J-Stability in p-adic Dynamics

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Notations

$\overline{D}_r(a)$: a closed disk centered at a with radius $r > 0$ in a given normed field.
$D_r(a)$: an open disk centered at a with radius $r > 0$ in a given normed field.
$\mathbb{S}(r)$: a circle centered at 0 with radius $r > 0$ in a given normed field.
$\mathcal{O}_K$: $\overline{D}_1(0)$ as a subring of the non-Archimedean field K.
$\mathcal{M}_K$: $D_1(0)$ as a maximal ideal of $\mathcal{O}_K$ in the non-Archimedean field K.
$\mathbb{N}$: the set of natural numbers.
$\mathbb{Z}$: the ring of integers.
$\mathbb{Z}_{\geq 0}$: the set of non-negative integers.
$\mathbb{Q}$: the field of rational numbers.
$\mathbb{R}$: the field of real numbers.
$\mathbb{R}_{\geq 0}$: the set of non-negative real numbers.
$\mathbb{R}_{>0}$: the set of positive real numbers.
$\mathbb{C}$: the field of complex numbers.
$\mathbb{Z}_p$: the ring of p-adic integers.
$\mathbb{Q}_p$: the field of p-adic rational numbers.
$\mathbb{C}_p$: the field of p-adic complex numbers.
$\mathbb{F}_p$: the quotient field $\mathbb{Z}/p\mathbb{Z}$.
$\mathbb{P}^1(K)$: the projective line over the field K.
$d|_A$: the restriction of a metric d to a subset A.
$\overline{A}$: the topological closure of a set A.
$\#(A)$: the cardinality of a set A.
ϕ^n : the n th iterate of a self-mapping ϕ.
$\mathrm{Fix}(\phi)$: the set of fixed points of a self-mapping ϕ.
$\mathrm{Per}(\phi)$: the set of all periodic points of a self-mapping ϕ.
$\deg(\phi)$: the degree of a polynomial map ϕ.
$\lambda_\phi(a)$: the multiplier of a polynomial map ϕ at a.
$|K^\times|$: the value group of value field $(K, |\cdot|)$.
π : the canonical projection from a set A to its quotient set $A/\sim$ where $\sim$ is some equivalence relation.

Preface

This is a master thesis on non-Archimedean dynamics, which the author studied during his master program in Nagoya University. The author found J-stable families in p-adic dynamical systems and calculated the Artin-Mazur zeta functions of rational maps over $\mathbb{C}_p$, motivated by analogous theorems in complex dynamical systems. The results can be found in 3.2 and 4.3 in this thesis. The other sections in this thesis are designed to provide basics of non-Archimedean dynamical systems.

Backgrounds of the research We say the pair (X, ϕ) is a dynamical system if X is a topological space and ϕ is a continuous self-mapping of X. For a given dynamical system (X, ϕ), one of our goals is to understand the behavior of each point in X by iteration of ϕ. To explain it more precisely, we shall use the notation

$$\phi^n := \overbrace{\phi \circ \phi \circ \cdots \circ \phi}^{n \text{ times}}.$$

Then, the goal is to understand the set $\{\phi^n(x) \mid x \in X\}$ for each $x \in X$.

One well-studied dynamical system is complex dynamical system, in which we consider the iteration of rational maps over $\mathbb{C}$ on the Riemann sphere. The theory of complex dynamics was first established by P. Fatou and G. Julia in the early 20th century. In the complex dynamical system of a given rational map, the Fatou set is defined as the largest open set in the Riemann sphere where small errors remain small under the iterations of the rational map. On the other hand, the Julia set, which is defined by the compliment of the Fatou set in the Riemann sphere, is the chaotic locus of the dynamical system. That is, after many iterations, any small error becomes arbitrary big. These two notions, the Fatou set and the Julia set, are essential in complex dynamical systems.

In this thesis, we focus on non-Archimedean dynamical systems. The theory of non-Archimedean dynamics is relatively new, and mostly developed in this century. In non-Archimedean dynamical systems, we consider the projective lines over non-Archimedean fields, especially algebraically closed complete non-Archimedean fields of characteristic zero, as an analogue of the Riemann sphere, and the iterations of rational maps over the field. As the Riemann sphere has the chordal metric, the projective line also has an analogue of the chordal metric. Moreover, we will consider the Fatou set and the Julia set as we do in complex dynamical systems. However, there are some differences from complex dynamical systems. For example, unlike the Riemann sphere, the projective line might not be compact, and is a totally disconnected topological space. In particular, this implies that the Julia set on the projective line might not be compact unlike the Julia set on the Riemann sphere.

There is a natural question in complex dynamical system: are there any relations of the Julia sets or the Fatou sets if two maps are close enough? In complex dynamical systems, the following theorem is well-known. The terminology used in the following theorem can be found in section 1.

Theorem 1. *Let d be a natural number with $d \geq 2$. Let $\{f_c \mid c \in \mathbb{C}\}$ be a family of the maps defined by*

$$\begin{aligned} f_c : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z^d + c \end{aligned}$$

with $c \in \mathbb{C}$. Suppose that c and c' in $\mathbb{C}$ satisfy

$$\lim_{n\to\infty} f_c^n(0) = \lim_{n\to\infty} f_{c'}^n(0) = \infty.$$

Then the dynamical systems $(f_c|_{\mathcal{J}(f_c)}, f_c)$ and $(f_{c'}|_{\mathcal{J}(f_{c'})}, f_{c'})$ are conjugate.

In fact, Theorem 1 can be explained by a theorem which is proved by R. Mañé, P. Sad, and D. Sullivan [MSS]. Roughly speaking, their theorem states that if two maps are close enough and have the same number of attracting cycles, then the dynamics on the Julia sets must be topologically the same. See Theorem 3.1.1 for more precise statement.

In the following theorem, $\mathbb{C}_p$ and $|\cdot|_p$ stand for a complex p-adic field and p-adic norm on $\mathbb{C}_p$, respectively.

Theorem 2. *Let d be a natural number with $d \geq 2$ and p be a prime number which is not divisible by d. Let $\{f_c \mid c \in \mathbb{C}_p\}$ be a family of the maps defined by*

$$\begin{aligned} f_c : \mathbb{C}_p &\to \mathbb{C}_p \\ z &\mapsto z^d + c \end{aligned}$$

with $c \in \mathbb{C}_p$. Suppose that c and c' in $\mathbb{C}_p$ satisfy

$$|c|_p > 1, \quad |c - c'|_p \leq |c|_p^{1/d}.$$

Then the dynamical systems $(f_c|_{\mathcal{J}(f_c)}, f_c)$ and $(f_{c'}|_{\mathcal{J}(f_{c'})}, f_{c'})$ are conjugate.

See Theorem 3.2.1 for more precise statement.

Brief summery of contents In section 1, we will review some basic notations of dynamical systems.

In section 2, basics of non-Archimedean dynamics will be focused on to understand the main results. To understand the dynamical systems, we will first prepare some facts of non-Archimedean analysis. We define the Fatou sets and Julia sets as for complex dynamical systems. In the latter part of this section, we will consider no wandering domains theorems.

In section 3, we will present the main result, the existence of J-stable families in p-adic dynamical systems. See Theorem 3.2.1. We also consider an application of the main result of this section.

In section 4, the Artin-Mazur zeta functions will be discussed. In particular, we will focus on rational maps over $\mathbb{C}$ or $\mathbb{C}_p$. We will review a result of A. Hinkkanen on the Artin-Mazur zeta functions of rational maps over $\mathbb{C}$ as a motivation of the author's main result. After then, we will prove the parallel result to Hinkkanen's theorem for rational maps over $\mathbb{C}_p$. See Theorem 4.3.1.

In section 5, we will summarize some facts that are used in this thesis.

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1 Introduction

In this section, we will give the definition and some basics of dynamical systems with some examples. We will mainly consider the properties of periodic points, which are the points of the space mapped to itself by some iteration of the map. This section is based on J. Milnor's textbook [M] and R. Devaney's textbook [RD].

1.1 Basics on Dynamical Systems

Definition 1.1.1 (Dynamical System)**.** Let X be a topological space and ϕ be a continuous map from X to itself. Then, the pair (X, ϕ) is called *a dynamical system*.

The following example is a typical example of a dynamical system.

Example 1.1.2. Let $\mathbb{C}$ be the complex field with the Euclidean metric and define the map

$$\begin{aligned} \phi_0 : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z^2. \end{aligned}$$

Then, $(\mathbb{C}, \phi_0)$ is a dynamical system.

To ease notation, for a given dynamical system (X, ϕ), we shall use notation

$$\phi^N := \overbrace{\phi \circ \phi \cdots \circ \phi}^{N\ times}$$

to denote the N-fold composition of ϕ with itself.

Definition 1.1.3 (Periodic Point)**.** Let N be a natural number and α be an element in X. Then, α is called *a periodic point of ϕ with period N* if

$$\phi^N(\alpha) = \alpha.$$

In particular, α is called *a fixed point of ϕ* if α is a periodic point of ϕ with period 1.

We shall use the following notation to denote the set of fixed points, periodic points with period $N \in \mathbb{N}$, and all periodic points, respectively.

$$\begin{aligned} Fix(\phi) &:= \{\alpha \in X \mid \phi(\alpha) = \alpha\}, \\ Per_N(\phi) &:= \{\alpha \in X \mid \phi^N(\alpha) = \alpha\}, \\ Per(\phi) &:= \bigcup_{N \in \mathbb{N}} Per_N(\phi). \end{aligned}$$

Definition 1.1.4. Let (X, ϕ) be a dynamical system and α be a periodic point. A natural number N is called *the prime period of α* if

$$\alpha \in Per_N(\phi), \quad \alpha \notin Per_M(\phi)$$

for all $M \in \{1, 2, \cdots, N-1\}$.

Example 1.1.5. Let $(\mathbb{C}, \phi_0)$ be the dynamical system defined as in Example 1.1.2. It is easy to check that for each $N \in \mathbb{N}$,

$$Fix(\phi) = \{1, 0\}, \quad Per_N(\phi) = \{0\} \cup \{\exp \frac{2\pi i}{2^N - 1} k \mid k = 0, 1, \cdots, 2^N - 2\}.$$

In particular, for each $N \in \mathbb{N}$,

$$\#(Per_N(\phi)) = 2^N.$$

Proposition 1.1.6. *Let N and M be natural numbers. If M is divisible by N, then*

$$Per_N(\phi) \subset Per_M(\phi).$$

Proof. In the case when

$$Per_N(\phi) = \emptyset,$$

the statement is clear. Let us assume

$$Per_N(\phi) \neq \emptyset.$$

Since M is divisible by N, there exists some k in N such that $M = k \cdot N$. Thus, for any x in $Per_N(\phi)$,

$$\phi^M(x) = \phi^{k \cdot N}(x) = \overbrace{\phi^N \circ \phi^N \cdots \circ \phi^N}^{k\ times}(x) = x.$$

□

Definition 1.1.7 (Conjugacy)**.** Let (X, ϕ) and (Y, ψ) be dynamical systems. The dynamical systems $(X,\ \phi)$ and $(Y,\ \psi)$ is called *conjugate* if there exists some homeomorphism

$$h : X \to Y$$

such that

$$\psi \circ h = h \circ \phi.$$

Moreover, h is called *a conjugacy from X to Y between ϕ and ψ.*

Example 1.1.8. Set the maps

$$\begin{aligned} \psi_r : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto r \cdot z^2 \end{aligned}$$

and

$$\mathbb{S}(r) := \{z \in \mathbb{C} \mid |z| = r\}$$

for each $r > 0$.

For any z in $\mathbb{S}(r)$, we have

$$|\psi_r(z)| = |r \cdot z^2| = r \cdot \frac{1}{r^2} = \frac{1}{r}.$$

This implies that

$$\psi_r(\mathbb{S}(r^{-1})) \subset \mathbb{S}(r^{-1}).$$

Claim For any r and R in $\mathbb{R}_{>0}$, the dynamical systems $(\mathbb{S}(r^{-1}), \psi_r|_{\mathbb{S}(r^{-1})})$ and $(\mathbb{S}(R^{-1}), \psi_R|_{\mathbb{S}(R^{-1})})$ are conjugate.

Proof. Considering the map

$$h_{r,R} : \mathbb{S}(r) \to \mathbb{S}(R)$$
$$z \mapsto \frac{r}{R}z,$$

we have that

$$\psi_R|_{\mathbb{S}(R^{-1})} \circ h_{r,R}(z) = \psi_R(\frac{r}{R}z) = R\frac{r^2}{R^2}z^2 = \frac{r^2}{R}z^2 = \frac{r}{R}rz^2 = h_{r,R}(rz^2) = h_{r,R} \circ \psi_r|_{\mathbb{S}(r^{-1})}(z).$$

□

This implies that for all r and R in $\mathbb{R}_{>0}$, the dynamical systems $(\mathbb{S}(r), \psi_r|_{\mathbb{S}(r)})$ and $(\mathbb{S}(R), \psi_R|_{\mathbb{S}(R)})$ are conjugate.

Proposition 1.1.9. *Let (X, ϕ) and (Y, ψ) be dynamical systems. If the dynamical systems (X, ϕ) and (Y, ψ) are conjugate, then for any $N \in \mathbb{N}$*

$$\#(Per_N(\phi)) = \#(Per_N(\psi)).$$

Proof. Let h be a conjugacy from X to Y between ϕ and ψ. It is sufficient to show that

$$h(Per_N(\phi)) \subset Per_N(\psi)$$

and

$$h^{-1}(Per_N(\psi)) \subset Per_N(\phi).$$

because the cardinality is invariant under the homeomorphism h.

Taking an arbitrary element x in $Per_N(\phi)$, we see that

$$\psi^N(h(x)) = h \circ \phi^N \circ h^{-1} \circ h(x) = h \circ \phi^N(x) = h(x).$$

On the other hand, for any arbitrary element $y \in Per_N(\psi)$, we have that

$$\phi^N(h^{-1}(y)) = h^{-1} \circ \psi^N \circ h \circ h^{-1}(y) = h^{-1} \circ \psi^N(y) = h^{-1}(y).$$

□

Corollary 1.1.10. *Let (X, ϕ) and (Y, ψ) be dynamical systems. Suppose that (X, ϕ) and (Y, ψ) are conjugate. Then*

$$\#(Per(\phi)) = \#(Per(\psi)).$$

Applying Proposition 1.1.9 to Example 1.1.8, we have the following example.

Example 1.1.11. Recall the dynamical systems defined in Example 1.1.8. For any arbitrary r in $\mathbb{R}_{>0}$, the dynamical system $(\mathbb{S}(r^{-1}), \psi_r|_{\mathbb{S}(r^{-1})})$ is conjugate to $(\mathbb{S}(1), \psi_1|_{\mathbb{S}(1)})$. By Example 1.1.5, for all $N \in \mathbb{N}$,

$$\#(Per_N(\psi_1|_{\mathbb{S}(1)})) = 2^N.$$

It follows from Proposition 1.1.9 that for all $N \in \mathbb{N}$,

$$\#(Per_N(\phi_r|_{\mathbb{S}(r^{-1})})) = \#(Per_N(\phi_1|_{\mathbb{S}(1)})) = 2^N.$$

Let us wrap up this section with the following proposition.

Proposition 1.1.12. *Let (X, ϕ) and (Y, ψ) be dynamical systems. Suppose that (X, ϕ) and (Y, ψ) are conjugate and h is a conjugacy from X to Y between ϕ and ψ. If $A \subset X$ is a dense subset in X, then $h(A)$ is dense in Y.*

Example 1.1.13. Recall the dynamical systems defined in Example 1.1.8. It is not difficult to show that

$$\overline{Per(\phi_1|_{\mathbb{S}(1)})} = \mathbb{S}(1).$$

By Example 1.1.8 and Proposition 1.1.9, we have that

$$\overline{Per(\phi_r|_{\mathbb{S}(r^{-1})})} = \mathbb{S}(r^{-1})$$

for any $r > 0$.

2 Dynamical Systems over Non-Archimedean Fields

2.1 Non-Archimedean Fields and Their Residue Fields

In 2.1, we will review the definition of non-Archimedean fields and their properties. We also see some examples of non-Archimedean fields, which will be called p-adic fields, with their constructions. One can find some interesting properties of non-Archimedean fields in this subsection. This subsection is based on the lecture notes written by A. Baker [AB], N. Koblitz's textbook [NK], and A. Robert's textbook [R].

Definition 2.1.1 (Normed field)**.** Let K be a field. The filed K is called *a normed field* if there exists a map

$$|\cdot| : K \to \mathbb{R}_{\geq 0}$$

satisfying the following properties.

(1) for any $k \in K$, $|k| \geq 0$,
(2) for any $k \in K$, $|k| = 0$ if and only if $k = 0$,
(3) for the unit $1 \in K$, $|1| = 1$,
(4) for any k_1 and k_2 in K, $|k_1 + k_2| \leq |k_1| + |k_2|$.

The map $|\cdot|$ is called *a norm over K*. Moreover, $(K, |\cdot|)$ is called a normed field if K is a field and $|\cdot|$ is a norm over K.

Definition 2.1.2 (Multiplicative norm)**.** Let $(K, |\cdot|)$ be a normed field. Then, the norm $|\cdot|$ is called *multiplicative* if for any k_1 and k_2 in K,

$$|k_1 k_2| = |k_1||k_2|.$$

The normed field $(K, |\cdot|)$ is called *a multiplicative normed field* if $|\cdot|$ is a multiplicative norm over K.

Example 2.1.3 (The complex field)**.** Let be $|\cdot|$ the Euclidean norm on the complex field $\mathbb{C}$. Then, $(\mathbb{C}, |\cdot|)$ is a multiplicative normed field.

Now we consider a property, which the complex field does not have.

Definition 2.1.4 (Non-Archimedean field)**.** Let $(K, |\cdot|)$ be a normed field. Then, the norm $|\cdot|$ is called *non-Archimedean* if it satisfies the following property.

$$|k_1 + k_2| \leq \max\{|k_1|, |k_2|\}$$

for all k_1 and k_2 in K.

The following proposition will be helpful to evaluate inequalities in non-Archimedean fields.

Proposition 2.1.5. *Let $(K, |\cdot|)$ be a non-Archimedean field and z and w be arbitrary elements of K. If $|z| < |w|$, then*

$$|z + w| = |w|.$$

Proof. It is clear from the ultra metric property that

$$|z + w| \leq \max\{|z|, |w|\} = |w|.$$

Assume that $|z + w| < |w|$. It follows that

$$|w| = |z + w - z| \leq \max\{|z + w|, |z|\} < \max\{|w|, |w|\} = |w|.$$

This is a contradiction. Hence, we have

$$|z + w| = |w|.$$

□

By this property, it is clear that the complex field $(\mathbb{C}, |\cdot|)$ with the Euclidean norm is not non-Archimedean. Moreover, the Euclidean norm over the set $\mathbb{Q}$ of the rational numbers is also not non-Archimedean. Now we give an example of non-Archimedean norm on $\mathbb{Q}$.

Example 2.1.6 (The p-Adic Norm)**.** Let p be a prime number, and define the map $|\cdot|_p$ from $\mathbb{Q}$ to $\mathbb{R}$ by

$$\left|\frac{m}{n}\right|_p := \begin{cases} p^{-k} & (m \neq 0), \\ 0 \quad (m = 0) \end{cases}$$

where k is an integer satisfying

$$\frac{m}{n} = p^k \frac{m'}{n'}$$

where m' and n' are integers which satisfy that m' and n' are not divisible by p. The map $|\cdot|_p$ is a norm on $\mathbb{Q}$ and it is called *the p-adic norm on* $\mathbb{Q}$. Moreover, $(\mathbb{Q}, |\cdot|_p)$ is a multiplicative non-Archimedean field.

See [AB, PROPOSITION 2.6] for the reason why $|\cdot|_p$ is a multiplicative non-Archimedean norm on $\mathbb{Q}$.

Definition 2.1.7 ($\mathbb{Q}_p$)**.** For a prime number p, $(\mathbb{Q}_p, |\cdot|_p)$ is defined as the pair of the completion of $\mathbb{Q}$ with respect to the p-adic norm on $\mathbb{Q}$ and the extended norm of the p-adic norm to the completion.

It is well known that the extended norm is non-Archimedean. The proofs can be found in [AB, $THEOREM$2.18]. The extended p-adic norm on $\mathbb{Q}_p$ is also denoted by the same symbol $|\cdot|_p$ of p-adic norm on $\mathbb{Q}$.

The following proposition will be helpful to understand the structure of $\mathbb{Q}_p$.

Proposition 2.1.8. *Let p be a prime number. For any $x \in \mathbb{Q}_p$, there exists some $N \in \mathbb{Z}$ and $\{a_i\}_{i=N}^{\infty} \subset \{0, 1, \cdots, p-1\}$ such that*

$$x = a_N p^N + a_{N+1} p^{N+1} + \cdots, \quad a_N \neq 0.$$

The proof of Proposition 2.1.8 can be found in [AB, THEOREM 2.29].

Corollary 2.1.9. *Let p be a prime number and n be a natural number. Then,*

$$\overline{D}_{p^n}(0) = \{a_{-n} p^{-n} + a_{-n+1} p^{-n+1} + \cdots \mid \{a_i\}_{i=-n}^{\infty} \subset \{0, 1, \cdots, p-1\}\}.$$

In particular,

$$\overline{D}_1(0) = \{a_0 p^0 + a_1 p^1 + \cdots \mid \{a_i\}_{i=0}^{\infty} \subset \{0, 1, \cdots, p-1\}\}.$$

In the rest of this subsection, we shall use $(K, |\cdot|)$ to denote a multiplicative non-Archimedean field and we consider the properties.

Proposition 2.1.10 ($\mathcal{O}_K$)**.** *Let $(K, |\cdot|)$ be a multiplicative non-Archimedean field and set*

$$\mathcal{O}_K := \{z \in K \mid |z| \leq 1\}.$$

The set $\mathcal{O}_K$ is a subring of K.

Proof. It is clear that

$$|0| = 0, \quad |1| = 1.$$

In particular, $1 \in \mathcal{O}_K$. Taking any z and$w$ in $\mathcal{O}_K$, we see that

$$|zw| = |z||w| \leq 1.$$

Hence, $z \cdot w \in \mathcal{O}_K$. Moreover, it follows immediately that

$$|z + w| \leq \max\{|z|, |w|\} \leq 1.$$

Thus, $z + w \in \mathcal{O}_K$. Hence, $\mathcal{O}_K$ is a subring of K. □

This is one of the special properties of non-Archimedean fields. One may easily find an example such that $\mathcal{O}_K$ is not a subring of K when K is not non-Archimedean.

Example 2.1.11 (p-Adic Integers)**.** Let p be a prime number and use the notation $(K, |\cdot|)$ to denote $(\mathbb{Q}_p, |\cdot|_p)$. As we saw in Corollary 2.1.9,

$$\mathcal{O}_K = \overline{D}_1(0) = \{a_0 p^0 + a_1 p^1 + \cdots \mid \{a_i\}_{i=0}^{\infty} \subset \{0, 1, \cdots, p-1\}\}.$$

By Proposition 2.1.10, the set

$$\{a_0 p^0 + a_1 p^1 + \cdots \mid \{a_i\}_{i=0}^{\infty} \subset \{0, 1, \cdots, p-1\}\}$$

is a ring. On the other hand, since

$$\frac{1}{p} \cdot \frac{1}{p} = \frac{1}{p^2},$$

we have that

$$\overline{D}_{p^{-1}}(0) = \{a_{-1} p^{-1} + a_0 p^0 + \cdots \mid \{a_i\}_{i=-1}^{\infty} \subset \{0, 1, \cdots, p-1\}\}$$

is not a ring.

Proposition 2.1.12. *Let us define a subset of $\mathcal{O}_K$ as follows.*

$$\mathcal{M}_K := \{z \in K \mid |z| < 1\}.$$

Then, the subset $\mathcal{M}_K$ is the maximal ideal of $\mathcal{O}_K$

Proof. Let us show the following claims.

Claim 1 The subset $\mathcal{M}_K$ is an ideal of $\mathcal{O}_K$.

Proof of Claim 1. Taking any arbitrary z and w in $\mathcal{M}_K$, we see that

$$|z+w| \leq \max\{|z|, |w|\} < 1.$$

That is, $z+w \in \mathcal{M}_K$. Moreover, for any k in $\mathcal{O}_K$, we have that

$$|k \cdot z| = |k||z| \leq |z| < 1.$$

Thus, $k \cdot z \in \mathcal{M}_K$. □

Claim 2 The subset $\mathcal{M}_K$ is maximal.

