Earthquake Maps of a Once-punctured Torus

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Abstract

The earthquake theorem states that for any orientation-preserving homeomorphism f of the boundary $\mathbf{S}^1_\infty$ of the Poincaré half-plane model $\mathbf{H}$, there uniquely exists a bijection from $\mathbf{H}$ to $\mathbf{H}$ which is discontinuous along some geodesics of $\mathbf{H}$ and the extension of f. The earthquake theorem can be applied to the case of surfaces. In particular, the earthquake theorem implies that for any two elements in the Teichmüller space of any complete hyperbolic surface, there uniquely exists an earthquake map from one to the other. Moreover, in [BKS], the earthquake theorem was proven in the enhanced Teichmüller space, which is the extension of the Teichmüller space. The enhanced Teichmüller space have the shear coordinates. We show the construction of an earthquake map for surfaces. We fix laminations of earthquake maps and calculate the domains where a base point moves by earthquake maps in the enhanced Teichmüller space of a once-punctured torus by using the shear coordinates.

1 Introduction

Thurston defined an earthquake map and proved the earthquake theorem for closed surfaces in a course at Princeton University during 1976-7. The contents of this course is summed up in [T1] and [T4]; however, these literatures do not mention the earthquake theorem. Kerckhoff was the first person to write Thurston's proof of the earthquake theorem for closed surfaces, applied this theorem to the Nielsen realization problem and answered it positively in [K]. Thurston gave a more elementary and more constructive proof of the earthquake theorem in [T3] later. This proof works in a more general context. Namely, Thurston proved the earthquake theorem for any complete hyperbolic surface.

On the other hand, Mess gave a new proof of the earthquake theorem for closed surfaces by using the Anti de Sitter space in [M]. The earthquake theorem is an intermediary between hyperbolic geometry and Anti de Sitter geometry.

An earthquake map of the Poincaré half-plane model $\mathbf{H}$ is a bijection from $\mathbf{H}$ to $\mathbf{H}$ which is discontinuous on some geodesics (we call the union of these geodesics a lamination) and locally an isometry of $\mathbf{H}$, is extended continuously to the boundary $\mathbf{S}^1_\infty$ of $\mathbf{H}$ and moves every point of $\mathbf{H}$ left (Definition 2). In [T3], the earthquake theorem for $\mathbf{H}$ states that for any orientation-preserving homeomorphism of $\mathbf{S}^1_\infty$, there uniquely exists an earthquake map which extends continuously to f on $\mathbf{S}^{\mathbf{1}}_{\infty}$. We explain the relation between the earthquake theorem and the universal Teichmüller space. Any quasiconformal mapping of $\mathbf{H}$ extends continuously to an orientation-preserving homeomorphism on $\mathbf{S}^{\mathbf{1}}_{\infty}$ which is quasisymmetric. We only consider quasiconformal mappings which map 0, 1, ∞ to 0, 1, ∞, respectively. Two quasiconformal mappings are equivalent if they extend to the same orientation-preserving homeomorphism on $\mathbf{S}^{\mathbf{1}}_{\infty}$. The set of the equivalence classes is the *universal Teichmüller space*. The earthquake theorem for $\mathbf{H}$ implies that for any two elements of the universal Teichmüller space, there uniquely exists an earthquake map which maps one to the other.

An earthquake map between two complete hyperbolic surfaces is an earthquake map of $\mathbf{H}$ which is equivariant with respect to the actions of the Fuchsian groups of the two surfaces. In [T3], the earthquake theorem for surfaces says that for any orientation-preserving homeomorphism of $\mathbf{S}^{\mathbf{1}}_{\infty}$ equivariant with respect to the actions of the Fuchsian groups of two surfaces, there uniquely exists an earthquake map equivariant with respect to the actions which extends continuously to f on $\mathbf{S}^{\mathbf{1}}_{\infty}$. In particular, the earthquake theorem for surfaces implies that for any two elements of the Teichmüller space of a surface, there uniquely exists an earthquake map of surfaces which maps one to the other. In addition, if a surface is of finite area, by choosing a base point and taking earthquake paths from this point, we consider the Thurston compactification of the Teichmüller space of the surface ([FLP]). From the above arguments, the earthquake theorem is important in the Teichmüller theory.

Note that there exists an earthquake map corresponding to an orientation-preserving homeomorphism of $\mathbf{S}^{\mathbf{1}}_{\infty}$ which is not quasisymmetric. We can consider an earthquake map which put out an element of the Teichmüller space. Therefore, we consider the enhanced Teichmüller space, which has the Teichmülle space as a subspace. The enhanced Teichmüller space of a surface of finite type with at least one puncture was introduced in [FG1]. This space is the natural extension of the Teichmüller space from the perspective of the representation theory and cluster algebra and has the shear coordinates, which is a diffeomorphism between the space and the product space of the spaces of all positive real numbers. It is proved in [BKS] that for any two elements of the enhanced Teichmüller space, there uniquely exists an earthquake map which maps one to the other. This proof is based on [M].

In this paper, we calculate some examples of an earthquake map in the enhanced Teichmüller space of a once-punctured torus. First, we show how we construct an earthquake map for surfaces which have a lamination intersecting a fundamental domain of a surface finite times. Next, We fix a base point and move it by earthquake maps of different laminations and search for the domains where it moves when we fix laminations. The domains are different from each other when the laminations are different from each other, for there uniquely exists an earthquake map from the base point to any point. Therefore, we give a decomposition of the enhanced Teichmüller space of a once-punctured torus. It will be useful to grasp the motions of earthquake maps in the enhanced Teichmüller space and understand the new developments of the Teichmüller theory and Anti de Sitter geometry related to the earthquake theorem.

2 Acknowledgements

I would like to thank my supervisor, Takuya Sakasai for much support for this paper. I would also like to thank Tsukasa Ishibashi for valuable advice. I would like to thank everyone in the laboratory I belong to for much useful advice. I would like to thank the members of Room 418 for correcting my English.

3 Preliminaries

Let $\mathbf{H}$ be the Poincaré half-plane model and $\mathbf{S}^{\mathbf{1}}_{\infty}$ be the boundary of $\mathbf{H}$.

Definition 1 ([T3] 2.2 Definition.) A closed subset of $\mathbf{H}$ which is the disjoint union of geodesics is called a *lamination*. The empty set is also a lamination. A geodesic of a lamination is called a *leaf*. A connected component of the complement of a lamination is called a *gap*. Both a leaf and a gap are called a *stratum*.

Since it is possible that a lamination is the union of uncountable geodesics, there can be several decompositions of a lamination into leaves. Throughout this paper, let a lamination have the information of a decomposition.

Definition 2 ([T3] 2.3 Definition.) Let λ be a lamination of $\mathbf{H}$. A map $E : \mathbf{H} \to \mathbf{H}$ is called an (λ-*left*) *earthquake map* if E meets the following conditions:

1. The map E is bijective and maps each stratum of λ to a stratum of another lamination by a restriction of an element of $\mathrm{PSL}(2, \mathbf{R})$. We call this restriction a *stratum map* for each stratum.

