Topology of the Julia sets of rational functions

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Preface

The dynamics of a rational function induces a subdivision of the Riemann sphere into the two complementary sets which are called the Fatou set and the Julia set. The behavior of iterations of the rational function on the Fatou set is regular. The behavior of iterations of the rational function on the Julia set is chaotic. The Julia sets are important objects in a sense of dynamics of the rational functions, and almost all the Julia sets take very complicated forms.

We are interested in topological structures of the Julia sets and the boundaries of Fatou components. Local connectivity can be an indicator of the complexity of the topological structures. There exist rational functions whose Julia sets are so complicate and not locally connected. The purpose of the paper is to clarify the topology of such complicated Julia sets.

This paper is organized as follows.

In Chapter 1, we consider biaccessible points in the Julia sets of some rational functions. Let Ω be a simply connected Fatou component and $z_0 \in \partial\Omega$. We are interested in knowing whether there exist at least two distinct external rays in Ω landing at z_0. We introduce a topological technique can be applied to the local dynamics. So we investigate which points in the Julia sets can be biaccessible by using the technique.

In Chapter 2, we consider periodic points on rotation domains under some conditions. Let Ω be a Fatou component on which the dynamics corresponds to irrational rotation. When the boundary $\partial\Omega$ fails to be locally connected, we are interested in the dynamics of the boundary $\partial\Omega$. We investigate whether there are periodic points on the boundary $\partial\Omega$.

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Contents

Chapter 1

On biaccessible points in the Julia sets of some rational functions

1.1 Introduction and results

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function of degree $d \geq 2$. We define the *Fatou set* of f as the union of all open sets $U \subset \widehat{\mathbb{C}}$ such that the family of iterates $\{f^{\circ n}|_U\}_{n \geq 0}$ forms a normal family, and the *Julia set* of f as the complement of the Fatou set of f. We denote the Julia set of f by J_f and the Fatou set of f by F_f. Clearly, the Fatou set F_f is open and the Julia set J_f is closed. A connected component of the Fatou set is called a *Fatou component*. Their fundamental properties can be found in [**Mi**].

For each fixed point z_0, the *multiplier* at z_0 is defined as $\lambda = f'(z_0)$ when $z_0 \neq \infty$ and is defined as $\lambda = \lim_{z \to \infty} 1/f'(z)$ when $z_0 = \infty$.

A fixed point z_0 is called *superattracting* if the multiplier λ is equal to zero, or equivalently z_0 is a critical point. Then the point z_0 is contained in the Fatou set F_f. The Fatou component containing the superattracting fixed point z_0 is called the *immediate basin* of z_0, and we denote by $\mathcal{A}_{z_0}$.

A fixed point z_0 is called *irrationally indifferent* if the multiplier λ satisfies $|\lambda| = 1$ but λ is not a root of unity, or equivalently there exists an irrational number θ such that $\lambda = e^{2\pi i\theta}$. So we distinguish between two possibilities.

If an irrationally indifferent fixed point z_0 lies in the Fatou set, the point

z_0 is called a *Siegel point*. The Fatou component containing a Siegel point z_0 is called the *Siegel disk* with *center* z_0, and we denote by $\mathcal{S}_{z_0}$.

If an irrationally indifferent fixed point z_0 belongs to the Julia set, the point z_0 is called a *Cremer point*. We say that a Cremer point z_0 has the *small cycles property* if every neighborhood of z_0 contains infinitely many periodic orbits. For quadratic polynomials, every Cremer point has the small cycles property [**Yo1**]. However, it is not known whether this is true for arbitrary rational functions.

An invariant Fatou component $\mathcal{H}$ is called a *Herman ring* if $\mathcal{H}$ is conformally isomorphic to some annulus. Then the dynamics of f on $\mathcal{H}$ corresponds to the dynamics of an irrational rotation on this annulus.