Proof of Claim 2. Let J be an ideal of $\mathcal{O}_K$ satisfying

$$\mathcal{M}_K \subsetneq J.$$

Thus, there exists at least one element $a \in J - \mathcal{M}_K$. It is clear that $|a| = 1$. Thus, a must have the inverse a^{-1} with $|a^{-1}| = 1$. Since J is an ideal of $\mathcal{O}_K$, we obtain that

$$1 = a \cdot a^{-1} \in J.$$

Thus, J must be equal to $\mathcal{O}_K$, that is, $\mathcal{M}_K$ is maximal. □

□

Example 2.1.13. Let p be a prime number and use the notation $(K, |\cdot|)$ to denote $(\mathbb{Q}_p, |\cdot|_p)$. It follows from Corollary 2.1.9 that

$$\mathcal{M}_K = D_1(0) = \{a_1 p^1 + a_2 p^2 + \cdots \mid \{a_i\}_{i=1}^{\infty} \subset \{0, 1, \cdots, p-1\}\}.$$

Proposition 2.1.10 and Proposition 2.1.12 imply that the quotient ring $\mathcal{O}_K/\mathcal{M}_K$ must be a field.

Definition 2.1.14 (Residue Field)**.** Let $(K, |\cdot|)$ be a multiplicative non-Archimedean field. The quotient field $\mathcal{O}_K/\mathcal{M}_K$ is called *the residue field of K.*

Example 2.1.15. Let p be a prime number and use the notation $(K, |\cdot|)$ to denote $(\mathbb{Q}_p, |\cdot|_p)$. Then, one may easily check that

$$\mathcal{O}_K/\mathcal{M}_K = \{0, 1, \cdots, p-1\} \cong \mathbb{F}_p$$

where $\mathbb{F}_p$ is the quotient field $\mathbb{Z}/p\mathbb{Z}$ and $\cong$ is the symbol of the field isomorphism.

Next we consider some topological properties of non-Archimedean fields. We shall use the notation

$$\overline{D}_r(a), \quad D_r(a)$$

to denote the set $\{z \in K || z-a| \leq r\}$ and $\{z \in K || z-a| < r\}$ for a in K and r in $\mathbb{R}_{>0}$, respectively. Note that

$$\overline{D}_r(a) \neq \overline{D_r(a)}$$

since $D_r(a)$ is closed with respect to $|\cdot|$.

Lemma 2.1.16. *Let a be an element of K and r be an element of $\mathbb{R}_{>0}$. If $b \in \overline{D}_r(a)$, then*

$$\overline{D}_r(a) = \overline{D}_r(b).$$

Proof. Let us choose any $z \in \overline{D}_r(a)$. Then, we have that

$$|z - b| = |z - a + a - b| \leq \max\{|z - a|, |a - b|\} \leq r.$$

That is,

$$\overline{D}_r(a) \subset \overline{D}_r(b).$$

It is clear that $a \in \overline{D}_r(b)$. Similarly, we can have that

$$\overline{D}_r(b) \subset \overline{D}_r(a).$$

□

The statement of Lemma 2.1.16 is also true for open disks with the same proof.

Corollary 2.1.17. *Let $a \in K$ and $r \in \mathbb{R}_{>0}$. If $b \in D_r(a)$, then we have*

$$D_r(a) = D_r(b).$$

Now we consider the applications of Lemma 2.1.16 and Corollary 2.1.17 to understand some topological properties of non-Archimedean fields. To consider the connectivity of non-Archimedean fields, we give a simple example of Lemma 2.1.16.

Example 2.1.18. It follows from Lemma 2.1.16 that for any $z \in \mathcal{O}_K$,

$$\overline{D}_1(0) = \overline{D}_1(z).$$

Corollary 2.1.19. *Let $(K, |\cdot|)$ be a non-Archimedean field. Then, $\overline{D}_1(0)$ is open with respect to $|\cdot|$.*

Proof. It is clear that $D_1(0)$ is a open set with respect to $|\cdot|$. Let us choose any $z \in \overline{D}_1(0)$ and $r \in \mathbb{R}_{>0}$ with $|z| = 1$ and $r < 1$. Then, it follows from Example 2.1.19 that

$$D_r(z) \subset \overline{D}_r(z) \subset \overline{D}_1(z) = \overline{D}_1(0).$$

That is, z is an interior point of $\overline{D}_1(0)$ with respect to $|\cdot|$. □

On the other hand, one can check easily that $\overline{D}_1(0)$ is a closed subset of K with respect to $|\cdot|$. Thus, we have the following result.

Theorem 2.1.20 (Disconnectedness)**.** *Every non-Archimedean field $(K, |\cdot|)$ is disconnected.*

Next we consider another application of Lemma 2.1.16 and Corollary 2.1.17.

Corollary 2.1.21. *Let a and b in K and r and s in $\mathbb{R}_{>0}$. If $\overline{D}_r(a) \cap \overline{D}_s(b) \neq \emptyset$, then*

$$\overline{D}_r(a) \subset \overline{D}_s(b) \quad \text{or} \quad \overline{D}_r(a) \supset \overline{D}_s(b).$$

Proof. We first assume that $s \geq r$. It follows from Lemma 2.1.16 that

$$\overline{D}_r(a) = \overline{D}_r(c) \subset \overline{D}_s(c) = \overline{D}_s(b)$$

for any $c \in \overline{D}_r(a) \cap \overline{D}_s(b)$. Thus, we have

$$\overline{D}_r(a) \subset \overline{D}_s(b).$$

Similarly, we can obtain that if $s \leq r$, then

$$\overline{D}_r(a) \supset \overline{D}_s(b).$$

□

To consider other topological properties of non-Archimedean fields, the following lemma is as important as Lemma 2.1.16. We will be very helpful to determine whether a given sequence is Cauchy or not. In the rest of this subsection, we shall use the notations $(K, |\cdot|)$ to denote a complete multiplicative non-Archimedean field.

Lemma 2.1.22. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in K. Then, $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if $\{a_n\}_{n\in\mathbb{N}}$ satisfies*

$$\lim_{n\to\infty} |a_{n+1} - a_n| = 0.$$

Proof. Let us begin with the following claim.

Claim If $\{a_n\}_{n\in\mathbb{N}}$ satisfies $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$, then $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Proof of Claim. Taking an arbitrary ϵ in $\mathbb{R}_{>0}$, we have $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_{n+1} - a_n| < \epsilon$. Moreover, for all $m_1 > m_2 \geq N$, we have

$$|a_{m_1} - a_{m_2}| = |a_{m_1} - a_{m_1-1} + a_{m_1-1} - \cdots + a_{m_2+1} - a_{m_2}| \leq \max_{n=m_2,\cdots,m_1-1}(|a_{n+1} - a_n|) < \epsilon.$$

This implies that $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. □

It is clear that if $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, then $\{a_n\}_{n\in\mathbb{N}}$ satisfies $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$. □

In particular, Lemma 2.1.22 will be powerful tool to consider the power series over non-Archimedean fields.

Corollary 2.1.23. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in K and set the sequence $\{s_i\}_{i=1}^{\infty}$ of partial sums*

$$s_i := \sum_{n=1}^{i} a_n.$$

Then, $\{s_i\}_{i=1}^{\infty}$ is convergent if

$$\lim_{n\to\infty} |a_n| = 0.$$

Example 2.1.24. Let us consider a sequence $\{p^n\}_{n\in\mathbb{N}}$ in $\mathbb{Q}_p$. Since $|p^n|_p = \dfrac{1}{p^n}$, we have $\lim_{n\to\infty} |p^n|_p = 0$. Thus,

$$\sum_{n=0}^{\infty} p^n = 1 + p + p^2 + \cdots$$

is convergent with respect to $|\cdot|_p$ on $\mathbb{Q}_p$.

Finally, we consider the value group of non-Archimedean fields. It will be important when we consider the analysis on non-Archimedean fields.

Definition 2.1.25 (Value Group)**.** Let L be a subset of K. Then, we consider the image

$$|L| := \{|l| \in \mathbb{R}_{\geq 0} \mid l \in L\}$$

of L by the norm $|\cdot|$. In particular, $|K^\times|$ is called *the value group of* $(K, |\cdot|)$ where $K^\times := K - \{0\}$.

One can check that $|K^\times|$ is a group and has the property

$$|K^\times| = |K| - \{0\}.$$

Example 2.1.26. Let p be a prime number and use the notation $(K, |\cdot|)$ to denote $(\mathbb{Q}_p, |\cdot|_p)$. Then, by Corollary 2.1.9, we obtain that
$$|\mathbb{Q}_p^\times| = \{p^n \mid n \in \mathbb{Z}\}.$$

Even in the complex field, we can define the value group as above. One can check that the value group of the complex field is equal to $\mathbb{R}_{>0}$.

Definition 2.1.27 (Rational Disk)**.** Let a be an element of K and r be an element of $\mathbb{R}_{>0}$. Then, the subset $\overline{D}_r(a)$ of K is called *a rational disk* if $r \in |K^\times|$. The subset $\overline{D}_r(a)$ is called *irrational* if it is not rational.

Definition 2.1.28 ($\mathbb{Q}_p^{alg}$)**.** For a prime number p, $(\mathbb{Q}_p^{alg}, |\cdot|_p)$ is defined as the pair of an algebraic closure of $\mathbb{Q}_p$ and the extended norm of $|\cdot|_p$ on $\mathbb{Q}_p$ to the algebraic closure.

The fact that $\mathbb{Q}_p$ is not algebraically closed and the existence of the algebraically closed field can be found in [AB, EXAMPLE 5.1]. In fact, the p-adic norm over $\mathbb{Q}_p$ can be extended uniquely to $\mathbb{Q}_p^{alg}$. We will denote the extended norm $|\cdot|_p$. Moreover, it is well-known that $(\mathbb{Q}_p^{alg}, |\cdot|_p)$ is a multiplicative non-Archimedean field.

Example 2.1.29. Let p be a prime number. Then,

$$|\mathbb{Q}_p^{alg} - \{0\}|_p = \{p^{m/n} \mid m \in \mathbb{Z}, n \in \mathbb{Z}^\times\}.$$

The proof can be found in [R, p.129, Proposition]. The value group of $\mathbb{Q}_p^\times$ is not equal to $\mathbb{R}_{>0}$ but is dense in $\mathbb{R}_{>0}$. However, it is well-known that $\mathbb{Q}_p^{alg}$ is not complete. Thus we consider the completion of $(\mathbb{Q}_p^{alg}, |\cdot|_p)$.

Definition 2.1.30 ($\mathbb{C}_p$)**.** For a prime number p, $(\mathbb{C}_p, |\cdot|_p)$ is defined as the pair of the completion of $\mathbb{Q}_p^{alg}$ with respect to $|\cdot|_p$ on $\mathbb{Q}_p^{alg}$ and the extended norm of $|\cdot|_p$ on $\mathbb{Q}_p^{alg}$ to the completion.

As we have constructed $(\mathbb{Q}_p, |\cdot|_p)$ from $(\mathbb{Q}, |\cdot|_p)$, we can also extended $|\cdot|_p$ over $\mathbb{Q}_p^{alg}$ to $\mathbb{C}_p$, uniquely. We use $|\cdot|_p$ to denote the extended norm over $\mathbb{C}_p$.

Theorem 2.1.31. *For any prime number p, $(\mathbb{C}_p, |\cdot|_p)$ is an algebraically closed complete non-Archimedean field of characteristic zero.*

The proof of Theorem 2.1.31 can be found in [R, p. 143, Theorem]. See also [AB, THEOREM 5.17].

Example 2.1.32. Let p be a prime number. Then,

$$|\mathbb{Q}_p^{alg}| = |\mathbb{C}_p|.$$

That is,
$$|\mathbb{C}_p| = \{p^{\frac{m}{n}} \mid m \in \mathbb{Z}, n \in \mathbb{Z}^\times\} \cup \{0\}.$$

In particular, we have
$$\overline{|\mathbb{C}_p^\times|} = \mathbb{R}_{\geq 0}.$$

The proof of Example 2.1.32 may found in [R, p. 138, Proposition 3].

2.2 Analysis on Non-Archimedean Fields

In 2.2, we will discuss analytic properties on non-Archimedean fields. In particular, we will focus on some properties of polynomial maps over non-Archimedean fields. This subsection is based on Silverman's textbook [S, Section 5.2]. Throughout this subsection, we shall use $(K, |\cdot|)$ to denote an algebraically closed complete multiplicative non-Archimedean field. We shall use the notation $Poly(K)$ and deg to denote the set of polynomial maps over K and the degree of a given polynomial map.

Example 2.2.1. For fixed $\alpha \in K$, we define

$$T_\alpha(z) := z + \alpha.$$

It is a polynomial map and $\deg(T_\alpha) = 1$.

Now we consider some properties of polynomial maps.

Proposition 2.2.2. *Let f be a polynomial map over K. Then, there exists some $\{\alpha_i\}_{i=1}^{\deg(f)}$ in K such that*

$$f(\alpha_i) = 0, \quad f(z) \neq 0$$

for all $i \in \{1, 2, \cdots, \deg(f)\}$, and all $z \in K - \{\alpha_i\}_{i=1}^{\deg(f)}$.

The proof is clear since K is an algebraic closed field.

Proposition 2.2.3. *Let f and g be polynomial maps over K. Then,*

$$\deg(f \circ g) = \deg(f) \cdot \deg(g).$$

The proof can be found in [B, p.32].

Proposition 2.2.4. *Let f be a non-constant polynomial map over K. Suppose 0 is not a zero of f. Then, there exists some $r > 0$ such that for any $z \in \overline{D}_r(0)$,*

$$|f(z)| = |f(0)|.$$

Proof. Suppose that the degree of f is $N \in \mathbb{N}$. There exists some $a_0, a_1, \cdots a_N$ in K such that

$$f(z) = a_0 + a_1 z + \cdots + a_N z^N, \quad a_0 = f(0), \quad a_N \neq 0.$$

Set

$$M := \max\{|a_j| \mid j \in \{0, 1, \cdots, N\}\}, \quad r := \min\{1, \frac{|a_0|}{2M}\}.$$

Claim For any $z \in \overline{D}_r(0)$,

$$|a_0| > |a_1 z + \cdots + a_N z^N|.$$

Proof of Claim. Since $|z| \leq r \leq 1$ and $|a_j| \leq M$ for any $j \in \{0, 1, \cdots, N\}$, we have

$$|a_j z^j| \leq M|z|^j \leq M|z|.$$

Moreover, since $|z| \leq \dfrac{|a_0|}{2M}$, we have

$$M|z| \leq \frac{|a_0|}{2}.$$

Thus, we have

$$|\sum_{i=1}^{N} a_i z^i| \leq \max\{|a_i z^i| \mid i = 1, 2, \cdots, N\} \leq \frac{|a_0|}{2} < |a_0|.$$

□

It follows from Proposition 2.1.5 that

$$|f(z)| = |a_0| = |f(0)|$$

for any $z \in \overline{D}_r(0)$. □

In fact, we can say more than this.

Proposition 2.2.5. *Let f be a non-constant polynomial map over K and denote it by*

$$f(z) = a_0 + a_1 z + \cdots + a_N z^N$$

for some $\{a_i\}_{i=1}^N$ with $a_N \neq 0$. Then, if j is the minimal number satisfying $a_j \neq 0$, then there exists some $r > 0$ such that

$$|f(z)| = |a_j||z|^j$$

for any $z \in \overline{D}_r(0)$.

The proof of Proposition 2.2.5 is similar to Proposition 2.2.4 so we omit the proof.

Example 2.2.6. Let p be a prime number. Define the map

$$f(z) := 1 + pz + p^2 z^2 + \cdots + p^{100} z^{100} \in Poly(\mathbb{C}_p).$$

It follows immediately that for any $z \in \overline{D}_1(0)$ and any $i \in \{1, 2, \cdots, 100\}$, we have

$$|p^i z^i| = p^{-i}|z|^i \leq p^{-i} < 1.$$

By Proposition 2.1.5, for any $z \in \overline{D}_1(0)$, we have

$$|f(z)| = 1.$$

Let us consider the formal derivative of the polynomial maps, and *the critical points* as we do in the real or complex analysis.

Definition 2.2.7 (Derivative)**.** Let f be a polynomial map over K. Suppose that there exists some $a_0, a_1, \cdots, a_N$ in K such that

$$f(z) = a_0 + a_1 + \cdots + a_N z^N, \quad a_N \neq 0.$$

Then, *the derivative f' of f* is defined by

$$f'(z) := a_1 + 2a_2 z + \cdots + N a_N z^{N-1}.$$

Moreover, a point $w \in K$ is called *a critical point of f* if $f'(w) = 0$.

One can check that the formal derivative of the polynomial maps is well-defined as a map from $Poly(K)$ to itself. Thus, we can consider the N fold derivative of polynomial maps.

Definition 2.2.8. Let f be a polynomial map over K. Then, we define *the N-th derivative of f* by

$$f^{(N)} := (f^{(N-1)})'$$

for any $N \in \mathbb{N}$ where $f^{(0)} := f$.

The following corollary follows from Proposition 2.2.5.

Corollary 2.2.9. *Let f be a non-constant polynomial map over K of $\deg(f) = N$. Then, there exists some $j \in \{1, 2, \cdots, N\}$ such that*

$$f^{(j)}(0) \neq 0, \quad f^{(i)}(0) = 0$$

for $i = 1, 2, \cdots j-1$. Moreover, there exists some $R > 0$ such that

$$|f(z) - f(0)| = |f^{(j)}(0)||z|^j$$

for any $z \in \overline{D}_R(0)$.

Proposition 2.2.10. *Let f be a polynomial map with $\deg(f) = N$. Then, for any $\alpha \in K$, there exists some $\{b_i\}_{i=0}^N \subset K$ such that*

$$f(z) = b_0 + b_1(z-\alpha) + \cdots + b_N(z-\alpha)^N.$$

The proof is similar to the case of the complex field and we omit it. See also [S, Proposition 5.8 (a)].

Corollary 2.2.11. *Let f be a non-constant polynomial map with $\deg(f) = N$, and α be an element of K. Then, there exists some $j \in \{1, 2, \cdots, N\}$ such that*

$$f^{(j)}(\alpha) \neq 0, \quad f^{(i)}(\alpha) = 0$$

for $i = 1, 2, \cdots j-1$. Moreover, there exists some $R > 0$ such that

$$|f(z) - f(\alpha)| = |f^{(j)}(\alpha)||z-\alpha|^j$$

for any $z \in \overline{D}_R(\alpha)$.

The proof is easily obtained from Corollary 2.2.9 and Proposition 2.2.10.

Now let us consider the continuity of polynomial maps as an application of Corollary 2.2.11.

Corollary 2.2.12. *Every polynomial map f is continuous on K with respect to $|\cdot|$.*

Proof. It is clear when f is a constant polynomial so let us suppose that

$$\deg(f) = N \geq 1.$$

Let us fix an arbitrary $\alpha \in K$. Then, it follows from Corollary 2.2.11 that there exists some $j = 1, 2, \cdots, N$ and $R > 0$ such that

$$f^{(j)}(\alpha) \neq 0, \quad |f(z) - f(\alpha)| = |f^{(j)}(\alpha)||z-\alpha|^j$$

for any $z \in \overline{D}_R(\alpha)$. Taking any $\epsilon > 0$, we set $\delta := \min\{r, \frac{\sqrt[j]{\epsilon}}{\sqrt[j]{|a_j|}}\} > 0$. Then for any $z \in \overline{D}_\delta(\alpha)$, we have

$$|f(z) - f(\alpha)| = |a_j||z|^j \leq \epsilon.$$

□

Since the formal derivative of a polynomial map is also a polynomial map, we have the following corollary immediately:

Corollary 2.2.13. *Let f be a polynomial map over K. Then, f' is a continuous map on K with respect to $|\cdot|$.*

Next we consider a theorem, which is an analogy to the Maximum Modulus Principle in complex analysis. In general, non-Archimedean fields may not be locally compact. Thus, the existence of the maximum in $\overline{D}_1(0)$ of a continuous map is not guaranteed. However, the following theorem tells us the existence of the maximum points in $\overline{D}_1(0)$ of a polynomial map over K.

Theorem 2.2.14. *Let f be a non-constant polynomial map on $\overline{D}_1(0)$. Assume that for all $i \in \mathbb{N}$, $\frac{|f^{(i)}(0)|}{i \cdot (i-1) \cdots 1} \leq 1$, and there exists some $j \in \mathbb{N}$ such that $\frac{|f^{(j)}(0)|}{j \cdot (j-1) \cdots 1} = 1$. Then, there exists some $z_0 \in \overline{D}_1(0)$ such that*

$$|f(z)| \leq |f(z_0)| = 1$$

for any $z \in \overline{D}_1(0)$. Moreover, it follows that

$$\sup\{|f(z)| \mid z \in \overline{D}_1(0)\} = \max\{\frac{|f^{(i)}(0)}{i \cdot (i-1) \cdots 1}| \mid i \geq 0\}.$$

Proof. Let us denote the degree of f by N, and write f as

$$f(z) := \sum_{i=0}^{N} a_i z^i.$$

Now we consider the induced polynomial map by f from $\mathcal{O}_K/\mathcal{M}_K$ to itself by

$$f^*(w) := \sum_{i=0}^{N} \pi(a_i) w^i$$

where $\pi : \mathcal{O}_K \to \mathcal{O}_K/\mathcal{M}_K$ is the canonical projection. Note that it might be $\pi(a_n) = 0$, and f^* is not a zero map since there exists some $j \in \{1, 2, \cdots, N\}$ such that

$$|a_j| = \frac{|f^{(j)}(0)}{j \cdot (j-1) \cdots 1} = 1 \neq 0.$$

Let us consider the zeros of f^*. Since $\mathcal{O}_K/\mathcal{M}_K$ is a field and f^* is a polynomial map over $\mathcal{O}_K/\mathcal{M}_K$, f^* has finitely many zeros in $\mathcal{O}_K/\mathcal{M}_K$.

Claim 1 $\mathcal{O}_K/\mathcal{M}_K$ is an algebraically closed field and every algebraically closed field contains infinitely many elements.

The proof of Claim 1 can be found Proposition 5.1.3 in APPENDIX A.

Moreover, since f^* is not a zero map, we can find w_0 such that $f^*(w_0) \neq 0$. Moreover, since π is surjective, we can also find z_0 satisfying

$$\pi(z_0) = w_0.$$

Moreover, since $|f^{(i)}(0)| \leq 1$ for all $i \in \mathbb{N}$, it follows that

$$|f(z_0)| \leq 1.$$

Claim 2 $|f(z_0)| = 1$.

Proof of Claim 2. (By contradiction) Assume that $|f(z_0)| < 1$. That is,

$$f(z_0) = \sum_{i=0}^{N} a_i z_0^i \in \mathcal{M}_K.$$

Thus, it follows that

$$\pi(f(z_0)) = 0 \in \mathcal{O}_K/\mathcal{M}_K.$$

On the other hand, since π is a ring homomorphism, we have

$$0 = \pi(f(z_0)) = f^*(w_0).$$

This implies that w_0 is a root of f^*. It is contradiction. □

Hence, we have proved our first statement.

Now let us consider our second statement.

Claim 3 $\max\{|a_i| \mid i \geq 0\} \geq 1$.

Proof of Claim 3. We have shown that for any $z \in \overline{D}_1(0)$, we have

$$|f(z)| \leq |f(z_0)| = 1$$

for some $z_0 \in \overline{D}_1(0)$. In particular, it follows that

$$\begin{aligned} 1 = |f(z_0)| = |\sum_{i=0}^{N} a_i z_0^i| &\leq \max\{|a_i||z_0|^i \mid i = 1, 2, \cdots, N\} \\ &\leq \max\{|a_i||1|^i \mid i = 1, 2, \cdots, N\} = \max\{|a_i| \mid i = 1, 2, \cdots, N\} \end{aligned}$$

since

$$|z_0|^i \leq 1$$

for all $i = 0, 1, \cdots, N$. □

Next we show the following claim.

Claim 4 $\max\{|a_i| \mid i \geq 0\} \leq 1$.

Proof of Claim 4. By our assumption, we have

$$|a_i| \leq 1$$

for all $i = 0, 1, \cdots, N$. This implies that

$$\max\{|a_i| \mid i = 0, 1, \cdots, N\} \leq 1.$$

□

□

In fact, it is not important that the given disk in Theorem 2.2.14 is the closed unit disk.