2. The map E is continuously extended to an orientation-preserving homeomorphism of $\mathbf{S}^{\mathbf{1}}_{\infty}$ by continuously extending each stratum map to the intersection of $\mathbf{S}^{\mathbf{1}}_{\infty}$ and the boundary of the stratum.

3. For two different strata A and B, let $E_A : \mathbf{H} \to \mathbf{H}$ be the element of $\mathrm{PSL}(2, \mathbf{R})$ such that the restriction of E_A to A is equal to the restriction of E to A. E_B is defined similarly. Then, $cmp(A, B) := E_A^{-1} \circ E_B : \mathbf{H} \to \mathbf{H}$ is a hyperbolic element of $\mathrm{PSL}(2, \mathbf{R})$ whose axis weakly separates A and B (i.e. any curve which joins a point of A to a point of B must cross the axis) and which moves B to the left in the view of A. The map $cmp(A, B)$ is called a *comparison isometry*.

We can assure that if we change A and B in the third condition, we get the same condition.

Thurston gave us the complete proof of the earthquake theorem for $\mathbf{H}$, however, he did not give the proof of the earthquake theorem for surfaces explicitly. We introduce Thurston's proof for $\mathbf{H}$ and give the proof for surfaces explicitly.

Theorem 3 ([T3] 3.1.Theorem) *For any orientation-preserving homeomorphism f of $\mathbf{S}^{\mathbf{1}}_{\infty}$, there uniquely exists an earthquake map E which is extended to f on $\mathbf{S}^{\mathbf{1}}_{\infty}$.*

Proof. Here, we only prove an existence of an earthquake for the proof of the next theorem. We take an orientation-preserving homeomorphism $f : \mathbf{S}^{\mathbf{1}}_{\infty} \to \mathbf{S}^{\mathbf{1}}_{\infty}$ and construct the earthquake map that extends to f on $\mathbf{S}^{\mathbf{1}}_{\infty}$. We fix a point $x_0 \in \mathbf{H}$.

We define

$$C := \{(x, \gamma \circ f) |\ x \in \mathbf{H},\ \gamma \in \mathrm{PSL}_2\mathbf{R},\ \gamma(x_0) = x)\}$$

and let $p : C \twoheadrightarrow \mathbf{H}$ be the projection. We endow C with the compact-open topology. For $(x, h) \in C$, $h : \mathbf{S}^{\mathbf{1}}_{\infty} \to \mathbf{S}^{\mathbf{1}}_{\infty}$ is called *extreme left* if h has at least one fixed point and $h(\theta) \geq \theta$ for any $\theta \in [0, 2\pi)$. Here, we take a fixed point w_0 of h and θ is the center angle of x from w_0 to a point of $\mathbf{S}^{\mathbf{1}}_{\infty}$. We define

$$XL := \{(x, g) \in C |\ g \text{ is extreme left}\}$$

and prove XL is homeomorphic to $\mathbf{H}$ by p. We take $(x, g) \in C$ and $y_0 \in \mathbf{S}^{\mathbf{1}}_{\infty}$. Let θ be the center angle of x from y_0 to a point of $\mathbf{S}^{\mathbf{1}}_{\infty}$. Since $\mathbf{S}^{\mathbf{1}}_{\infty}$ is compact, there is a minimum value T of the function $h(\theta) - \theta$. The function $h(\theta) + T$ is the unique extreme left homeomorphism. XL is proven to be homeomorphic to $\mathbf{H}$ by p.

For $(x, g) \in XL$, let $\mathrm{Fix}(g)$ be the set of all fixed points of g and $H(g)$ be the convex hull of $\mathrm{Fix}(g)$ in $\mathbf{H} \cup \mathbf{S}^{\mathbf{1}}_{\infty}$. We show that for any two distinct extreme left homeomorphisms g_1 and g_2,

$$(H(g_1) \cap H(g_2)) \cap \mathbf{H} = \emptyset. \tag{1}$$

Suppose that there are two geodesics l_1 and l_2 such that $l_i \in H(g_i)$ $(i = 1, 2)$ and $l_1 \cap l_2 \neq \emptyset$. Let x_1 and x_2 be the endpoints of l_1 and y_1 and y_2 be the endpoints of l_2. The movements of $x_1,\ x_2,\ g_1(y_1),\ g_1(y_2)$ on $\mathbf{S}^{\mathbf{1}}_{\infty}$ by $g_2 \circ g_1^{-1}$ are counterclockwise, counterclockwise, clockwise, clockwise, respectively, for both of g_1 and g_2 are extreme left. Therefore, $g_2 \circ g_1^{-1}$ has four fixed points by the intermediate value theorem and

$$g_2 \circ g_1^{-1} = \mathrm{id}$$

is proven. We have shown (1).

We induce the topology by Hausdorff metric in the set of all closed subsets in $\mathbf{H} \cup \mathbf{S}^{\mathbf{1}}_{\infty}$. Then, H is continuous.

XL is homeomorphic to $\mathbf{H}$, and the compactification $\overline{XL}$ of XL is considered to be $\mathbf{H} \cup \mathbf{S}^{\mathbf{1}}_{\infty}$. We extend H to $\overline{H}$ by assigning $x \in \mathbf{S}^{\mathbf{1}}_{\infty}$ to $\overline{H}(x) := \{x\} \subset \mathbf{H} \cup \mathbf{S}^{\mathbf{1}}_{\infty}$. Note that the element $(x', \gamma \circ f)$ of XL which is near a point b in the boundary of XL have all fixed points near b in $\mathbf{S}^{\mathbf{1}}_{\infty}$. It shows that $\overline{H}$ is continuous.

We will show that for any $x \in \mathbf{H}$, there is an element $(x', \gamma \circ f)$ in XL such that $x \in H(\gamma \circ f)$. We define

$$h : \overline{XL} \to \mathbf{H} \cup \mathbf{S}^{\mathbf{1}}_{\infty},\ g \mapsto \text{the center of gravity of } H(g).$$

We take a positive real number ϵ and let

$$\beta : \overline{XL} \to \mathbf{R}_{\geq \mathbf{0}}$$

be the bump function with compact support in ϵ-neighborhood of x_0 and the integral value 1. The convolution $\beta \star h$ is continuous and the restriction of $\beta \star h$ to $\mathbf{S}^{\mathbf{1}}_{\infty}$ is the identity. Therefore, $\beta \star h$ is surjective. Let $\beta = \beta_i,\ \epsilon = \epsilon_i$ and consider the function sequence $\beta_i \star h$ such that $\epsilon_i \to 0$. Let g be the limit of the function sequence g_i such that $\beta_i \star h(g_i) = x$. Since $\overline{XL}$ is compact, there exists a converging subsequence of g_i . It is shown that $x \in H(g)$.