Let $\Omega \subset \widehat{\mathbb{C}}$ be a simply connected domain. Assume that the boundary $\partial\Omega$ contains at least two points. For the sake of convenience, we assume that Ω contains infinity ∞, and consider a conformal isomorphism $\Phi : \widehat{\mathbb{C}} - \overline{\mathbb{D}} \to \Omega$ such that $\Phi(\infty) = \infty$. For each *angle* $t \in \mathbb{R}/\mathbb{Z}$, the *external ray* is defined as

$$R_t = \{\Phi(re^{2\pi it}) : r > 1\}.$$

For each *radius* $r > 1$, the *equipotential curve* is defined as

$$E_r = \{\Phi(re^{2\pi it}) : t \in \mathbb{R}/\mathbb{Z}\}.$$

If there exists a point $z \in \partial\Omega$ such that $\lim_{r\searrow 1} \Phi(re^{2\pi it}) = z$, then we say that the external ray R_t *lands* at the point z. A point $z \in \partial\Omega$ is called *accessible* from Ω if there exists a continuous curve $\gamma : [0, 1) \to \Omega$ such that $\lim_{s\nearrow 1} \gamma(s) = z$. Then there exists an external ray landing at z (see for example [**Mc**, Corollary 6.4]).

Definition 1.1.1 We say that a point $z \in \partial\Omega$ is *biaccessible* from Ω if there exist at least two distinct external rays landing at z (see Figure 1.1).

In the above definition, the biaccessibility from Ω does not depend on the choice of the Riemann maps Φ. In fact, it depends only the topology of the boundary $\partial\Omega$. By a theorem of F. and M. Riesz (see [**Mi**]), $\partial\Omega - \{z\}$ is disconnected whenever $z \in \partial\Omega$ is biaccessible from Ω. Moreover, the converse is true (see [**Mc**, Theorem 6.6]). Therefore, $z \in \partial\Omega$ is biaccessible from Ω if and only if $z \in \partial\Omega$ is a *cut point* of $\partial\Omega$, namely $\partial\Omega - \{z\}$ is disconnected.

We are interested in the topological structures of the Julia sets and the boundaries of Fatou components. There are some results about local connectivity (see for example [**Mi**, **Pe**, **R**, **Ra**]) and (bi)accessibility (see for

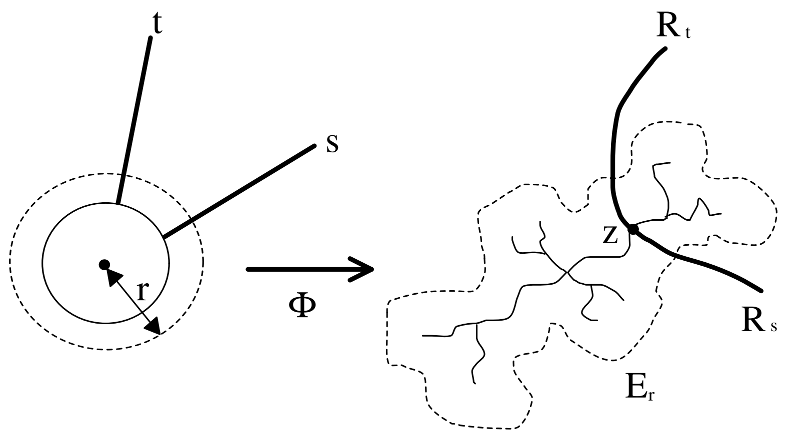

Figure 1.1

example [**P**, **Sch**, **Smi**, **Zd**]). As for Siegel disks, the location of biaccessible points is well known as given in the following proposition.

Proposition 1.1.1 *Let f be a rational function of degree $d \geq 2$. Assume that infinity ∞ is a Siegel point. Let $\mathcal{S}_\infty$ be the Siegel disk with center ∞. If z is biaccessible from $\mathcal{S}_\infty$, then it is a periodic point of f.*

Proof. We take a conformal isomorphism $\Phi : \widehat{\mathbb{C}} - \overline{\mathbb{D}} \to \mathcal{S}_\infty$ such that $\Phi(\infty) = \infty$ and $\Phi^{-1} \circ f \circ \Phi(w) = \lambda w$, where λ is the multiplier at ∞. So λ is written as $e^{2\pi i\theta}$ with an irrational number θ. We consider the dynamics of external rays in the Siegel disk $\mathcal{S}_\infty$. It is easy to see $f^{\circ n}(R_t) = R_{t+n\theta}$ for all $n \geq 0$.