Corollary 2.2.15 (The Maximal Modulus Principle). *Let $\overline{D}_r(a)$ be a rational closed disk and f be a holomorphic map on $\overline{D}_r(a)$. Then, there exists a $z_0 \in \overline{D}_r(a)$ such that for all $z \in \overline{D}_r(a)$,*

$$|f(z)| \leq |f(z_0)|.$$

Moreover, we have that

$$\sup\{|f(z)| \mid z \in \overline{D}_r(a)\} = \max\{\frac{|f^{(i)}(a)}{i \cdot (i-1) \cdots 1} r^i \mid i \geq 0\}.$$

The proof can be found in [S, Theorem 5.13 (a)]. Before moving on to the next property of polynomial maps, let us clarify the significance of the assumption that K is an algebraically closed field by the following example:

Example 2.2.16. Let p be a prime number, and consider the map defined by $F(z) := z^p - z \in Poly(z)$. Then, it follows from Theorem 5.1.2 that for all $z \in \overline{D}_1(0)$,

$$|F(z)|_p \leq \frac{1}{p}.$$

On the other hand, we have that

$$\max\{|1|_p, |-1|_p\} = 1.$$

Next we will consider the theorem, which is an analogue of the Minimal Modulus Principle in complex analysis.

Theorem 2.2.17. *Let f be a polynomial map over K. If f has no zeros in $\overline{D}_1(0)$, then $|f(z)|$ must be constant on $\overline{D}_1(0)$.*

Proof. Without loss of generality, we may assume that f is monic. Let us denote f by

$$f(z) := a_0 + a_1 z + \cdots + z^N$$

for some $\{a_i\}_{i=0}^{N-1} \subset K$. Then, we have

Claim 1 $|a_0| > |a_i|$ for all $i = 1, 2, \cdots, N$.

Proof of Claim 1. Let us denote the zeros of f by

$$\alpha_1, \ \alpha_2, \ \cdots, \ \alpha_N.$$

That is, we may write f by

$$f(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_N).$$

It follows from Vieta's formula that

$$a_0 = (-1)^N \alpha_1 \cdots \alpha_N, \quad a_{N-i} = \sum_{1 \leq k_1 < k_2 \cdots < k_i \leq N} (-1)^i \alpha_{k_1} \cdots \alpha_{k_i}$$

for all $i = 1, 2, \cdots, N$. It follows from the multiplicativity of $|\cdot|$ that

$$|a_i| \leq \max\{|\alpha_{k_1} \alpha_{k_2} \cdots \alpha_{k_{N-i}}|\}$$

for any $i = 1, 2, \cdots, N$. Since we assumed $|\alpha_i| > 1$ for all $i = 1, 2, \cdots, N$, we have

$$|a_0| = |\alpha_1 \cdots \alpha_N| > \max\{|\alpha_{k_1} \cdots \alpha_{k_{N-i}}|\} \geq |a_i|.$$

for all $i = 1, 2, \cdots, N$. □

Claim 2 For all $z \in \overline{D}_1(0)$, $|f(z)| = |a_0|$.

Proof of Claim 2. It follows from Claim 1 that

$$|a_i||z|^i \leq |a_i| < |a_0|$$

for all $z \in \overline{D}_1(0)$, $i = 1, 2, \cdots, N$. Thus, it follows from Proposition 2.1.5 that

$$|f(z)| = |a_0|$$

for all $z \in \overline{D}_1(0)$. □

□

In fact, it is also not essential that the given disk is the closed unit disk.

Corollary 2.2.18 (the Minimal Modulus Principle)**.** *Let $\overline{D}_r(a)$ be a rational closed disk and f be a non-constant polynomial map over K. If f has no zeros in $\overline{D}_r(a)$, then $|f(z)|$ must be constant on $\overline{D}_r(a)$.*

Proof. Let us consider the map

$$\begin{aligned} T_a : K &\to K \\ z &\mapsto cz + a \end{aligned}$$

where $c \in K$ satisfies $|c| = r$, and $a \in K$. Moreover, we define

$$g := f \circ T_a.$$

It is clear that g is a polynomial map over K. Furthermore, we have the following claim.

Claim g has no zeros in $\overline{D}_1(0)$.

Proof of Claim. (By contradiction) Let us assume that there exists some $\alpha \in \overline{D}_1(0)$ such that

$$g(\alpha) = 0.$$

This implies immediately that

$$0 = g(\alpha) = f \circ T_a(\alpha) = f(c\alpha + a).$$

However, it is clear that

$$c\alpha + a \in \overline{D}_r(a).$$

Thus, it is contradiction to our assumption. □

Now we apply Theorem 2.2.17 to g. Then, we have $|g(z)|$ is constant on $\overline{D}_1(0)$. That is, there exists some $C \in K^\times$ such that

$$|g(w)| = |f \circ T_a(w)| = C$$

for all $w \in \overline{D}_1(0)$. Since T_a is bijective between $\overline{D}_1(0)$ and $\overline{D}_r(a)$, we have for all $z \in \overline{D}_r(a)$, there exists $w \in \overline{D}_1(0)$ such that

$$|f(z)| = |g(w)| = C.$$

□

Let us finish this subsection with an application of the Maximum Modulus Principle. Let us begin with an interesting topological property.

Proposition 2.2.19. *Let f be a polynomial map over K with $f(0) = 0$. Then,*

$$f(\overline{D}_1(0)) = \overline{D}_s(0)$$

where $s := \sup\{|f(z)| \mid z \in \overline{D}_1(0)\}$. Moreover, we have

$$|f'(0)| \leq s.$$

The proof can be found in [S, Proposition 5.16(b)].

The following corollary implies us that we do not have to assume that 0 is a fixed point of the polynomial map.

Corollary 2.2.20. *Let f be a polynomial map over K and $\overline{D}_r(a)$ be a rational closed disk. Then*

$$f(\overline{D}_r(a)) = \overline{D}_s(f(a))$$

where $s := \sup |f(z)| \mid z \in \overline{D}_r(a)$. Moreover,

$$|f'(a)| \leq \frac{s}{r}.$$

Proof. Suppose that $\deg(f) = N$ and define

$$\begin{aligned} f_1 : K \to K, \quad & f_2 : K \to K \\ z \mapsto cz + a, \quad & z \mapsto z - f(a) \end{aligned}$$

where $c \in K$ satisfies $|c| = r$. Note that the existence of c is guaranteed by $r \in |K^\times|$. Now we set

$$g := f_2 \circ f \circ f_1.$$

Then, it is clear that $g \in Poly(K)$ and $\deg(g) = N$.

Claim 0 is a fixed point of g. Moreover, we have

$$s = \sup\{|f(z)| \mid z \in \overline{D}_r(a)\} = \sup\{|g(z)| \mid z \in \overline{D}_1(0)\}.$$

Proof of Claim. It follows immediately that

$$g(0) = f_2 \circ f \circ f_1(0) = f_2 \circ f(a) = 0.$$

On the other hand, it follows from Corollary 2.2.18 that

$$\sup\{|f(z)| \mid z \in \overline{D}_r(a)\} = \max\{\frac{|f^{(i)}(a)|}{i \cdot (i-1) \cdots 1}|r|^i \mid i \geq 0\},$$
$$\sup\{|g(z)| \mid z \in \overline{D}_1(0)\} = \max\{\frac{|g^{(i)}(0)|}{i \cdot (i-1) \cdots 1}|1|^i \mid i \geq 0\}.$$

It follows from the chain rule that

$$|g^{(i)}(0)| = |f^{(i)}(a)||c|^i.$$

This implies that

$$\sup\{|f(z)| \mid z \in \overline{D}_r(a)\} = \sup\{|g(z)| \mid z \in \overline{D}_1(0)\}.$$

□

It follows that

$$f_2 \circ f(\overline{D}_r(a)) = f_2 \circ f \circ f_1(\overline{D}_1(0)) = g(\overline{D}_1(0)) = \overline{D}_s(0).$$

Since f_2 is bijective, we have

$$f(\overline{D}_r(a)) = f_2^{-1}(\overline{D}_s(0)) = \overline{D}_s(f(a)).$$

Furthermore, it follows from the chain rule that

$$|1 \cdot f'(a) \cdot c| = |(f_2 \circ f \circ f_1)'(0)| = |g'(0)| \leq s.$$

Thus,

$$|\phi'(a)| \leq \frac{s}{r}.$$

□

The following corollary follows immediately from Corollary 2.2.20.

Corollary 2.2.21. *Let $(K, |\cdot|)$ be an algebraically closed complete multiplicative non-Archimedean field and $\overline{D}_r(a)$ be rational. If f is a polynomial map over K, then f is an open map.*

2.3 The Projective Line and Its Topology

In 2.3, we will discuss the projective line and the chordal metric of non-Archimedean fields, which is another metric on the non-Archimedean field $(K, |\cdot|)$, analog to the Riemann sphere and its chordal metric in complex dynamics. See [B, p. 28] for the Riemann sphere and its choral metric. Moreover, we will see some topological properties of the projective line and the chordal metric of non-Archimedean fields. This subsection is based on Silverman's textbook [S, Section 2.1].

Definition 2.3.1 (The Projective Line)**.** Let $(K, |\cdot|)$ be a multiplicative normed field. Then, *the projective line over K* is defined by

$$\mathbb{P}^1(K) := (K \times K)^{\times} / \sim$$

where $(K \times K)^{\times} = K \times K - \{(0,0)\}$, and $(X, Y) \sim (X', Y')$ if there exists some $k \in K^{\times}$ such that $X = kX'$ and $Y = kY'$.

We denote an element, of which the representative element is (X, Y) by $[X, Y]$.

By considering the inclusion map

$$\begin{aligned} \iota : K &\to \mathbb{P}^1(K) \\ z &\mapsto [z, 1], \end{aligned}$$

it is clear that the original field K is naturally included in the projective line $\mathbb{P}^1(K)$.

Proposition 2.3.2. *Let $(K, |\cdot|)$ be a non-Archimedean multiplicative field. Then,*

$$\mathbb{P}^1(K) = \{[z, 1] \mid z \in K\} \cup \{[1, 0]\}.$$

Proof. It is sufficient to show that for any

$$[X, Y] \in \mathbb{P}^1(K) - \{[1, 0]\},$$

we have

$$[X, Y] \in \{[X, 1] \mid X \in K\}.$$

If $Y = 0$, we have $[X, 0] = [1, 0]$. This implies that

$$[X, Y] = [Y^{-1}X, 1] \in \{[X, 1] \mid X \in K\}.$$

□

The projective line $\mathbb{P}^1(K)$ has a natural decomposition $\mathbb{P}^1(K) = K \cup \{\infty\}$ where ∞ is an element which is not contained in K and corresponding to $[1, 0] \in \mathbb{P}^1(K)$.

Definition 2.3.3 (The Chordal Metric)**.** Let $(K, |\cdot|)$ be a non-Archimedean field and $\mathbb{P}^1(K)$ be the projective line over K. Then, we define *the chordal metric on* $\mathbb{P}^1(K)$ by

$$\rho([X, Y], [X', Y']) := \frac{|XY' - X'Y|}{\max\{|X|, |Y|\} \cdot \max\{|X'|, |Y'|\}}$$

for all $[X, Y], [X', Y'] \in \mathbb{P}^1(K)$.

One can check that this metric is well-defined and some interesting properties such as the non-Archimedean property. See [S, Proposition 2.4]. Let us see some other properties of the chordal metric. Let us begin with the following proposition.

Proposition 2.3.4. *Let* $(K, |\cdot|)$ *be a non-Archimedean multiplicative field. Then, we have*

$$\mathbb{P}^1(K) = \{[X, 1] \mid |X| \leq 1\} \cup \{[1, Y] \mid |Y| \leq 1\}.$$

Moreover, $(\{[X, 1]| \mid X \in K\}, \rho)$ *is isometric to* $(\mathcal{O}_K, |\cdot|)$*.*

Proof. For any $[X, Y] \in \mathbb{P}^1(K)$, we have either $|X| \leq |Y|$ or $|X| \geq |Y|$. Let us assume that $|X| \leq |Y|$. Then, we have

$$[X, Y] = [XY^{-1}, 1], \quad |X||Y^{-1}| \leq 1.$$

This implies that

$$[X, Y] \in \{[X, 1] \mid |X| \leq 1\}.$$

If $|X| \geq |Y|$, then we have that

$$[X, Y] = [1, X^{-1}Y], \quad |X^{-1}||Y| \leq 1.$$

This implies that

$$[X, Y] \in \{[1, Y] \mid |Y| \leq 1\}.$$

Thus, we have complete the proof of the first statement.

Let us choose arbitrary two elements $[X_1, 1], [X_2, 1] \in \{[X, 1] \mid |X| \leq 1\}$. Then, we have

$$\rho([X_1, 1], [X_2, 1]) = \frac{|X_1 1 - X_2 1|}{\max\{|X_1|, 1\} \max\{|X_2|, 1\}} = |X_1 - X_2|.$$

□

The following proposition follows easily.

Proposition 2.3.5. $\{\overline{D}_r(a) \mid a \in K, r > 0\} \cup \{\mathbb{P}^1(K) - D_s(b) \mid b \in K, s > 0\}$ *is a family of open sets and forms an open base of* $(\mathbb{P}^1(K), \rho_K)$.

Each polynomial map over K induces a well-defined map on $\mathbb{P}^1(K)$ as follows.

Proposition 2.3.6. *Let K be a field and f be a polynomial map over K. Then, the map*

$$f([X,Y]) := \begin{cases} [f(X/Y), 1] & (Y \neq 0), \\ [1, 0] & (Y = 0). \end{cases}$$

is well-defined on $\mathbb{P}^1(K)$.

Let us close this subsection with a property of polynomial maps with respect to ρ.

Theorem 2.3.7. *Let $(K, |\cdot|)$ be an algebraically closed complete non-Archimedean field and f be a polynomial map over K. Then, there exists some $C > 0$ such that*

$$\rho(f([X,Y]), f([X',Y'])) \leq C\rho([X,Y],[X',Y'])$$

for any $[X,Y],[X',Y'] \in \mathbb{P}^1(K)$.

The proof can be found in [S, Theorem 2.14].

2.4 Dynamics of Polynomial Maps of Degree One

In 2.4, we will consider the dynamics of polynomial maps, whose the degree is one, over non-Archimedean fields. We mainly consider an analogue of the Classification Theorem of Mobius Transformation in complex dynamical systems. This subsection is based on the lecture notes written by Tomoki Kawahira. Let us fix an algebraically closed complete non-Archimedean field $(K, |\cdot|)$ of characteristic zero. To ease notation, we shall use

$$Poly_1(K) := \{f \in Poly(K) \mid \deg(f) = 1\}$$

to denote *the set of polynomial maps of the degree* 1. One may easily check that $Poly_1(K)$ is a group with respect to the composition $\circ$ of maps.

Example 2.4.1. Let p be a prime number, and $a \in \mathbb{C}_p^\times$ satisfy $|a|_p \neq 1$. Considering

$$f : \mathbb{C}_p \to \mathbb{C}_p$$
$$z \mapsto az,$$

we easily see that

$$Per_n(f) = \{0\}.$$

Moreover, one can show that

$$\lim_{n\to\infty} f(z) = \begin{cases} 0 & (|a|_p < 1), \\ \infty & (|a|_p > 1) \end{cases}$$

for all $z \in \mathbb{C}_p^\times$.

Example 2.4.2. Let us consider the dynamical systems $(\mathbb{C}_p, \phi_{1,b})$ where $\phi_{1,b}(z) = z + b$ and $b \in \mathbb{C}_p^{\times}$. Then, it easily follows that

$$\phi_{1,b}^n(z) = z + nb$$

for any $z \in \mathbb{C}_p$. In particular, we have

$$|\phi_{1,b}^{p^i}(z) - z| = |z + p^i b - z| \le |p^i||b| \to 0 \quad (i \to \infty)$$

for any $z \in \mathbb{C}_p$. On the other hand, we have

$$|\phi_{1,b}^{(p-1)^i}(z) - z| = |z + (p-1)^i b - z| = |b| \neq 0$$

for any $i \in \mathbb{N}$ since b is not zero.

In fact, it is sufficient to consider just two cases to understand the dynamics of the polynomial maps of degree 1 by the following theorem.

Theorem 2.4.3. *If $f \in Poly_1(K)$, then there exists some $g \in Poly_1(K)$ and $\lambda \in K^{\times}$ such that*

$$\lambda z = g^{-1} \circ f \circ g(z),$$

or

$$z + 1 = g^{-1} \circ f \circ g(z).$$

Proof. We assume that $\phi \neq Id_K$ since the case when $\phi = Id_K$ is clear. Since K is algebraically closed and f is a polynomial map with $\deg(f) = 1$, f has at most one fixed point in K.

Case 1: One Fixed Point We first assume there exists only one fixed point $\alpha \in K$. Then, we consider

$$g(z) := z - \alpha.$$

Note that the inverse map, g^{-1}, of g maps 0 to α. Now we consider

$$f^g(z) := g \circ f \circ g^{-1}(z).$$

Then, it follows that

Claim 1 $f^g \in Poly_1(K)$. Moreover, $f^g(0) = 0$.

Proof of Claim 1. It is clear that $f^g \in Poly_1(K)$ since f and $g \in Poly_1(K)$ and $\deg(f \circ g) = \deg(f) \cdot \deg(g)$ by Proposition 2.2.3. Moreover, it follows immediately that

$$f^g(0) = g \circ f \circ g^{-1}(0) = g \circ f(\alpha) = g(\alpha) = 0.$$

□

Hence, it follows immediately that

$$f^g(z) = g \circ f \circ g^{-1}(z) = \lambda z$$

for some $\lambda \in K^{\times}$.

Case 2: No Fixed Points Let us assume that f has no fixed points in K. Then, it is not difficult to check that

$$f(z) = z + c$$

for some $c \in K^{\times}$.

Claim 2 There exists some $g \in Poly_1(K)$ such that

$$g \circ f \circ g^{-1}(z) = z + 1.$$

Proof of Claim 2. Let us consider

$$g(z) := \frac{1}{c} z.$$

One may easily check that

$$g^{-1}(z) = cz.$$

It easy to check that

$$g \circ f \circ g^{-1}(z) = g \circ f(cz) = g(c(z+1)) = z + 1.$$

□

□

In the rest of this subsection, we will focus on the invariance of ρ_K under some maps. Let us define

$$T : K \times K \times K \to K$$
$$(z, a, c) \mapsto az + c,$$

and denote

$$T_{a,c}(z) := T(z, a, c).$$

Proposition 2.4.4. *For any $a, c \in \mathcal{O}_K$ with $|a| = 1$, we have*

$$\rho_K(T_{a,c}([X, Y]), T_{a,c}([X', Y'])) = \rho_K([X, Y], [X', Y'])$$

for any $[X, Y], [X', Y'] \in \mathbb{P}^1(K)$.

Proof. One may easily check that

$$T_{a,c}([X, Y]) = [aX + cY, Y]$$

for any $[X, Y] \in \mathbb{P}^1(K)$. On the other hand, we have

$$|aX + cY| \le \max\{|aX|, |cY|\} \le \max\{|X|, |Y|\},$$
$$|X| = |aX| = |aX + cY - cY| \le \max\{|aX + cY|, |cY|\} \le \max\{|aX + cY|, |Y|\}$$

for any $[X, Y] \in \mathbb{P}^1(K)$. This implies that

$$\max\{|X|, |Y|\} = \max\{|aX + cY|, |Y|\}$$

Thus, it follows that

$$\begin{aligned}\rho_K(T_{a,c}([X,Y]),T_{a,c}([X',Y'])) &= \rho_K([X+cY,Y],[X'+cY',Y']) \\ &= \frac{|(X+cY)Y'-Y(X'+cY')|}{\max\{|X+cY|,|Y|\}\max\{|X'+cY'|,|Y'|\}} \\ &= \frac{|XY'-YX'|}{\max\{|X|,|Y|\}\max\{|X'|,|Y'|\}} = \rho_K([X,Y],[X',Y']).\end{aligned}$$

□

Definition 2.4.5. We define

$$\begin{aligned} S : \mathbb{P}^1(K) &\to \mathbb{P}^1(K) \\ [X,Y] &\mapsto [Y,X]. \end{aligned}$$

Then, we have the following proposition.

Proposition 2.4.6. *For any* $[X,Y],[X',Y'] \in \mathbb{P}^1(K)$,

$$\rho_K(S([X,Y]),S([X',Y'])) = \rho_K([X,Y],[X',Y']).$$

The proof is straightforward and we omit it.

2.5 The Fatou Set and the Julia Set

In 2.5, we will define the Fatou set and the Julia set on non-Archimedean fields and see some properties of them, which are similar to the properties of the complex dynamical systems. See [B, Definition 3.1.3] for the complex cases. As an important example, we will see a dynamical system, which has an empty Julia Set. This subsection is based on J. Silverman's textbook [S, Section 5.4]. Let us begin with the definitions of the equicontinuity and uniform Lipschitzness in metric spaces.

Definition 2.5.1 (Equicontinuity)**.** Let (X,d) be a metric space, and U be an open set of X, and Φ be a collection of maps from X to itself. Then, we say Φ is *equicontinuous on* U if for any $\epsilon > 0$ and $x \in U$, there exists $\delta > 0$ such that if $d(x,y) < \delta$, then $d(\phi(x),\phi(y)) < \epsilon$ for any $\phi \in \Phi$.

Definition 2.5.2 (Uniform Lipschitzness)**.** Let (X,d) be a metric space, and U be an open set of X, and Φ be a collection of maps from X to itself. Then, we say Φ is *uniformly Lipschitz on* U if there exists $C > 0$ such that for any $x,y \in U$

$$d(\phi(x),\phi(y)) \le Cd(x,y).$$

The next proposition shows us a relation between the above two definitions.

Proposition 2.5.3. *Let* (X,d) *be a metric space,* U *be an open set and* Φ *be a collection of maps from* X *to itself. If* Φ *is uniformly Lipschitz on* U*, then* Φ *is equicontinuous on* U*.*

The proof may follow immediately and we omit it.

Now let us focus on the projective lines over non-Archimedean fields and their chordal metrics and consider the dynamical systems of them. We first define the Fatou set and the Julia set as follows.

Definition 2.5.4 (The Fatou Set and The Julia set)**.** Let $(K, |\cdot|)$ be a non-Archimedean field and ϕ be a map from $\mathbb{P}^1(K)$ to itself. We define *the Fatou set* $\mathcal{F}(\phi)$ *of* ϕ as the maximal open set on which the a family of iteration $\{\phi^n\}_{n\in\mathbb{N}}$ of ϕ is equicontinuous, and *the Julia set* $\mathcal{J}(\phi)$ *of* ϕ as the complement of the Fatou set of ϕ of $\mathbb{P}^1(K)$.

Note that it is clear from the definition that the Fatou set is always open and the Julia set is closed.

Example 2.5.5. Let us consider a map $f(z) := z^2 \in \mathbb{C}_2[z]$. Then, we have that

$$\overline{D}_1(0) \subset \mathcal{F}(f).$$

Indeed, by Proposition 2.3.5, $\overline{D}_1(0)$ is an open subset on $\mathbb{P}^1(K)$. Moreover, one may easily check that

$$\begin{aligned}\rho_2(f([z,1]), f([w,1])) = \rho_2([z^2,1],[w^2,1]) &= \frac{|z^2-w^2|_2}{\max\{|z^2|_2, |1|_2\}\max\{|w^2|_2, |1|_2\}} \\ &= |z^2-w^2|_2 = |z+w|_2 \cdot |z-w|_2 \le |z-w|_2 = \rho_2([z,1],[w,1])\end{aligned}$$

for any $z, w \in \overline{D}_1(0)$. It follows immediately that for any $n \in \mathbb{N}$

$$\rho_2([f^n(z),1],[f^n(w),1]) \le \rho_2([z,1],[w,1]).$$

That is, $\Phi := \{f^n\}_{n\in\mathbb{N}}$ is uniformly Lipschitz on $\overline{D}_1(0)$. By Proposition 2.5.3, we have $\overline{D}_1(0) \subset \mathcal{F}(f)$.

Let us close this subsection with some properties of the Fatou set and the Julia set. Let us fix an algebraic closed complete field $(K, |\cdot|)$ of characteristic zero.

Proposition 2.5.6 (Complete Invariance)**.** *Let f be a polynomial map over K. Then,*

$$f(\mathcal{F}(f)) = \mathcal{F}(f), \quad f(\mathcal{J}(f)) = \mathcal{J}(f).$$

Proposition 2.5.7. *Let f be a polynomial map over K. Then,*

$$\mathcal{F}(f^n) = \mathcal{F}(f), \quad \mathcal{J}(f^n) = \mathcal{J}(f)$$

for all $n \in \mathbb{N}$.

The proofs are the same as those in complex dynamical system. See [S, Proposition 5.18].

Proposition 2.5.8. *Let f be a polynomial map over K and S be a map defined as in Proposition 2.4.5. Then,*

$$\mathcal{F}(S\circ f\circ S^{-1}) = S(\mathcal{F}(f)), \quad \mathcal{J}(S\circ f\circ S^{-1}) = S(\mathcal{J}(f)).$$

It follows immediately from Proposition 2.4.6.

