the base graph for $J\left(\frac{3}{15}\right)$ is mapped to P_C by an element of $\langle \alpha_1, \delta\rangle$.

Proof. Define $\xi := (\alpha_1^{-1})^{\delta} \in \langle \alpha_1, \delta\rangle$. We give each edge lying on upper side of the vertex A a label by elements of $\{0,1\}$ as in Figure 16.

The element α_1 sents the edge labeled 0 to $\overline{AB_0}$ and each edge having a label $0w_1w_2w_3\cdots$ to an edge whose label is $w_1w_2w_3\cdots$. In the same manner, ξ sends the edge labeled 1 to $\overline{AB_0}$ and each edge having a label $1w_1w_2w_3\cdots$ to an edge with a label $w_1w_2w_3\cdots$. Therefore, an edge labeled $w_0w_1\cdots w_n$ is sent to $\overline{AB_0}$ by $\xi_n\xi_{n-1}\cdots\xi_0$ where $\xi_j = \alpha_1$ if $w_j = 0$ and $\xi_j = \xi$ if $w_j = 1$.

For edges lying on the lower side of B_0, we can use the same argument by conjugating δ. $\square$

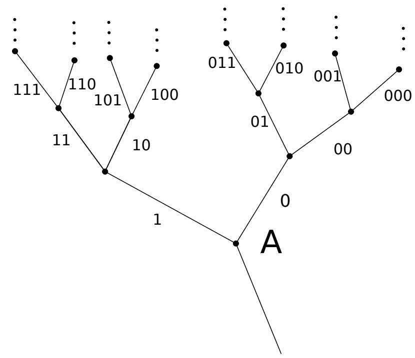


Figure 16: The base tree with labels

Definition 4.3.5. Let e be an edge in an finite representing graph and we denote two vertices of e by v_1, v_2. Then a **path** for e is a regulated path in $K\left(\frac{3}{15}\right)$ connecting v_1 and v_2. The **main path** is the path corresponding to $\overline{AB_0}$.

Lemma 4.3.6. Every branchpoint lying on a path for an edge in the base tree is mapped to A by an element of $\langle \alpha_1, \alpha_2, \delta \rangle$, and every pool which has an intersection with a path is mapped to P_C by an element of $\langle \alpha_1, \alpha_2, \delta \rangle$.

Proof. To show the first half, let B be a branchpoint lying on a path for an edge e in the base tree and B is of generation $n \in \mathbb{N}$. By Lemma 4.3.4 we may assume B lies on the main path $\overline{AB_0}$ without loss of generality.

If $n = 0$, then the equality $B = A$ follows and the claim holds trivially. When $n = 1$, we see that $B = B_0$ and $\delta(B) = A$. Next we suppose $n = 3$ (n cannot be equal 2). Let B_L be the child of the critical pool P_C lying between A and P_C, and B_R be that lying between B_0 and P_C. In this case, $B = B_L$ or B_R. If $B = B_L$, $\alpha_2^{-1}(B) = A$. If $B = B_R$, $\epsilon^\delta(B) = A$.

Now we suppose $n > 3$ and we use induction on n. Then B has either B_L or B_R as an ancestor.

<u>Case1</u> Assume that B lies between A and B_L. Since $\epsilon(B_L) = B_0$ and $\epsilon(B_0)$ is not an ancestor of $\epsilon(B)$, $\epsilon(B)$ is at most of generation $n - 1$.

We use similar arguments for remaining cases.

<u>Case2</u> Assume that B lies between P_C and B_L. Since $\alpha_2^{-1}(B_L) = A$ and $\alpha(A)$ is not an ancestor of $\alpha(B)$, $\alpha_2^{-1}(B)$ is at most of generation $n - 1$.

<u>Case3</u> Assume that B lies between P_C and B_R. Since $(\alpha_2^{-1})^\delta(B_R) = B_0$ and $\alpha_2^\delta(B_0)$ is not an ancestor of $\alpha_2^\delta(B)$, $\alpha_2^\delta(B)$ is at most of generation $n - 1$.

<u>Case4</u> Assume that B lies between B_R and B_0. Since $\epsilon^\delta(B_L) = B_0$ and $\epsilon^\delta(A)$ is not an ancestor of $\epsilon^\delta(B)$, $\epsilon^\delta(B)$ is at most of generation $n - 1$.

For the latter half, let P be a pool lying on the main path and be of generation n. If $n = 2$, then $P = P_C$. Suppose $n = 4$ and let P_L and P_R be dominant pools for segments bounded by A, B_L and B_0, B_R respectively. Then $P = P_L$ or P_R. If $P = P_L$, $\epsilon(P) = P_C$ and if $P = P_R$, $\epsilon^\delta(P) = P_C$.

Now we suppose $n > 4$ and use induction on n again. Let P' be a parent of P, then P' is of generation $n-1$ and it must be a branchpoint. The above argument shows that there exists an element $\varphi \in \langle \alpha_1, \alpha_2, \delta \rangle$ such that $\varphi(P')$ is at most of generation $n-2$, then $\varphi(P)$ is of at most generation $n-1$. $\square$

Using the same argument in the proof of Proposition 3.2.2, we can show the following.

Proposition 4.3.7. The group $\langle \beta, \gamma, \delta \rangle$ is isomorphic to the Thompson group T.

Lemma 4.3.8. Every branch point in $J\left(\frac{3}{15}\right)$ is mapped to A by an element of $\langle \alpha_1, \alpha_2, \beta, \gamma, \delta \rangle$ and every pool in $J\left(\frac{3}{15}\right)$ is mapped to the critical pool P_C by an element of $\langle \alpha_1, \alpha_2, \beta, \gamma, \delta \rangle$.

Proof. Let us consider the first half. Let B be a branchpoint and be of generation n. By an action of $\varphi \in \langle \alpha_1, \alpha_2, \delta \rangle$, we may assume that B lies in a segment S_0 bounded by A and B_0. If B lies on the main path, the claim is already shown. We consider the case where B lies on a branch arm or a pool arm but does not lie on the main path. If B is on a branch arm rooted at the main path, we send this arm to a branch arm rooted at A by some $\varphi_1 \in \langle \alpha_1, \alpha_2, \delta \rangle$ by Lemma 4.3.6. $\varphi_1(B)$ is at most of generation $n-3$. Assume $\varphi_1(B)$ lies on a segment bounded by two branchpoints corresponding to the vertices of an edge e in the base tree. Using Lemma 4.3.4, we send e to AB_0 by an element $\varphi_2 \in \langle \alpha_1, \delta \rangle$. Then $\varphi_2\varphi_1(B)$ is at most of generation $n-3$ and lies in S_0.

Let us consider the case where B is on a pool arm rooted at a pool which intersects with the main path. We may assume that this pool arm is rooted at the critical pool. We send this arm to the dominant arm rooted at $\left(\frac{2}{15}; \frac{9}{15}\right)$ by some element $\psi \in \langle \beta, \gamma, \delta \rangle \cong T$ since T acts on the set of dyadic points on the unit circle transitively. $\psi(P)$ is at most of generation $n-3$. As above, we may suppose that $\psi(B)$ lies on S_0. Iterating this argument finite times, we can send B to a point with sufficiently small generation which must be on the main path.

We can show the latter half by a similar argument. $\square$

Theorem 4.3.9. The rearrangement group $T\left(\frac{3}{15}\right)$ is generated by $\{\alpha_1, \alpha_2, \beta, \gamma, \delta\}$.

Proof. Let $f \in T\left(\frac{3}{15}\right)$. By Lemma 4.3.4, we may assume $f(P_C) = P_C$. Furthermore, since the group $\langle \beta, \gamma, \delta \rangle$ acts on pool arcs for P_C, there exists an element $g \in \langle \beta, \gamma, \delta \rangle$ such that $g \circ f$ fixes each pool arc for P_C. From now on we assume that f fixes each pool arc for P_C. Let D_f, R_f be the reduced domain and range diagram of f and n be the number of loci in D_f. We use induction on n.

If $n = 1$, $f = \mathrm{id}$ and we are done. Suppose $n > 1$. Let $a_1, \ldots, a_m$ be pool arcs for P_C that have some loci of D_f on corresponding pool arms or dominant arms. Then we can decompose f as $f = f_m \circ \cdots \circ f_1$ where f_j is a rearrangement that D_{f_j} has only loci on the pool arc corresponding to a_j except for A.

Assume $m > 1$. Then each D_{f_j} has at most $n-1$ loci, then we are done.

Next we assume $m = 1$. Since $\langle \beta, \gamma, \delta \rangle$ acts on pool arcs for P_C, we may assume that a_1 corresponds to $\left(\frac{19}{30}; \frac{1}{10}\right)$. Then D_f and R_f must contain B_L or B_1 or B_2. Composing α_2^{-1} and α_2 to f finitely many times from the left and the right respectively, we may suppose that D_f and R_f do not have B_L. The number of loci of D_f does not increase under this operation.

Case 1 If $B_1, B_2 \in D_f$, let f_i be the rearrangement such that D_{f_i} has only loci of D_f which lie in the branch arm $[2^{i-1}/7, 2^i/7]$ for $i = 1, 2$. Then each D_{f_i} has at most $n-1$ loci and $f = f_2 \circ f_1$.