From the above arguments, for any $x \in \mathbf{H}$, there uniquely exists an extreme left homeomorphism g such that $x \in H(g)$. Therefore, we get the lamination which is the union of all the boundaries of $H(g)$ $(g \in XL)$. The definition of $E : \mathbf{H} \to \mathbf{H}$ is as follows: for any stratum A, we take $g_A \in XL$ and $\gamma_A \in \mathrm{PSL}_2\mathbf{R}$ such that $A \subset H(g_A)$ and $g_A = \gamma_A^{-1} \circ f$. Define the restriction of E to A as γ_A. When we consider the extension E to $\mathbf{S}^{\mathbf{1}}_{\infty}$, note that for any point x in the intersection of $\mathbf{S}^{\mathbf{1}}_{\infty}$ and the closure of A, $\gamma_A^{-1} \circ f(x) = x$ and $E(x) = \gamma_A(x) = f(x)$. All we have to prove is that the comparison isometry of E meets the condition of an earthquake map. We take two distinct strata A and B, and consider the comparison isometry

$$cmp(A, B) = \gamma_A^{-1} \circ \gamma_B = g_A \circ g_B^{-1}.$$

Let g_A and $g_B \in XL$ satisfying $A \in H(g_A)$ and $B \in H(g_B)$. Let ∂A and ∂B be the boundaries of A and B, respectively. $H(g_A)$ and $H(g_B)$ are closed and have no intersection. There are $a_1,\ a_2,\ b_1$ and $b_2 \in \mathbf{S}^{\mathbf{1}}_{\infty}$ such that a_1 and a_2 are the nearest points to the endpoints of the interval which is the connected component of $\mathbf{S}^{\mathbf{1}}_{\infty} - \partial B$ and includes ∂A and that b_1 and b_2 are the nearest points to the endpoints of the interval which is a connected component of $\mathbf{S}^{\mathbf{1}}_{\infty} - \partial A$ and includes ∂B. The movements of $g_B(a_1),\ g_B(a_2),\ g_B(b_1) = b_1$ and $g_B(b_2) = b_2$ by $cmp(A, B)$ are clockwise, clockwise, counterclockwise, counterclockwise, respectively. There are at least two fixed points of $cmp(A, B)$ between ∂A and ∂B by the intermediate value theorem. Therefore, the map $cmp(A, B)$ is a hyperbolic element which has an axis between A and B and moves left. We have proved that E is an earthquake map. ■

Theorem 4 ([T3] 5.4.Corollary) *Let M and N be complete hyperbolic surfaces which are allowed to be of infinite type. For an orientation-preserving homeomorphism $f : M \to N$ whose lift $\widetilde{f} : \mathbf{H} \to \mathbf{H}$ can be extended to an orientation-preserving homeomorphism $\bar{f}$ of $\mathbf{S}^{\mathbf{1}}_{\infty}$, there uniquely exists an earthquake map $\widetilde{E}$ which is continuously extended to $\bar{f}$ on $\mathbf{S}^{\mathbf{1}}_{\infty}$. Furthermore, the lamination λ of $\widetilde{E}$ is projected on a lamination of M and $\widetilde{E}(\lambda)$ is projected on a lamination* *of N. The map $\widetilde{E}$ induces a map $E : M \to N$ such that $E \circ \pi_M = \pi_N \circ \widetilde{E}$. We also call E an (λ-left) earthquake map.*

Proof. Let Γ and Γ' be the Fuchsian groups of M and N, respectively. All we have to prove is that the lamination λ on $\mathbf{H}$ is projected on the lamination of M. Suppose that it is not projected on the lamination of M, that is to say, there are two leaves of the lamination on $\mathbf{H}$ which are projected and have a transitive intersection on M. Then, as we constructed a left earthquake as above, there are $\gamma \in \Gamma$, g and $g' \in XL$ such that

$$\gamma(H(g)) \cap \gamma(H(g')) \neq \emptyset \ \text{ and } \ \gamma(H(g)) \neq H(g').$$

We fix these γ, g and g'. In other words, there are $y_1,\ y_2 \in \gamma(\mathrm{Fix}(g)), x'_1,\ x'_2 \in \gamma(\mathrm{Fix}(g))$ such that the geodesic joining y_1 to y_2 and the geodesic joining x'_1 to x'_2 have a transitive intersection. Here, $\gamma(\mathrm{Fix}(g)) =\mathrm{Fix}(\gamma g \gamma^{-1})$ and we can represent $g = \gamma_{XL} \circ \widetilde{f},\ \gamma_{XL} \in \mathrm{PSL}_2\mathbf{R}$. By the symmetry of $\widetilde{f}$ of the Fuchsian groups, there is $\gamma' \in \Gamma'$ such that $\widetilde{f} \circ \gamma^{-1} \circ \widetilde{f}^{-1} = \gamma'$. Therefore, we calculate

$$\gamma \circ g \circ \gamma^{-1} = \gamma \circ \gamma_{XL} \circ \gamma' \circ \widetilde{f}$$

and understand $\gamma \circ g \circ \gamma^{-1} \in C$. We examine whether $\gamma \circ g \circ \gamma^{-1}$ is extreme left. The map $\gamma \circ g \circ \gamma^{-1}$ has $y_1,\ y_2$ as fixed points. Since $g(x) \geq x$ ($x \in \mathbf{S}^{\mathbf{1}}_{\infty}$) and γ and γ^{-1} are orientation-preserving, $\gamma \circ g \circ \gamma^{-1}(x) \geq x$ ($x \in \mathbf{S}^{\mathbf{1}}_{\infty}$), that is to say, $\gamma \circ g \circ \gamma^{-1}$ is extreme left. By $\gamma(H(g)) = H(\gamma \circ g \circ \gamma^{-1})$, the convex hulls of these extreme left homeomorphisms $\gamma \circ g \circ \gamma^{-1}$ and g' have an intersection. It is a contradiction. ■

Next, let S be an oriented topological surface of finite type of genus g with s punctures. Suppose that $s > 0$ and the Euler number $2 - 2g - n < 0$.

Definition 5 An element C of the fundamental group $\pi_1 S$ of S is called *peripheral* if C is homotopic to a puncture. C is called *essential* if C is neither trivial nor peripheral.

Definition 6 Let X be the set of all representations $\rho : \pi_1 S \to \mathrm{PSL}_2\mathbf{R}$ meeting the conditions (i) ρ is faithful, (ii) the image of ρ is a discrete subgroup of $\mathrm{PSL}_2\mathbf{R}$ and (iii) ρ maps every peripheral element of $\pi_1 S$ to a parabolic element and every essential one to a hyperbolic one. We get a quotient space of X by identifying elements ρ and ρ' of X if there is an element γ of $\mathrm{PSL}_2\mathbf{R}$ such that $\rho' = \gamma \circ \rho \circ \gamma^{-1}$. This quotient space is called the *Teichmüller space* $\mathcal{T}(S)$ of S.