Example 2.5.9 (No Julia Set)**.** Let us consider $f(z) := z^2 \in \mathbb{C}_2[z]$. Then, we have shown that

$$\overline{D}_1(0) \subset \mathcal{F}(f).$$

On the other hand, one may easily check that

$$S(\mathbb{P}^1(\mathbb{C}_2) - \overline{D}_1(0)) = D_1(0), \quad S\circ f\circ S^{-1}([X,Y]) := [X^2, Y^2].$$

Similarly, we can have $D_1(0) \subset \mathcal{F}(S\circ f\circ S^{-1})$. By Proposition 2.5.8, we have

$$D_1(0) \subset S(\mathcal{F}(f)).$$

Since $S\circ S = Id_{\mathbb{P}^1(\mathbb{C}_2)}$, we have

$$\mathbb{P}^1(\mathbb{C}_2) - \overline{D}_1(0) \subset \mathcal{F}(f).$$

To sum up, we have

$$\mathbb{P}^1(\mathbb{C}_2) \subset \mathcal{F}(f).$$

2.6 Multiplier

In 2.6, we will consider the multiplier which is an analogue of the multiplier in complex dynamical systems. After defining the multiplier, we will consider some properties of it. In particular, we will see the relation with the Fatou set and the Julia set. This subsection is based on Silverman's textbook [S, Section 5.4].

Definition 2.6.1 (Multiplier)**.** Let $(K, |\cdot|)$ be a non-Archimedean field, f be a polynomial map over K with $\deg(f) \geq 2$, and $\alpha \in \mathbb{P}^1(K)$ be a periodic point with prime period N. *The multiplier* $\lambda_f(\alpha)$ *of* f *at* α is defined by

$$\lambda_f(\alpha) := \begin{cases} (f^N)'(\alpha) & (\alpha \neq \infty), \\ 0 & (\alpha = \infty). \end{cases}$$

One can check that the definition of the multiplier is well-defined. Using the multiplier, we classify periodic points of non-Archimedean dynamical systems.

Definition 2.6.2 (Classification of Periodic Points)**.** Let $(K, |\cdot|)$ be a non-Archimedean field, ϕ be a polynomial map over K, and $\alpha \in \mathbb{P}^1(K)$ be a periodic point of f with period N. Then, we call α

an attracting periodic point if $|\lambda_\phi(a)| < 1$,
a repelling periodic point if $|\lambda_\phi(a)| > 1$,
a neutral periodic point if $|\lambda_\phi(a)| = 1$.

Example 2.6.3. Let us consider $f(z) := z^2$ on $\mathbb{C}_3$. Then, one may easily check that

$$Fix(f) = \{0, 1, \infty\}.$$

Then, 0 is an attracting fixed point. Indeed, it is easy to check that $\lambda_f(0) = 0$. Moreover, ∞ is also an attracting fixed point since $\lambda_f(\infty) = 0$. On the other hand, 1 is a neutral fixed point since $|\lambda_f(1)|_3 = |2|_3 = 1$.

Let us wrap up this subsection with some propositions. Let $(K, |\cdot|)$ be an algebraically closed complete non-Archimedean field of characteristic zero, and f be a polynomial map over K with $\deg(f) \geq 2$.

Proposition 2.6.4. *The non-repelling periodic points are in the Fatou set.*

Proof. Let $\alpha \in \mathbb{P}^1(K)$ be a non-repelling periodic point. We will prove this statement in three steps for two distinct cases.

Case 1: $\lambda_f(\alpha) \neq 0$.

Then, by Corollary 2.2.11, there exists some $1 \geq R > 0$ such that

$$|f(z) - f(w)| = |f'(\alpha)||z - w|$$

for any $z, w \in \overline{D}_R(\alpha)$.

Step 1: $\alpha \in Fix(f)$, and $|\alpha| \leq 1$.

Since $|\alpha| \leq 1$ and $\alpha \in Fix(f)$, we have

$$|f(z)| = |f(z) - f(\alpha) + f(\alpha)| \leq \max\{|f'(\alpha)||z-\alpha|, |\alpha|\} \leq 1$$

for any $z \in \overline{D}_R(\alpha)$. This implies that

$$\begin{aligned}\rho_K(f([z,1]), f([w,1])) = \rho_K([f(z),1],[f(w),1]) &= \frac{|f(z)-f(w)|}{\max\{|f(z)|,1\}\max\{|f(w)|,1\}} \\ &= |f'(\alpha)|\frac{|z-w|}{\max\{|z|,1\}\max\{|w|,1\}} \leq \rho_K([z,1],[w,1])\end{aligned}$$

for any $z, w \in \overline{D}_R(\alpha)$. It is not difficult to show that for any $n \in \mathbb{N}$ and $z, w \in \overline{D}_R(\alpha)$,

$$\rho_K(f^n([z,1]), f^n([w,1])) \leq \rho_K([z,1],[w,1]).$$

This implies that $\{\phi_f^i\}_{i\in\mathbb{N}}$ is uniformly Lipschitz on $\overline{D}_R(\alpha)$.

Step 2: $\alpha \in Fix(f)$, and $|\alpha| > 1$.

Since

$$|\alpha| > 1 \geq |f'(\alpha)||z-\alpha| = |f(z) - f(\alpha)|$$

for all $z \in \overline{D}_R(\alpha)$, we have

$$|f(z)| = |f(z) - f(\alpha) + f(\alpha)| = \max\{|f(z)-f(\alpha)|, |f(\alpha)|\} = |\alpha|$$

for all $z \in \overline{D}_R(\alpha)$. Moreover, since

$$|\alpha| > 1 \geq |z-\alpha|$$

for all $z \in \overline{D}_R(\alpha)$, we have

$$|z| = |z-\alpha+\alpha| = \max\{|z-\alpha|, |\alpha|\} = |\alpha|$$

for all $z \in \overline{D}_R(\alpha)$. Thus, it follows that

$$\begin{aligned}\rho_K(f([z,1]), f([w,1])) &= \frac{|f(z)-f(w)|}{\max\{|f(z)|,1\}\max\{|f(w)|,1\}} = \frac{|f(z)-f(w)|}{\max\{|\alpha|,1\}\max\{|\alpha|,1\}} \\ &= |f'(\alpha)|\frac{|z-w|}{\max\{|z|,1\}\max\{|w|,1\}} \leq \rho_K([z,1],[w,1])\end{aligned}$$

for all $z, w \in \overline{D}_R(\alpha)$. It is not difficult to show that for any $n \in \mathbb{N}$ and $z, w \in \overline{D}_R(\alpha)$,

$$\rho_K(f^n([z,1]), f^n([w,1])) \leq \rho_K([z,1],[w,1]).$$

This implies that $\{\phi_f^i\}_{i\in\mathbb{N}}$ is uniformly Lipschitz on $\overline{D}_R(\alpha)$.

Step 3: $\alpha \in Per(f) - Fix(f)$.

Suppose that α is a periodic point of f with period $N \in \mathbb{N}$. Then, by Step 1 and 2 and Proposition 2.5.7, we conclude

$$\alpha \in \mathcal{F}(f^N) = \mathcal{F}(f).$$

Case 2: $\lambda_f(\alpha) = 0$.

Let us prove it in the two steps.

Step 1: $\alpha \in K$.

The proof proceeds in the same way as that of Case 1 since there exists some $1 \geq R > 0$ such that

$$|f(z) - f(w)| \leq |z - w|$$

for all $z, w \in \overline{D}_R(\alpha)$ so we omit the details. See [S, Proposition 5.20].

Step 2: $\alpha = \infty$.

Suppose that

$$f(z) := a_0 + a_1 z + \cdots + a_N z^N \in Poly(K)$$

where $\{a_i\}_{i=0}^N \subset K$ and $a_N \neq 0$.

Then, one can check that

$$S \circ \phi_f \circ S^{-1}[X, Y] = [X^N, a_N Y^N + a_{N-1} Y^{N-1} X + \cdots + a_0 X^N] \quad (X \neq 0).$$

Considering

$$g(z) := a_N + a_{N-1} z + \cdots + a_0 z^N \in Poly(K),$$

it is clear that there exists some $r > 0$ such that

$$g(z) \neq 0$$

for all $z \in \overline{D}_r(0)$. It follows from Corollary 2.2.18 that there exists some $c > 0$ such that for all $z \in \overline{D}_r(0)$,

$$|g(z)| = c.$$

Now we set $C := \dfrac{r^N}{c} > 0$ and $R := \min\{1, C^{-1}, r\} > 0$. Then, one may show that

$$\frac{|z^N g(w) - w^N g(z)|}{|g(w)||g(z)|} \leq |z - w|$$

for all $z, w \in \overline{D}_R(0)$. See [S, Proposition 5.8, Proposition 5.10]. Hence, we have

$$\begin{aligned}
\rho_K(S \circ f \circ S^{-1}[z, 1], S \circ f \circ S^{-1}[w, 1]) &= \rho_K([z^N, g(z)], [w^N, g(w)]) \\
&= \frac{|z^N g(w) - w^N g(z)|}{\max\{|z^N|, |g(z)|\} \max\{|w|^N, |g(w)|\}} \\
&\leq \frac{|z - w|}{\max\{|z^N|, 1\} \max\{|w|^N, 1\}} \\
&= \frac{|z - w|}{\max\{|z|, 1\} \max\{|w|, 1\}} = \rho_K([z, 1], [w, 1]).
\end{aligned}$$

It is easy to check that for all $n \in \mathbb{N}$,

$$\rho_K(S \circ f \circ S^{-1}[z,1], S \circ f \circ S^{-1}[w,1]) \leq \rho_K([z,1],[w,1]).$$

Thus, $\{S \circ f^i \circ S^{-1}\}_{i \in \mathbb{N}}$ is uniformly continuous on $\overline{D}_R(0)$. It follows from Proposition 2.5.8 that $\infty \in \mathcal{F}(\phi_f)$.

□

In particular, we have the following corollary.

Corollary 2.6.5. *Let f be a polynomial map over K with* $\deg(f) \geq 2$*. Then, the Fatou set $\mathcal{F}(f)$ is not empty-set.*

Proof. f has a non-repelling fixed point at ∞. Moreover, by Proposition 2.6.4, $\infty \in \mathcal{F}(\phi_f)$. □

Similarly, one can check the following proposition. See also [S, Proposition 5.20 (b)].

Proposition 2.6.6. *Let f be a polynomial map over K. Then, the repelling periodic points of f are in the Julia set.*

2.7 Montel's Theorems

In 2.7, we will see an analogue of Montel's theorem of complex analysis for non-Archimedean fields. See [B, Theorem 3.3.4] or [M, Theorem 3.7] for Montel's Theorem in complex analysis. L-C. Hsia, who proved the theorem initially, has shown that it holds for a collection of rational maps over K in his paper [H, MAIN THEOREM] and considers the relation between repelling periodic points and Julia set of non-Archimedean fields as Fatou and Julia do in the complex dynamical systems. We will see it in 2.10 as an application of Montel's theorem for non-Archimedean fields. This subsection is based on L-C. Hsia's paper [H] and J. Silverman's textbook [S, Section 5.6].

Let us fix an algebraically closed complete non-Archimedean field $(K, |\cdot|)$ of characteristic zero throughout this subsection. Let us begin with a simple example.

Example 2.7.1. Suppose that $F \subset Poly(K)$ is a collection of polynomial maps over K such that

$$f(\overline{D}_1(0)) \subset \overline{D}_1(0)$$

for any $f \in F$. Then, for each $f \in F$,

$$\begin{aligned}|f(z) - f(w)| = |z-w| \cdot |f'(0) + f^{(2)}(0)(z+w) + \cdots +\\ f^{(N)}(0)(z^{N-1} + z^{N-2}w + \cdots + w^{N-1})| \leq |z-w|\end{aligned}$$

for any z and $w \in \overline{D}_1(0)$ where N is the degree of f. Moreover, since $|f(z)| \leq 1$ for any $z \in \overline{D}_1(0)$, we have that

$$\begin{aligned}\rho_K(f([z,1]), f([w,1])) &= \frac{|f(z)-f(w)|}{\max\{|f(z)|,1\}\max\{|f(w)|,1\}} \leq \frac{|z-w|}{\max\{|f(z)|,1\}\max\{|f(w)|,1\}}\\ &= |z-w| = \frac{|z-w|}{\max\{|z|,1\}\max\{|w|,1\}} = \rho_K([z,1],[w,1])\end{aligned}$$

for any $z, w \in \overline{D}_1(0)$ and $f \in F$. This implies that F is uniformly Lipschitz on $\overline{D}_1(0)$.

Theorem 2.7.2. *Let F be a collection of polynomial maps over K. Suppose that there exists at least one element $\alpha \in K$ such that*

$$\bigcup_{f \in F} f(\overline{D}_1(0)) \cap \{\alpha\} = \emptyset.$$

Then, F is uniformly Lipschitz on $\overline{D}_1(0)$ with respect to the chordal metric ρ_K.

Proof. Let us consider the following two cases.

Case 1: The omitted point α is zero.

Let us fix any $f \in F$. Then, it follows immediately that f has no zeros in $\overline{D}_1(0)$. Thus, by Theorem 2.2.17, for any $z \in \overline{D}_1(0)$,

$$|f(z)| = |f(0)|.$$

On the other hand, it is easy to check from the proof of Theorem 2.2.17 that

$$\begin{aligned} |f(z) - f(w)| = |z - w| \cdot |f'(0) + f^{(2)}(0)(z + w) + \cdots + \\ f^{(N)}(0)(z^{N-1} + z^{N-2}w + \cdots + w^{N-1})| \leq |f(0)||z - w| \end{aligned}$$

for any z and $w \in \overline{D}_1(0)$ where N is the degree of f. Let us dentate $|f(0)|$ by C. It follows that

$$\rho_K(f([z,1]), f([w,1])) \leq \frac{C|z-w|}{\max\{1, |f(z)|\} \max\{1, |f(w)|\}} = \frac{C|z-w|}{\max\{1, C\} \max\{1, C\}}$$

for any $z, w \in \overline{D}_1(0)$.

Now we first assume that $C \leq 1$. Then, we immediately have that

$$\begin{aligned} \rho_K(f([z,1]), f([w,1])) &\leq \frac{C|z-w|}{\max\{1, |f(z)|\} \max\{1, |f(w)|\}} = \frac{C|z-w|}{\max\{1, C\} \max\{1, C\}} \\ &\leq 1 \cdot |z - w| = \frac{|z-w|}{\max\{|z|, 1\}, \max\{|w|, 1\}} = \rho_K([z,1],[w,1]) \end{aligned}$$

for any $z, w \in \overline{D}_1(0)$.

Next we assumed that $C > 1$. Then, we also have that

$$\begin{aligned} \rho_K(f([z,1]), f([w,1])) &\leq \frac{C|z-w|}{\max\{1, |f(z)|\} \max\{1, |f(w)|\}} = \frac{C|z-w|}{\max\{1, C\} \max\{1, C\}} \\ &= \frac{C \cdot |z-w|}{C \cdot C} = \frac{\rho_K([z,1],[w,1])}{C} \leq \rho_K([z,1],[w,1]) \end{aligned}$$

for any $z, w \in \overline{D}_1(0)$.

Since we choose an arbitrary $f \in F$, this implies that F is uniformly Lipschitz on $\overline{D}_1(0)$.

Case 2: The omitted point α is not 0.

Now we consider

$$\begin{aligned} T_\alpha : \mathbb{P}^1(K) &\to \mathbb{P}^1(K) \\ [X, Y] &\mapsto [X - \alpha Y, Y] \end{aligned}$$

and define

$$F_\alpha := \{T_\alpha \circ f \mid f \in F\}.$$

Then we show that

Claim 1 F_α is uniformly Lipschitz on $\overline{D}_1(0)$.

Proof of Claim 1. For any $f \in F$, it follows easily that

$$T_\alpha \circ f([z, 1]) \neq 0.$$

It follows from Case 1 that F_α is uniformly Lipschitz on $\overline{D}_1(0)$. □

Claim 2 F is also uniformly Lipschitz on $\overline{D}_1(0)$.

Proof of Claim 2. It follows from Claim 1 that there exists some $C > 0$ such that

$$\rho_K(T_\alpha \circ f([z, 1]), T_\alpha \circ f([w, 1])) \leq C\rho_K([z, 1], [w, 1])$$

for all $z, w \in \overline{D}_1(0)$. Since T_α is bijective, it follows from Theorem 2.3.7 that there exists some $C' > 0$ such that

$$\begin{aligned}\rho_K(f([z, 1]), f([w, 1])) &\leq \rho_K(T_\alpha^{-1} \circ T_\alpha \circ f([z, 1]), T_\alpha^{-1} \circ T_\alpha \circ f([w, 1])) \\ &\leq C'\rho_K(T_\alpha \circ f([z, 1]), T_\alpha \circ f([w, 1])) \leq C'C\rho_K([z, 1], [w, 1])\end{aligned}$$

for any $z, w \in \overline{D}_1(0)$ and $f \in F$. □

□

As an application of Theorem 2.7.2, Montel's theorem in non-Archimedean fields can be obtained.

Corollary 2.7.3 (Montel's theorem)**.** *Let $\overline{D}_r(a)$ be a rational closed disk, and F be a collection of polynomial maps over K. Suppose that there exists at least one element $\alpha \in K$ such that*

$$\bigcup_{f \in F} f(\overline{D}_r(a)) \cap \{\alpha\} = \emptyset.$$

Then F is uniformly Lipschitz on $\overline{D}_r(a)$ with respect to the chordal metric ρ_K.

Proof. Let us consider

$$\begin{aligned}T : K &\to K \\ z &\mapsto c \cdot z + a\end{aligned}$$

where $c \in K$ satisfies $|c| = r$. It is clear that T is bijective from $\overline{D}_1(0)$ to $\overline{D}_r(a)$. Considering

$$F := \{f \circ T \mid f \in F\},$$

we obtain the following claim.

Claim $F \subset Poly(K)$. Moreover, F is uniformly Lipschitz on $\overline{D}_1(0)$.

Proof of Claim 1. The first statement is clear since T is a polynomial map over K. To show the second statement, we show that

$$\bigcup_{f \in F} f \circ T(\overline{D}_1(0)) \cap \{\alpha\} = \emptyset$$

by contradiction. Let us assume that there exists some $g \in F$ and $w \in \overline{D}_1(0)$ such that $g(cw+a) = g \circ T(w) = \alpha$. Since $cw + a \in \overline{D}_r(a)$, This is a contradiction. The statement can be proved by Theorem 2.7.2. □

Thus, there exists some $C > 0$ such that

$$\begin{aligned}\rho_K([f([z,1])],[f([w,1])]) &= \rho_K([f(z),1],[f(w),1]) \\ &= \rho_K([f \circ T \circ T^{-1}(z),1],[f \circ T \circ T^{-1}(w),1]) \\ &\leq C\rho_K([T^{-1}(z),1],[T^{-1}(w),1])\end{aligned}$$

for any $z, w \in \overline{D}_r(a)$. On the other hand, by Theorem 2.3.7, there exists some $C' > 0$ such that

$$\rho_K([T^{-1}(z),1],[T^{-1}(w),1]) \leq C'\rho_K([z,1],[w,1])$$

for any $z, w \in \overline{D}_r(a)$. This implies that F is uniformly Lipschitz on $\overline{D}_r(a)$ with respect to ρ_K. □

Motel's theorem is very helpful to determine if a given open set is in the Fatou as we do in complex dynamical systems. In the following three subsections, we will see several applications of Motel's theorem.

2.8 An Application of Montel's Theorem I: Properties of the Julia Sets

In 2.8, we will see an application of Hsia's theorem to non-Archimedean dynamical systems. In particular, one will notice the statements in this subsection are true for complex dynamical systems. See [M, Corollary 4.13, Corollary 4.14] for an application of Motel's theorem to complex dynamical systems. This subsection is based on L-C. Hsia's paper [H] and J. Silverman's textbook [S, Section 5.6]. Let $(K, |\cdot|)$ be a non-Archimedean field and f be a polynomial map over K with $\deg(f) \geq 2$.

Proposition 2.8.1. *Suppose that $\mathcal{J}(\phi_f) \neq \emptyset$. Then, the backward orbit of any point of the Julia set is dense in the Julia set with respect to ρ_K. That is,*

$$\overline{\bigcup_{n \in \mathbb{N}} f^{-n}(\{\alpha\})} = \mathcal{J}(f)$$

for all $\alpha \in \mathcal{J}(f)$.

Proof. It follows from Proposition 2.5.6 that for any $\alpha \in \mathcal{J}(f)$ and any $n \in \mathbb{N}$,

$$f^{-n}(\alpha) \subset \mathcal{J}(f).$$

Moreover, since the Julia set is closed, we have that the closure of the backward orbit of any point of the Julia set is contained in the Julia set.

Now let us fix an arbitrary element $z \in \mathcal{J}(f)$. By Corollary 2.7.3, for any closed rational disk $\overline{D}_r(z)$,

$$\bigcup_{n \in \mathbb{N}} f^n(\overline{D}_r(z)) = K.$$

On the other hand, by Proposition 2.6.4, we have that

$$\alpha \in \mathcal{J}(f) \subset K = \bigcup_{n \in \mathbb{N}} f^n(\overline{D}_r(z)).$$

Hence, there exists some $N \in \mathbb{N}$ such that

$$\alpha \in f^N(\overline{D}_r(z)).$$

That is,

$$f^{-N}(\{\alpha\}) \subset \overline{D}_r(z).$$

□

Proposition 2.8.2. *There are no isolated points in the Julia set.*

The proof can be found in [S, Corollary 5.32 (c)].

Corollary 2.8.3. *Suppose that $\mathcal{J}(f) \neq \emptyset$. Then, the Julia set is uncountable.*

Proof. (By contradiction) Assume that the Julia set is countable, or finite. Then, the Julia set can be written as

$$\mathcal{J}(f) = \{x_i\}_{i \in \mathbb{N}} \subset K.$$

Then, since the Julia set has no isolated points, we have

$$\overline{X_i} = \mathcal{J}(f)$$

where $X_i = \mathcal{J}(f) - \{x_i\}$ for all $i \in \mathbb{N}$ with respect to ρ_K. Since the Julia set is a closed subset of $\mathbb{P}^1(K)$ and $\mathbb{P}^1(K)$ is a complete metric space, the Julia set is also a complete metric space. See Proposition 5.2.1. It follows from the Theorem 5.2.2 that

$$\overline{\bigcap_{i \in \mathbb{N}} X_i} = \mathcal{J}(f).$$

However, it is easy to check that

$$\bigcap_{i \in \mathbb{N}} X_i = \emptyset.$$

Thus, the Julia set must be empty. This is a contradiction to our assumption. □

2.9 An Application of Montel's Theorems II: The Fatou Set of Quadratic Maps

In 2.9, we will see another application of Montel's theorem to non-Archimedean dynamical systems. We will try to understand the dynamic systems generated by quadratic polynomial maps over $\mathbb{C}_p$. A few results in this subsection are some of the original results of the author. See Proposition 2.9.2 to

Proposition 2.9.5. Throughout this subsection, we fix a prime number p and focus on the dynamics of the quadratic maps

$$\begin{aligned} f : \mathbb{C}_p \times \mathbb{C}_p &\to \mathbb{C}_p \\ (z, c) &\mapsto z^2 + c. \end{aligned}$$

To ease notation, we shall use the notation $f_c(z) := f(z, c)$. As we have seen in Example 2.5.9, it is possible to have empty Julia sets in non-Archimedean dynamical systems. In fact, one of interesting results of this subsection is that there are many non-Archimedean dynamical systems with no Julia sets.

Theorem 2.9.1. *Let f be a polynomial map over $\mathbb{C}_p$ with $\deg(f) \geq 2$. Suppose that $S \subset K$ satisfies that*

$$S \neq \emptyset, \quad f^{-1}(S) \subset S, \quad \overline{S} = S$$

where $\overline{S}$ is the topological closure of S with respect to ρ_p. Then, $\mathcal{J}(f) \subset S$.

Proof. It is clear if $\mathcal{J}(f) = \emptyset$. Let us assume that $\mathcal{J}(f) \neq \emptyset$, and set $U := K - S$. It follows immediately that

$$f(U) \subset U, \quad U \subset K\text{: open},$$

which gives

$$f^n(U) \subset U$$

for any $n \in \mathbb{N}$. Moreover, since S is non-empty, there exists some $\alpha \in S \subset K$ such that

$$\bigcup_{n \in \mathbb{N}} f^n(U) \cap \{\alpha\} = \emptyset.$$

By Montel's theorem and Proposition 2.3.5, we obtain that $U \subset \mathcal{F}(f)$. Thus, $\mathcal{J}(f) \subset S$. □

Using this theorem, we analyse the dynamics of $(\mathbb{P}^1(\mathbb{C}_p), f_c)$.