Case 2 If $B_1 \in D_f$ but $B_2 \notin D_f$, there exists an integer $k \geq 0$ such that the locus diagram $D' = (\epsilon^\delta)^k \circ \alpha_1(D_f)$ of has loci only on $[1/7, 4/7]$ except for B_0, and the number of loci of D' is not larger than that of D_f. In the same manner, the locus diagram there exists an integer $l \geq 0$ such that the locus diagram $R' = (\epsilon^\delta)^l \circ \alpha_1(R_f)$ of has loci only on $[1/7, 4/7]$ except for B_0, and the number of loci of R' is not larger than that of R_f. Then the locus pair diagram (D', R') corresponds to a rearrangement $f' = (\epsilon^\delta)^l \circ \alpha_1 \circ f \circ \alpha_1^{-1} \circ (\epsilon^\delta)^{-k}$, and we can reduce the locus B_0. Hence f' is in $\langle \alpha_1, \alpha_2, \beta, \gamma, \delta \rangle$ by induction. $\square$

4.4 Properties of $T\left(\frac{3}{15}\right)$

Let G be a finite representing graph for a finite locus diagram (we also denote this diagram by G). We want to define a **local address** of each edge of G. A local address is a finite or infinite sequence of elements of $\mathbb{Z}_3$. First we give the edge $\overline{AB_0}$ of the base tree for $J\left(\frac{3}{15}\right)$ a local address $a \in \mathbb{Z}_3$.

The graph G is obtained by adding finitely many vertices and edges to the graph consisting of the single vertex A without edges. Assume G has at least one edge. Essentially we have just four ways to add vertices or edges beginning with the graph consisting of vertices A and B_0 and an edge $\overline{AB_0}$. Let H be a finite representing graph for a finite locus diagram.

1° Let $\overline{BB'}$ be an edge of H with a local address $w = w_0 w_1 \dots w_n$, and let the vertex $B (\neq A)$ be of valency at most 2, of generation n and corresponding to a branch point. Let the vertex B' be a branch point and a parent of B. Then we may add at most two edges connecting B and their children with local addresses $w_0 w_1 \dots w_{n-1}(w_n + 1)$ and $w_0 w_1 \dots w_{n-1}(w_n + 2)$ to H (indicated as dotted lines in Figure 17). For example, if B coincides with A, set $B' = B_0$ and we may add edges $\overline{AB_1}$ and $\overline{AB_2}$.

By definition, if a local address of $\overline{AB_0}$ is given, then local addresses of all edges of the base tree are determined automatically.

2° Let $\overline{BB'}$ be an edge of H with a local address $w = w_0 w_1 \dots w_n$, and B and B' are both corresponding to branch points. Then we may add two edges corresponding to the dominant pool for $\overline{BB'}$ with a local address $w(w_n + 2)(w_n + 4) \dots (w_n + 2m) \dots$, an infinite sequence of $\mathbb{Z}_3$, and two edges corresponding to half segments are given an address w. All addresses of edges of the base graph are determined automatically if a local address of $\overline{AB_0}$ is given.

3° Let $\overline{BB'}$ be an edge of H with a local address $w = w_0 w_1 \dots w_n$, and assume B and B' are corresponding to a branch and a pool point respectively. Then we may add vertices and edges which are children or grandchildren of B' with local addresses as indicated in Figure 19.

4° Let e be an edge of H with a local address $w = w_0 w_1 \dots w_n$ which has a dominant point of a pool P as a vertex. Let $\overline{BB'}$ be an edge of H corresponding to dyadic interval of a pool P with a local address $w' = w(w_n+2)(w_n+4)\dots$. Then we add a pool vertex C corresponding to a child of B or B' and the branch vertex C' corresponding to the child of C, and an edge $\overline{CC'}$ with a local address $w = w_0 w_1 \dots w_n$ (see Figure 20).

By the construction, if a local address of an edge of H which has a branchpoint A as a vertex and lies on the main path is given, all addresses of the other edges of H are determined automatically. We call this edge a **reference edge**. In this situation, we call H an **addressed graph** and the set of all pairs of an edge of H and its local address is called an **address** of H.

We make some observations for relationships between addressed graphs and elements of $T\left(\frac{3}{15}\right)$. It is clear that the elements β, γ, δ do not change addresses of their domain graphs. Let $w = w_0 w_1 \dots w_n$ be a local address of the edge of H which has a branchpoint B_1 as a vertex and lying on a path corresponding to $\overline{AB_1}$. The element α_1 changes address of its domain graph. A local address of the reference edge of the range graph is obtained by adding $1 \in \mathbb{Z}_3$ to its each term of w. Figure 21 shows an example.

Also α_2 changes local addresses of edges of the domain graph. The local address of the reference edge of the range graph is obtained by appending $w_0 \in \mathbb{Z}_3$ to just the left of the first term of w. Figure 22 shows an example.

Now we define a group homomorphism

$$\begin{array}{cccc} \phi_1\colon & T\left(\frac{3}{15}\right) & \longrightarrow & \mathbb{Z}_3 \\ & \cup\!\!\!\!\;\raisebox{0pt}{} & & \cup \\ & f & \longmapsto & d-r \end{array},$$

where d and r are the first terms of the reference edge of the domain and the range graph of f respectively. For example, $\phi_1(\alpha_1) = \phi_1(\alpha_2) = 1$, and $\phi_1(\beta) = \phi_1(\gamma) = \phi_1(\delta) = 0$.

Next we define another homomorphism ϕ_2 from $T\left(\frac{3}{15}\right)$ to $\mathbb{Z}$. Let D_f and R_f be the domain and the range diagram of $f \in T\left(\frac{3}{15}\right)$ respectively, and let P be a pool lying on D_f. Let S_q be the half segment rooted at a pool vertex q for P in D_f such that if q corresponds to a dominant point then the other end of S_q is an end of the segment dominated by P, otherwise it is the child of q. Such a half segment S_q is uniquely determined. Let $\mathcal{D}_{S_q}$ be the set of all vertices of D_f corresponding to the descendants of P lying on $S_q - \{\text{ends of } S_q\}$. If $\mathcal{D}_{S_q}$ is not empty, there exists a unique sequence of branch vertices $d_1 \succeq d_2 \succeq \cdots \succeq d_m$ in $\mathcal{D}_{S_q}$ such that d_{i+1} is a child of d_i. We denote the length m of the sequence by $l_q = m$. For a pool vertex r which is not lying on D_f, we formally define $l_r = 0$.

Set $\mathcal{A} := \left\{q \in J\left(\frac{3}{15}\right) \;\middle|\; q : \text{pool point}\right\}$ and define a map

$$\begin{array}{cccc} h\colon & \mathcal{A} \times T\left(\frac{3}{15}\right) & \longrightarrow & \mathbb{Z} \\ & \cup & & \cup \\ & (q, f) & \longmapsto & l_{f(q)} - l_q \end{array}.$$

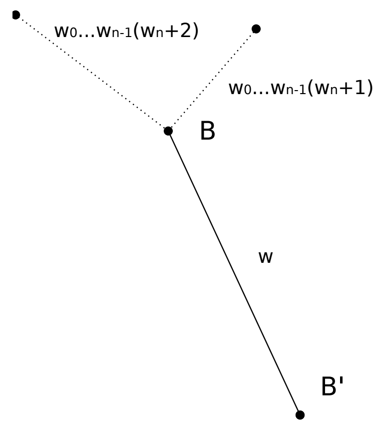


Figure 17:

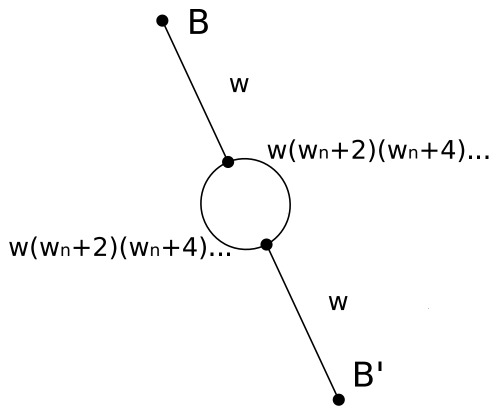


Figure 18:

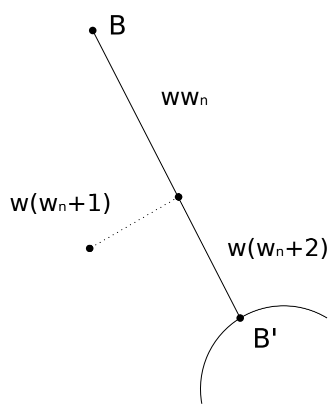


Figure 19:

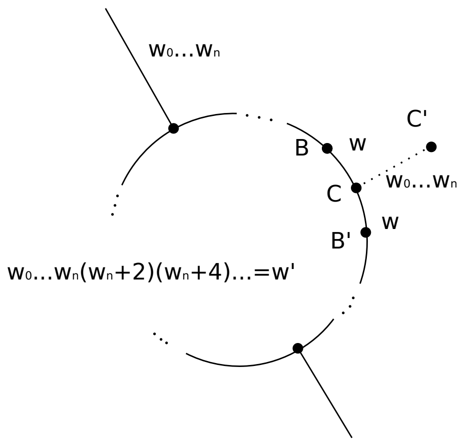


Figure 20:

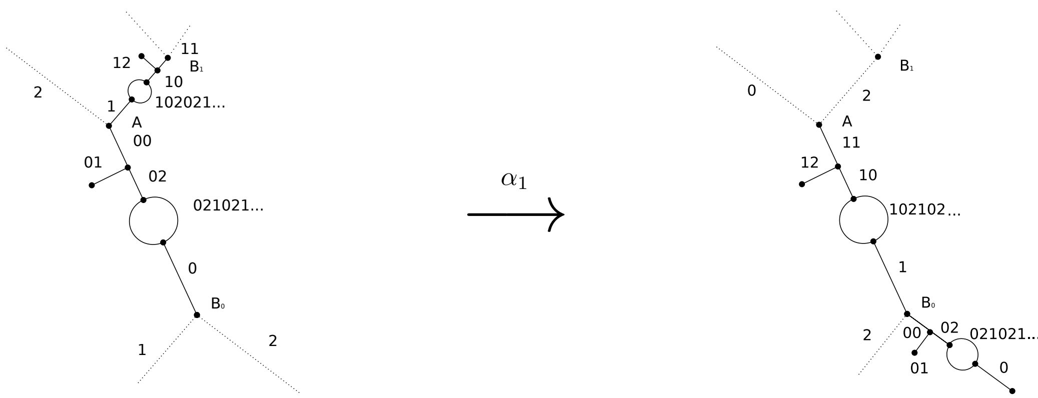


Figure 21:

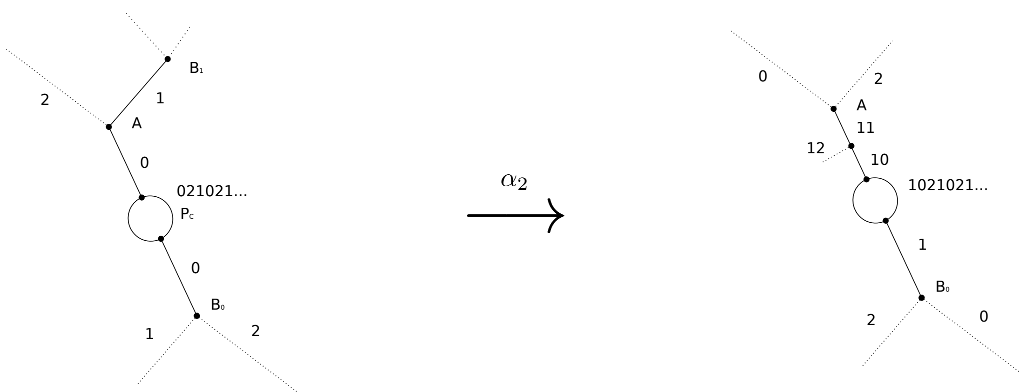


Figure 22:

Since for each $f \in T\left(\frac{3}{15}\right)$, $l_q = 0$ for almost every pool point q, the following function is well-defined:

$$\begin{array}{cccc} \phi_2\colon & T\left(\frac{3}{15}\right) & \longrightarrow & \mathbb{Z} \\ & \Cup & & \Cup \\ & f & \longmapsto & \displaystyle\sum_{q \in D_f \cap \mathcal{A}} h(q, f) \end{array}.$$

It is easy to see ϕ_2 is a homomorphism. For example, $\phi_2(\alpha_2) = 1$, and $\phi_2(\alpha_1) = \phi_2(\beta) = \phi_2(\gamma) = \phi_2(\delta) = 0$. Set

$$\phi\colon T\left(\tfrac{3}{15}\right) \to \mathbb{Z}_3 \times \mathbb{Z};\ \ f \mapsto (\phi_1(f), \phi_2(f)).$$

Theorem 4.4.1. $\phi\colon T\left(\frac{3}{15}\right) \to \mathbb{Z}_3 \times \mathbb{Z}$ induces a group isomorphism

$$T\left(\tfrac{3}{15}\right) / \left[T\left(\tfrac{3}{15}\right), T\left(\tfrac{3}{15}\right)\right] \xrightarrow{\cong} \mathbb{Z}_3 \times \mathbb{Z}.$$

Proof. It is clear that ϕ is surjective and $\left[T\left(\frac{3}{15}\right), T\left(\frac{3}{15}\right)\right] < \ker\phi$. It remains to show $\ker\phi < \left[T\left(\frac{3}{15}\right), T\left(\frac{3}{15}\right)\right]$. Set $\epsilon_1 = \delta\alpha_1$, $\epsilon_2 = \alpha_1^{-1}\alpha_2$, $G = T\left(\frac{3}{15}\right)$, $H = \ker\phi$ and $S = \{\epsilon_1, \epsilon_2, \beta, \gamma, \delta\}$. The set S is also the generating set of $T\left(\frac{3}{15}\right)$. We should note that $\phi(\epsilon_1) = (1, 0)$, $\phi(\epsilon_2) = (0, 1)$ and $\epsilon_1^3 = \mathrm{id}$. We define a section σ as follows:

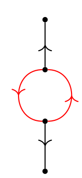
$$\sigma\colon G/H \to G;\ \ f \mapsto \epsilon_1^k \epsilon_2^l \text{ where } \phi(f) = (k, l).$$

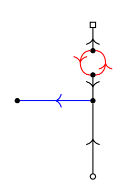
Set $R = \sigma(G/H) = \left\{\epsilon_1^k\epsilon_2^l \in G \;\middle|\; k \in \mathbb{Z}_3, l \in \mathbb{Z}\right\}$ and $\sigma(gH) = \overline{g}$ for $g \in G$. By calculation, we can see that $(\overline{\eta\epsilon_1^k\epsilon_2^l})^{-1}\eta\epsilon_1^k\epsilon_2^l = \eta^{\epsilon_1^k\epsilon_2^l}$ where $\eta \in \{\beta, \gamma, \delta\}$, $(\overline{\epsilon_1\epsilon_1^k\epsilon_2^l})^{-1}\epsilon_1\epsilon_1^k\epsilon_2^l = \mathrm{id}$, $(\overline{\epsilon_2\epsilon_1^k\epsilon_2^l})^{-1}\epsilon_2\epsilon_1^k\epsilon_2^l = [\epsilon_2^{-1}, (\epsilon_1^k\epsilon_2^l)^{-1}]$. It follows that $\left\{(\overline{sr})^{-1}sr \;\middle|\; s \in S, r \in R\right\} \subset [G, G]$ i.e. $H < [G, G]$ by Schreier's lemma. $\square$

4.5 Another definition of $T\left(\frac{3}{15}\right)$ using replacement systems

We will define the colored replacement system $(G_0, \mathcal{R} = \{e_c \to R_c\}_{c \in C})$ for $J\left(\frac{3}{15}\right)$ according to [BF15a]. For notations and definitions, see Appendix A.

Set $C = \{\text{black, red, blue}\}$ and define the base graph G_0 and replacement rules as follows:

$G_0 \quad = $  ,

$e_{\text{black}} =$  $\longrightarrow$  $= R_{\text{black}},$

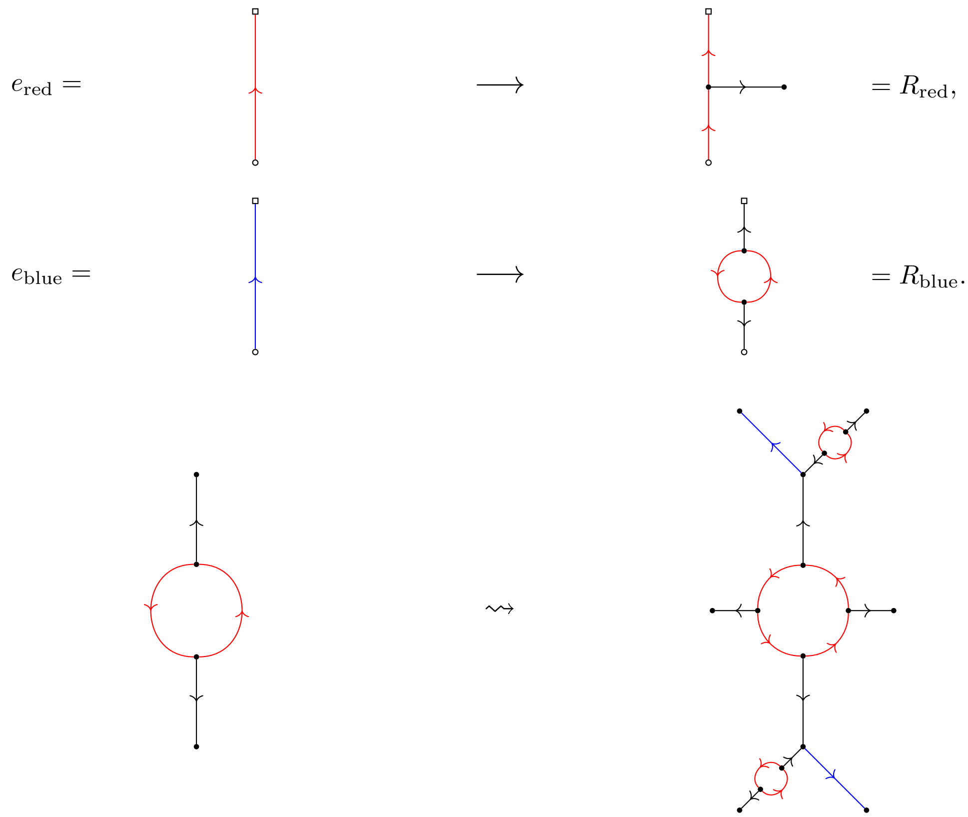


Figure 23: The base graph G_0 and its full expansion G_1

For example, the full expansion G_1 of G_0 is shown in Figure 23. This replacement rule $(G_0, \mathcal{R})$ is clearly expanding and finite branching. Let $X = X(G_0, \mathcal{R})$ be the limit space and let $\Gamma = \Gamma(G_0, \mathcal{R})$ be the graph family for $(G_0, \mathcal{R})$. Since we can consider each graph $G \in \Gamma$ to be a finite representing graph, for a rearrangement $f \colon X \to X$ there is a corresponding rearrangement in $T\left(\frac{3}{15}\right)$. Conversely, for a rearrangement $f \in T\left(\frac{3}{15}\right)$ we can construct a rearrangement $f \colon X \to X$. Therefore the rearrangement group for $(G_0, \mathcal{R})$ coincides with $T\left(\frac{3}{15}\right)$.