Definition 7 [FG1] Let X' be the set of all representations $\rho : \pi_1 S \to \mathrm{PSL}_2\mathbf{R}$ meeting the conditions (i) ρ is faithful, (ii) the image of ρ is a discrete subgroup of $\mathrm{PSL}_2\mathbf{R}$ and (iii) ρ maps every peripheral element of $\pi_1 S$ to a parabolic or hyperbolic element and every essential one to a hyperbolic one. We get a quotient space of X' by identifying elements ρ and ρ' of X' if there is an element γ of $\mathrm{PSL}_2\mathbf{R}$ such that $\rho' = \gamma \circ \rho \circ \gamma^{-1}$. This quotient space is called the *enhanced Teichmüller space* (or the *holed Teichmüller space*) $\hat{\mathcal{T}}(S)$ of S.

Definition 8 We define the correspondence between 'a pair of a square and its diagonal line in $\mathbf{H} \cup \mathbf{S}^{\mathbf{1}}_{\infty}$ with every vertex in $\mathbf{S}^{\mathbf{1}}_{\infty}$' and 'a positive real number'. Given a square and a diagonal line of the square, let D be the diagonal line. Let x be one endpoint of D and z be the other endpoint. Let y and w be the other two vertices of a square so that the order of x, y, z, and w is clockwise. We define

$$X_D := \frac{w-x}{w-z} \cdot \frac{z-y}{y-x} = -[x : y : z : w],$$

where $[x : y : z : w]$ is the cross ratio. Since the cross ratio is invariant under Möbius transformations, X_D does not change if we choose the other endpoint as x.

Definition 9 An *ideal arc* of S is an arc embedded to S with both endpoints in punctures. An *ideal triangulation* of S is a collection of ideal arcs whose complement in S is a collection of triangles whose edges are ideal arcs.

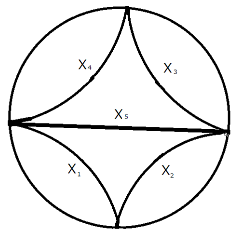
Theorem 10 (Thurston[T2], Fock-Goncharov[FG1] Section 4.1, see also Penner[P] Chapter 2 Theorem 4.4) *We fix an ideal triangulation of S. It has $6g-6+3s$ ideal arcs $v_1,\ v_2, \ldots,\ v_{6g-6+3s}$. For $\rho \in \hat{\mathcal{T}}(S)$, we consider the projection from $\mathbf{H}$ to S. For each pair of a square and its diagonal line v_j consisting of ideal arcs, we take a lift of the pair in $\mathbf{H}$ and compute a positive real number X_{v_j}. Then, we get a real analytic homeomorphism*

$$\hat{\mathcal{T}}(S) \to (\mathbf{R}_{>0})^{6g-6+3s},\ \rho \mapsto (X_{v_1},\ X_{v_2},\ \ldots,\ X_{v_{6g-6+3s}}).$$

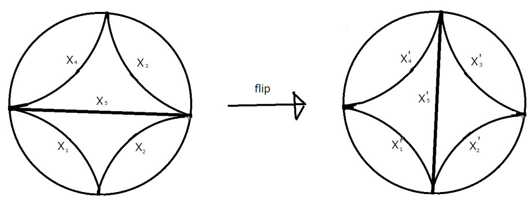
Penner calls $(X_{v_1},\ X_{v_2},\ \ldots,\ X_{v_{6g-6+3s}})$ the shear coordinates. Moreover, an element of $\hat{\mathcal{T}}(S)$ is in $\mathcal{T}(S)$ if and only if for each puncture, the product of all the shear coordinates of the edges which have at least one endpoint in the puncture is equal to 1. Here, if an edge has both endpoints in the puncture, we square the shear coordinate of the edge.

Note that since the cross ratio is invariant under Möbius transformations, X_{v_j} does not depend on the choice of a lift.

Theorem 11 ([FG1] Section4.1 and [P] Chapter 2 Theorem 4.7.) *We take an ideal triangulation of S. We choose the edges $v_1,\ v_2,\ v_3,\ v_4$, and v_5 on $\mathbf{H}$ which are the lifts of the edges of the ideal triangulation and construct two ideal triangles as in the following picture. Let $X_1,\ X_2,\ X_3,\ X_4$, and X_5 be the shear coordinates of the edges $v_1,\ v_2,\ v_3,\ v_4$, and v_5, respectively.*



We make a new triangulation and new shear coordinates on S by the flip of v_5 as in the following picture. Let the new edges be $v_1',\ v_2',\ v_3',\ v_4'$ and v_5' and its shear coordinates be $X_1',\ X_2',\ X_3',\ X_4'$ and X_5', respectively.



Then, we get the relation between these old and new shear coordinates. If $v_1,\ v_2,\ v_3$ and v_4 are different from each other, then

$$\begin{aligned} X_1' &= X_1(1+X_5^{-1})^{-1} && (1)\\ X_2' &= X_2(1+X_5) && (2)\\ X_3' &= X_3(1+X_5^{-1})^{-1} && (3)\\ X_4' &= X_4(1+X_5) && (4)\\ X_5' &= X_5^{-1} && (5) \end{aligned}$$

If $v_1 = v_3$, then (1) and (3) are modified to $X_1' = X_1(1+X_5^{-1})^{-2}$ and $X_3' = X_3(1+X_5^{-1})^{-2}$. If $v_2 = v_4$. If (2) and (4) are modified to $X_2' = X_2(1+X_5)^2$ and $X_4' = X_4(1+X_5)^2$. If $v_1 = v_2$, then (1) and (2) are modified to $X_1' = X_1X_5$ and $X_2' = X_2X_5$. If $v_1 = v_4$, then (1) and (4) are modified to $X_1' = X_1X_5$ and $X_4' = X_4X_5$. If $v_2 = v_3$, then (2) and (3) are modified to $X_2' = X_2X_5$ and $X_3' = X_3X_5$. If $v_3 = v_4$, then (3) and (4) are modified to $X_3' = X_3X_5$ and $X_4' = X_4X_5$.

We state the earthquake theorem for the enhanced Teichmüller space.