If z is biaccessible from $\mathcal{S}_\infty$, then there exist two distinct external rays R_s and R_t landing at z. Since θ is irrational, we may suppose that

$$s < s + N\theta < t < t + N\theta < s + 1,$$

where N is some number. Let U_1 and U_2 be two distinct components of $\mathbb{C} - (R_s \cup \{z\} \cup R_t)$. So we may assume that $f^{\circ N}(R_s) \subset U_1$ and $f^{\circ N}(R_t) \subset U_2$ (see Figure 1.2).

Both $f^{\circ N}(R_s)$ and $f^{\circ N}(R_t)$ land at $f^{\circ N}(z)$ by the continuity of $f^{\circ N}$. Therefore, $f^{\circ N}(z) \subset \overline{U_1} \cap \overline{U_2}$, and thus $f^{\circ N}(z) = z$.

We consider which points can be biaccessible from the immediate basins of superattracting fixed points. For quadratic polynomials with irrationally

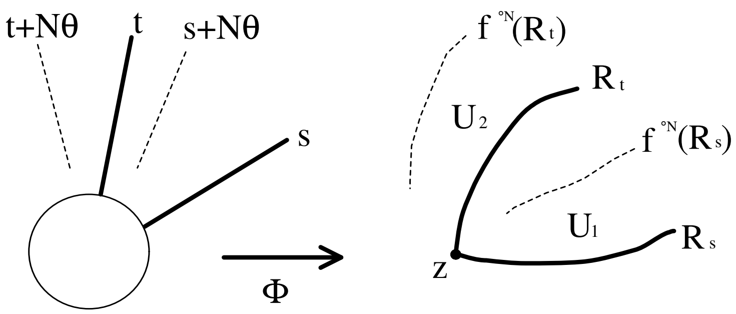

Figure 1.2

indifferent fixed points, S. Zakeri [**Za**] showed the following proposition which is an improvement of [**SZ**, Theorem 3].

Proposition 1.1.2 *Let $f_c(z) = z^2 + c$ be a quadratic polynomial with an irrationally indifferent fixed point α. Assume that z_0 is biaccessible from the immediate basin $\mathcal{A}_\infty$ of infinity. Then:*

- *if α is a Siegel point, the critical point 0 is contained in the forward orbit $\{f_c^{\circ n}(z_0)\}_{n\geq 0}$ of z_0;*
- *if α is a Cremer point, then the point α is contained in the forward orbit $\{f_c^{\circ n}(z_0)\}_{n\geq 0}$ of z_0.*

In the above proposition, if α is a Cremer point, we are interested in whether the point α is accessible or not. In fact, this is an open problem. If the point α is accessible, then it follows from the Snail Lemma that infinitely many external rays land at the point.

In this paper, we shall extend Proposition 1.1.2 for more general polynomials and some rational functions of degree 3. In fact, such functions are well known and selected so as to have simple locations of critical points. However, we deal with the biaccessibility of Fatou components of genuine rational functions, which probably has not been studied as yet.

First, we will show the following which is a small extension of the proposition for polynomials with only one critical point in $\mathbb{C}$.

Theorem 1.1.1 *Let $f_c(z) = z^d + c$ be a polynomial of degree $d \geq 2$ with an irrationally indifferent fixed point α. Assume that z_0 is biaccessible from the immediate basin $\mathcal{A}_\infty$ of infinity. Then:*

- *if α is a Siegel point, the critical point 0 is contained in the forward orbit $\{f_c^{\circ n}(z_0)\}_{n\geq 0}$ of z_0;*

- *if α is a Cremer point, either the point α is contained in the forward orbit $\{f_c^{\circ n}(z_0)\}_{n\geq 0}$ of z_0 or the critical point 0 is contained in the forward orbit $\{f_c^{\circ n}(z_0)\}_{n\geq 0}$ of z_0.*

In the above theorem, if α is a Cremer point which has the small cycles property, then the critical point 0 is not accessible from $\mathcal{A}_\infty$ [**Ki**, Theorem 1.1]. Then $0 \notin \{f_c^{\circ n}(z_0)\}_{n\geq 0}$, and so we can conclude that $\alpha \in \{f_c^{\circ n}(z_0)\}_{n\geq 0}$. According to [**Yo1**], every Cremer point of quadratic polynomials has the small cycles property, so the conclusion of the second part in Proposition 1.1.2 is just $\alpha \in \{f_c^{\circ n}(z_0)\}_{n\geq 0}$.