Theorem 4.5.1. The rearrangement group $T\left(\frac{3}{15}\right)$ is of type F_∞.

Proof. By Theorem A.1.10, it is enough to show that for every $m \geq 1$, the set

$$\Gamma_{<m} = \{G \in \Gamma(G_0, \mathcal{R}) \mid G \text{ has less than } m \text{ collapsible subgraphs}\}$$

is finite. We will show that if we expand a graph G in $\Gamma_{<m}$ then the number of collapsible subgraphs strictly increases. For an edge e of G colored with red, locally we only have to consider the situation indicated in Figure 24.

There are just two collapsible subgraphs overlapping the edge e, however, we find three collapsible subgraphs lying on the right-hand side graph in Figure 24. Therefore if we expand an edge colored with red, then the number of collapsible subgraphs strictry increases.

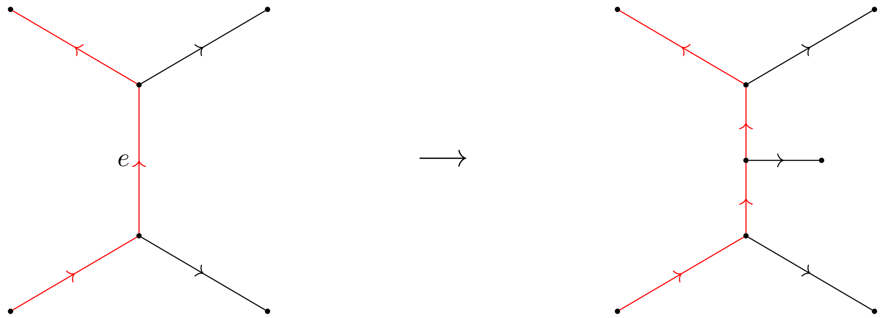


Figure 24:

For an edge e of G colored with blue, also we have to consider the unique case as in Figure 25. Possibly there is one collapsible subgraph overlapping an edge e, see Figure 26, however there are two collapsible subgraphs in the right-hand side graph in Figure 25, then it follows that the number of collapsible subgraphs strictly increases if we expand a blue edge.

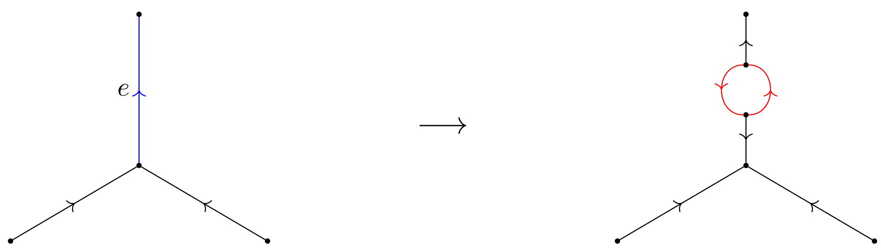


Figure 25:

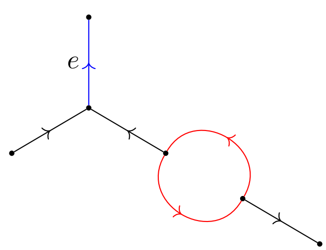


Figure 26:

Finally we consider the expansion of a black edge e. There are just three cases indicated in Figures 27, 28 and 29.

In the case of Figure 27, there are no collapsible subgraphs overlapping e, and the expanded graph has two new collapsible subgraphs. In the case of Figures 28 and 29, there is just one collapsible subgraph overlapping e for each case (indicated in the left-hand side of Figure 29 and in Figure 30 respectively). However each expanded graph has two new collapsible subgraphs.

From the above argument, the number of collapsible subgraphs increases by at least one when we perform a simple expansion, then the set $\Gamma_{<m}$ must be finite. $\square$

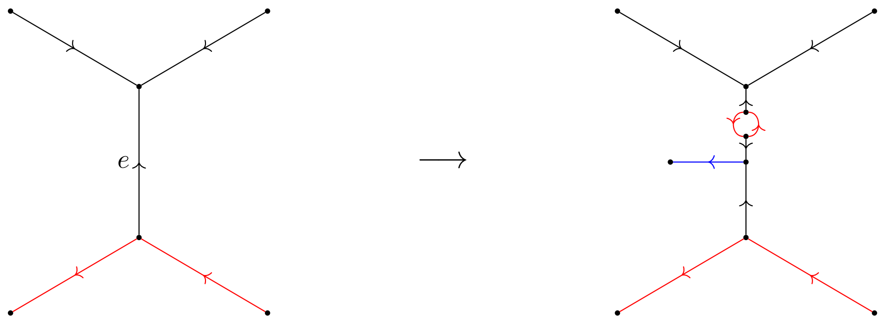


Figure 27:

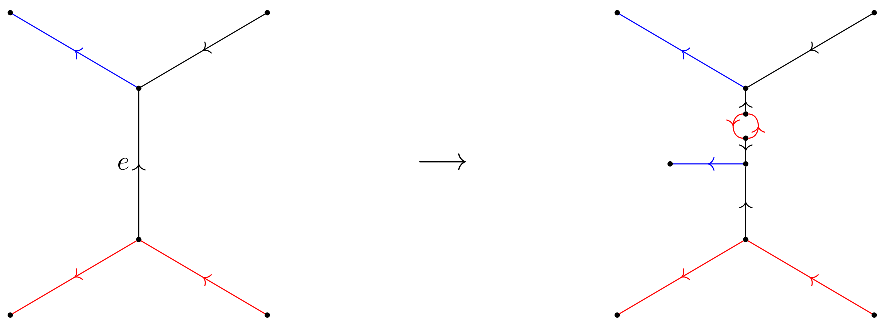


Figure 28:

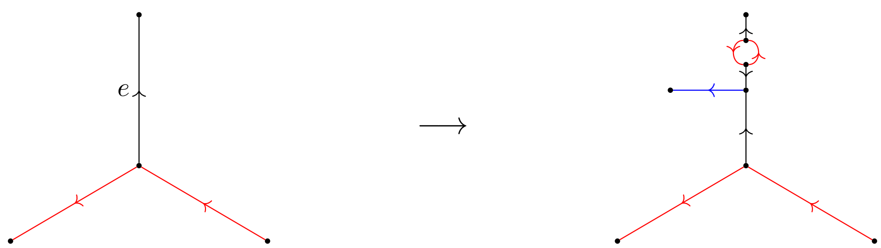


Figure 29:

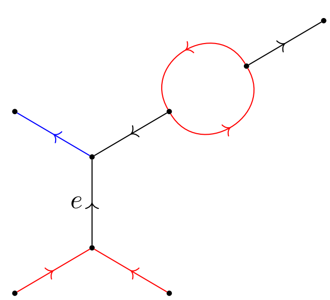


Figure 30:

4.6 Thompson-like groups for some other primitive components

We also define Thompson-like groups $T\left(\frac{3}{2^n-1}\right)$ in a way similar to $T\left(\frac{3}{15}\right)$. Let $\mathcal{P}_n^1 = \{\{1/(2^n-1), \ldots, 2^{n-1}/(2^n-1)\}$ and $\mathcal{P}_n^2 = \{\{3/(2^n-1), 4/(2^n-1)\}, \{6/(2^n-1), 8/(2^n-1)\}, \ldots, \{3\cdot 2^{n-2}/(2^n-1), 1/(2^n-1)\}\}$ be orbit portraits.

Definition 4.6.1.

- A point $p \in J\left(\frac{3}{2^n-1}\right)$ is called a **branch point** if there exist n dynamical rays $\mathcal{R}_{a_1}, \ldots, \mathcal{R}_{a_n}$ landing on p such that the set $\{a_1, \ldots, a_n\}$ is sent to an element of $\mathcal{P}_n$ under finitely many times of angle doubling $t \to t^2$.
- A point $p \in J\left(\frac{3}{2^n-1}\right)$ is called a **pool point** if there exist two dynamical rays $\mathcal{R}_a, \mathcal{R}_b$ landing on p such that the set $\{a, b\}$ is sent to an element of $\mathcal{P}_n^2$ under finitely many times of doubling.

Now we define segments, half segments, pools, pool points, branch arms and pool arms for $J\left(\frac{3}{2^n-1}\right)$ by slightly changing their definitions for $J\left(\frac{3}{15}\right)$. Then we also define the rearrangement group $T\left(\frac{3}{2^n-1}\right)$. As well as the group $T\left(\frac{3}{15}\right)$, $T\left(\frac{3}{2^n-1}\right)$ is generated by five elements, and there is a surjective group homomorphism $\phi\colon T\left(\frac{3}{2^n-1}\right) \to \mathbb{Z}_{n-1}\times\mathbb{Z}$ which induces an isomorphism $T\left(\frac{3}{2^n-1}\right) \Big/ \left[T\left(\frac{3}{2^n-1}\right), T\left(\frac{3}{2^n-1}\right)\right] \cong \mathbb{Z}_{n-1} \times \mathbb{Z}$.

Theorem 4.6.2. The rearrangement group $T\left(\frac{3}{2^n-1}\right)$ is generated by five elements and its abelianization $T\left(\frac{3}{2^n-1}\right) \Big/ \left[T\left(\frac{3}{2^n-1}\right), T\left(\frac{3}{2^n-1}\right)\right]$ is isomorphic to $\mathbb{Z}_{n-1} \times \mathbb{Z}$ for every $n \geq 3$.