Theorem 12 ([BKS] Theorem 1.4.) *For any two points in the enhanced Teichmüller space $\hat{\mathcal{T}}(S)$, there uniquely exists an earthquake map from one to the other.*

4 Results

Let M and N be complete hyperbolic surfaces allowed to be of infinite type. By Theorem 4, if we consider all the earthquake maps from $\mathbf{H}$ to $\mathbf{H}$ which are equivariant with respect to the actions of the Fuchsian groups of M and N, we consider all the earthquake maps from M to N. For the later calculation, we construct earthquake maps from $\mathbf{H}$ to $\mathbf{H}$, equivariant with respect to the actions of the Fuchsian groups M and N, and with a lamination whose lift λ on $\mathbf{H}$ crosses a fundamental domain of M finite times. Then, a leaf of the lamination must be adjacent to two different gaps. Let Γ be the Fuchsian group of M. Let $l_1,\ l_2,\dots,\ l_n$ be leaves of λ such that $\{\gamma(x)\mid\gamma\in\Gamma,\ x\in\cup_{j=1}^{n}l_j\}=\lambda$, no element of Γ maps l_{j_1} to l_{j_2} $(j_1\neq j_2)$ and $l_1,\ l_2,\dots,\ l_n$ intersect the same fundamental domain F_0. We suppose that the boundary of F_0 crosses the lamination transversally. Let $g_1,\ g_2,\dots,\ g_m$ be all the gaps adjacent to $l_1,\ l_2,\dots,$ or l_n (if $j_1\neq j_2$, then $g_{j_1}\neq g_{j_2}$). The strata $l_1,\ l_2,\dots,\ l_n$ and $g_1,\ g_2,\dots,\ g_m$ cover F_0. First, we choose one gap g_{first} of $g_1,\ g_2,\dots,\ g_m$ and give the identity to g_{first}. We give each gap g_j a hyperbolic element h_j whose axis is the geodesic adjacent to g_j and weakly separating g_j and g_{first} and which moves g_j left in the view of g_{first}. Next, we define a stratum map on g_j. We define the identity as the stratum map of g_{first}. When $j\neq first$, we take a path γ of finite length and not tangent to the leaves $l_1,\ l_2,\dots,$ or l_n joining g_{first} to g_j in F_0. Let $g_{first}=:g_{j_0},\ g_{j_1},\dots,\ g_{j_{m'}},\ g_{j_{m'+1}}:=g_j$ be the gaps which γ passes through. We may impose the condition that γ should pass through l_{j_i} when γ gets in each gap g_{j_i}. Then, the order of the gaps which γ passes through is unique. We define a stratum map

$$E_{g_j}:=h_{j_1}\circ h_{j_2}\circ\cdots\circ h_{j_{m'+1}}$$

for each gap g_j. We define the same stratum map of each l_j as of g_j. We have defined an earthquake map on the strata which have an intersection with F_0. Finally, we recursively define an earthquake map on $\mathbf{H}$. If there is a leaf l between a gap Δ_α which a stratum map E_α has been given to and a gap Δ_β which a stratum map has not been given to, we take the element γ_α of Γ which maps some g_α to Δ_α and some g_β to Δ_β $(g_\alpha,\ g_\beta\in\{g_1,\ g_2,\dots,\ g_m\})$. Then, we define a stratum map

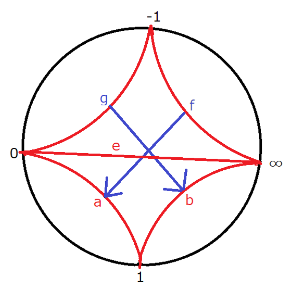
$$E_\beta:=E_\alpha\circ\gamma_\alpha\circ E_{g_\alpha}^{-1}\circ E_{g_\beta}\circ\gamma_\alpha^{-1}$$

as the stratum map of g_β. Repeating the above construction of a stratum map, we construct a left earthquake on $\mathbf{H}$.

In the following, we consider the case where S is a once-punctured torus. First, we fix the base point ρ_0 of $\hat{\mathcal{T}}(S)$ as follows: the fundamental domain F_0 of the base point $\rho_0(\pi_1 S)$ is the square whose vertices are $-1,\ 0,\ 1,\ \infty$ and whose edges are geodesics in $\mathbf{H}$. Let $\rho_0(\pi_1 S)$ be the free group of rank two generated by automorphisms f and g of $\mathbf{H}$, where

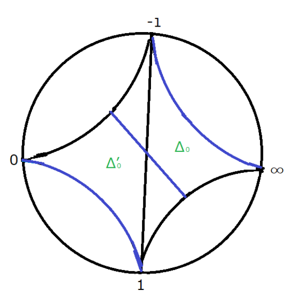
$$f(z) := \frac{z+1}{z+2},\ g(z) := \frac{2z+1}{z+1}.$$

Second, we fix an ideal triangulation of S as follows: we give the fundamental domain F_0 an ideal triangulation whose triangles are two geodesic triangles, one of which has vertices in $-1,\ 0,\ \infty$ and the other of which has vertices in $0,\ 1,\ \infty$. It also gives S an ideal triangulation. Moreover, we consider the shear coordinates on $\hat{\mathcal{T}}(S)$. Let a be the edge joining 0 and 1. Let b be the edge joining 1 and ∞. Let e be the edge joining 0 and ∞. The coordinates (X_a, X_b, X_e) give the shear coordinates on $\hat{\mathcal{T}}(S)$. We calculate that the shear coordinates of ρ_0 are $(1, 1, 1)$. We consider the changes of the shear coordinates from the base point ρ_0 by left earthquakes.



A triangulation of F_0

Example 13 (i) Let a lamination λ_1 on F_0 be a collection of three geodesics whose endpoints are 0 and 1, -1 and ∞, and $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$ (it is a simple closed geodesic on the surface of ρ_0). Furthermore, we define a lamination λ_1 on $\mathbf{H}$ by taking all images of the lamination on F_0 by $\rho(\pi_1 S)$.



We define

$$\begin{aligned} E_1(z) &:= \frac{2az-a+\sqrt{5a^2-4}}{(-a+\sqrt{5a^2-4})z+3a-\sqrt{5a^2-4}}, \\ E_2(z) &:= \frac{\lambda z}{2(\lambda-1)z-\lambda+2}. \end{aligned}$$

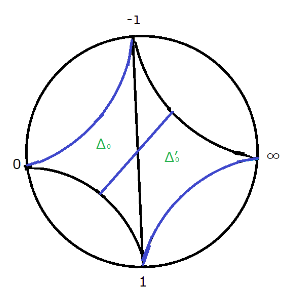
Since E_1 and E_2 must move left, we need the condition that $1 \leq a$ and $1 \leq \lambda < 2$. E_1 is a hyperbolic element which has $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$ as fixed points. E_2 is a hyperbolic element which has 0 and 1 as fixed points and maps $1/2$ to $\lambda/2$. Let Δ_0 and Δ_0' be the gaps as the above picture. We define $\Delta_f := f(\Delta_0)$, $\Delta_{gf} := g \circ f(\Delta_0)$ and so forth. Now, we define an earthquake map E by $E|_{\Delta_0} := \mathrm{id}$, $E|_{\Delta_0'} := E_1$, $E|_{\Delta_f} := E_1 \circ E_2$, $E|_{\Delta_f'} := E_1 \circ E_2 \circ f \circ E_1 \circ f^{-1}$, $E|_{\Delta_g} := \mathrm{id}$, $E|_{\Delta_g'} := E_1$ and so forth. This construction conforms to the above construction in this section. The points $0,\ 1/2,\ 1,\ 2,\ \infty$ and -1 are mapped to $(a-\sqrt{-4+5a^2})/(-3a+\sqrt{-4+5a^2})$, $(a^2(8-5\lambda)+a\sqrt{-4+5a^2}(-4+\lambda)+2(-2+\lambda))/(4-3a\sqrt{-4+5a^2}(-2+\lambda)-2\lambda+a^2(-14+5\lambda))$, $(a+\sqrt{-4+5a^2})/(2a)$, 2, ∞ and -1 by E, respectively. We calculate that these shear coordinates are

$$\left(\frac{4a^2\lambda}{(3a-\sqrt{5a^2-4})^2(2-\lambda)},\ \frac{(3a-\sqrt{5a^2-4})^2}{4},\ \frac{1}{a^2}\right),\quad (1 \leq a,\ 1 \leq \lambda < 2).$$