The following theorem gives an extension for some polynomials having more than one critical point in $\mathbb{C}$. However, we can make use to the symmetrical locations of critical points.

Theorem 1.1.2 *Let $g_\theta(z) = e^{2\pi i\theta}z + z^d$ be a polynomial of degree $d \geq 2$ so that the origin is an irrationally indifferent fixed point. Let $c_0, c_1, \cdots, c_{d-2}$ be all critical points of g_θ in $\mathbb{C}$. Assume that z_0 is biaccessible from the immediate basin $\mathcal{A}_\infty$ of infinity. Then:*

- *if the origin is a Siegel point, there exists a critical point c_{j_0} which is contained in the forward orbit $\{g_\theta^{\circ n}(z_0)\}_{n\geq 0}$ of z_0;*

- *if the origin is a Cremer point, either the origin is contained in the forward orbit $\{g_\theta^{\circ n}(z_0)\}_{n\geq 0}$ of z_0 or there exists a critical point c_{j_0} which is contained in the forward orbit $\{g_\theta^{\circ n}(z_0)\}_{n\geq 0}$ of z_0.*

In the above theorem, if the origin is a Cremer point which has the small cycles property, then there exists a critical point c_{j_0} which is not accessible from $\mathcal{A}_\infty$ [**Ki**, Theorem 1.1]. In addition, the symmetry of the Julia set implies that every critical point c_j is not accessible from $\mathcal{A}_\infty$ (see Section 1.5). Therefore, $c_j \notin \{g_\theta^{\circ n}(z_0)\}_{n\geq 0}$ for all j, and so we can conclude that $0 \in \{g_\theta^{\circ n}(z_0)\}_{n\geq 0}$.

Finally, we will consider some rational functions of degree 3 which are corresponding to quadratic polynomials with irrationally indifferent fixed points in a sense. Indeed, the dynamics of analytic circle diffeomorphisms with irrational rotation numbers and the local dynamics of irrationally indifferent fixed points are similar in certain respects. So we will suggest a new application of Herman compacta to the proof of the following theorem.

Theorem 1.1.3 *Let* $h(z) = h_{\theta,a}(z) = e^{2\pi i\theta} z^2(z-a)/(1-\bar{a}z)$ *be a rational function so that* $|a| > 3$ *and the rotation number* $\mathrm{Rot}(h|_{\mathbb{S}^1})$ *is irrational. Let* c *be the critical point of* h *such that* $|c| > 1$. *Assume that* z_0 *is biaccessible from the immediate basin* $\mathcal{A}_\infty$ *of infinity. Then the critical point* c *is contained in the forward orbit* $\{h^{\circ n}(z_0)\}_{n \geq 0}$ *of* z_0.

In the above theorem, we fix $|a| > 3$ and consider the one-parameter family $h_{\theta,a}(z) = e^{2\pi i\theta} z^2(z-a)/(1-\bar{a}z)$ with θ of rational functions. From the continuity and the monotonous increasing of the rotation function $\theta \mapsto \mathrm{Rot}(h_{\theta,a}|_{\mathbb{S}^1})$, we can adjust the rotation number to be any desired irrational constant (see [**MS**, Section I.4]).

1.2 Local dynamics

In this section, we suppose that f is a rational function of degree $d \geq 2$ and consider the local dynamics of f. We introduce Siegel compacta and Herman compacta. They are essential for the proofs of the theorems. First, we mention about the linearizability.

Definition 1.2.1 Let z_0 be an irrationally indifferent fixed point of f. Let λ be the multiplier at z_0, so it is written as $e^{2\pi i\theta}$ with an irrational number θ. If there exists a local holomorphic change of coordinate $z = \Phi(w)$, with $\Phi(0) = z_0$, such that $\Phi^{-1} \circ f \circ \Phi$ is the irrational rotation $w \mapsto e^{2\pi i\theta} w$ near the origin, then we say that f is *linearizable* at the point z_0.

An irrationally indifferent fixed point z_0 of f is either a Siegel point or a Cremer point, according to whether f is linearizable at the point z_0 or not. There are some results about the linearizability of irrationally indifferent fixed points (see for example [**Mi**, Section 11]).