Let us see some examples for the rearrangement group $T\left(\frac{3}{7}\right)$ for the "airplane" Julia set $J\left(\frac{3}{7}\right)$. Set $\mathcal{P}_2^1 = \{\{1/3, \ldots, 2/3\}$ and $\mathcal{P}_2^2 = \{\{3/7, 4/7\}, \{6/7, 1/7\}, \{5/7, 2/7\}\}$. The element α_1 is indicated in Figure 32. The loci coming from $\mathcal{P}_2$ are colored with blue.

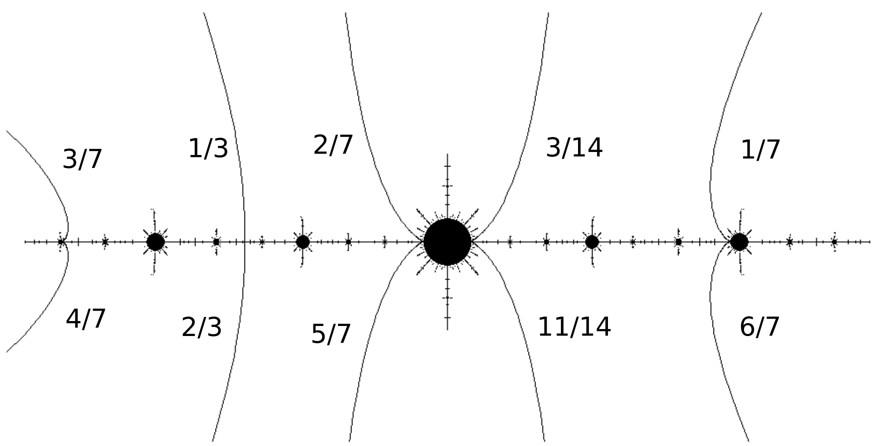


Figure 31: The filled airplane Julia set $K\left(\frac{3}{7}\right)$ with dynamical rays

Also we define the replacement rule for the airplane analogically, see Figure 33. We should remark that our replacement rule is different from that for the airplane introduced in [BF15a].

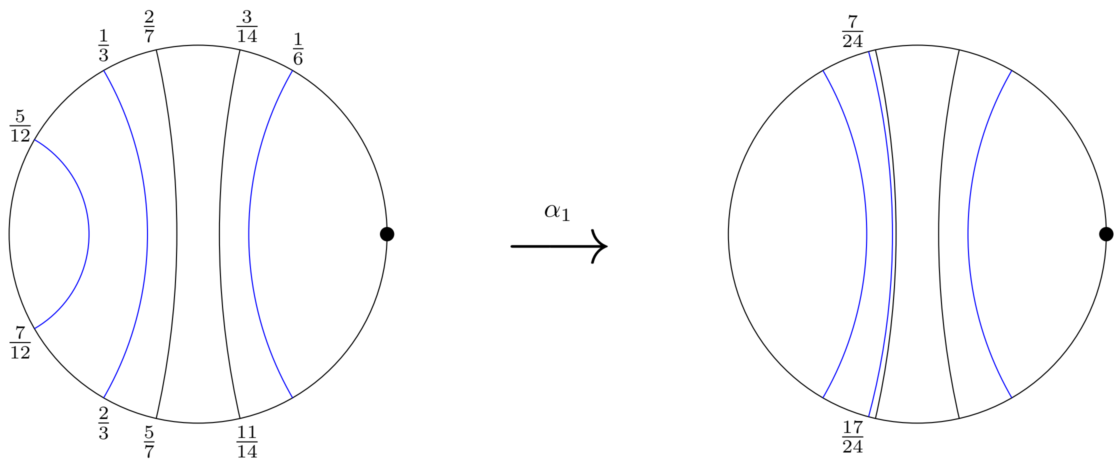


Figure 32:

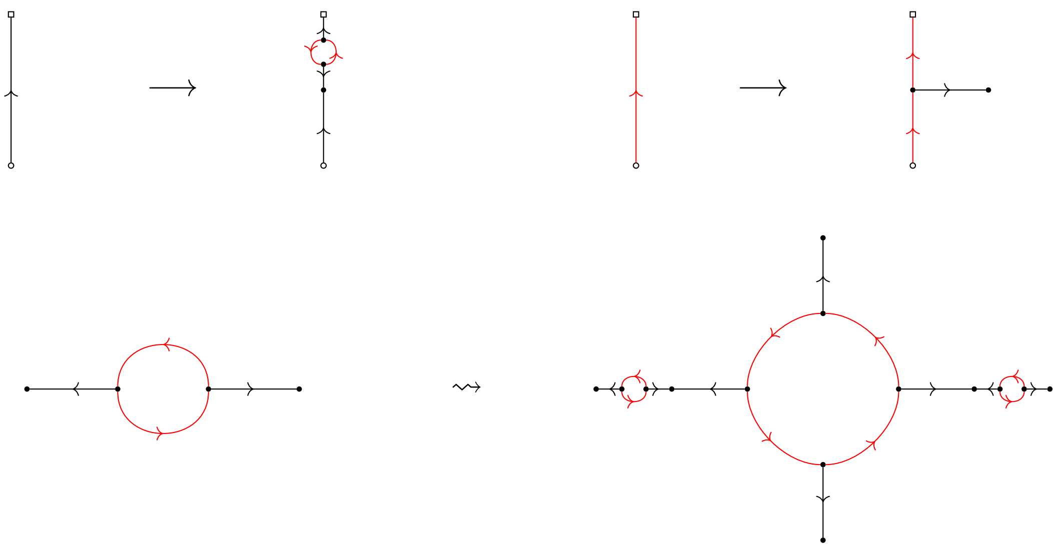


Figure 33: The base graph for the airplane and its full expansion

5 Thompson-like groups for tuned Julia sets

In this section, we will consider rearrangement groups for tuned Julia sets. First we consider the rearrangement group for the angle $22/63 = 1/3 * 1/7$, which is the (smaller) characteristic angle corresponding to the Julia set for the basilica tuned by the rabbit (see Figure 4). The corresponding angled internal address and the orbit portraits are

$$1_{1/2} \to 2_{1/3} \to 6, \text{ and}$$
$$\mathcal{P}^1 = \{\{1/3, 2/3\}\} \text{ and } \mathcal{P}^2 = \{\{22/63, 25/63, 37/63\}, \{44/63, 50/63, 11/63\}\}.$$

5.1 Thompson-like groups for the angle $\frac{22}{63}$

Let $K = K_c = K\left(\frac{1}{3}\right)$ and $L = K_{c_R} = K\left(\frac{1}{7}\right)$ be the filled basilica and the rabbit Julia sets (see Figures 2 and 3). Let $\{U_i\}_{i\in\mathbb{N}}$ be the family of Fatou components of K and assume U_0 contains $0 \in \mathbb{C}$. According to Theorem 2.4.8 and its proof in [Hs00], when we replace each component U_i by a copy L_i of L to obtain the filled Julia set K_L, a point in ∂U_i at an internal angle t corresponds to the point of L_i at which the dynamical ray of L_i at the angle t lands.

Let C_i be the gap of $\mathcal{L}^{(2)}$ corresponding to U_i. The map $\tau\colon \mathcal{L}^{(n)}_{C_0} \to S^1$ defined in the proof of Proposition 3.2.2 is extended continuously to the map $\tau_0\colon \partial C_0 \to S^1$ since the set of all dyadic points is dense in S^1. Also we define a map $\tau_i\colon \partial U_i \to S^1$ for $i \in \mathbb{N}$ in the same manner. In particular the root of U_i is sent to $0 \in S^1$ by τ_i.

For an end $e \in \mathbb{Q}/\mathbb{Z} \subset S^1$ of a locus in $\mathcal{L}^{(3)}$, there exists a unique point $e' \in \partial C_i \cap S^1$ such that $\tau_i(e') = e$. Then we define a map

$$\sigma_i\colon \mathcal{E}^{(3)} \to \partial C_i \cap S^1; \quad e \mapsto \sigma_i(e) = e'$$

where $\mathcal{E}^{(n)}$ is the set of all ends of loci in $\mathcal{L}^{(n)}$. For a locus $L = (a_1; a_2; a_3)$ in $\mathcal{L}^{(3)}$, the images of the vertices by σ_i yield the locus $(\sigma_i(a_1); \sigma_i(a_2); \sigma_i(a_3))$, and we denote this locus by $\sigma_i(L)$. For each map σ_i, every locus in $\mathcal{L}^{(3)}$ is embedded into each gap C_i of $\mathcal{L}^{(2)}$.

The pinching lamination $\mathcal{L}\left(\frac{22}{63}\right) = \mathcal{L}\left(\frac{1}{3} * \frac{1}{7}\right) = \mathcal{L}\left(\frac{1}{2^2-1} * \frac{1}{2^3-1}\right)$ coincides with the set $\mathcal{L}^{(2)} \cup \bigcup_{i\in\mathbb{N}} \sigma_i(\mathcal{L}^{(3)}) = \mathcal{L}\left(\frac{1}{3}\right) \cup \bigcup_{i\in\mathbb{N}} \sigma_i(\mathcal{L}\left(\frac{1}{7}\right))$.