Proposition 2.9.2. *If $|c|_p \leq 1$, then $\mathcal{J}(f_c) = \emptyset$.*

Proof. We will show the following claims.

Claim 1 $\mathcal{J}(f_c) \subset \overline{D}_1(0)$.

Proof of Claim 1. One may easily check that

$$f_c(\mathbb{P}^1(\mathbb{C}_p) - \overline{D}_1(0)) \subset \mathbb{P}^1(\mathbb{C}_p) - \overline{D}_1(0),$$

which gives us

$$f_c^{-1}(\overline{D}_1(0)) \subset \overline{D}_1(0).$$

Moreover, it is clear that $\overline{D}_1(0)$ is non-empty and closed. Thus, by Theorem 2.9.1, we have $\mathcal{J}(f_c) \subset \overline{D}_1(0)$. □

Claim 2 $f_c(\overline{D}_1(0)) \subset \overline{D}_1(0)$.

Proof of Claim 2. Since $|c|_p \leq 1$, it follows that for any $z \in \overline{D}_1(0)$

$$|f_c(z)|_p = |z^2 + c|_p \leq \max\{|z|_p^2, |c|_p\} \leq 1.$$

□

One may inductively show from Claim 2 that for any $n \in \mathbb{N}$

$$f_c^n(\overline{D}_1(0)) \subset \overline{D}_1(0).$$

Thus, by Montel's theorem, $\overline{D}_1(0) \subset \mathcal{F}(f_c)$. □

In fact, we can say more than this. Let us begin with the following lemma.

Lemma 2.9.3. *Let g be a non-constant monic polynomial over $\mathbb{C}_p$. Suppose that $|g^{(i)}(0)|_p \leq 1$ for all $i \in \mathbb{N}$. Then, all roots of g must be in $\overline{D}_1(0)$.*

See Proposition 5.1.4 for the proof of Lemma 2.9.3.
Now we can show the following proposition.

Proposition 2.9.4. *Let g be a non-constant monic polynomial over $\mathbb{C}_p$. Suppose that $|g^{(i)}(0)|_p \leq 1$ for all $i \in \mathbb{N}$. Then, $\mathcal{J}(g) = \emptyset$.*

The proof is easily obtained from Corollary 2.7.3 or Example 2.7.1 and we omit it.

Proposition 2.9.5. *Suppose that $p \neq 2$ and $|c|_p > 1$. Then $\mathcal{J}(f_c) \neq \emptyset$. Moreover, $\mathcal{J}(f_c) \subset \mathbb{S}(|c|_p^{\frac{1}{2}})$.*

Proof. Since $\mathbb{C}_p$ is an algebraically closed field and $\deg(f_c) = 2$, there exists $\{\alpha, \beta\} \subset K$ such that

$$f_c(\alpha) = \alpha, \quad f_c(\beta) = \beta.$$

Note that α may coincidence β.

Claim 1 The elements α, β are repelling fixed points.

Proof of Claim 1. Let us first assume that $|\alpha|_p < |c|_p^{1/2}$. It follows that

$$|\alpha|_p^2 < |c|_p, \quad |\alpha|_p < |c|_p^{1/2} < |c|_p.$$

Thus, it follows from Proposition 2.1.5 that

$$|\alpha^2 - \alpha + c|_p = |c|_p > 0.$$

This is a contradiction to $\alpha^2 - \alpha + c = 0$.

Next we assume that $|\alpha|_p > |c|_p^{1/2}$. It follows that

$$|\alpha|_p^2 > |c|_p, \quad |\alpha|_p^2 > |\alpha|_p.$$

Thus, it follows from Proposition 2.1.5 that

$$|\alpha^2 - \alpha + c|_p = |\alpha|_p^2 > 0$$

because $c \neq 0$ means that $\alpha \neq 0$. This is also a contradiction to $\alpha^2 - \alpha + c = 0$.

Furthermore, we easily check that

$$|f_c'(\alpha)|_p = |2|_p|\alpha|_p = |c|_p^{1/2} > 1.$$

This implies that α is a repelling fixed point of f_c. One can check that the proof proceeds in the same way for β. □

By Proposition 2.6.6, $\alpha, \beta \in \mathcal{J}(f_c)$. In particular, this implies that $\mathcal{J}(f_c) \neq \emptyset$. Next, we show the following claim.

Claim 2 For any $z \in \mathbb{S}(|c|_p^{1/2})$, $f_c^{-1}(\{z\})$ is in $\mathbb{S}(|c|_p^{\frac{1}{2}})$.

The proof proceeds in the same way as that of Claim 1 so we omit it. On the other hand, it is clear that $\mathbb{S}(|c|_p^{1/2})$ is non-empty and backward invariant under ϕ_f. Thus, by Proposition 2.9.1, we have

$$\mathcal{J}(f_c) \subset \mathbb{S}(|c|_p^{1/2}).$$

□

2.10 An Application of Montel's Theorem III: The Julia Set and Periodic Points

In 2.10, we will see the relationship between the Julia set and the repelling periodic points of rational maps as another application of Hsia's theorem to non-Archimedean dynamical systems. We will omit the proof of the main theorem in this subsection, but we will see some examples related to the main theorem. This subsection is based on L-C. Hsia's paper [H] and J. Silverman's textbook [S, Section 5.7].

Let us begin with a motivation of this subsection. The following theorem was proved by G. Julia and P. Fatou.

Theorem 2.10.1. *Let f be a rational map over $\mathbb{C}$ with $\deg(f) \geq 2$. Then, the set of the repelling periodic points of f is dense in the Julia set of f.*

One can find the proof in [M, Theorem 14.1] or [B, Theorem 6.9.2]. The analogue of Theorem 2.10.1 in non-Archimedean dynamics was conjectured by L-C. Hsia in his paper [H, CONJECTURE 4.3]. Let $(K, |\cdot|)$ be an algebraically closed complete non-Archimedean field of characteristic zero.

Conjecture 2.10.2. *Let f be a rational map over $\mathbb{C}_p$ with $\deg(f) \geq 2$. Then, the set of the repelling periodic points of f is dense in the Julia set of f.*

It is still an open problem in non-Archimedean dynamical systems. However, Hsia has succeed to prove a close statement.

Theorem 2.10.3. *Let f be a rational map over K with $\deg(f) \geq 2$. Then,*

$$\mathcal{J}(f) \subset \overline{Per(f)}.$$

We will omit the proof. See [S, Theorem 5.37] or [H, THEOREM 3.1] for the proof of Theorem 2.10.3. Now we consider the reasons why we cannot extend Theorem 2.10.3 as in complex dynamics. One of reasons is the number of non-repelling periodic points. In complex dynamics, the number of the non-repelling periodic points in the Fatou set is finite. See [M, Corollary 10.16]. If the number of the non-repelling periodic points in the Fatou set is finite, we can easily prove Conjecture 2.10.2. However, although the number of the super-attracting periodic points, which is attracting periodic points with multiplier 0, is finite in non-Archimedean dynamical systems, it is possible to have attracting periodic points.

Example 2.10.4 (Only Attracting Periodic Points)**.** Let p be a prime number, and consider

$$\begin{aligned} f : \mathbb{C}_p &\to \mathbb{C}_p \\ z &\mapsto z^p. \end{aligned}$$

Then, it is easy to check that

$$\#(Per_N(f^N)) = p^N + 1$$

for any $N \in \mathbb{N}$. This implies that

$$\#(Per(f)) = \infty.$$

Moreover, one may prove that

$$Per(f) \subset \overline{D}_1(0) \cup \{\infty\}.$$

This implies that for any $\alpha \in Per(f) - \{\infty\}$, there exists some smallest $M \in \mathbb{N}$ such that $f^M(\alpha) = \alpha$, and, by the chain rule, we have

$$\lambda_f(\alpha) = |(f^M)'(\alpha)|_p = |p\alpha_1^{p-1}|_p|p\alpha_2^{p-1}|_p \cdots |p\alpha_M^{p-1}|_p \leq \frac{1}{p^M} < 1$$

where $\alpha_i := f^i(\alpha)$ for $i = 1, 2, \cdots, M$. Moreover, it follows from Definition 2.6.1 that ∞ is an attracting fixed point. Thus, we conclude that all periodic points of ϕ_f are attracting periodic points. One can check that

$$\lambda_f(0) = \lambda_f(\infty) = 0, \quad \lambda_f(\alpha) \neq 0$$

for all a in $Per(f) - \{0, \infty\}$.

Let us wrap up this subsection with a theorem proven by J. Bezivin in [JB, THÈORÉM 3], which almost gives the answer to Conjecture 2.10.2.

Theorem 2.10.5. *Let f be a rational map over $\mathbb{C}$ with $\deg(f) \geq 2$. Suppose that there exists at least one repelling periodic point of f. Then, the set of the repelling periodic points of f is dense in the Julia set of f.*

2.11 Disk Components over Non-Archimedean Fields

In 2.11, we will define new domains, which will be called disk components, and see some properties of it. This subsection is based on Silverman's textbook [S, Section 5.8] and [RB01].

Let $(K, |\cdot|)$ be an algebraically closed complete non-Archimedean field of characteristic zero throughout this subsection.

Let us define disk component, which is an analogue of path-connected component of the complex dynamics.

Definition 2.11.1 (Disk Component)**.** Let X be a topological space, $\mathcal{D}$ be a collection of subsets of X, and U be a non-empty open set of X. Then, $V \subset U$ is called *a disk component of U with respect to $\mathcal{D}$* if V is a non-empty open subset of U and satisfy that for any $v_1, v_2 \in V$, there exist some $N \in \mathbb{N}$ and a sequence $\{D^i\}_{i=1}^N$ in $\mathcal{D}$ in V such that

$$v_1 \in D^1, \quad v_2 \in D^N, \quad D^i \cap D^{i+1} \neq \emptyset$$

for any $i \in \{1, 2, \cdots, N\}$.

Example 2.11.2. Let us consider $\mathbb{C}$ with the Euclidean topology. Then, $D_r(a)$ is a disk component of $\mathbb{C}$ for all $a \in \mathbb{C}$ and $r > 0$. On the other hand, $(D_r(a) - \overline{D}_{r_1}(a)) \cup D_{r_2}(a)$ is not a disk component of $\mathbb{C}$ for all $a \in \mathbb{C}$ and $r > r_1 > r_2 > 0$.

One may notice that the following proposition is true.

Proposition 2.11.3. *Let U be a non-empty open subset of $\mathbb{C}$. Then, U is a path-connected component if and only if U is a disk-connected component.*

The proof follows easily and we omit it.

Example 2.11.4 (Non-Archimedean Case)**.** Let p be a prime number and Let us consider $(\mathbb{C}_p, |\cdot|_p)$. Then, $D_r(a)$ and $\overline{D}_r(a)$ are disk components of $\mathbb{C}_p$ for all $a \in \mathbb{C}$ and $r > 0$.

Proposition 2.11.5. *Let U be a non-empty open subset of K. If V is a non-empty disk component of U, then there exists some $a \in K$, and $r > 0$ such that either*

$$V = K, \quad \overline{D}_r(a), \quad \text{or} \quad D_r(a).$$

Proof. Let us fix an arbitrary $z_0 \in V$ and set

$$r_0 := \sup\{r > 0 \mid \overline{D}_r(z_0) \subset V\}.$$

Note that r_0 is well-defined since $z_0 \in V \subset U$ is an interior point. It is clear that if $r_0 = \infty$, then $V = K$ so we assume that $r_0 < \infty$.

Claim $D_{r_0}(z_0) \subset V \subset \overline{D}_{r_0}(z_0)$.

Proof of Claim. We first consider the proof of $D_{r_0}(z_0) \subset V$. For any $z \in D_{r_0}(z_0)$, we have

$$|z - z_0| < r_0.$$

It follows from the construction of r_0 that

$$z \in D_{|z-z_0|}(z_0) \subset \overline{D}_{r_0}(z_0) \subset V.$$

Next, we consider the proof of $V \subset \overline{D}_{r_0}(z_0)$. For any $z \in V$, there exists some $N \in \mathbb{N}$ and $\{\overline{D}_{r_i}(a_i) \subset V\}_{i=1}^N$ such that

$$z_0 \in \overline{D}_{r_1}(a_1), \quad z \in \overline{D}_{r_N}(a_N), \quad \overline{D}_{r_i}(a_i) \cap \overline{D}_{r_{i+1}}(a_{i+1}) \neq \emptyset$$

for any $i = 1, 2, \cdots, N-1$. Moreover, by using Corollary 2.1.21 inductively, we have that there exists some $R > 0$ such that

$$\overline{D}_R(z_0) = \bigcup_{i=1}^N \overline{D}_{r_i}(a_i) \subset V.$$

In particular, by the construction of r_0, $R \leq r_0$. Hence, we have

$$|z - z_0| \leq R \leq r_0.$$

□

Now we consider the following two cases.

Case 1: There exists some $w_0 \in V$ such that $|w_0 - z_0| = r_0$.

In this case, we have the following claim.

Claim 2 $V = \overline{D}_{r_0}(z_0)$.

Proof of Claim 2. By Claim 1, it is sufficient to show that

$$\overline{D}_{r_0}(z_0) \subset V.$$

Let us take an arbitrary $z \in \overline{D}_{r_0}(z_0)$. Since $w_0 \in V$, it follows from Definition 2.11.1 that there exists some $N \in \mathbb{N}$ and $\{\overline{D}_{r_i}(a_i)\}_{i=1}^{N}$ in U such that

$$z_0 \in \overline{D}_{r_1}(a_1), \quad w_0 \in \overline{D}_{r_N}(a_N), \quad \overline{D}_{r_i}(a_i) \cap \overline{D}_{r_{i+1}}(a_{i+1}) \neq \emptyset$$

for any $i = 1, 2, \cdots, N-1$.

Moreover, by using Corollary 2.1.21 inductively, we have that there exists some $R' > 0$ such that

$$\overline{D}_{R'}(z_0) = \bigcup_{i=1}^{N} \overline{D}_{r_i}(a_i) \subset V.$$

In particular, by the construction of r_0, $R' \leq r_0$. On the other hand,

$$w_0 \in \bigcup_{i=1}^{N} \overline{D}_{r_i}(a_i) = \overline{D}_{R'}(z_0).$$

implies that $r_0 = |z_0 - w_0| \leq R'$. Thus, we have

$$z \in \overline{D}_{r_0}(z_0) = \overline{D}_{R}(z_0) \subset V.$$

□

Case 2: Every $w \in V$ satisfies $|z_0 - w| \neq r_0$.

In this case, we have the following claim.

Claim 3 $V = D_{r_0}(z_0)$.

Proof of Claim 3. By Claim 1, it is sufficient to show that

$$V \subset D_{r_0}(z_0).$$

Let us choose any $z \in V$ and we show that $|z - z_0| < r_0$ by contradiction. By our assumption of Case 2, we may assume that

$$|z - z_0| > r_0.$$

Then, since V is a disk component of U, there exists some $N \in \mathbb{N}$ and $\{\overline{D}_{r_i}(a_i)\}_{i=1}^{N}$ in V such that

$$z_0 \in \overline{D}_{r_1}(a_1), \quad z \in \overline{D}_{r_N}(a_N), \quad \overline{D}_{r_i}(a_i, r_i) \cap \overline{D}_{r_{i+1}}(a_{i+1}) \neq \emptyset$$

for any $i = 1, 2, \cdots, N-1$. Moreover, by using Corollary 2.1.21 inductively, there exists some $R > 0$ such that

$$\overline{D}_{R}(z_0) = \bigcup_{i=1}^{N} \overline{D}_{r_i}(a_i) \subset V.$$

Furthermore, since

$$z \in \overline{D}_{r_N}(a_N) \subset \bigcup_{i=1}^{N} \overline{D}_{r_i}(a_i) = \overline{D}_R(z_0),$$

we have

$$r_0 < |z - z_0| \leq R.$$

This implies that

$$\overline{D}_R(z_0) \subset V, \quad \textit{and} \quad r_0 < R.$$

This is a contradiction to the construction of r_0. □

□

Now we consider the set of disk components of $\mathbb{P}^1(K)$. Let us begin with the definition of the set of disks on projective lines.

Definition 2.11.6. Let us define *the collection of closed disks of* $\mathbb{P}^1(K)$ by

$$\mathcal{D}_{closed} := \{\overline{D}_r(a) \subset K \mid a \in K,\ r \in \mathbb{R} \cap |K^\times|\} \cup \{\mathbb{P}^1(K) - D_r(a) \mid a \in K,\ r \in \mathbb{R} \cap |K^\times|\}.$$

Similarly, we define *the collection of open disks of* $\mathbb{P}^1(K)$ by

$$\mathcal{D}_{open} := \{D_r(a) \subset K \mid a \in K,\ r \in \mathbb{R} \cap |K^\times|\} \cup \{\mathbb{P}^1(K) - \overline{D}_r(a) \mid a \in K,\ r \in \mathbb{R} \cap |K^\times|\}.$$

The following proposition gives the shape of disk components of $\mathbb{P}^1(K)$. The proof is the same as Proposition 2.11.5 and we omit it. See [S, Proposition 5.45].

Proposition 2.11.7. *Let U be an non-empty open set and V be a disk component of U with respect to $\mathcal{D}_{closed}$. Then, V is either*

$$\mathcal{P}^1(K), \quad \mathcal{P}^1(K) - \{P\}, \quad \textit{or} \quad V \in \mathcal{D}_{closed} \cup \mathcal{D}_{open}$$

where $P \in \mathbb{P}^1(K)$.

2.12 p-adically Hyperbolic Maps

In 2.12, we consider an analogue of hyperbolic maps in complex dynamical systems. In complex dynamical systems, a rational map ϕ over $\mathbb{C}$ is called *a hyperbolic map* if its Julia set does not contain any critical points of ϕ. This subsection is based on R. Benedetto's paper [RB01] and Silverman's textbook [S, Section 5.8].

Let $(K, |\cdot|)$ be a finite extension of $(\mathbb{Q}_p, |\cdot|_p)$. Note that $(K, |\cdot|)$ is a locally compact and complete non-Archimedean fields of characteristic zero.

Definition 2.12.1. Let f be a polynomial map on $\mathbb{C}_p$ over K with $\deg(f) \geq 2$. Then, f is called *a (p-adically) hyperbolic map* if there is no critical point in $\mathcal{J}(f) \subset \mathbb{C}_p$.

One can find the definition of critical points of a given polynomial map in Definition 2.2.7.

Example 2.12.2. Let p be a prime number and let us consider the map

$$f : \mathbb{C}_p \to \mathbb{C}_p$$
$$z \mapsto z^2 + \frac{1}{p}.$$

It is clear that $f \in \mathbb{Q}_p[z]$. Moreover, the only critical point of f is $0 \in \mathcal{F}(f)$ since

$$\lim_{n\to\infty} f^n(0) = \infty \in \mathcal{F}(f).$$

Example 2.12.3. Let p be an odd prime number and let us consider the map

$$F : \mathbb{C}_p \to \mathbb{C}_p$$
$$z \mapsto \frac{z^p - z^{p-1}}{p} + 1.$$

It is clear that $F \in \mathbb{Q}_p[z]$ and 0 is a critical point of F. However, it follows easily that

$$F(0) = 1, \quad F(1) = 1, \quad F'(1) = 1, \quad |F'(1)|_p = p > 1.$$

In particular, this implies that

$$1 \in \mathcal{J}(F), \quad 0 \in F^{-1}(\mathcal{J}(F)).$$

By Proposition 2.5.6, we have $0 \in \mathcal{J}(F)$. Thus, F is not a hyperbolic map.

Theorem 2.12.4 (A Equivalent Theorem for Hyperbolic Maps)**.** *Let f be a polynomial map over K with $\deg(f) \geq 2$. Then, f is a (p-adically) hyperbolic map if and only if for any finite extension field L of K, there exists some $M \in \mathbb{N}$ such that*

$$|(f^M)(z)| \geq 2$$

for all $z \in \mathcal{J}(f) \cap L$.

We omit the proof. It can be found in [RB01, MAIN THEOREM] or [S, Theorem 5.46].

2.13 No Wandering Domains Theorems

In 2.13, we will see two non-Archimedean no wandering domains theorems, which are analogues of Sullivan's no wandering domains theorem in complex dynamics, proved by R. Benedetto. One of them is related to hyperbolic maps, and we will see its proof in this subsection. We will omit the proof of the other one, but compare with two theorems. This subsection is based on R. Benedetto's papers [RB00], [RB01]. Let us begin with a motivation.

Theorem 2.13.1 (Sullivan's No Wandering Domains Theorems)**.** *Let f be a rational map over $\mathbb{C}$ with $\deg(f) \geq 2$. Then, the Fatou set of f has non-wandering components. That is, for any component U of the Fatou set of f, there exists some $n > m \in \mathbb{N}$ such that $f^n(U) = f^m(U)$.*

See [B, Theorem 8.1.2] or [M, Theorem F.1] for the proof of Theorem 2.13.1. The following conjecture is a natural question in the non-Archimedean fields.

Conjecture 2.13.2. *Let p be a prime number and f be a rational map over $\mathbb{C}_p$ with $\deg(f) \geq 2$. Then, the Fatou set of f has no wandering disk components.*

R. Benedetto has proved partly this conjecture in his paper [RB01]. Moreover, he also proved that this conjecture fails for some polynomial maps over $\mathbb{C}_p$ in [THEOREM 1.1][RB02]. We will consider it in the next subsection.

In this subsection, we will consider Benedetto's no wandering domains theorem for polynomial maps. In fact, he proved it for p-adically hyperbolic rational maps. See [RB01, COROLLARY 3.1]. Let $(K, |\cdot|)$ be a finite extension field of $(\mathbb{Q}_p, |\cdot|_p)$. Note that $(K, |\cdot|_p)$ is a locally compact and complete non-Archimedean field of characteristic zero.

Theorem 2.13.3. *Let f be a polynomial map over K on $\mathbb{C}_p$ with $\deg(f) \geq 2$. If there are no critical points in $\mathcal{J}(f)$ and $\mathcal{J}(f) \subset \overline{D}_1(0)$, then $\mathcal{F}(f)$ has no wandering disk components.*

Proof. (By contradiction) Let us assume that there exists a wandering domain $U \neq \emptyset$ of $\mathcal{F}(f)$. Without loss of generality, we may assume that

$$U \subset \overline{D}_1(0), \quad f^n(U) \subset \overline{D}_1(0)$$

for all $n \in \mathbb{N}$. Let us choose any element α_1 in U and $\gamma_1 > 0$ such that

$$\gamma_1 \in |\mathbb{C}_p^\times|_p, \quad \overline{D}_{\gamma_1}(\alpha_1) \subset U.$$

Setting $L := K(\alpha_1)$, it is clear that

$$\alpha_1 \in L \cap \overline{D}_1(0), \quad f^n(\alpha_1) \in L \cap \overline{D}_1(0)$$

for all $n \in \mathbb{N}$. Moreover, since f is a p-adically hyperbolic map on K, by Theorem 2.12.4, there exists some $M \in \mathbb{N}$ such that

$$|(f^M)'(w)|_p \geq 2$$

for all $w \in \mathcal{J}(\phi_f) \cap L$. To ease notation, we shall use

$$g := f^M,$$

and consider the dynamics of g. It is clear that U is also a wandering domain of g. Now we define $\{(\alpha_i, \gamma_i)\}_{i \in \mathbb{N}}$ as

$$\alpha_i := g^{i-1}(\alpha_1), \quad \overline{D}_{\gamma_i}(\alpha_i) = g^{i-1}(\overline{D}_{\gamma_1}(\alpha_1))$$

for each $i \in \mathbb{N}$. It is clear that $\gamma_i \in |\mathbb{C}_p^\times|_p$ for all $i \in \mathbb{N}$. Then, since $L \cap \mathcal{O}_K$ is compact, this implies that for any subsequence $\{\alpha_{i_j}\}_{j \in \mathbb{N}}$ of $\{\alpha_i\}_{i \in \mathbb{N}}$, there exists some $\beta \in L \cap \mathcal{O}_K$ such that

$$\lim_{j \to \infty} |\alpha_{i_j} - \beta|_p = 0.$$

Moreover, we obtain the following claims.