The domain where ρ_0 moves by the earthquake of λ_0 is

$$\left\{(X_a, X_e, X_e)\middle| \frac{4}{(3-\sqrt{5-4X_e})^2} \leq X_a,\ X_b = \frac{(3-\sqrt{5-4X_e})^2}{4X_e}, 0 < X_e \leq 1\right\}.$$

(ii) Let a lamination λ_2 on F_0 be a collection of three geodesics whose endpoints are 0 and -1, 1 and ∞), and $(-1+\sqrt{5})/2$ and $(-1-\sqrt{5})/2$ (it is a simple closed geodesic on the surface of ρ_0). Furthermore, we define a lamination λ_1 on $\mathbf{H}$ by taking all images of the lamination on F_0 by $\rho(\pi_1 S)$.



We define

$$\begin{aligned} E_1(z) &:= \frac{2az + a - \sqrt{5a^2-4}}{(a-\sqrt{5a^2-4})z + 3a - \sqrt{5a^2-4}}, \\ E_2(z) &:= \lambda z - \lambda + 1. \end{aligned}$$

Since E_1 and E_2 must move left, we need the condition that $1 \le a$ and $1 \le \lambda$. E_1 is a hyperbolic element which has $(-1+\sqrt{5})/2$ and $(-1-\sqrt{5})/2$ as fixed points. E_2 is a hyperbolic element which has 1 and ∞ as fixed points. Let Δ_0 and Δ_0' be the gaps as the above picture. We define $\Delta_f := f(\Delta_0)$ and so forth. Now, we define an earthquake map E by $E|_{\Delta_0} := \mathrm{id}$, $E|_{\Delta_0'} := E_1$, $E|_{\Delta_f} := \mathrm{id}$, $E|_{\Delta_f'} := E_1$, $E|_{\Delta_g} := E_1 \circ E_2$, $E|_{\Delta_g'} := E_1 \circ E_2 \circ g \circ E_1 \circ g^{-1}$ and so forth. The points $0,\ 1/2,\ 1,\ 2,\ \infty$ and -1 are mapped to $0,\ 1/2,\ (-3a+\sqrt{-4+5a^2})/(2(-2a+\sqrt{-4+5a^2}))$, $(2+3a\sqrt{-4+5a^2}-a^2(7+2\lambda))/(4+a\sqrt{-4+5a^2}(5+\lambda)-a^2(11+\lambda))$, $-2a/(-a+\sqrt{-4+5a^2})$ and -1 by E, respectively. We calculate that these shear coordinates are

$$\left(\frac{1}{a^2},\ \frac{4a^2\lambda}{(3a-\sqrt{5a^2-4})^2},\ \frac{(3a-\sqrt{5a^2-4})^2}{4}\right), \quad (1 \le a,\ 1 \le \lambda).$$

The domain where ρ_0 moves by the earthquake of λ_0 is

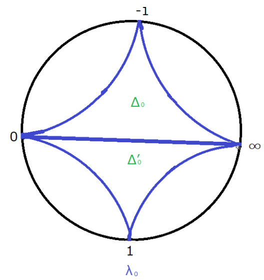
$$\left\{(X_a, X_e, X_e) \middle| 0 < X_a \le 1,\ \frac{4}{(3-\sqrt{5-4X_a})^2} \le X_b,\ X_e = \frac{(3-\sqrt{5-4X_a})^2}{4X_a}\right\}.$$

We calculate the product of the shear coordinates of Example 13 (i) and (ii)

$$\text{(i)}\ \left(\frac{\lambda}{2-\lambda}\right)^2,\ (1 \le \lambda < 2) \qquad \text{(ii)}\ \lambda^2,\ (1 \le \lambda).$$

As for Example 13, an earthquake map along a geodesic which flows in a puncture puts the base point out of the Teichmüller space $\mathcal{T}(S)$ and an earthquake map along a geodesic which is closed deforms the base point within the Teichmüller space $\mathcal{T}(S)$. There is no quasiconformal mapping of S corresponding to the former earthquake map.

Example 14 Let a lamination λ_0 on F_0 be a collection of five geodesics whose endpoints are 0 and 1, -1 and ∞, 1 and ∞, 0 and -1, and ∞ and 0. Furthermore, we define a lamination λ_0 on $\mathbf{H}$ by taking all images of the lamination on F_0 by $\rho(\pi_1 S)$.



We define

$$E_1(z) := \lambda_1 z,$$

$$E_2(z) := \frac{\lambda_2 z}{2(\lambda_2 - 1)z - \lambda_2 + 2}\ ,$$

$$E_3(z) := (2\lambda_3 - 1)z + 2(1 - \lambda_3).$$

Since $E_1,\ E_2,\ E_3$ must move left, we need the condition that $1 \le \lambda_1,\ 1 \le \lambda_2 < 2$ and $1 \le \lambda_3$. E_1 is a hyperbolic element which has 0 and ∞ as fixed points and maps 1 to λ_1. E_2 is a hyperbolic element which has 0 and 1 as fixed points and maps 1/2 to $\lambda_2/2$. E_3 is a hyperbolic element which has 1 and ∞ as fixed points and maps 2 to $2\lambda_3$. Let Δ_0 and Δ'_0 be the triangle whose vertices are -1, 0 and ∞ and the triangle whose vertices are 0, 1 and ∞, respectively. We define $\Delta_f := f(\Delta_0)$, $\Delta_{gf} := g \circ f(\Delta_0)$ and so forth. Now, we define an earthquake map E by $E|_{\Delta_0} := \mathrm{id}$, $E|_{\Delta'_0} := E_1$, $E|_{\Delta_f} := E_1 \circ E_2$, $E|_{\Delta'_f} := E_1 \circ E_2 \circ f \circ E_1 \circ f^{-1}$, $E|_{\Delta_g} := E_1 \circ E_3$, $E|_{\Delta'_g} := E_1 \circ E_3 \circ g \circ E_1 \circ g^{-1}$ and so forth. The points $0,\ 1/2,\ 1,\ 2,\ \infty$ and -1 are mapped to $0, \lambda_1\lambda_2/2,\ \lambda_1,\ 2\lambda_1\lambda_3,\ \infty$ and -1 by E, respectively. We calculate that these shear coordinates are

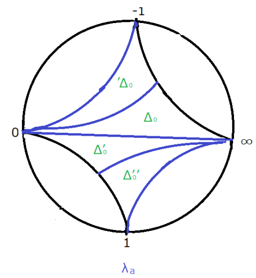
$$\left(\frac{\lambda_2}{2-\lambda_2},\ 2\lambda_3 - 1,\ \lambda_1\right),\ (1 \le \lambda_1,\ 1 \le \lambda_2 < 2,\ 1 \le \lambda_3).$$

The domain where ρ_0 moves by the earthquake of λ_0 is

$$\{(X_a, X_b, X_e) \mid X_a \ge 1,\ X_b \ge 1,\ X_e \ge 1\}.$$

We calculate the earthquakes of different laminations which arise from λ_0 by flips.