Definition 1.2.2 Assume that $f|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$ is an analytic circle diffeomorphism whose rotation number $\mathrm{Rot}(f|_{\mathbb{S}^1})$ is irrational. If there exists an analytic circle diffeomorphism $\Phi : \mathbb{S}^1 \to \mathbb{S}^1$ such that $\Phi^{-1} \circ f \circ \Phi$ is the irrational rotation $w \mapsto e^{2\pi i \mathrm{Rot}(f|_{\mathbb{S}^1})} w$, then we say that f is *linearizable* on $\mathbb{S}^1$.

For a general theory on analytic circle diffeomorphisms, we refer to [**MS**]. There are some results about the linearizability for analytic circle diffeomorphisms with irrational rotation numbers (see for example [**Yo2**]). In addition, there are fine theorem correspondences between the linearizability of irrationally indifferent fixed points and the linearizability for analytic circle diffeomorphisms with irrational rotation numbers (see [**PM**, Theorem I.4.1]).

The following two propositions will be used for the proofs of Theorem 1.1.1 and Theorem 1.1.2.

Proposition 1.2.1 *Let z_0 be an irrationally indifferent fixed point of f. Let U be a bounded neighborhood of z_0 so that the boundary ∂U is a Jordan closed curve. Assume that f is univalent on a neighborhood of $\overline{U}$. Then there exists a set S with the following properties:*

- *S is compact, connected, and $\widehat{\mathbb{C}} - S$ is connected;*
- *$z_0 \in S \subset \overline{U}$, $S \cap \partial U \neq \emptyset$, and $f(S) = S$.*

Moreover, f is linearizable at z_0 if and only if the interior $\mathrm{Int}S$ *of S contains z_0.*

We say that such a set S is a *Siegel compactum* for (f, U). Its applications can be found in [**PM**, Section IV]. The above proposition is described in [**PM**, Theorem 1], however, we do not assume that f^{-1} is defined and univalent on a neighborhood of $\overline{U}$. In fact, the condition leaves no impression on the results.

Proposition 1.2.2 *Assuming the hypothesis in Proposition 1.2.1, let S be a Siegel compactum for (f, U). Then:*

- *if z_0 is a Siegel point, there are no points which are biaccessible from $\widehat{\mathbb{C}} - S$;*
- *if z_0 is a Cremer point, then the point z_0 is the only possible point which is biaccessible from $\widehat{\mathbb{C}} - S$.*

Proof. This proof is referred from the explanations of [**Za**, Proposition 1] and [**SZ**, Proposition 2]. We use proof by contradiction.

First, assume that z_0 is a Siegel point and there exists a point z which is biaccessible from $\widehat{\mathbb{C}} - S$. Let $\Phi : \widehat{\mathbb{C}} - \overline{\mathbb{D}} \to \widehat{\mathbb{C}} - S$ be a conformal isomorphism such that $\Phi(\infty) = \infty$. So $g = \Phi^{-1} \circ f \circ \Phi$ is univalent on an outer neighborhood of $\mathbb{S}^1$. Then g is extended and univalent on a neighborhood of $\mathbb{S}^1$ by the reflection principle. Furthermore, the rotation number $\mathrm{Rot}(g|_{\mathbb{S}^1})$ corresponds to the irrational number θ which satisfies $\lambda = e^{2\pi i\theta}$, where λ is the multiplier at z_0 [**PM**, Theorem 2].

Let R_s and R_t be two distinct external rays land at z. Let X be the component of $\mathrm{Int}S$ which contains the Siegel point z_0. Clearly, $f(X) = X$. Let V be the component of $\mathbb{C} - (R_s \cup \{z\} \cup R_t)$ which does not contain X. We cut off V along an equipotential curve E_r, and thus have the Jordan domain W which is contained in V. Then $D = \Phi^{-1}(W - S)$ has the interval $I \subset \mathbb{S}^1$ as a part of its boundary (see Figure 1.3).