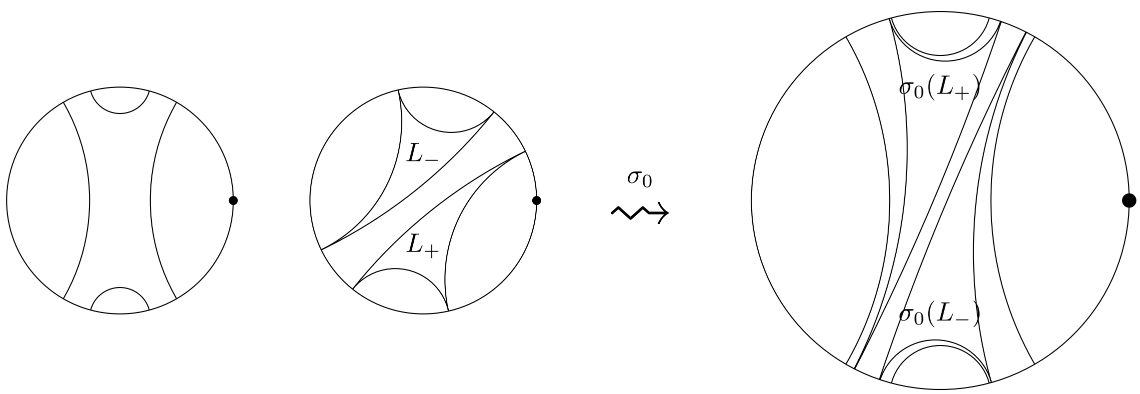


Figure 34: Embedding primary loci of $\mathcal{L}^{(3)}$ into the critical gap of $\mathcal{L}^{(2)}$ by σ_0

Definition 5.1.1. A **finite locus diagram** for $J\left(\frac{22}{63}\right) = J\left(\frac{1}{3} * \frac{1}{7}\right)$ is the closed unit disc $\overline{\Delta}$ with the following elements from (1) to (4):

Let S be a finite subset of $\mathbb{N}$ containing $0 \in \mathbb{N}$, and let D_i be a finite locus diagram for $J\left(\frac{1}{7}\right)$ for each $i \in S$.

(1) Primary loci for $J\left(\frac{1}{3}\right)$: $\left(\frac{1}{3}; \frac{2}{3}\right)$ and $\left(\frac{1}{2} + \frac{1}{3}; \frac{1}{2} + \frac{2}{3}\right)$.

(2) Images of all loci of D_i by σ_i for each $i \in S$.

(3) For each locus $L = (a_1; a_2; a_3)$ in $\sigma_i(D_i)$ $(i \in S)$ and for each arc $(a_k; a_{k+1})$ (with subscripts modulo 3), we add the locus which has the longest standard interval among the loci of $\mathcal{L}^{(2)}$ surrounded by S^1 and $(a_k; a_{k+1})$.

(4) A finite number of loci so that the set of loci added in (3) with themselves yields a finite locus diagram for $\mathcal{L}^{(2)}$.

As in Section 3.1, we also define **locus pair diagrams** for $J\left(\frac{22}{63}\right)$ and each of them corresponds to a unique piecewise linear homeomorphism on S^1 preserving $\mathcal{L}\left(\frac{1}{3} * \frac{1}{7}\right)$, and this map induces an orientation preserving homeomorphism on $J\left(\frac{22}{63}\right)$ which is called an **rearrangement**.

Proposition 5.1.2. Every locus pair diagram for $J\left(\frac{22}{63}\right)$ has a unique reduced locus pair diagram.

Proof. Let $f \in T\left(\frac{22}{63}\right)$, and let D_f and R_f be a domain and a range diagram for f. Each standard interval of the domain diagram D_f of f must be regular with respect to f. Let S be a finite subset of $\mathbb{N}$ and $D_i(i \in S)$ be a finite locus diagram for $J^{(3)}$ that determine D_f as in Definition 5.1.1.

A locus pair diagram for f is reduced if and only if:

(1) A locus pair diagram $(D_f \cap \mathcal{L}^{(2)}, R_f \cap \mathcal{L}^{(2)})$ is reduced as an element of $T^{(2)}$, and

(2) for every $i \in S$, each regular interval in $D_f \cap \sigma_i(D_i)$ is maximal under inclusion.

Since any two maximal regular intervals have disjoint interiors, there can be only one subdivision of the circle into regular intervals. □

Definition 5.1.3. The set

$$T\left(\tfrac{1}{3} * \tfrac{1}{7}\right) = T\left(\tfrac{22}{63}\right) = \left\{f \colon J\left(\tfrac{22}{63}\right) \to J\left(\tfrac{22}{63}\right) \,\middle|\, f \text{ is a rearrangement for } J\left(\tfrac{22}{63}\right)\right\}$$

has a group structure under composition. It is called a **rearrangement group** for $J\left(\frac{22}{63}\right) = J\left(\frac{1}{3} * \frac{1}{7}\right)$.

The gap C in $\mathcal{L}\left(\frac{1}{3} * \frac{1}{7}\right)$ corresponding to the Fatou component which contains 0 is called the **critical gap** for $\mathcal{L}\left(\frac{1}{3} * \frac{1}{7}\right)$. A locus surrounding the critical gap is called a **critical locus** and let $\mathcal{L}_C\left(\frac{1}{3} * \frac{1}{7}\right)$ be the set of all critical loci in $\mathcal{L}\left(\frac{1}{3} * \frac{1}{7}\right)$.

Every element g of $T^{(3)}$ is represented by an pair of finite locus diagrams of embedded locus lamination $\sigma_i(\mathcal{L}^{(3)}) \subset \overline{C_i}$, then we denote this embedded element by $\sigma_i(g)$.

Definition 5.1.4.

- The **stabilizer** $\operatorname{stab}(C) = \left\{f \in T\left(\frac{1}{3} * \frac{1}{7}\right) \,\middle|\, f(C) = C\right\}$ is the group of elements of $T\left(\frac{1}{3} * \frac{1}{7}\right)$ which sent C to itself.
- The **rigid stabilizer** $\operatorname{rist}(C)$ is the subgroup of elements of $T\left(\frac{1}{3} * \frac{1}{7}\right)$ such that for each element $f \in \operatorname{rist}(C)$ there exists a rearrangement $f' \in T^{(3)}$ which belongs to the rigid stabilizer for $T^{(3)}$ and $\sigma_0(f') = f$.

By Theorem 3.2.6, $T^{(n)}$ is generated by four elements α_1, β, γ and δ. We rewrite them $\alpha_n, \beta_n, \gamma_n$ and δ_n respectively.

Remark 5.1.5. The elements $\alpha_2, \delta_2 \in T^{(2)}$ are naturally regarded as elements of $T\left(\frac{1}{3} * \frac{1}{7}\right)$. However, β_2 and γ_2 cannot be regarded as elements of $T\left(\frac{1}{3} * \frac{1}{7}\right)$ in a natural way, since they do not preserve the lamination $\mathcal{L}\left(\frac{1}{3} * \frac{1}{7}\right)$. On the other hand, for each $i \in \mathbb{N}$, $T\left(\frac{1}{7}\right)$ is embedded by σ_i as a subgroup of $T\left(\frac{1}{3} * \frac{1}{7}\right)$, since every element of $T^{(n)}$ has the form $2^p\left(x + \frac{q}{2^r}\right)$ $(p, q, r \in \mathbb{Z})$ in its linear segments.

Set $\sigma = \sigma_0$, and for each element $\eta_3 \in T\left(\frac{1}{7}\right)$ (where $\eta_3 \in \{\alpha_3, \beta_3, \delta_3, \delta_3\}$), the embedded element $\sigma(\eta_3)$ is denoted by $\eta_{3,2}$. The next proposition is shown in the same way as in the proof of Proposition 3.2.2.

Proposition 5.1.6. (1) Each element of $\mathrm{stab}(C)$ acts on $\mathcal{L}_C\left(\frac{1}{3} * \frac{1}{7}\right)$ as an element of T.

(2) The rigid stabilizer $\mathrm{rist}(C)$ acts on $\mathcal{L}_C\left(\frac{1}{3} * \frac{1}{7}\right)$ as an isomorphic copy of T.

Corollary 5.1.7. The rigid stabilizer $\mathrm{rist}(C)$ is genarated by $\beta_{3,2}, \gamma_{3,2}$, and $\delta_{3,2}$.

Lemma 5.1.8. The group $\langle \alpha_2, \alpha_{3,2}, \beta_{3,2}, \gamma_{3,2}, \delta_{3,2} \rangle$ acts transitively on the gaps of $\mathcal{L}\left(\frac{1}{3} * \frac{1}{7}\right)$.

Proof. Let G be a gap of $\mathcal{L}\left(\frac{1}{3} * \frac{1}{7}\right)$. First we assume that G lies on the critical gap C_0 of $\mathcal{L}^{(2)}$. Then there exists an element f of the group $\langle \alpha_{3,2}, \beta_{3,2}, \delta_{3,2}, \gamma_{3,2} \rangle \cong T^{(3)}$ such that $f(G) = C$ by Lemma 3.2.5.

Next we suppose there is the gap H_1 of $\mathcal{L}^{(2)}$ on which G lies, and suppose H_1 is of depth 1 as a gap of $\mathcal{L}^{(2)}$. Let L_1 be the locus lying on $\sigma(\mathcal{L}^{(3)})$ which has the longest standard interval among the loci separating G_1 from the critical gap C of $\mathcal{L}\left(\frac{1}{3} * \frac{1}{7}\right)$. By Lemma 3.2.12, there is an element g of $\langle \alpha_2, \alpha_{3,2}, \beta_{3,2}, \gamma_{3,2}, \delta_{3,2} \rangle$ such that $g(I_1) = [1/3, 2/3]$ or $[5/24, 7/24]$, where I_1 is the standard interval of the locus of $\mathcal{L}^{(2)}$ separating G_0 from the critical gap of $\mathcal{L}^{(2)}$. Since $\sigma((\delta_3\alpha_3)^{-1})([5/24, 7/24]) = \zeta^{-1}([5/24, 7/24]) = [1/3, 2/3]$, we may assume $g(I_1) = [1/3, 2/3]$. The gap $\alpha_2(g(G))$ lies on the critical gap of $\mathcal{L}^{(2)}$, then the problem is reduced to the first case.