Example 15 (i) Let λ_a be the lamination by a flip on a over λ_0. In other words, on F_0, λ_a has five geodesics whose endpoints are 0 and -1, 1 and ∞, 0 and -2, 0 and ∞, and 1/2 and ∞. Furthermore, we define a lamination on $\mathbf{H}$ by taking all images of the lamination on F_0 by $\rho(\pi_1 S)$.



We define

$$\begin{aligned}
E_1(z) &:= \lambda_1 z, \\
E_2(z) &:= (2\lambda_2 z - 1)z - \lambda_2 + 1, \\
E_3(z) &:= (2\lambda_3 - 1)z - 2\lambda_3 + 2.
\end{aligned}$$

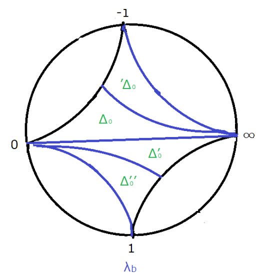
Since $E_1,\ E_2,\ E_3$ must move left, we need the condition that $1 \leq \lambda_1,\ 1 \leq \lambda_2$ and $1 \leq \lambda_3$. E_1 is a hyperbolic element which has $0,\ \infty$ as fixed points and maps 1 to λ_1. E_2 is a hyperbolic element which has $1/2,\ \infty$ as fixed points and maps 1 to λ_2. E_3 is a hyperbolic element which has $1,\ \infty$ as fixed points and maps 2 to $2\lambda_3$. Let $'\Delta_0,\ \Delta_0,\ \Delta_0',\ \Delta_0''$ be the gaps as the above picture. Let $\Delta_f := f(\Delta_0)$ and so forth. We define an earthquake map E by $E|_{'\Delta_0} := f^{-1} \circ E_2^{-1} \circ f,\ E|_{\Delta_0} := \mathrm{id},\ E|_{\Delta_0'} := E_1,\ E|_{\Delta_0''} := E_1 \circ E_2,\ E|_{'\Delta_g} := E_1 \circ E_2 \circ E_3,\ E|_{\Delta_0} := E_1 \circ E_2 \circ E_3 \circ g \circ f^{-1} \circ E_2 \circ f \circ g^{-1}$ and so forth. The points $0,\ 1/2,\ 1,\ 2,\ \infty,\ -1$ are mapped to $0,\ \lambda_1/2,\ \lambda_1\lambda_2,\ \lambda_1(-1+\lambda_2(4-8\lambda_3)+2\lambda_3+\lambda_2^2(-3+8\lambda_3))/\lambda_2,\ \infty,\ -1/\lambda_2$. We calculate that the shear coordinates are

$$\left(\frac{1}{2\lambda_2 - 1},\ \frac{(2\lambda_2-1)^2(2\lambda_3-1)}{\lambda_2^2},\ \lambda_1\lambda_2^2\right),\quad (1 \leq \lambda_1,\ 1 \leq \lambda_2,\ 1 \leq \lambda_3).$$

The domain where ρ_0 moves by the earthquake of λ_a is

$$\left\{(X_a,\ X_b,\ X_e) \mid 0 < X_a \leq 1,\ \frac{4}{(X_a+1)^2} \leq X_b,\ \frac{(X_a+1)^2}{4X_a^2} \leq X_e\right\}.$$

(ii) Let λ_b be the lamination by a flip on b over λ_0. In other words, on F_0, λ_b has five geodesics whose endpoints are -1 and ∞, 0 and 1, $-1/2$ and ∞, 0 and -1, and -1 and 0. Furthermore, we define a lamination on $\mathbf{H}$ by taking all images of the lamination on F_0 by $\rho(\pi_1 S)$.



We define

$$E_1(z) := \lambda_1 z,$$

$$E_2(z) := \frac{\lambda_2 z}{(2\lambda_2 - 1)z - \lambda_2 + 2}$$

$$E_3(z) := \frac{\lambda_3 z}{(\lambda_3 - 1)z - \lambda_3 + 2}.$$

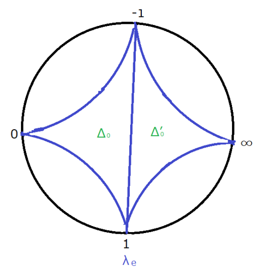
Since $E_1,\ E_2,\ E_3$ must move left, we need the condition that $1 \leq \lambda_1,\ 1 \leq \lambda_2$ and $1 \leq \lambda_3$. E_1 is a hyperbolic element which has $0,\ \infty$ as fixed points and maps 1 to λ_1. E_2 is a hyperbolic element which has $1/2,\ \infty$ as fixed points and maps 1 to λ_2. E_3 is a hyperbolic element which has $1,\ \infty$ as fixed points and maps 2 to $2\lambda_3$. Let $'\Delta_0,\ \Delta_0,\ \Delta_0',\ \Delta_0''$ be the gaps as the above picture. Let $\Delta_f := f(\Delta_0)$ and so forth. We define an earthquake map E by $E|_{'\Delta_0} := f^{-1} \circ E_2^{-1} \circ f,\ E|_{\Delta_0} := \mathrm{id},\ E|_{\Delta_0'} := E_1,\ E|_{\Delta_0''} := E_1 \circ E_2,\ E|_{'\Delta_g} := E_1 \circ E_2 \circ E_3,\ E|_{\Delta_0} := E_1 \circ E_2 \circ E_3 \circ g \circ f^{-1} \circ E_2 \circ f \circ g^{-1}$ and so forth. The points $0,\ 1/2,\ 1,\ 2,\ \infty,\ -1$ are mapped to $0,\ \lambda_2/2,\ \lambda_1\lambda_2,\ \lambda_1(-1+\lambda_2(4-8\lambda_3)+2\lambda_3+\lambda_2^2(-3+8\lambda_3))/\lambda_2,\ \infty,\ -1/\lambda_2$. We calculate the shear coordinates are

$$\left(\frac{\lambda_2}{(2-\lambda_2)(2-\lambda_3)^2},\ \frac{2-\lambda_3}{\lambda_3},\ \lambda_1\lambda_3^2\right),\ (1 \leq \lambda_1,\ 1 \leq \lambda_2 < 2,\ 1 \leq \lambda_3 < 2).$$

The domain where ρ_0 moves by the earthquake of λ_1 is

$$\left\{(X_a, X_b, X_e)\ \middle|\ \frac{(X_b+1)^2}{4X_b^2} \leq X_a,\ 0 < X_b \leq 1,\ \frac{4}{(1+X_b)^2} \leq X_e\right\}.$$

(iii) Let λ_e be the lamination by a flip on e over λ_0. In other words, on F_0, λ_a has five geodesics whose endpoints are 0 and 1, -1 and ∞, 1 and ∞, 0 and -1, and -1 and 1. Furthermore, we define a lamination on $\mathbf{H}$ by taking all images of the lamination on F_0 by $\rho(\pi_1 S)$.