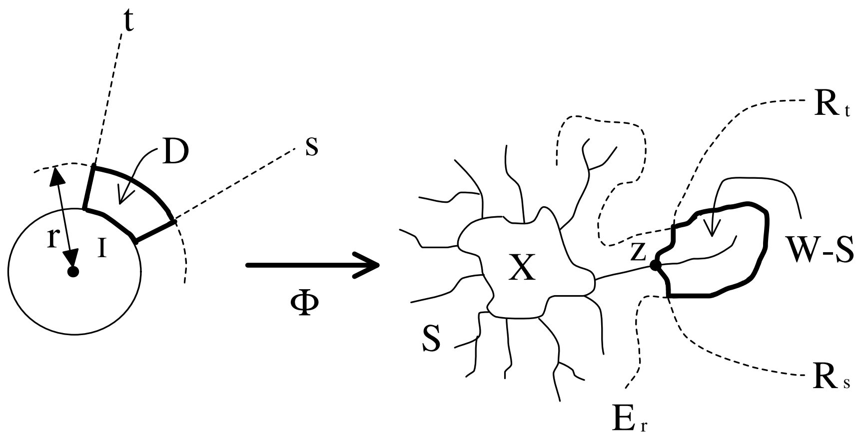

Figure 1.3

Since the rotation number $\mathrm{Rot}(g|_{\mathbb{S}^1})$ is irrational, there exists N such that $\bigcup_{j=0}^{N} g^{\circ j}(I) = \mathbb{S}^1$. We could take a more smaller $r > 1$, so that $g, g^{\circ 2}, \cdots, g^{\circ N}$ are univalent on D, and furthermore, $\bigcup_{j=0}^{N} g^{\circ j}(D)$ is an outer neighborhood of $\mathbb{S}^1$.

Then $f, f^{\circ 2}, \cdots, f^{\circ N}$ are univalent on $W - S$, and thus $\bigcup_{j=0}^{N} f^{\circ j}(W - S)$ is an outer neighborhood of S. So any point of the boundary $\partial X \subset \partial S$ can be approximated by some sequence in $\bigcup_{j=0}^{N} f^{\circ j}(W - S)$. Now the injectivity of f implies that each Jordan domain $f^{\circ j}(W)$ does not intersect X, therefore, $\overline{f^{\circ j}(W)} \cap \overline{X}$ contains at most one point $f^{\circ j}(z)$. This contradicts that ∂X has infinitely many points.

Now, assume that z_0 is a Cremer point and there exists a point $z \neq z_0$ which is biaccessible from $\widehat{\mathbb{C}} - S$. Let $\Phi : \widehat{\mathbb{C}} - \overline{\mathbb{D}} \to \widehat{\mathbb{C}} - S$ be a conformal isomorphism such that $\Phi(\infty) = \infty$. So $g = \Phi^{-1} \circ f \circ \Phi$ is univalent on an outer neighborhood of $\mathbb{S}^1$. Then g is extended and univalent on a neighborhood of $\mathbb{S}^1$ by the reflection principle. Furthermore, the rotation number $\mathrm{Rot}(g|_{\mathbb{S}^1})$ corresponds to the irrational number θ which satisfies $\lambda = e^{2\pi i\theta}$, where λ is the multiplier at z_0 [**PM**, Theorem 2].

Let R_s and R_t be two distinct external rays land at z. Let V be the component of $\mathbb{C} - (R_s \cup \{z\} \cup R_t)$ which does not contain z_0. We cut off V along an equipotential curve E_r, and thus have the Jordan domain W which is contained in V. Then $D = \Phi^{-1}(W - S)$ has the interval $I \subset \mathbb{S}^1$ as a part of its boundary (see Figure 1.4).