If G lies on a gap of $\mathcal{L}^{(2)}$ of depth $n \geq 2$, we can also show the claim by taking a sequence of gaps and using induction on n as in the proof of Lemma 3.2.5. □

Theorem 5.1.9. The group $T\left(\frac{1}{3} * \frac{1}{7}\right)$ is generated by the elements $\alpha_2, \alpha_{3,2}, \beta_{3,2}, \delta_{3,2}$ and $\gamma_{3,2}$.

Proof. Let f be an element of $T\left(\frac{1}{3} * \frac{1}{7}\right)$. By Lemma 5.1.8 and Proposition 5.1.6, we may assume that $f \in \mathrm{rist}(C)$. Let $m(\geq 6)$ be the number of the loci of the reduced domain diagram D_f of f. We use induction on m. If $m = 6$, then $f = \mathrm{id}$. Suppose $m \geq 7$. Assume that the reduced domain diagram D_f of f contains critical loci $L_1, \ldots, L_k \in \mathcal{L}\left(\frac{1}{3} * \frac{1}{7}\right)$.

<u>Case1</u> Assume D_f has loci which are not lying on $\mathcal{L}^{(2)}$ in more than one standard interval for L_i. Then we write

$$f = f_1 \circ f_2 \circ \cdots \circ f_k$$

where each $f_i \in T\left(\frac{22}{63}\right)$ is an rearrangement which has the same critical loci as f but has non-critical loci only in the standard interval for L_i. Each f_i has less than m loci, then by induction, $f_i \in \langle \alpha_2, \beta_{3,2}, \gamma_{3,2}, \delta_{3,2} \rangle$.

Case2 Assume D_f has non-critical loci only behind the critical locus $L = L_i$ except loci lying on $\mathcal{L}^{(2)}$. By Proposition 5.1.6, we may assume $L = \sigma(L_+) = \sigma((9/14; 11/14; 1/14))$ (see Figure 34).

1° If there are just two loci on the critical gap C_0 of $\mathcal{L}^{(2)}$, D_f must contain the locus $(1/3; 2/3)$ and the domain diagram $\alpha_2^{-1} D_f \alpha_2$ of the rearrangement $\alpha_2^{-1} f \alpha_2$ with m loci can be reduced: at least three loci $\alpha_2((1/6; 5/6)) = (1/12; 11/12), \alpha_2(\sigma(L))$, and $\alpha_2(\sigma(\delta_3(L)))$ can be erased. Since the reduced domain diagram for $\alpha_2^{-1} f \alpha_2$ has fewer than m loci, $\alpha_2^{-1} f \alpha_2 \in \langle \alpha_2, \beta_{3,2}, \gamma_{3,2}, \delta_{3,2} \rangle$ by induction.

2° If there are more than two loci on the critical gap C_0 of $\mathcal{L}^{(2)}$, D_f must contain at least one locus $\alpha_{3,2}(L)$ or $\alpha_{3,2}\delta_{3,2}\alpha_{3,2}(L)$. The domain diagram $\alpha_{3,2}^{-1} D_f \alpha_{3,2}$ of $\alpha_{3,2}^{-1} f \alpha_{3,2}$ with m loci can also be reduced: the loci $\alpha_{3,2}(L)$ or $\alpha_{3,2}\delta_{3,2}\alpha_{3,2}(L)$ can be erased. Since the reduced domain diagram for $\alpha_{3,2}^{-1} f \alpha_{3,2}$ has fewer than m loci, $\alpha_{3,2}^{-1} f \alpha_{3,2} \in \langle \alpha_2, \beta_{3,2}, \gamma_{3,2}, \delta_{3,2} \rangle$ by induction. $\square$

We consider a coloring of gaps of $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(3)}$ with elements of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ respectively. Also we give a color $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_3$ to each gap $\sigma_i(G)$ of $\mathcal{L}\left(\frac{22}{63}\right)$ where the gap C_i of $\mathcal{L}^{(2)}$ is colored with $a \in \mathbb{Z}_2$ and the gap G of $\mathcal{L}^{(3)}$ is colored with $b \in \mathbb{Z}_3$. Now we define a homomorphism

$$\phi\colon T\left(\tfrac{1}{3} * \tfrac{1}{7}\right) \to \mathbb{Z}_2 \times \mathbb{Z}_3$$

where $\phi(f) = (k, l)$ if $f \in T\left(\frac{1}{3} * \frac{1}{7}\right)$ changes the color (a, b) of the critical gap C of $\mathcal{L}\left(\frac{1}{3} * \frac{1}{7}\right)$ to $(a + k, b + l)$.

Theorem 5.1.10. The homomorphism ϕ induces a group isomorphism

$$\phi\colon T\left(\tfrac{1}{3} * \tfrac{1}{7}\right) \big/ \left[T\left(\tfrac{1}{3} * \tfrac{1}{7}\right), T\left(\tfrac{1}{3} * \tfrac{1}{7}\right)\right] \xrightarrow{\cong} \mathbb{Z}_2 \times \mathbb{Z}_3.$$

Proof. We use the similar argument to the proof of Theorem 3.2.8. Set $\epsilon = \delta_{3,2}\alpha_2$ and $\zeta = \delta_{3,2}\alpha_{3,2}$. Set $G = T\left(\frac{1}{3} * \frac{1}{7}\right)$, $H = \ker \phi$, $S = \{\epsilon, \zeta, \beta_{3,2}, \gamma_{3,2}, \delta_{3,2}\}$. The set S also generates $T\left(\frac{1}{3} * \frac{1}{7}\right)$. Let $\rho\colon G/H \to G$ be the section of the quotient map defined by $\overline{g} = \rho(gH) = \epsilon^k \zeta^l$ if $\phi(g) = (k, l)$, and $\rho(G/H)$ is denoted by R. Since $\epsilon^2 = \zeta^3 = \mathrm{id}$, ρ is well-defined. Set $U = \left\{ (\overline{sr})^{-1} sr \;\middle|\; s \in S, r \in R \right\}$. We have to show that $H = \left[T\left(\frac{1}{3} * \frac{1}{7}\right), T\left(\frac{1}{3} * \frac{1}{7}\right)\right]$. Since $\mathbb{Z}_2 \times \mathbb{Z}_3$ is abelian, $\left[T\left(\frac{1}{3} * \frac{1}{7}\right), T\left(\frac{1}{3} * \frac{1}{7}\right)\right] < H$ is trivial.

For the converse direction, it is enough to show that $U \subset \left[T\left(\frac{1}{3} * \frac{1}{7}\right), T\left(\frac{1}{3} * \frac{1}{7}\right)\right]$ since U generates H by Schreier's lemma. By calculation, we find that $(\overline{\eta\epsilon^k\zeta^l})^{-1}\eta\epsilon^k\zeta^l = \eta^{\epsilon^k\zeta^l}$ where $\eta \in \{\mathrm{id}, \beta_{3,2}, \gamma_{3,2}, \delta_{3,2}\}$, and $(\overline{\epsilon\epsilon^k\zeta^l})^{-1}\epsilon\epsilon^k\zeta^l = \mathrm{id}$, and $(\overline{\zeta\epsilon^k\zeta^l})^{-1}\zeta\epsilon^k\zeta^l = [\zeta^{-1}, (\epsilon^k\zeta^l)^{-1}]$. Since $\langle \beta_{3,2}, \gamma_{3,2}, \delta_{3,2} \rangle \cong T = [T, T]$, the elements $\beta_{3,2}, \gamma_{3,2}$ and $\delta_{3,2}$ are in $\left[T\left(\frac{1}{3} * \frac{1}{7}\right), T\left(\frac{1}{3} * \frac{1}{7}\right)\right]$. Then it follows that $\eta^{\epsilon^k\zeta^l}$, $[\zeta^{-1}, (\epsilon^k\zeta^l)^{-1}] \in \left[T\left(\frac{1}{3} * \frac{1}{7}\right), T\left(\frac{1}{3} * \frac{1}{7}\right)\right]$, and we see $U \subset \left[T\left(\frac{1}{3} * \frac{1}{7}\right), T\left(\frac{1}{3} * \frac{1}{7}\right)\right]$. $\square$

The above theorems are easily generalized to the following form.

Theorem 5.1.11. The rearrangement group $T\left(\frac{1}{2^n-1} * \frac{1}{2^m-1}\right)$ is generated by five elements, and the following holds:

$$T\left(\frac{1}{2^n - 1} * \frac{1}{2^m - 1}\right) \bigg/ \left[T\left(\frac{1}{2^n - 1} * \frac{1}{2^m - 1}\right), T\left(\frac{1}{2^n - 1} * \frac{1}{2^m - 1}\right)\right] \cong \mathbb{Z}_n \times \mathbb{Z}_m.$$

A Appendix

A.1 Replacement systems

In this appendix, we recall replacement systems defined in [BF15a]. For a graph G, $V(G)$ and $E(G)$ mean the vertex and edges sets of G.

Definition A.1.1. An **(edge) replacement rule** is a pair $e \to R$, for e a non-loop directed edge with the initial vertex v and the terminal vertex w, and a finite directed graph R with $v, w \in V(R)$ called a **replacement graph**.