Claim 1

$$\lim_{i \to \infty} \gamma_i = 0.$$

Proof of Claim 1. (By contradiction) Let us assume that

$$\lim_{i \to \infty} \gamma_i \neq 0.$$

That is, there are some $\epsilon > 0$ and $\{\gamma_{i_j}\}_{j \in \mathbb{N}}$ such that

$$\gamma_{i_j} \geq \epsilon.$$

This implies that

$$\bigcup_{j=1}^{\infty} \overline{D}_{\gamma_{i_j}}(\alpha_{i_j}) \subset L \cap \overline{D}_1(0).$$

On the other hand, since $L \cap \overline{D}_1(0)$ is a topological compact space with respect to $+$, there exists the Haar Measure μ on $L \cap \overline{D}_1(0)$. See Theorem 5.4.4. Thus, it follows from Theorem 5.4.4 that

$$\mu(\overline{D}_\epsilon(0)) > 0, \quad \mu(\overline{D}_\epsilon(\alpha)) = \mu(\overline{D}_\epsilon(0))$$

for all $\alpha \in L$. Since the disks is disjoint, we have that

$$\infty = \infty \cdot \mu(\overline{D}_\epsilon(0)) \leq \mu(\sum_{j=1}^{\infty} \overline{D}_{\gamma_{i_j}}(\alpha_{i_j})) = \sum_{j=1}^{\infty} \mu(\overline{D}_{\gamma_{i_j}}(\alpha_{i_j})) < \mu(L \cap \overline{D}_1(0)) = 1.$$

This is a contradiction. $\square$

Claim 2 There exists some $\{\alpha_{i_j}\}_{j\in\mathbb{N}}$ such that $|g'(\alpha_{i_j})|_p < 1$ for all $j \in \mathbb{N}$.

Proof of Claim 2. It follows from Claim 1 that there exists some subsequence $\{\gamma_{i_j}\}_{j\in\mathbb{N}}$ of $\{\gamma_i\}_{i\in\mathbb{N}}$ such that for each $j \in \mathbb{N}$,

$$\gamma_{i_j+1} < \gamma_{i_j}.$$

On the other hand, since

$$g(\overline{D}_{\gamma_i}(\alpha_i)) = \overline{D}_{\gamma_{i+1}}(\alpha_{i+1})$$

for all $i \in \mathbb{N}$, it follows from Corollary 2.2.20 that for all $i \in \mathbb{N}$,

$$|g'(\alpha_i)|_p \leq \frac{\gamma_{i+1}}{\gamma_i}.$$

In particular, we have

$$|g'(\alpha_{i_j})|_p \leq \frac{\gamma_{i_j+1}}{\gamma_{i_j}} < 1.$$

for any $\{\alpha_{i_j}\}_{j\in\mathbb{N}}$. □

Claim 3 g' is a continuous map on $\mathcal{J}(\phi)$ with respect to $|\cdot|$.

The proof is clear so we omit it. See Corollary 2.2.12.

Now we fix the subsequence $\{\alpha_{i_j}\}_{j\in\mathbb{N}}$ obtained in Claim 2. Since $L \cap \overline{D}_1(0)$ is compact, there exists some $\beta \in L \cap \overline{D}_1(0)$ and subsequence $\{\alpha_{i_{j_k}}\}_{k\in\mathbb{N}}$ of $\{\alpha_{i_j}\}_{j\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} |\beta - \alpha_{i_{j_k}}|_p = 0.$$

Claim 4 $\beta \in \mathcal{F}(\phi_g) = \mathcal{F}(\phi_f)$.

Proof of Claim 4. (By contradiction) Let us assume that $\beta \notin \mathcal{F}(g)$. That is,

$$\beta \in \mathcal{J}(g) = \mathcal{J}(g) \cap L.$$

It follows from Theorem 2.12.4 that

$$|g'(\beta)| \geq 2.$$

On the other hand, it follows from Claim 2 that $|g'(\alpha_{i_j})| < 1$ for all $j \in \mathbb{N}$. In particular, $|g'(\alpha_{i_{j_k}})| < 1$ for all $k \in \mathbb{N}$. Moreover, since g' is continuous on $\mathcal{J}(f)$ and $\beta \in \mathcal{J}(f)$, we have that

$$|g'(\beta)| = |g'(\lim_{k\to\infty} \alpha_{i_{j_k}})| = \lim_{k\to\infty} |g'(\alpha_{i_{j_k}})| \leq 1.$$

This is a contradiction. □

Now let V be the disk component of $\mathcal{F}(f)$ containing β.

Claim 5 U is a non-wandering disk component of $\mathcal{F}(f)$.

Proof of Claim 5. Since

$$\lim_{k\to\infty} |\alpha_{i_{j_k}} - \beta| = 0,$$

there exists some $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $\alpha_{i_{j_k}} \in V$. Let us fix two distinct $m > n \geq k_0$. Considering

$$h := g^{i_{jm} - i_{jn}},$$

it is clear that

$$h(\alpha_{i_{j_m}}) = \alpha_{i_{j_n}}.$$

This implies that V is equal to the disk component of $\mathcal{F}(f)$ containing α_{i_m} and also to the disk component of $\mathcal{F}(f)$ containing the image of α_{i_n} by $f^{Mi_{jm} - Mi_{jn}}$. Hence, U is a non-wandering disk component of $\mathcal{F}(f)$. □

This is a contradiction to the assumption that U is a wandering domain of $\mathcal{F}(f)$. □

One can easily check the following corollary.

Corollary 2.13.4. *Let f be a polynomial map over K on $\mathbb{C}_p$ with $\deg(f) \geq 2$. If there are no critical points in $\mathcal{J}(f)$, then $\mathcal{F}(f)$ has no wandering disk components.*

In fact, R. Benedetto has also proved a stronger 'no wandering domains theorem' in his paper [RB00, THEOREM 1.2]. We will see the statement and compare it with Theorem 2.13.3. To understand the statement, let us introduce some terminology.

Definition 2.13.5. Let f be a polynomial map over $\mathbb{C}_p$ with $\deg(f) \geq 2$ and P is a point in $\mathbb{P}^1(\mathbb{C}_p)$. Then, P is called

Julia if P is in the Julia set of f,
recurrent if $P \in \overline{\{f^n(P)\}}_{n\in\mathbb{N}}$,
wildly critical if there exists some $m \in \mathbb{N}$ such that for all $n \in \{1, 2, \cdots, m\}$

$$f^{(m)}(P) \neq 0, \quad f^{(n)}(P) = 0.$$

There exists an obvious relation between wildly critical points and critical points.

Proposition 2.13.6. *Let f be a polynomial map over $\mathbb{C}_p$ with $\deg(f) \geq 2$ and P is a point in $\mathbb{P}^1(\mathbb{C}_p)$. If P is wildly critical, then P is critical.*

Now let us see the statement of the stronger "no wandering domains theorem".

Theorem 2.13.7. *Let f be a polynomial map over K with $\deg(f) \geq 2$ on $\mathbb{C}_p$. If f has no wildly critical recurrent Julia points, then the Fatou set of f has no wandering domains.*

Note that the original statement, proved by R. Benedetto, holds not only for polynomial maps over K, but also for rational maps over K.

Theorem 2.13.7 is stronger that Corollary 2.13.4 but not the same. Indeed, if f has a wildly critical recurrent Julia point, then this point is also a critical Julia point. This implies that if f has no critical point Julia point, f has also no wildly critical recurrent Julia point. However, the converse might be false. See the following example.

Example 2.13.8. Let p be an odd prime number and let us consider

$$\begin{aligned} F : \mathbb{C}_p &\to \mathbb{C}_p \\ z &\mapsto \frac{z^p - z^{p-1}}{p} + 1. \end{aligned}$$

As we checked in Example 2.12.3, F is not hyperbolic map. We check that F has no widely critical recurrent Julia point. We easily obtain that

$$F'(z) = \frac{pz^{p-1} - (p-1)z^{p-2}}{p} = z^{p-2}\frac{pz - (p-1)}{p}.$$

Thus, it follows that

$$\{z \in \mathbb{C}_p \mid F'(z) = 0\} = \{0, \frac{p-1}{p}\}.$$

Now let us show the following claims.

Claim 1 If $|z|_p > 1$, then $|F(z)|_p > 1$.

Proof of Claim 1. It follows immediately that

$$|z|_p^p > |z|_p^{p-1}.$$

Thus, by Proposition 2.1.5, we have that

$$\left| \frac{z^p - z^{p-1}}{p} \right|_p = p|z|_p^p > 1.$$

This implies that for any $|z|_p > 1$,

$$|F(z)|_p = \left| \frac{z^p - z^{p-1}}{p} + 1 \right|_p = p|z|_p^p > 1.$$

□

Claim 2 $\mathbb{P}^1(\mathbb{C}_p) - \overline{D}_1(0) \subset \mathcal{F}(F)$.

The proof of Claim 2 follows easily from Claim 1 and Theorem 2.7.2 so we omit it.

Claim 3

$$\frac{p-1}{p} \in \mathcal{F}(F).$$

Proof of Claim 3. It follows from Proposition 2.1.5 that

$$|p-1|_p = \max\{|p|_p, |1|_p\} = 1.$$

Thus, we have

$$\left| \frac{p-1}{p} \right|_p = |\frac{1}{p}|_p = p > 1.$$

By Claim 2, we have

$$\frac{p-1}{p} \in \mathcal{F}(F).$$

□

On the other hand, 0 is not recurrent because

$$0 \mapsto^F 1 \mapsto^F 1 \mapsto^F \cdots .$$

This implies that F has no critical recurrent Julia points, in particular, F has no wildly critical recurrent Julia points.

2.14 An Example of Wandering Domains

In complex dynamical systems, by Theorem 2.13.1, every polynomial maps has no wandering domains. However, in the non-Archimedean dynamical systems, there exists a polynomial map with wandering domain. In 2.14, we will see an example of polynomial maps, which have wandering domains, proposed by R.L.Benedetto [RB02, THEOREM1.1].

Theorem 2.14.1. *There exists some rational map over $\mathbb{C}_p$ with a wandering domain. More precisely, there is some $a \in \mathbb{C}_p$ such that the polynomial map*

$$F_a(z) := (1-a)z^p + z^{p-1}$$

has no critical Julia points but has a wandering disk component.

R. Benedetto proved that F_a has a wandering domain in [RB02, THEOREM 1.1]. It follows from Corollary 2.13.4 that $a \in \mathbb{C}_p$ cannot be an element of a finite extension field of $\mathbb{Q}_p$. Indeed, if a is in some finite extension field K of $\mathbb{Q}_p$, F_a must be a polynomial map over K. One can check that F_a is hyperbolic over K.

On the other hand, J. Rivera-Letelier has suggested a question as follows.

Question 2.14.2. *Is there any rational map over a finite extension field of $\mathbb{Q}_p$ on $\mathbb{C}_p$ with wandering domains?*

Of course, by Theorem 2.13.4, the rational map cannot be p-adically hyperbolic. This conjecture is still an open problem in non-Archimedean dynamical systems.

3 J-Stability in p-adic Dynamics

In this section, we will see the stability of Julia set. In the complex dynamics, Mañé, Sad, and Sullivan proved a theorem which gives a condition for J-stability of rational maps over $\mathbb{C}$. See Theorem 3.1.1. The author proved that some simple families of polynomial maps over $\mathbb{C}_p$ have J-stability. The result and proof will be described in 3.2, and a simple application of the main result will be described in 3.3.

3.1 J-Stability in Complex Dynamics

In 3.1, we will see a motivation for the main result. To ease notation, we shall use the notations Rad, Rad_D to denote the set of rational maps over $\mathbb{C}$ and the set of rational maps over $\mathbb{C}$ of degree D for some $N \in \mathbb{N}$, respectively.

In complex dynamical systems, the following theorem was proved by Mañè, Sad, and Sullivan in their paper [MSS].

Theorem 3.1.1. *Let D be a number which is greater than 2 and f be a rational map of degree D. If f has a connected neighborhood $U \subset Rat_D$ such that each $g \in U$ has the same number of attracting cycles as f, then for each $g \in U$ there exists a unique quasi-conformal conjugacy $h_{g,f} : J(g) \to J(f)$ such that*

$$f \circ h_{g,f} = h_{g,f} \circ g.$$

Now we will consider an analogue of MSS for the p-adic dynamical systems.

3.2 J-Stability in p-adic Dynamics

In this subsection, we consider J-stable families in p-adic dynamics. Let us fix a prime number p and $d \in \mathbb{N}$ with $p \nmid d$. Then, we define

$$\begin{aligned} \phi(\cdot,\cdot) : \mathbb{C}_p \times \mathbb{C}_p &\to \mathbb{C}_p \\ (z, c) &\mapsto z^d + c. \end{aligned}$$

To ease notation, we shall use

$$\phi_c(z) := \phi(z, c) = z^d + c.$$

The main result is as follows.

Theorem 3.2.1. *For any $c \in \mathbb{C}_p$ with $|c|_p > 1$, suppose that $c' \in \mathbb{C}_p$ satisfies $|c' - c|_p \leq |c|_p^{1/d}$. Then, there exists a local isomeric homeomorphism $h_{c,c'} : \mathcal{J}(\phi_c) \to \mathcal{J}(\phi_{c'})$ such that*

$$\phi_{c'} \circ h_{c,c'} = h_{c,c'} \circ \phi_c$$

on $\mathcal{J}(\phi_c)$.

Let us begin with some key lemmas. Let us fix $c \in \mathbb{C}_p$ with $|c|_p > 1$ and set $\lambda := |c|_p^{(d-1)/d}$.

Lemma 3.2.2. *$\mathcal{J}(\phi_c) \neq \emptyset$ and $\mathcal{J}(\phi_c)$ has no critical poins. Moreover,*

$$\mathcal{J}(\phi_c) \subset \mathbb{S}(|c|_p^{1/d}), \quad \phi_c^{-1}(\mathbb{S}(|c|_p^{1/d})) \subset \mathbb{S}(|c|_p^{1/d}).$$

Proof. Let us begin with the following claim.

Claim 1 There exists some $\alpha \in K$ such that

$$\phi_c(\alpha) = \alpha, \quad |\phi_c'(\alpha)|_p > 1.$$

Proof of Claim 1. Since $\mathbb{C}_p$ is algebraically closed, there exists some $\{\alpha_i\}_{i=1}^d$ such that

$$\phi_c(\alpha_1) - \alpha_1 = \phi_c(\alpha_2) - \alpha_2 = \cdots = \phi_c(\alpha_n) - \alpha_n = 0.$$

We will prove that

$$|\alpha_1|_p = |\alpha_2|_p = \cdots = |\alpha_d|_p = |c|_p^{1/d}$$

by contradiction. Let us assume that there exists some $|\alpha_j| \neq |c|_p^{1/d}$. Then, we consider the following cases.

Case 1: $|\alpha_j|_p < |c|_p^{1/d}$.

In this case, we have that

$$|\alpha_j|_p^d < |c|_p, \quad |\alpha_j|_p < |c|_p^{\frac{1}{d}} < |c|_p.$$

It follows from Proposition 2.1.5 that

$$|\phi_c(\alpha_j) - \alpha_j|_p = |\alpha_j^d + c - \alpha_j|_p = \max\{|\alpha_j|_p^d, |\alpha_j|_p, |c|_p\} = |c|_p > 1.$$

On the other hand, we have that

$$|\phi_c(\alpha_j) - \alpha_j|_p = |0|_p = 0.$$

This is a contradiction. Hence,

$$|\alpha_j|_p \geq |c|_p^{1/d}.$$

Case 2: $|\alpha_j|_p > |c|_p^{1/d}$.

In this case, since $|c|_p > 1$, we have that

$$|\alpha_j|_p^d > |c|_p, \quad |\alpha_j|_p^d > |\alpha|_p.$$

It follows from Proposition 2.1.5 that

$$|\phi_c(\alpha_j) - \alpha_j|_p = |\alpha_j^d + c - \alpha_j|_p = \max\{|\alpha_j|_p^d, |\alpha_j|_p, |c|_p\} = |\alpha_j|_p^d > 1.$$

On the other hand, we have that

$$|\phi_c(\alpha_j) - \alpha_j|_p = |0|_p = 0.$$

This is a contradiction. Hence,

$$|\alpha_j|_p = |c|_p^{1/d}.$$

Thus, for all $i = 1, 2, \cdots, d$,

$$|\alpha_i|_p = |c|_p^{1/d}.$$

In particular, since $d \nmid p$, we have that for any $i = 1, 2, \cdots, d$,

$$|\phi_c'(\alpha_i)|_p = |d \cdot \alpha_i^{d-1}|_p = |d|_p|\alpha_i^{d-1}|_p = |c|_p^{(d-1)/d}.$$

This implies that every fixed point of ϕ_c is repelling. □

Thus, ϕ_c has a repelling fixed point so it follows from Proposition 2.6.6 that

$$\mathcal{J}(\phi_c) \neq \emptyset.$$

Next, we see the following claim.

Claim 2 0 is the only critical point of ϕ_c.

The proof follows immediately so we omit it. Finally, let us prove the following claim.

Claim 3 $\phi_c^{-1}(\mathbb{S}(|c|_p^{1/d})) \subset \mathbb{S}(|c|_p^{1/d})$.

Proof of Claim 3. Let us take an arbitrary $w \in \mathbb{S}(|c|_p^{1/d})$. Then, we will show that if $z \in K$ satisfies

$$\phi_c(z) - w = 0,$$

then $|z|_p = |c|_p^{1/d}$ by contradiction. Let us assume that $|z|_p \neq |c|_p^{1/d}$. Then, we consider the following cases.

Case 1: $|z|_p < |c|_p^{1/d}$.

In this case, we have that

$$|z|_p^d < |c|_p, \quad |w|_p = |c|_p^{1/d} < |c|_p.$$

It follows from Proposition 2.1.5 that

$$|\phi_c(z) - w|_p = |z^d + c - w|_p = \max\{|z|_p^d, |w|_p, |c|_p\} = |c|_p > 1.$$

On the other hand, we have that

$$|\phi_c(z) - w|_p = |0|_p = 0.$$

This is a contradiction. Hence,

$$|z|_p \geq |c|_p^{1/d}.$$

Case 2: $|z|_p > |c|_p^{1/d}$.

In this case, since $|c|_p > 1$, we have that

$$|z|_p^d > |c|_p > |c|_p^{1/d} = |w|_p.$$

It follows from Proposition 2.1.5 that

$$|\phi_c(z) - w|_p = |z^d + c - w|_p = \max\{|z|_p^d, |w|_p, |c|_p\} = |z|_p^d > 1.$$

On the other hand, we have that

$$|\phi_c(z) - w|_p = |0|_p = 0.$$

This is a contradiction. Hence,

$$|z|_p = |c|_p^{1/d}.$$

This implies that

$$\phi_c^{-1}(\mathbb{S}(|c|_p^{1/d})) \subset \mathbb{S}(|c|_p^{1/d}).$$

□

In particular, it is clear that $\mathbb{S}(|c|_p^{1/d}) \subset K$ is non-empty closed with respect to ρ_p so it follows from Theorem 2.9.1 that

$$\mathcal{J}(\phi_c) \subset \mathbb{S}(|c|_p^{1/d}).$$

□

Lemma 3.2.3. *For any $r \in [0, |c|_p^{1/d}] \cap |\mathbb{C}_p^\times|_p$ and $a \in \mathbb{S}(|c|_p^{\frac{1}{d}})$, there exists some $\{\overline{D}_{r/\lambda}(b_i)\}_{i=1}^d$ such that*

$$\phi_c^{-1}(\overline{D}_r(a)) = \bigsqcup_{i=1}^{d} \overline{D}_{r/\lambda}(b_i).$$

Moreover,

$$\phi_c|\overline{D}_{r/\lambda}(b_i) \to \overline{D}_r(a)$$

is homeomorphic for each $i \in \{1, 2, \cdots, d\}$.

Proof of Lemma 3.2.3. By Lemma 3.2.2, a is not critical point so there exists $\{b_i\}_{i=1}^d$ such that for all $i \neq j = 1, 2, \cdots, d$,

$$\phi_c(b_i) = a, \quad b_i \neq b_j.$$

Now let us fix $i \in \{1, 2, \cdots, d\}$, and show the following claims.

Claim 1 For any $k = 2, 3, \cdots, d$, we have

$$\left| \frac{\phi_c^{(k)}(b_i)}{k!} \right|_p < |\phi_c'(b_i)|_p.$$

Proof of Claim 1. It follows immediately that for $k = 1, 2, \cdots, d$,

$$\frac{\phi_c^{(k)}(b_i)}{k!} = \frac{d \cdot (d-1) \cdot \cdots \cdot (d-k+1)}{k!} b_i^{d-k} = \binom{d}{k} b_i^{d-k}.$$

Thus, for every $k = 2, 3, \cdots, d$, we have that

$$\left| \frac{\phi_c^{(k)}(b_i)}{k!} \right|_p = \left| \binom{d}{k} b_i^{d-k} \right|_p \leq |b_i|_p^{d-k} = |d|_p |b_i|_p^{d-k} < |d \cdot b_i^{d-1}|_p = |\phi_c'(b_i)|_p.$$

□

Claim 2 For any $z, w \in \overline{D}_{\frac{r}{\lambda}}(b_i)$,

$$|\phi_c(z) - \phi_c(w)|_p = \lambda |z - w|_p.$$

Proof of Claim 2. We can write ϕ_c as follows.

$$\phi_c(z) = z^d + c = \sum_{k=0}^{d} \frac{\phi_c^{(k)}(b_i)}{k!} (z - b_i)^k.$$

It follows from Claim 1 that for any $k = 2, 3, \cdots, d$,

$$\begin{aligned}
&\left| \frac{\phi_c^{(k)}(b_i)}{k!} \{(z-b_i)^{k-1} + (z-b_i)^{k-2}(w-b_i) + \cdots + (w-b_i)^{k-1}\} \right|_p \\
&< \lambda |(z-b_i)^{k-1} + (z-b_i)^{k-2}(w-b_i) + \cdots + (w-b_i)^{k-1}|_p \\
&\leq \lambda \left(\frac{r}{\lambda}\right)^{k-1} \leq \lambda \frac{r^{d-1}}{\lambda} \leq |c|_p^{\frac{d-1}{d}} = \lambda = |\phi'(b_i)|_p. \qquad (3.1)
\end{aligned}$$

Moreover, we have

$$
\begin{aligned}
|\phi_c(z)-\phi_c(w)|_p &= \left|\sum_{k=0}^{d}\frac{\phi_c^{(k)}(b_i)}{k!}(z-b_i)^k - \sum_{k=0}^{d}\frac{\phi_c^{(k)}(b_i)}{k!}(w-b_i)^k\right|_p \\
&= \left|\sum_{k=1}^{d}\frac{\phi_c^{(k)}(b_i)}{k!}\{(z-b_i)^k-(w-b_i)^k\}\right|_p \\
&= |z-w|_p\cdot\max\{|\phi_c'(b_i)|_p, \left|\frac{\phi_c^{(2)}(b_i)}{2!}\right|_p |(z-b_i)+(w-b_i)|_p, \cdots, \left|\frac{\phi_c^{(d)}(b_i)}{d!}\right|_p \cdot \\
&\qquad |(z-b_i)^{d-1}+(z-b_i)^{d-2}(w-b_i)+\cdots+(w-b_i)^{d-1}|_p\}.
\end{aligned}
$$

Thus, it follows from (3.1) and Proposition 2.1.5 that

$$
\begin{aligned}
|\phi_c(z)-\phi_c(w)|_p &= |z-w|_p\cdot\max\{|\phi_c'(b_i)|_p, \left|\frac{\phi_c^{(2)}(b_i)}{2!}\right|_p |(z-b_i)+(w-b_i)|_p, \cdots, \left|\frac{\phi_c^{(d)}(b_i)}{d!}\right|_p \cdot \\
&\qquad |(z-b_i)^{d-1}+(z-b_i)^{d-2}(w-b_i)+\cdots+(w-b_i)^{d-1}|_p\} \\
&= |z-w|_p\,|\phi_c'(b_i)|_p = \lambda|z-w|_p.
\end{aligned}
$$

for any $z, w \in \overline{D}_{r/\lambda}(b_i)$. □

It follows from Theorem 5.4.5 that ϕ_c is bijective from $\overline{D}_{r/\lambda}(b_i)$ to $\overline{D}_r(a)$.

Claim 3 For any $i \neq j = 1, 2, \cdots, d$, we have

$$\overline{D}_{r/\lambda}(b_i) \cap \overline{D}_{r/\lambda}(b_j) = \emptyset.$$

Proof of Claim 3. (By contradiction) Let us assume that there exist two distinct i and j in $\{1, 2, \cdots, d\}$ such that

$$\overline{D}_{r/\lambda}(b_i) \cap \overline{D}_{r/\lambda}(b_j) \neq \emptyset.$$

It follows from Corollary 2.1.21 that

$$\overline{D}_{r/\lambda}(b_i) = \overline{D}_{r/\lambda}(b_j).$$

In particular, this implies that $b_j \in \overline{D}_{r/\lambda}(b_i)$. Moreover, since ϕ_c is bijective from $\overline{D}_{r/\lambda}(b_i)$ to $\overline{D}_r(a)$, we have that $\phi_c(b_i) \neq \phi_c(b_j)$. It is a contradiction to the fact that

$$\phi_c(b_i) = \phi_c(b_j) = a.$$

□

Since ϕ_c is a polynomial, it follows from Corollary 2.2.12 and Corollary 2.2.21 that ϕ_c is homeomorphic from $\overline{D}_{r/\lambda}(b_i)$ to $\overline{D}_r(a)$.