We define

$$E_1(z) := \frac{-\lambda_1 z + 1}{z - \lambda_1},$$

$$E_2(z) := \frac{\lambda_2 z}{2(\lambda_2 - 1)z - \lambda_2 + 2},$$

$$E_3(z) := (2\lambda_3 - 1)z - 2\lambda_3 + 2.$$

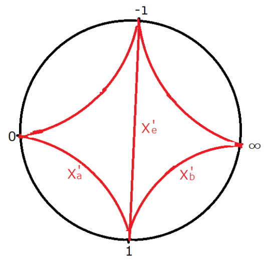
Since $E_1,\ E_2,\ E_3$ must move left, we need the condition that ($1 < \lambda_1$ or $\lambda_1 = \infty$), $1 \leq \lambda_2 < 2$ and $1 \leq \lambda_3$. E_1 is a hyperbolic element which has -1 and 1 as fixed points and maps ∞ to $-\lambda_1$. E_2 is a hyperbolic element which has 0 and 1 as fixed points and maps $1/2$ to $\lambda_2/2$. E_3 is a hyperbolic element which has 1 and ∞ as fixed points and maps 2 to $2\lambda_3$. Let Δ_0 and Δ_0' be the gaps as in the above picture. Let $\Delta_f := f(\Delta_0)$ and so forth. We define an earthquake map E by $E|_{\Delta_0} := \mathrm{id},\ E|_{\Delta_0'} := E_1,\ E|_{\Delta_f'} := E_2,\ E|_{\Delta_f} := E_2 \circ f \circ E_1^{-1} \circ f^{-1},\ E|_{\Delta_g} := E_1 \circ E_3,\ E|_{\Delta_g'} := E_1 \circ E_3 \circ g \circ E_1 \circ g^{-1}$ and so forth. The points $0,\ 1/2,\ 1,\ 2,\ \infty$ and -1 are mapped to $0,\ (\lambda_1+1)\lambda_2/(2\lambda_1+\lambda_2),\ 1,\ (2\lambda_1^2\lambda_3 - 2\lambda_1 + 1)/(\lambda_1^2 - 2\lambda_1\lambda_3 - \lambda_1 + 1),\ -\lambda_1$ and -1 by E, respectively. We calculate that the shear coordinates are

$$\left(\frac{(\lambda_1+1)^2\lambda_2}{\lambda_1^2(2-\lambda_2)},\ \frac{\lambda_1^2(2\lambda_3-1)}{(\lambda_1-1)^2},\ \frac{\lambda_1-1}{\lambda_1+1}\right),\ (1<\lambda_1 \text{ or } \lambda_1=\infty,\ 1\leq\lambda_2<2,\ 1\leq\lambda_3).$$

The domain where ρ_0 moves by the earthquake of λ_1 is

$$\left\{(X_a, X_b, X_e)\ \middle|\ \frac{4}{(X_e+1)^2} \leq X_a,\ \frac{(X_e+1)^2}{4X_e^2} \leq X_b,\ 0 < X_e \leq 1\right\}.$$

We make another calculation by Theorem 11. We take another triangulation with the edges $a',\ b',\ e'$ as in the following picture.



The points $0,\ 1/2,\ 1,\ 2,\ \infty$ and -1 are mapped to $0,\ (\lambda_1+1)\lambda_2/(2\lambda_1+\lambda_2),\ 1,\ (2\lambda_1^2\lambda_3-2\lambda_1+1)/(\lambda_1^2-2\lambda_1\lambda_3-\lambda_1+1),\ -\lambda_1$ and -1 by E, respectively We calculate another shear coordinates

$$(X'_a,\ X'_b,\ X'_e)=\left(\frac{4\lambda_2}{2-\lambda_2},\ \frac{2\lambda_3-1}{4},\ \frac{\lambda_1+1}{\lambda_1-1}\right)$$

By Theorem 10,

$$\begin{aligned}(X_a,\ X_b,\ X_e) &= (X'_a(1+X_e'^{-1})^{-2},\ X'_b(1+X'_e)^2,\ X_e'^{-1}) \\ &= \left(\frac{(\lambda_1+1)^2\lambda_2}{\lambda_1^2(2-\lambda_2)},\ \frac{\lambda_1^2(2\lambda_3-1)}{(\lambda_1-1)^2},\ \frac{\lambda_1-1}{\lambda_1+1}\right).\end{aligned}$$

We get the same result.

We calculate the products of the shear coordinates of Example 14 and Examples 15 (i), (ii) and (iii)

Example 14 $\quad \left\{\frac{\lambda_1\lambda_2(2\lambda_3-1)}{2-\lambda_2}\right\}^2,\ (1\le\lambda_1,\ 1\le\lambda_2<2,\ 1\le\lambda_3)$

Example 15 (i) $\quad \{\lambda_1(2\lambda_2-1)(2\lambda_3-1)\}^2,\ (1\le\lambda_1,\ 1\le\lambda_2,\ 1\le\lambda_3),$

(ii) $\quad \left\{\frac{\lambda_1\lambda_2\lambda_3}{(2-\lambda_2)(2-\lambda_3)}\right\}^2,\ (1\le\lambda_1,\ 1\le\lambda_2<2,\ 1\le\lambda_3<2)$

(iii) $\quad \left\{\frac{(\lambda_1+1)\lambda_2(2\lambda_3-1)}{(\lambda_1-1)(2-\lambda_2)}\right\}^2,\ (1<\lambda_1$ or $\lambda_1=\infty,\ 1\le\lambda_2<2,\ 1\le\lambda_3).$

As for Examples 14 and 15, an earthquake map along a geodesic which flows in a puncture puts the base point out of the Teichmüller space $\mathcal{T}(S)$. Note that there is a plane which Example 14 and Example 15 (i) share. The plane is the place where the earthquake maps along other two leaves than a leaf which changes by a flip moves the base point. Then, a flip of a lamination of an earthquake map changes a domain where the base point moves by an earthquake map to another domain adjacent to the original domain. The relation between Example 14 and Examples 15 (ii) and (iii) is the same.

By Theorem 12, if laminations are different, then the domains where the base point moves are different. Therefore, we decompose the enhanced Teichmüller space into domains where the base point moves by earthquake maps of fixed laminations. The dimension of each domain is equal to the number of leaves of its lamination. We have calculated some domains in Examples 14 and 15.

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