Definition A.1.2. A **colored replacement system** $(G_0, \mathcal{R} = \{e_c \to R_c\}_{c \in C})$ consists of the following data:

(1) A finite set C of **colors**.

(2) A directed **base graph** G_0, whose edges have been colored by the elements of C.

(3) For each $c \in C$, a directed replacement rule $e_c \to R_c$, where e_c is a directed edge colored by c, and R_c is a colored replacement graph.

A colored replacement system with a single color is simply called a **replacement system**.

Replacing an edge $\epsilon \in E(G_0)$ by (a copy of) R attaching the initial and terminal vertices of R respectively to the initial and terminal vertices of e, we obtain a new finite directed graph $G_0 \triangleleft \epsilon$, say **simple expansion** of G_0. A graph obtained from G_0 through a sequence of simple expansions is called an **expansion** of G_0. We will always replace an edge ϵ colored by c by R_c. The reverse of a (simple) expansion is called a **(simple) contraction**. For a graph G, the graph obtained by expanding each edge of G is called the **full expansion** of G. For each $n \in \mathbb{N}_{>0}$ let G_n be the full expansion of G_{n-1}.

Let $(G_0, \mathcal{R} = \{e_c \to R_c\}_{c \in C})$ be a colored replacement system.

Definition A.1.3. The **graph family** $\Gamma(G_0, \mathcal{R})$ is the set of all finite directed graphs obtained by expanding G_0 by the colored replacement system $\mathcal{R}$.

Definition A.1.4. (1) A colored replacement system $(G_0, \mathcal{R})$ is **expanding** if G_0 has no isolated vertices, the initial point v and the terminal point w of R_c does not share an edge in R_c, and $|E(R_c)| \geq 3$ and $|V(R_c)| \geq 2$ for each color $c \in C$.

(2) The replacement system $\mathcal{R}$ is said to be **finite branching** if there exists an upper bound on the degrees of vertices in the full expansion sequence for $\mathcal{R}$.

Hereafter we always assume replacement systems are expanding.

Definition A.1.5. Define the **symbol space** for $(G_0, \mathcal{R})$ by

$$\Omega = \left\{ \epsilon_0 \epsilon_1 \epsilon_2 \cdots \in E(G_0) \times (\bigcup_{c \in C} E(R_c))^{\infty} \;\middle|\; \begin{array}{l} \text{for each } j \in \mathbb{N}, \\ \text{if } \epsilon_j \text{ can be replaced by } R_c \text{ then } \epsilon_{j+1} \in E(R_c) \end{array} \right\}$$

endowed with the product topology. The **gluing relation** $\sim$ on Ω is the equivalence relation defined as follows: for $\epsilon_0\epsilon_1\epsilon_2\cdots$ and $\epsilon'_0\epsilon'_1\epsilon'_2\cdots$ in Ω,

$$\epsilon_0\epsilon_1\epsilon_2\cdots \sim \epsilon'_0\epsilon'_1\epsilon'_2\cdots \Leftrightarrow \begin{array}{l} \forall n \in \mathbb{N},\ \text{the edges of } G_n \text{ with } \textbf{addresses} \\ \epsilon_0\epsilon_1\epsilon_2\cdots\epsilon_n \text{ and } \epsilon'_0\epsilon'_1\epsilon'_2\cdots\epsilon'_n \text{ share at least one vertex} \end{array}$$

The **limit space** is the quotient space $X = \Omega/\sim$.

The gluing relation is not necessarily an equivalence relation if $\mathcal{R}$ is not expanding.

Definition A.1.6. For an edge $e = \epsilon_0\cdots\epsilon_n \in E(G_n)$, let $\Omega(e)$ be the set of all points in Ω that have $\epsilon_0\cdots\epsilon_n$ as a prefix. The **cell** $C(e)$ is the image of $\Omega(e)$ in the limit space X. A vertex of $C(e)$ corresponding to an end of an edge e is called a **boundary point** of $C(e)$. The complement of the boundary points is called the **interior** of the cell.

Let $e \in E(G_n)$ and $e' \in E(G_{n'})$ be edges of the same color that are either loops or not loops. Then there is a **canonical homeomorphism**

$$\Phi\colon \Omega(e) \to \Omega(e'); \quad e\zeta_1\zeta_2\cdots \mapsto e'\zeta_1\zeta_2\cdots$$

for every (sequence of) edges $\zeta_1, \zeta_2, \ldots$ in some R_c.

Definition A.1.7. (1) A **cellular partition** of X is a cover of X by finitely many cells whose interiors are disjoint.

(2) A homeomorphism $f\colon X \to X$ is called a **rearrangement** if there exists a cellular partition of X such that f restricts to a canonical homeomorphism on each cell of the partition.

Proposition A.1.8. The rearrangements of X form a group under composition. We call this group the **rearrangement group** for $(G_0, \mathcal{R})$.

Let R_c^{loop} denote the graph obtained from R by gluing the initial and terminal vertices.

Definition A.1.9. Let G be a directed graph. A **characteristic map** for R_c in G is an isomorphism $\chi\colon R_c \to S$ or $\chi\colon R_c^{\text{loop}} \to S$, where S is a subgraph of G such that for each interior vertex $v \in R_c$, every edge of G incident on $\chi(v)$ lies in S. This subgraph S is called the **collapsible subgraph** of G.

The replacement system $\mathcal{R}$ is said to be **finite branching** if there exists an upper bound on the degrees of vertices in the full expansion sequence for $\mathcal{R}$.

Theorem A.1.10 ([BF15a])**.** Assume that the replacement system is finite branching and its replacement graph is connected. If for every $m \geq 1$, all but finitely many graphs in $\Gamma(G_0, \mathcal{R})$ have at least m different collapsible subgraphs, then the corresponding rearrangement group is of type F_∞.

References

[BF15a] J. Belk and B. Forrest. Rearrangement Groups of Fractals. *ArXiv e-prints*, October 2015.

[BF15b] James Belk and Bradley Forrest. A Thompson group for the basilica. *Groups Geom. Dyn.*, 9(4):975–1000, 2015.

[CFP96] J. W. Cannon, W. J. Floyd, and W. R. Parry. Introductory notes on Richard Thompson's groups. *Enseign. Math. (2)*, 42(3-4):215–256, 1996.

[DH84] A. Douady and J. H. Hubbard. *Étude dynamique des polynômes complexes. Partie I*, volume 84 of *Publications Mathématiques d'Orsay [Mathematical Publications of Orsay]*. Université de Paris-Sud, Département de Mathématiques, Orsay, 1984.

[DH85a] A. Douady and J. H. Hubbard. *Étude dynamique des polynômes complexes. Partie II*, volume 85 of *Publications Mathématiques d'Orsay [Mathematical Publications of Orsay]*. Université de Paris-Sud, Département de Mathématiques, Orsay, 1985. With the collaboration of P. Lavaurs, Tan Lei and P. Sentenac.

[DH85b] Adrien Douady and John Hamal Hubbard. On the dynamics of polynomial-like mappings. *Ann. Sci. École Norm. Sup. (4)*, 18(2):287–343, 1985.

[Dou86] A. Douady. Algorithms for computing angles in the Mandelbrot set. In *Chaotic dynamics and fractals (Atlanta, Ga., 1985)*, volume 2 of *Notes Rep. Math. Sci. Engrg.*, pages 155–168. Academic Press, Orlando, FL, 1986.

[Hs00] Peter Haï ssinsky. Modulation dans l'ensemble de Mandelbrot. In *The Mandelbrot set, theme and variations*, volume 274 of *London Math. Soc. Lecture Note Ser.*, pages 37–65. Cambridge Univ. Press, Cambridge, 2000.

[Hub16] John Hamal Hubbard. *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 2.* Matrix Editions, Ithaca, NY, 2016. Surface homeomorphisms and rational functions.

[Lav86] Pierre Lavaurs. Une description combinatoire de l'involution définie par M sur les rationnels à dénominateur impair. *C. R. Acad. Sci. Paris Sér. I Math.*, 303(4):143–146, 1986.

[McM94] Curtis T. McMullen. *Complex dynamics and renormalization*, volume 135 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1994.

[Mil00a] John Milnor. Local connectivity of Julia sets: expository lectures. In *The Mandelbrot set, theme and variations*, volume 274 of *London Math. Soc. Lecture Note Ser.*, pages 67–116. Cambridge Univ. Press, Cambridge, 2000.

[Mil00b] John Milnor. Periodic orbits, externals rays and the Mandelbrot set: an expository account. *Astérisque*, (261):xiii, 277–333, 2000. Géométrie complexe et systèmes dynamiques (Orsay, 1995).

[Mil06] John Milnor. *Dynamics in one complex variable*, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, third edition, 2006.

[Sch04] Dierk Schleicher. On fibers and local connectivity of Mandelbrot and Multibrot sets. In *Fractal geometry and applications: a jubilee of Benoît Mandelbrot. Part 1*, volume 72 of *Proc. Sympos. Pure Math.*, pages 477–517. Amer. Math. Soc., Providence, RI, 2004.

[Sch17] Dierk Schleicher. Internal addresses of the Mandelbrot set and Galois groups of polynomials. *Arnold Math. J.*, 3(1):1–35, 2017.

[Ser03] Ákos Seress. *Permutation group algorithms*, volume 152 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2003.

[Smi13] Will Smith. Thompson-Like Groups for Dendrite Julia Sets. 2013. A Senior Project submitted to Bard Collage, unpublished.

[WZ16] S. Witzel and M. C. B. Zaremsky. The Basilica Thompson group is not finitely presented. *ArXiv e-prints*, March 2016.