□

Finally, we prove Theorem 3.2.1.

Proof of Theorem 3.2.1. Let us begin with the construction of sets $\{\Omega_c^n\}_{n\geq 0}$.

- **The Construction of Sets**

For every $c \in \mathbb{C}_p$ with $|c|_p > 1$, we define $\{\Omega_c^n\}_{n\in\mathbb{N}}$ as follows.

$$\begin{aligned}\Omega_c^0 &:= \mathbb{S}(|c|_p^{1/d}),\\ \Omega_c^1 &:= \phi_c^{-1}(\Omega_c^0),\\ &\cdots,\\ \Omega_c^n &:= \phi_c^{-n}(\Omega_c^0),\\ &\cdots.\end{aligned}$$

It follows from Lemma 3.2.2 and Proposition 2.5.6 that for any $n \in \mathbb{N}$

$$\Omega_c^n \subset \Omega_c^{n-1}, \quad \mathcal{J}(\phi_c) \subset \Omega_c^n.$$

Moreover, Setting

$$\Omega_c^\infty := \bigcap_{n\in\mathbb{N}} \Omega_c^{n-1},$$

we obtain that

$$\mathcal{J}(\phi_c) \subset \Omega_c^\infty.$$

In particular, by Lemma 3.2.2, we have that $\Omega_c^\infty \neq \emptyset$.

- **The Construction of Homeomorphisms**

Let us fix $c \in \mathbb{C}_p$ with $|c|_p > 1$ and choose $c' \in \mathbb{C}_p$ satisfying $|c - c'|_p \leq |c|_p^{1/d}$ and set

$$\delta_i := \frac{|c|_p^{1/d}}{\lambda^i} > 0$$

for all $i \in \mathbb{N}$. Then, we have the following claim.

Claim 1 $\Omega_c^0 = \Omega_{c'}^0$.

Proof of Claim 1. It follows immediately from Proposition 2.1.5 that

$$|c|_p = |c - c' + c'|_p = \max\{|c - c'|_p, |c'|_p\} = |c'|_p$$

since $|c - c'|_p \leq |c|_p^{1/d} < |c|_p$. Thus, we have that

$$|c|_p^{1/d} = |c'|_p^{1d}.$$

□

Thus, we define $h_0 : \Omega_c^0 \to \Omega_{c'}^0$ as the identity map on Ω_c^0. Now we consider the following claim.

Claim 2 For any $z \in \Omega_c^1$, there exists a unique $w \in \phi_{c'}^{-1}(\{h_0 \circ \phi_c(z)\})$ such that

$$|w - z|_p \leq \delta_1.$$

Proof of Claim 2. it follows immediately that

$$|h_0 \circ \phi_c(z) - \phi_{c'}(z)|_p = |\phi_c(z) - \phi_{c'}(z)|_p = |c - c'|_p \leq |c|_p^{\frac{1}{d}}.$$

This implies that

$$\phi_{c'}(z) \in \overline{D}_{\delta_0}(h_0 \circ \phi_c(z)).$$

It follows from Lemma 3.2.2 that there exists the unique $w \in \phi_{c'}^{-1}(\{h_0 \circ \phi_c(z)\})$ such that

$$z \in \phi_{c'}^{-1}(\overline{D}_{\delta_0}(h_0 \circ \phi_c(z))) = \overline{D}_{\delta_1}(w).$$

□

We define $h_1 : \Omega_c^1 \to \Omega_{c'}^1$ as $h_1(z) := w$. Then, h_1 satisfies

$$|h_1(z) - h_0(z)|_p \leq \delta_1, \quad h_0 \circ \phi_c(z) = \phi_{c'} \circ h_1(z)$$

for all $z \in \Omega_c^1$. Now let us construct $\{h_{i+1}\}_{i \in \mathbb{N}}$, inductively. Let us assume that for $k \geq 1$, h_k have been already constructed and satisfy

$$|h_k(z) - h_{k-1}(z)|_p \leq \delta_k, \quad h_{k-1} \circ \phi_c(z) = \phi_{c'} \circ h_k(z)$$

for all $z \in \Omega_c^k$. We have the following claim.

Claim 3 For any $z \in \Omega_c^{k+1}$, there exists the unique $w \in \phi_{c'}^{-1}(\{h_{k+1} \circ \phi_c(z)\})$ such that

$$|w - h_k(z)|_p \leq \delta_{k+1}.$$

Proof of Claim 3. It follows immediately that

$$|\phi_{c'} \circ h_k(z) - h_k \circ \phi_c(z)|_p = |h_{k-1} \circ \phi_c(z) - h_k \circ \phi_c(z)|_p \leq \delta_k.$$

This implies that

$$\phi_{c'}(h_k(z)) \in \overline{D}_{\delta_k}(h_k \circ \phi_c(z)).$$

It follows from Lemma 3.2.2 that there exists the unique $w \in \phi_{c'}^{-1}(\{h_k \circ \phi_c(z)\})$ such that

$$h_k(z) \in \phi_{c'}^{-1}(\overline{D}_{\delta_k}(h_k \circ \phi_c(z))) = \overline{D}_{\delta_{k+1}}(w).$$

□

Claim 4 For any $k \in \mathbb{N}$, we have

$$h_{k-1} \circ \phi_c = \phi_{c'} \circ h_k \quad \text{on } \Omega_c^k.$$

This is clear from the construction of $\{h_i\}_{i \geq 0}$ so we omit it.

Claim 5 For any $k \in \mathbb{N}$, $h_k : \Omega_c^k \to \Omega_{c'}^k$ is a homeomorphism.

Proof of Claim 5. As we constructed $\{h_k : \Omega_c^k \to \Omega_{c'}^k\}_{k\in\mathbb{N}}$ in Claim 2 and 3, we can also construct $\{\tilde{h}_n : \Omega_{c'}^n \to \Omega_c^n\}_{n\in\mathbb{N}}$ satisfying

$$|\tilde{h}_k(w) - \tilde{h}_{k-1}(w)|_p \leq \delta_1, \quad \tilde{h}_{k-1} \circ \phi_c(w) = \phi_{c'} \circ \tilde{h}_k(w)$$

for all $w \in \Omega_{c'}^k$. Moreover, it is easy to check that $h_k \circ \tilde{h}_k = \tilde{h}_k \circ h_k$ is equal to the identity map on $\Omega_{c'}^n$ for all $n \in \mathbb{N}$. □

Claim 6 There exists a homeomorphism $h_\infty : \Omega_c^\infty \to \Omega_{c'}^\infty$ such that for all $z \in \Omega_c^\infty$

$$\lim_{k\to\infty} |h_\infty(z) - h_k(z)|_p = 0.$$

Proof of Claim 6. It follows from Claim 3 and Lemma 3.2.2 that for any $z \in \Omega_c^\infty$, we have

$$|h_{k+1}(z) - h_k(z)|_p \leq \delta_{k+1}.$$

Moreover, since

$$\lim_{k\to\infty} \delta_{k+1} = 0,$$

it follows from Lemma 2.1.22 that there exists some $w \in \mathbb{C}_p$ such that

$$\lim_{k\to\infty} |w - h_k(z)|_p = 0.$$

Setting $h_\infty(z) := w$ for each $z \in \Omega_c^\infty$, we easily see that $h_\infty : \Omega_c^\infty \to \Omega_{c'}^\infty$ is a continuous map since $\{h_k\}_{k\in\mathbb{N}}$ is uniformly convergence. Similarly, we can find a continuous map $\tilde{h}_\infty : \Omega_{c'}^\infty \to \Omega_c$ such that for each $z \in \Omega_{c'}^\infty$,

$$\lim_{k\to\infty} |\tilde{h}_\infty(z) - \tilde{h}_k(z)|_p = 0.$$

Moreover, for any $z \in \Omega_c^\infty$, we have

$$\tilde{h}_\infty \circ h_\infty(z) = \lim_{k\to\infty} \tilde{h}_k \circ h_k(z) = \lim_{k\to\infty} z = z.$$

Hence, $h_\infty : \Omega_c^\infty \to \Omega_{c'}^\infty$ is a homeomorphism. □

- **Some Properties of h_∞**

Claim 7 h_∞ is a topological conjugacy between ϕ_c and $\phi_{c'}$ on Ω_c^∞.

Proof of Claim 7. For any $z \in \Omega_c^\infty$, we have

$$h_\infty \circ \phi_c(z) = \lim_{k\to\infty} h_k \circ \phi_c(z) = \lim_{k\to\infty} \phi_{c'} \circ h_{k+1}(z) = \phi_{c'} \circ \lim_{k\to\infty} h_{k+1}(z) = \phi_{c'} \circ h_\infty(z).$$

□

Claim 8 $h_\infty(\mathcal{J}(\phi_c)) = \mathcal{J}(\phi_{c'})$.

Proof of Claim 8. Let α be a repelling fixed point of ϕ_c. See the proof of Lemma 3.2.2 for the existence of α. By Lemma 3.2.2 and Clam 7, $h_\infty(\alpha)$ is a repelling fixed point of $\phi_{c'}$. Applying Proposition 2.8.1, we have that

$$h_\infty(\mathcal{J}(\phi_c)) = h_\infty(\overline{\bigcup_{n\in\mathbb{N}} \phi_c^{-n}(\{\alpha\})}) = \overline{\bigcup_{n\in\mathbb{N}} h_\infty \circ \phi_c^{-n}(\{\alpha\})} = \overline{\bigcup_{n\in\mathbb{N}} \phi_{c'}^{-n}(\{h_\infty(\alpha)\})} = \mathcal{J}(\phi_{c'}).$$

□

Hence, by considering the restriction of h_∞ to $\mathcal{J}(\phi_c)$ as $h_{c,c'}$, it is clear that $h_{c,c'}$ is a homeomorphism and satisfies that

$$\phi_{c'} \circ h_{c,c'} = h_{c,c'} \circ \phi_c.$$

on $\mathcal{J}(\phi_c)$.

Finally, let us prove $h_{c,c'}$ is a local isometry.

Claim 9 For any $\alpha \in \mathcal{J}(\phi_c)$ and $z, w \in \overline{D}_1(\alpha) \cap \mathcal{J}(\phi_c)$ and $n \in \mathbb{N}$,

$$|h_{n-1}(z) - h_{n-1}(w)|_p = |h_n(z) - h_n(w)|_p.$$

The proof follows immediately by induction on $n \in \mathbb{N}$.

In particular, this implies that for any $\alpha \in \mathcal{J}(\phi_c)$ and $z, w \in \overline{D}_1(\alpha) \cap \mathcal{J}(\phi_c)$ and $n \in \mathbb{N}$,

$$|h_n(z) - h_n(w)|_p = |z - w|_p.$$

We obtain that

$$|h_\infty(z) - h_\infty(w)| = \lim_{n\to\infty} |h_n(z) - h_n(w)|_p = |z - w|_p$$

for any $z, w \in \overline{D}_1(\alpha) \cap \mathcal{J}(\phi_c)$. □

3.3 An Application of the Main Result

Let us close this section with an application of the main theorem. The following proposition might be easily derived from [S, Corollary 5.25].

Corollary 3.3.1. *Let us fix an odd prime number p and consider the polynomial $\phi_c(z) := z^2 + c$ over $\mathbb{C}_p$. If $|c + \frac{1}{2p} + \frac{1}{4p^2}| \leq p$, then $\mathcal{J}(\phi)$ is compact.*

4 The Artin-Mazur Zeta Functions

In this section, we will see the Artin-Mazur zeta functions of some dynamical systems.

4.1 Definitions of Artin-Mazur Zeta Functions

In 4.1, we give the definition of the Artin-Mazur zeta function of a given dynamical system with some examples. This subsection is based on M. Artin and B. Mazur's paper [AM].

Definition 4.1.1. Let us define

$$\exp(T) := \sum_{n=0}^{\infty} \frac{1}{n!} T^n \in \mathbb{C}[[T]], \quad \log(1+T) := \sum_{n=1}^{\infty} \frac{(-1)^n}{n} T^n \in \mathbb{C}[[T]]$$

as formal power series over $\mathbb{C}$.

The definition of the Artin-Mazur zeta function was introduced by Artin-Mazur [AM, p.84].

Definition 4.1.2 (Artin-Mazur Zeta Function)**.** Let (X, f) be a dynamical system. Assume that the number N_n of the isolated fixed points of f^n for each $n \in \mathbb{N}$ is finite. *The Artin-Mazur zeta function of f over X* is defined by

$$Z_f(T) = \exp(\sum_{n=1}^{\infty} \frac{N_n}{n} T^n) \in \mathbb{C}[[T]]$$

as a formal power series.

One of the reasons why we consider the Artin-Mazur zeta function is that it is invariant under conjugacy. That is, if two dynamical systems are conjugate, then the Artin-Mazur zeta functions must be the same because the number of periodic points for each period is the same. See Proposition 1.1.9.

We will give some examples of the Artin-Mazur zeta functions from complex dynamical systems. When we consider the Artin-Mazur zeta functions of rational maps over $\mathbb{C}$, it is important to consider 'parabolic' periodic points.

Definition 4.1.3 (Parabolic Periodic Point)**.** Let f be a rational map over $\mathbb{C}$ and $\alpha \in \mathbb{C}$ be a fixed point of f. Then, α is called *parabolic* if there exists some $q \in \mathbb{N}$ such that

$$(f'(\alpha))^q = 1.$$

Example 4.1.4. Let p be a natural number with $p \geq 2$ and let us consider

$$\begin{aligned} f : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto \frac{-1+\sqrt{-3}}{2} z + z^p. \end{aligned}$$

Then, 0 is a fixed point of f. Moreover, we have

$$f'(0) \cdot f'(0) \cdot f'(0) = (\frac{-1+\sqrt{-3}}{2})^3 = 1.$$

This implies that 0 is a parabolic fixed point of f.

Example 4.1.5. Let a be a complex number with $|a| \neq 1$ and let us define

$$L : \mathbb{C} \to \mathbb{C}$$
$$z \mapsto a \cdot z.$$

One may easily check that the only fixed point of L is 0. Thus, we have

$$Z_L(T) = \exp(\sum_{n=1}^{\infty} \frac{1}{n} T^n) = \exp \circ \log(1-T)^{-1} = \frac{1}{1-T} \in \mathbb{C}(T).$$

4.2 The Artin-Mazur Zeta Function of Rational Maps over $\mathbb{C}$

In 4.2, we will focus on the Artin-Mazur zeta functions of complex dynamical systems. In particular, we give the result, which was proved by by A. Hinkkanen, that the Artin-Mazur zeta functions of rational maps over $\mathbb{C}$ are rational. This subsection is based on A. Hinkkanen's paper [AH].

Let us begin with the Hinkkanen's theorem [AH, THEOREM 1]. In the following theorem, we shall use the notation $\deg(f)$ to denote degree of a give rational map f over $\mathbb{C}$.

Theorem 4.2.1. *Let f be a rational map over $\mathbb{C}$ with $\deg(f) \geq 2$. Then, we have*

$$Z_f(T) = (1-dT)^{-1}(1-T)^{-1} \prod_{i=1}^{N} (1-T^{p_i q_i})^{l_i} \in \mathbb{C}[[T]]$$

where N is the number of the distinct parabolic cycles of f and p_i, q_i, l_i are natural numbers depending on the parabolic cycles for each $i = 1, 2, \cdots, N$, and $d = \deg(f)$.

In particular, Theorem 4.2.1 implies that if f has a rational map over $\mathbb{C}$ with no parabolic cycles and $d := \deg(f) \geq 2$, then

$$Z_f(T) = (1-dT)^{-1}(1-T)^{-1} \in \mathbb{Q}(T).$$

In the statement, it is not trivial that the number of parabolic cycles of rational maps is finite. The following theorem is well-known for complex dynamical systems.

Theorem 4.2.2. *Suppose that f is a rational map over $\mathbb{C}$ and the degree of f is $d \in \mathbb{N}$. The number of non-repelling cycles is less than $2d-2$. In particular, the number of parabolic periodic points is finite.*

See [M, Corollary 10.16] for the proof.

Example 4.2.3. Let us denote the Riemann sphere by $\hat{\mathbb{C}}$ and consider the map

$$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$
$$z \mapsto z^2.$$

It is easy to check that

$$Per(f) \subset \{0\} \cup \{\infty\} \cup \{z \in \mathbb{C} \mid |z| = 1\}.$$

It is clear that 0 and ∞ are fixed points. Moreover, we obtain that for any $n \in \mathbb{N}$,

$$f^n(z) - z = z^{2^n} - z = z(z^{2^n - 1} - 1), \quad (f^n(z) - z)' = 2^n z^{2^n - 1} - 1.$$

This implies that $f^n(z) - z \in Poly(\mathbb{C})$ does not have any multiple zeros in $\{z \in \mathbb{C} \mid |z| = 1\}$. Thus, we have

$$Z_f(T) = \exp(\frac{2^n + 1}{n} T^n) = \exp(\frac{(2T)^n}{n}) \exp(\frac{T^n}{n}) = (1-2T)^{-1}(1-T)^{-1}.$$

4.3 The Artin-Mazur Zeta Function of Rational Maps over $\mathbb{C}_p$

In 4.3, we will focus on the Artin-Mazur zeta functions of p-adic dynamical systems. In particular, we will see a result, which is calculated by the author, that the Artin-Mazur zeta functions of rational maps over $\mathbb{C}_p$ are rational. In this subsection, we will consider the dynamics of rational maps over $\mathbb{C}_p$ on $\mathbb{P}^1(\mathbb{C}_p)$, which we did not consider in this thesis. One can find basics of dynamics of rational maps over $\mathbb{C}_p$ in [S, Section 5.2, 5.3].

Let us fix a prime number p and begin with the main result.

Theorem 4.3.1. *Let $f \in \mathbb{C}_p(T)$ be a rational map with $\deg(f) \geq 2$. Then, we have*

$$Z_f(T) = (1 - dT)^{-1}(1 - T)^{-1} \prod_{i=1}^{N} (1 - T^{p_i q_i})^{l_i}$$

where N is the number of the distinct parabolic cycles of f and p_i, q_i, l_i are natural numbers depending on the parabolic cycles for each $i = 1, 2, \cdots, N$, and $d = \deg(f)$.

As we did in the complex case, we have to consider the finiteness of parabolic cycles of rational maps over $\mathbb{C}_p$. In fact, we have the follows theorem.

Lemma 4.3.2. *Let $f \in \mathbb{C}_p(z)$ be a rational map with $\deg(f) \geq 2$. Then, the number of parabolic cycles is finite.*

Proof. Let α be a parabolic fixed point of f and λ be the multiplier. Suppose that $\alpha \in \mathbb{C}_p$, and λ is a primitive q th root of unity. By Theorem 5.4.3, there exists some ring isomorphic map

$$\iota : \mathbb{C}_p \to \mathbb{C}.$$

Let us show the following claim.

Claim 1 $\mu := \iota(\lambda)$ is a primitive q th root of unity.

Proof of Claim 1. It follows immediately that

$$\mu^q = (\iota(\lambda))^q = \iota(\lambda^q) = \iota(1) = 1.$$

Moreover, if there exists a $0 < j < q$ such that $\mu^j = 1$, then

$$1 = \mu^j = \iota(\lambda)^j = \iota(\lambda^j).$$

Since i is injective, we have $\lambda^j = 1$. It is a contradiction to our assumption that λ is a primitive q th root of unity. □

We denote

$$f(z) = \frac{f_1(z)}{f_2(z)}, \quad f_1(z) = a_0 + a_1 z + \cdots + a_N z^N, f_2(z) = b_0 + b_1 z + \cdots + b_M z^M \in \mathbb{C}[z]$$

where $\max\{\deg(f_1), \deg(f_2)\} \geq 2$ and $f_2(\alpha) \neq 0$.

Consider

$$g(z) := \frac{\iota(a_0) + \iota(a_1)z + \cdots + \iota(a_N)z^N}{\iota(b_0) + \iota(b_1)z + \cdots \iota(b_M)z^M} \in \mathbb{C}(z).$$

Claim 2 $\iota(\alpha)$ is a fixed point of g, and the multiplier of g at $\iota(\alpha)$ is μ.

Proof of Claim 2. It follows immediately that

$$g(\iota(\alpha)) = \frac{\iota(f_1(\alpha))}{\iota(f_2(\alpha))} = \iota(f(\alpha)) = \iota(\alpha) = 0.$$

Moreover, it follows from Claim 1 that

$$g'(\iota(\alpha)) = \frac{\iota(f_1'(\alpha)f_2(\alpha)) - \iota(f_1(\alpha)f_2'(\alpha))}{(\iota(f_2(\alpha)))^2} = \iota(f'(\alpha)) = \iota(\lambda) = \mu.$$

□

The other cases that $\alpha = \infty$ or α is parabolic periodic can be reduced to this case. Thus, we have that a parabolic periodic points in $\mathbb{C}_p$ corresponds to a parabolic periodic points in $\mathbb{C}$. Thus, it follows from Theorem 4.2.2 that the number of parabolic periodic points of g is finite. Hence, the number of parabolic cycles of rational maps over $\mathbb{C}_p$ must be finite. □

In fact, we can say more than this. The following result was proved by J. Rivera-Letelier in his paper [RL].

Theorem 4.3.3. *Let f be a rational map over $\mathbb{C}_p$ with $\deg(f) \geq 2$. Then, the number of the super-attracting cycles and the parabolic cycles is less than $2d-2$ where $d = \deg(f)$.*

The proof is fundamentally the same as the above. See [RL, THÉORÈM 4.1] for the proof.

To prove Theorem 4.3.1, we need to prepare some propositions and lemmas.

Proposition 4.3.4. *Let $\lambda \in \mathbb{C}_p$ is a primitive q th root of unity. Then, $n \in \mathbb{N}$ satisfies $\lambda^n = 1$ if and only if n is divisible by q.*

It is clear so we omit it.

Proposition 4.3.5. *Let $f \in \mathbb{C}_p(T)$ be a rational map with $\deg(f) \geq 2$ and q be a natural number. Suppose that 0 is a fixed point of f. Then, there exists some $N \geq 2$ and $A \in \mathbb{C}_p^\times$ such that*

$$f(z) = \lambda_f(0)z + Az^N + O(z^{N+1}) \quad (z \to 0).$$

Moreover, for any $k \in \mathbb{N}$, we have

$$f^k(z) = \lambda_f(0)^k z + A\lambda_f(0)^{k-1}(1 + \lambda_f(0)^{N-1} + \cdots + \lambda_f(0)^{(k-1)(N-1)})z^N + O(z^{N+1}) \quad (z \to 0).$$

Proof. To ease notation, we shall use

$$\lambda := \lambda_f(0)$$

in this proof. Since f is a rational map and 0 is not pole of f, there exists some $r > 0$ and $\{a_i\}_{i \in \mathbb{N}} \subset \mathbb{C}_p$ such that for any $z \in \overline{D}_r(0)$,

$$f(z) = a_1 z + a_2 z^2 + \cdots, \quad \lim_{i \to \infty} |a_i|_p r^i = 0.$$

In particular, it is clear that

$$a_1 = f'(0) = \lambda.$$

Moreover, since $\deg(f) \geq 2$, there exists sone $j \geq 2$ such that $a_j \neq 0$. Thus, setting

$$N := \min\{n \geq 2 \mid a_n \neq 0\}, \quad A := a_N,$$

we have

$$f(z) = \lambda z + Az^N + O(z^{N+1}) \quad (z \to 0).$$

Now let us prove the second statement by an induction on $k \in \mathbb{N}$. The case when $k = 1$ is clear. Assume that it holds for the case when $k = i$. Then it follows immediately that

$$\begin{aligned} f^{i+1}(z) = f^i \circ f(z) &= \lambda^i f(z) + A\lambda^{i-1}(1 + \lambda^{n-1} + \cdots + \lambda^{(i-1)(n-1)}) f(z)^n + O(z^{n+1}) \\ &= \lambda^i(\lambda + Az^n) + A\lambda^{i-1}(1 + \lambda^{n-1} + \cdots + \lambda^{(i-1)(n-1)})(\lambda z + Az^n)^n + O(z^{n+1}) \\ &= \lambda^{i+1} z + A\lambda^i z^n + A\lambda^{i-1}\lambda^n(1 + \lambda^{n-1} + \cdots + \lambda^{(i-1)(n-1)}) z^n + O(z^{n+1}) \\ &= \lambda^{i+1} z + A\lambda^i(1 + \lambda^{n-1} + \cdots + \lambda^{(i-1)(n-1)} + \lambda^{i(n-1)}) z^n + O(z^{n+1}) \quad (z \to 0). \end{aligned}$$

Thus, the statement holds for every $k \in \mathbb{N}$. □

Proposition 4.3.6. *Let $f \in \mathbb{C}_p(T)$ be a rational map with $\deg(f) \geq 2$ and q be a natural number. Suppose that 0 is a fixed point of f and $\lambda_f(\alpha)$ is a q th root of unity. Then, there exist some $B \in \mathbb{C}_p^\times$ and $M \geq 2$ such that*

$$f^q(z) = z + Bz^M + O(z^{M+1}) \quad (z \to 0).$$

Moreover, we have

$$q \mid M - 1.$$

Proof. To ease notation, we shall use

$$\lambda := \lambda_f(0)$$

in this proof. It follows from Proposition 4.3.5 that

$$f^q(z) = \lambda^q z + A\lambda^{q-1}(1 + \lambda^{N-1} + \cdots + \lambda^{(q-1)(N-1)}) z^N + O(z^{N+1}) \quad (z \to 0)$$

for some $A \in \mathbb{C}_p^\times$ and $N \geq 2$. Thus, it follows from $\deg(f^q) \geq 2$ that there exist some $B \in \mathbb{C}_p^\times$ and $M \geq 2$ such that

$$f^q(z) = z + Bz^M + O(z^{M+1}) \quad (z \to 0).$$

Next, we show the second statement. Suppose that

$$f(z) = \lambda z + Az^N + a_1 z^{N+1} + \cdots + a_{M-N} z^M + O(z^{M+1}) \quad (z \to 0)$$

where $\{a_i\}_{i=1}^{M-N} \subset \mathbb{C}_p$. Then we first have that

$$\begin{aligned} f^q \circ f(z) = f(z) + B(f(z))^M + O(z^{M+1}) &= \lambda z + Az^N + a_1 z^{N+1} + \cdots + a_{M-N} z^M \\ &\quad + B(\lambda z + Az^N + a_1 z^{N+1} + \cdots + a_{M-N} z^M)^M + O(z^{M+1}) \\ &= \lambda z + Az^N + a_1 z^{N+1} + \cdots + a_{M-N-1} z^{M-1} + (a_{M-N} + B\lambda^M) z^M + O(z^{M+1}) \quad (z \to 0). \end{aligned}$$

On the other hand, we obtain that

$$\begin{aligned} f \circ f^q(z) &= \lambda f^q(z) + A(f^q(z))^N + a_1(f^q(z))^{N+1} + \cdots + a_{M-N}(f^q(z))^M + O(z^{M+1}) \\ &= \lambda(z + Bz^M) + A(z + Bz^M)^N + a_1(z + Bz^M)^{N+1} + \cdots + a_{M-N}(z + Bz^M)^M + O(z^{M+1}) \\ &= \lambda z + Az^N + \cdots + a_{M-N-1} z^{M-1} + (a_{M-N} + B\lambda) z^M + O(z^{M+1}) \quad (z \to 0). \end{aligned}$$

Since there is an obvious functional equation

$$f^q \circ f = f^{q+1} = f \circ f^q,$$

we have that

$$a_{M-N} + B\lambda^m = a_{M-N} + B\lambda.$$

It follows from Proposition 4.3.4 that

$$q \mid M - 1.$$

□

Lemma 4.3.7. *Let f be a rational map over $\mathbb{C}_p$ with $\deg(f) \geq 2$. Suppose that α is a periodic point of f with prime period r. The multiplicity of α of $f^{n\cdot r}(z) - z$ is greater than 1 if and only if $\lambda_f(\alpha)$ is a primitive q th root of unity for some $q \in \mathbb{N}$ and n is divisible by q. In particular, if the multiplicity of α of $f^{n\cdot r}(z) - z$ is greater than 1 for some $n \in \mathbb{N}$, then α must be parabolic. Moreover, if the multiplicity of α of $f^{k\cdot r}(z) - z$ is greater than 1, then the multiplicity of α of $f^{k\cdot r\cdot n}(z) - z$ is equal to the multiplicity of α of $f^{k\cdot r}(z) - z$ for all $n \in \mathbb{N}$.*

Proof. We consider the following cases.

Case 1: $\alpha = 0$.

Let us first assume that $\alpha = 0$. To ease notation, we shall use

$$\lambda := \lambda_f(0).$$

Then, it follows from Proposition 4.3.5 that there exists some $N \geq 2$ and $A \in \mathbb{C}_p^\times$ such that

$$f^r(z) = \lambda z + Az^N + O(z^{N+1}) \quad (z \to 0).$$

Suppose that the multiplicity of α of $f^{n\cdot r}(z) - z$ is greater than 1. It follows from Proposition 4.3.5 that

$$f^{n\cdot r}(z) - z = (\lambda^n - 1)z + A\lambda^{n-1}(1 + \lambda^{N-1} + \cdots + \lambda^{(n-1)(N-1)})z^N + O(z^{N+1}) \quad (z \to 0).$$

This implies that

$$\lambda^n - 1 = 0.$$

That is, λ must be a primitive q th root of unity for some $q \in \mathbb{N}$. Moreover, it follows from Proposition 4.3.4 that n must be divisible by q.

Now suppose that λ is a primitive q th root of unity for some $q \in \mathbb{N}$ and n is divisible by q. Then, it follows from Proposition 4.3.5 that

$$f^{n\cdot r}(z) - z = (\lambda^n - 1)z + A\lambda^{n-1}(1 + \lambda^{N-1} + \cdots + \lambda^{(n-1)(N-1)})z^N + O(z^{N+1}) \quad (z \to 0).$$

By Proposition 4.3.4, we have

$$f^{n\cdot r}(z) - z = A\lambda^{n-1}(1 + \lambda^{N-1} + \cdots + \lambda^{(n-1)(N-1)})z^N + O(z^{N+1}) \quad (z \to 0).$$

Since $N \geq 2$, this implies that the multiplicity of α of $f^{n\cdot r}(z) - z$ is greater than 1.

Next, we suppose that the multiplicity of α of $f^r(z) - z$ is greater than 1. This implies that

$$f^{k\cdot r}(z) = z + Az^N + O(z^{N+1}) \quad (z \to 0).$$

Then, it follows from Proposition 4.3.5 that

$$\begin{aligned} f^{r\cdot n}(z) &= z + Az^N(1+1+\cdots+1) + O(z^{N+1}) \\ &= z + A\cdot n\cdot z^N + O(z^{N+1}) \quad (z\to 0) \end{aligned}$$

Case 2: $\alpha \in \mathbb{C}_p^\times$.

By considering the conjugation $T^{-1}\circ f\circ T$ by

$$\begin{aligned} T : \mathbb{C}_p &\to \mathbb{C}_p \\ z &\mapsto z-\alpha, \end{aligned}$$

we may reduce the argument to Case 1.

Case 3: $\alpha = \infty$.

By considering the conjugation $T^{-1}\circ f\circ T$ by

$$\begin{aligned} T : \mathbb{C}_p &\to \mathbb{C}_p \\ z &\mapsto \frac{1}{z}, \end{aligned}$$

we may reduce the argument to Case 1.

□

Now let us show Theorem 4.3.1.

Proof of Theorem 4.3.1. Setting $d = \deg(f)$, we have that for any $N\in\mathbb{N}$, the number of fixed points of $f^N(z) - z$ is equal to $d^N + 1$, counted with multiplicity.

- **The Multiplicity of Each Fixed Point**

Let α be an element of $\mathbb{P}^1(\mathbb{C}_p)$. Suppose that there exists some $p_\alpha\in\mathbb{N}$ such that for all $0 < i < p_\alpha$,

$$f^{p_\alpha}(\alpha) = \alpha, \quad f^i(\alpha) \neq \alpha.$$

Case 1: $\lambda_f(\alpha)$ is a primitive q_α th root of unity.

Then

$$\lambda_f(\alpha)^{q_\alpha} = 1, \quad \lambda_f(\alpha)^j \neq 1$$

for all $0 < j < q_\alpha$. It is easy to check that

$$\lambda_f(\alpha) = \lambda_f(f(\alpha)) = \cdots = \lambda_f(f^{p_\alpha - 1}(\alpha)).$$

Moreover, it follows from Proposition 4.3.5 and Lemma 4.3.7 that there exists some $l_\alpha\in\mathbb{N}$ such that $f^{k\cdot p_\alpha\cdot q_\alpha}(z) - z$ has a root of multiplicity $q_\alpha\cdot l_\alpha$ at α for any $k\in\mathbb{N}$. Thus, $f^{k\cdot p_\alpha\cdot q_\alpha}(z) - z$ has a root of multiplicity $p_\alpha\cdot q_\alpha\cdot l_\alpha$ on the cycle $\{\alpha, f(\alpha), \cdots, f^{p_\alpha-1}(\alpha)\}$ for any $k\in\mathbb{N}$.

Case 2: $\lambda_f(\alpha)$ is not a root of unity.

In this case, it follows from Lemma 4.3.7 that $f^{p_\alpha}(z) - z$ has a root of multiplicity 1 at α.

- **The Calculation of the Artin-Mazur Zeta Function**

By Lemma 4.3.2, there exist finitely many parabolic cycles. That is, there exists some $N \in \mathbb{N}$ and $\{z_i\}_{i=1}^N \subset \mathbb{P}^1(\mathbb{C}_p)$ such that

$$\begin{aligned} C_1 &:= \{z_1, f(z_1), \cdots, f^{p_1 - 1}(z_1)\}, \\ C_2 &:= \{z_2, f(z_2), \cdots, f^{p_2 - 1}(z_2)\}, \\ &\cdots \\ C_N &:= \{z_N, f(z_N), \cdots, f^{p_N - 1}(z_N)\}, \end{aligned}$$

where p_i satisfies

$$f^{p_i}(z_i) = z_i, \quad f^j(z_i) \neq z_i \quad (\forall j = 1, 2, \cdots p_i - 1)$$

for each $i = 1, 2, \cdots N$, and $C_i \cap C_j = \emptyset$ for each $i \neq j = 1, 2, \cdots, N$. Since each C_i is a parabolic cycle, there exists some $q_i \in \mathbb{N}$ such that

$$\lambda_f(z_i)^{q_i} = 1, \quad \lambda_f(z_i)^k \neq 1$$

for each $0 < k < q_i$. It follows from Case 1 that

$$\lambda_f(z_i) = \lambda_f(f(z_i)) = \cdots = \lambda_f(f^{p_i - 1}(z_i)).$$

Moreover, there exists $l_i \in \mathbb{N}$ such that $f^{k \cdot p_i \cdot q_i}(z) - z$ has a root of multiplicity $p_i \cdot q_i \cdot l_i$ on the cycle $\{z_i, f(z_i), \cdots, f^{p_i - 1}(z_i)\}$ for any $k \in \mathbb{N}$. Hence, we obtain the following calculation from Lemma 4.3.7.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{N_n}{n} T^n &= \sum_{n=1}^{\infty} \frac{d^n + 1}{n} T^n - \sum_{i=1}^{N} \sum_{k_i=1}^{\infty} \frac{p_i q_i l_i}{p_i q_i k_i} T^{p_i q_i k_i} \\ &= \sum_{n=1}^{\infty} \frac{d^n}{n} T^n + \sum_{n=1}^{\infty} \frac{1}{n} T^n - \sum_{i=1}^{N} l_i \sum_{k_i=1}^{\infty} \frac{1}{k_i} T^{p_i q_i k_i} \\ &= \log(1 - dT)^{-1} + \log(1 - T)^{-1} + \sum_{i=1}^{N} l_i \log(1 - T^{p_i q_i}) \\ &= \log\{(1 - dT)^{-1}(1 - T)^{-1} \prod_{i=1}^{N} (1 - T^{p_i q_i})^{l_i}\}. \end{aligned}$$

This implies that

$$Z_f(T) = \exp(\sum_{n=1}^{\infty} \frac{N_n}{n} T^n) = (1 - dT)^{-1}(1 - T)^{-1} \prod_{i=1}^{N} (1 - T^{p_i q_i})^{l_i}.$$

□

5 APPENDIX

In this section, we will give some results from other mathematical fields to help read this thesis.

5.1 APPENDIX A: Some Results from Algebra

Let us fix a prime number p and denote the quotient filed $\mathbb{Z}/\mathbb{Z}_p$ by $\mathbb{F}_p$.

Theorem 5.1.1 (Fermat's Little Theorem in $\mathbb{F}_p$)**.** *Let F be the polynomial map*

$$\begin{aligned} F : \mathbb{F}_p &\to \mathbb{F}_p \\ z &\mapsto z^p - z. \end{aligned}$$

Then, for any $w \in \mathbb{F}_p$, we have

$$F(w) = 0.$$

As an application of Theorem 5.1.1, we have the following theorem. Note that $\mathbb{Z}_p$ is the p-adic integers defined by $\{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$.

Theorem 5.1.2. *Let F be the polynomial map*

$$\begin{aligned} F : \mathbb{Z}_p &\to \mathbb{Z}_p \\ z &\mapsto z^p - z. \end{aligned}$$

Then, for any $w \in \mathbb{Z}_p$, we have

$$|F(w)|_p \leq \frac{1}{p}.$$

Proof. By Proposition 2.1.8, there exists some $\{p_i\}_{i\in\mathbb{N}}$ with $p_i \in \{0, 1, ..., p-1\}$ such that

$$w = p_0 + p_1 p + p_2 p^2 + \cdots .$$

Considering the canonical projection $\pi : \mathbb{Z}_p \to \mathbb{F}_p$, we see that

$$\pi(w) = p_0, \quad \pi(w^p) = p_0^p.$$

By Theorem 5.1.1, we have

$$\pi(F(w)) = \pi(w^p - w) = p_0^p - p_0 = 0.$$

This implies that

$$|F(w)|_p \leq \frac{1}{p}.$$

□

Proposition 5.1.3. *If K be an algebraically closed field, then K must be infinite.*

Proof. (By contradiction) Assume that K is a finite field. Then, there exist some $N \in \mathbb{N}$ and $\{p_i\}_{i1}^N$ such that

$$\{p_i\}_{i=1}^N = K.$$

Considering a monic polynomial

$$P(z) := \prod_{i=1}^{d}(z - p_i) + 1 \in Poly(K),$$

we see that $P(z)$ has no roots in K. This is a contradiction to the fact that K is algebraically closed. □

For the following proposition, we recall our notation

$$\mathcal{O}_K := \{z \in K \mid |z| \leq 1\}.$$

Proposition 5.1.4. *Let $(K, |\cdot|)$ be an algebraically closed non-Archimedean field. If P is a non-constant monic polynomial over $\mathcal{O}_K$, the roots must be in $\mathcal{O}_K$.*

One may use the Newton polygon to prove this proposition. However, an alternative proof by contradiction is given in this thesis.

Proof. (By contradiction) Let us fix a polynomial

$$P(z) = a_0 + a_1 z + \cdots + a_d z^d \in Poly(\mathcal{O}_K), \quad a_d = 1.$$

Since K is algebraically closed, the polynomial P must have a root ξ in K. Let us assume that $|\xi| > 1$. This implies that $|\xi|^n > 1$ for any $n = 1, 2, \cdots, d$. Moreover, $|\xi|^{n+1} > |\xi|^n$ for all $n = 0, 1, \cdots, d-1$. Thus, we obtain that for all $n = 0, 1, \cdots, d-1$,

$$|a_n \xi^n| < |\xi^d|.$$

It follows from Proposition 2.1.5 that

$$|P(\xi)| = |\sum_{i=0}^{d} a_i \xi^i| = |\xi|^d > 1.$$

On the other hand, it is clear that

$$|P(\xi)| = |0| = 0.$$

This is a contradiction. □

5.2 APPENDIX B: Some Results from Real Analysis

Proposition 5.2.1. *Let (X, d) be a complete metric space. If the subset A of X is non-empty and closed, then $(A, d|_A)$ is also a complete metric space where $d|_A$ is the restriction of d to A.*

Proof. Let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in A. Since (X, d) is complete, there exists $a \in X$ such that

$$\lim_{n\to\infty} a_n = a.$$

However, since A is a closed subset, it follows immediately that a must be in A. This implies that $\{a_n\}_{n\in\mathbb{N}}$ is a convergent sequence in A. □

Theorem 5.2.2 (Baire Category Theorem)**.** *Let (X, d) be a complete metric space. Suppose that $\{X_i\}_{i\in\mathbb{N}}$ is a sequence of open subsets of X such that $\overline{X_i} = X$ for all $i \in \mathbb{N}$. Then*

$$\overline{\bigcap_{i\in\mathbb{N}} X_i} = X.$$

Proof. It is clear that

$$\overline{\bigcap_{i\in\mathbb{N}} X_i} \subset X.$$

Let us prove that

$$X \subset \overline{\bigcap_{i\in\mathbb{N}} X_i}.$$

Taking an arbitrary $x \in X$ and $r > 0$, we will show that

$$D_r(x) \cap \bigcap_{i\in\mathbb{N}} X_i \neq \emptyset.$$

Since $\overline{X_1} = X$, we can take

$$x_1 \in D_r(x) \cap X_1 \neq \emptyset.$$

Moreover, since $D_r(x) \cap X_1 \subset X$ is open, there exists some $r_1 > 0$ such that

$$\overline{D}_{r_1}(x_1) \subset D_r(x) \cap X_1, \quad r_1 \leq \frac{r}{2}.$$

Let us assume that $\{(x_i, r_i)\}_{i=1}^N$, which satisfy

$$\overline{D}_{r_{i+1}}(x_{i+1}) \subset D_{r_i}(x_i) \cap X_{i+1}, \quad r_{i+1} \leq \frac{r_i}{2}$$

for all $i \in \{1, 2, \cdots, N-1\}$, have already been constructed. Since $\overline{X_{N+1}} = X$, we can also take

$$x_{N+1} \in D_{r_N}(x_N) \cap X_{N+1} \neq \emptyset.$$

Moreover, since $D_{r_N}(x_N) \cap X_{N+1} \subset X$ is open, there exists some $r_{N+1} > 0$ such that

$$\overline{D}_{r_{N+1}}(x_{N+1}) \subset D_{r_N}(x_N) \cap X_{N+1}, \quad r_{N+1} \leq \frac{r_N}{2}.$$

It is clear that for all $k \in \mathbb{N}$

$$r_k \leq \frac{r}{2^k}, \quad \overline{D}_{r_{k+1}}(x_{k+1}) \subset D_{r_k}(x_k) \subset D_r(x).$$

In particular, one may easily check that $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is a complete metric space, there exists some $x_\infty \in X$ such that

$$\lim_{k\to\infty} |x_\infty - x_k| = 0, \quad |x - x_\infty| < r.$$

Moreover, for all $n \in \mathbb{N}$,

$$x_\infty \in X_n.$$

This implies that

$$D_r(x) \cap \bigcap_{i\in\mathbb{N}} X_i \neq \emptyset.$$

Since $x \in X$ and $r > 0$ are arbitrary, we have

$$X \subset \overline{\bigcap_{i\in\mathbb{N}} X_i}.$$

□

5.3 APPENDIX C: Some Results from Complex Analysis

Theorem 5.3.1 (Riemann-Hurwitz Formula)**.** *Suppose that f is a polynomial map over $\mathbb{C}$ and the degree of f is $D \in \mathbb{N}$. Then,*

$$\sum_{\alpha \in \mathbb{C}} (e_\alpha(f) - 1) = D - 1$$

where $e_\alpha(f)$ is the order of $f(z) - f(\alpha)$ at α, that is,

$$e_\alpha(f) := \min\{n \geq 0 \mid f^{(n)}(\alpha) \neq 0\}$$

where $f^{(n)}(\alpha)$ is a n-th derivative of f at α for each $n \in \mathbb{N}$ and $f^{(0)}(\alpha) = f(\alpha)$.

The most well-known proof of this theorem may be a topological proof by using Euler's number. In this thesis, an algebraic proof is given to use this theorem not only in the complex field, but also in non-Archimedean fields. One will notice that the proof can be applied to rational maps and that can be found in [S, Theorem 1.1].

Proof. Let us begin with the following claim.

Claim 1 $e_\alpha(f') = e_\alpha(f) - 1$.

Proof of Claim 1. Fixing an arbitrary $\alpha \in \mathbb{C}$. We shall use notation E to denote $e_\alpha(f)$. There exist some $R > 0$ and $\{a_i\}_{i=1}^{D}$ in $\mathbb{C}$ such that for all $z \in \overline{D}_R(\alpha)$,

$$f(z) = f(\alpha) + a_E(z - \alpha)^E + a_{E+1}(z - \alpha)^{E+1} + \cdots + a_D(z - \alpha)^D.$$

Thus, we have that

$$f'(z) = a_E \cdot E \cdot (z - \alpha)^{E-1} + \cdots + Da_D(z - \alpha)^{D-1}.$$

Since the characteristic of $\mathbb{C}$ is 0, this implies that $e_\alpha(f') = e_\alpha(f) - 1$. □

Claim 2 $e_\alpha(f) = 0$ if and only if $f(\alpha) = 0$.

The proof of Claim 2 follows immediately so we omit it. Since $\mathbb{C}$ is an algebraically closed field and $\deg(f') = D - 1$, there exist $D - 1$ zeros of f in $\mathbb{C}$, counted with multiplicity. Thus, we have

$$\sum_{\alpha \in \mathbb{C}} (e_\alpha(f) - 1) = \sum_{\alpha \in \mathbb{C}} e_\alpha(f') = \sum_{\alpha\text{: zeros of } f'} e_\alpha(f') = \deg(f') = D - 1.$$

□

One may notice that the properties, which have been used in Theorem 5.3.1, are true for $\mathbb{C}_p$. See Theorem 2.1.31. Hence, we have the following corollary.

Theorem 5.3.2 (Riemann-Hurwitz Formula)**.** *Let p be a prime number. Suppose that f is a polynomial map over $\mathbb{C}_p$ and the degree of f is $D \in \mathbb{N}$. Then,*

$$\sum_{\alpha \in \mathbb{C}_p} (e_\alpha(f) - 1) = D - 1.$$

5.4 APPENDIX D: p-adic Fields

In this subsection, we give some properties of p-adic fields. First of all, we consider a motivation why we consider the p-adic norm on $\mathbb{Q}$. In fact, we will see that the essential norm on $\mathbb{Q}$ is either the Euclidean norm or the p-adic norm for some prime number $p \in \mathbb{N}$. Let us begin with a terminology.

Definition 5.4.1 (Trivial Norm)**.** A norm $|\cdot|$ defined on $\mathbb{Q}$ is *trivial* if it satisfy

$$|0| = 0, \quad |x| = 1$$

for all $x \in \mathbb{Q} - \{0\}$.

Theorem 5.4.2 (Ostrowski's Theorem)**.** *Let $|\cdot|$ be a non-trivial multiplicative norm on $\mathbb{Q}$. Then, $|\cdot|$ is equivalent to the Euclidean norm or the p-adic norm for some prime number $p \in \mathbb{N}$ where $|\cdot|_1$ is equivalent to $|\cdot|_2$ if there exists some $C > 0$ such that for all $x \in \mathbb{Q}$, we have*

$$|x|_1 = |x|_2^C.$$

Secondly, we give a theorem, which tells us the algebraic relationship between $\mathbb{C}$ and $\mathbb{C}_p$.

Theorem 5.4.3. *Let p be a prime number. There exists some field isomorphic map ι between the field $\mathbb{C}_p$ and the complex field $\mathbb{C}$.*

The proof can be found in [R, p.145, Theorem].
Thirdly, let us consider a measure on p-adic fields.

Theorem 5.4.4 (Haar Measure)**.** *Let $(K, \cdot)$ be a compact topological group. Then, there exists a unique Borel measure μ such that*

(1) $\mu(K) = 1$,
(2) *If U is non-empty Borel set of K, then $\mu(U) > 0$,*
(3) *μ is invariant under $\cdot$.*

This unique measure μ is called the Haar measure on K.

The proof can be found in [JC, CHAPTER VI]. In particular, since $\mathbb{Q}_p$ or a finite extension field K of $\mathbb{Q}_p$ is a locally compact field, the closed unit disk of it must be a compact ring. Moreover, the closed disk is a topological group with respect to $+$. Thus, we can find the Haar measure on the closed unit disk with respect to $+$.
Finally, we give an equivalent theorem of bijectivity of polynomial maps over $\mathbb{C}_p$.

Theorem 5.4.5. *Let p be a prime number and f be a polynomial map over $\mathbb{C}_p$. Suppose that a and b in $\mathbb{C}_p$ satisfy $b = f(a)$. Then f maps the rational closed disk $\overline{D}_r(a)$ bijectively onto the rational closed disk $\overline{D}_s(b)$ if and only if*

$$|f(z) - f(w)|_p = \frac{s}{r}|z - w|_p$$

for all $z, w \in \overline{D}_r(a)$.

See [RB01] or [S, Exercise 5.4 (c)] for the proof.

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