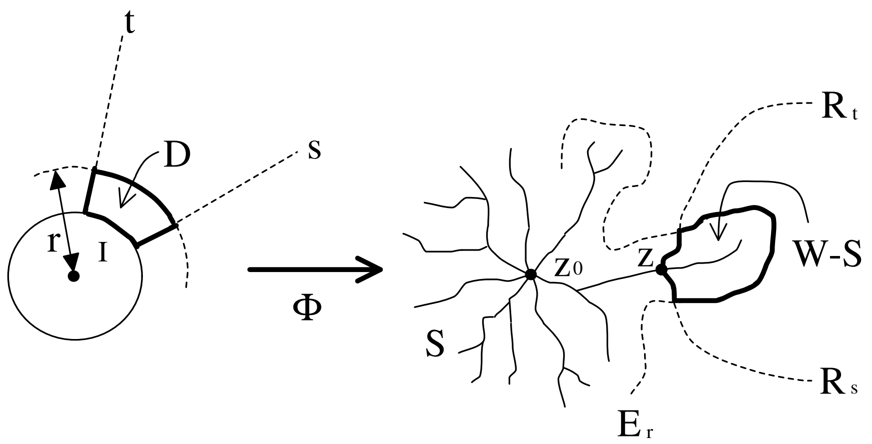

Figure 1.4

Since the rotation number $\mathrm{Rot}(g|_{\mathbb{S}^1})$ is irrational, there exists N such that $\bigcup_{j=0}^{N} g^{\circ j}(I) = \mathbb{S}^1$. We could take a more smaller $r > 1$, so that $g, g^{\circ 2}, \cdots, g^{\circ N}$ are univalent on D, and furthermore, $\bigcup_{j=0}^{N} g^{\circ j}(D)$ is an outer neighborhood of $\mathbb{S}^1$.

Then $f, f^{\circ 2}, \cdots, f^{\circ N}$ are univalent on $W - S$, and thus $\bigcup_{j=0}^{N} f^{\circ j}(W - S)$ is an outer neighborhood of S. So the Cremer point $z_0 \in \partial S$ can be approximated by some sequence in $\bigcup_{j=0}^{N} f^{\circ j}(W - S)$. However, the injectivity of f implies that each Jordan domain $f^{\circ j}(W)$ does not contain z_0 and each $f^{\circ j}(z)$ is not z_0, therefore, $\overline{f^{\circ j}(W)} \cap \{z_0\} = \emptyset$.

The following two propositions will be used for the proof of Theorem 1.1.3.

Proposition 1.2.3 *Let U be a bounded annular neighborhood of $\mathbb{S}^1$ such that the boundary ∂U consists of two Jordan closed curves $\gamma_1 \subset \mathbb{C}-\overline{\mathbb{D}}$ and $\gamma_2 \subset \mathbb{D}$. Assume that f is univalent on a neighborhood of $\overline{U}$ and $f|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$ is an analytic circle diffeomorphism whose rotation number $\mathrm{Rot}(f|_{\mathbb{S}^1})$ is irrational. Assume that $f(U)$ does not contain the bounded component of $\widehat{\mathbb{C}} - \gamma_2$. Then there exists a set H with the following properties:*

- *H is compact, connected, and $\widehat{\mathbb{C}}-H$ has just two connected components;*
- *$\mathbb{S}^1 \subset H \subset \overline{U}$, $H \cap \gamma_1 \neq \emptyset$, $H \cap \gamma_2 \neq \emptyset$, and $f(H) = H$.*

Moreover, f is linearizable on $\mathbb{S}^1$ if and only if the interior $\mathrm{Int}H$ of H contains $\mathbb{S}^1$.

We say that such a set H is a *Herman compactum* for (f, U). The above proposition is described in [**PM**, Theorem V.1.1]. We do not assume that f^{-1} is defined and univalent on a neighborhood of $\overline{U}$, however, we add the assumption that $f(U)$ does not contain the bounded component of $\widehat{\mathbb{C}} - \gamma_2$.

Proposition 1.2.4 *Assuming the hypothesis in Proposition 1.2.3, let H be a Herman compactum for (f, U). Then there are no points which are biaccessible from the unbounded component of $\widehat{\mathbb{C}} - H$.*

In the rest of this section, we shall show the above two propositions.

Lemma 1.2.1 *Let U be a bounded annular neighborhood of $\mathbb{S}^1$ such that the boundary ∂U consists of two Jordan closed curves $\gamma_1 \subset \mathbb{C} - \overline{\mathbb{D}}$ and $\gamma_2 \subset \mathbb{D}$. Assume that f is univalent on a neighborhood of $\overline{U}$ and $f|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$ is an analytic circle diffeomorphism whose rotation number $\mathrm{Rot}(f|_{\mathbb{S}^1})$ is Diophantine. Then the Herman ring $\mathcal{H}$ intersects both γ_1 and γ_2.*

Proof. This proof is referred from the proof of [**PM**, Theorem II.3.1]. Since the rotation number $\mathrm{Rot}(f|_{\mathbb{S}^1})$ is Diophantine, f is linearizable on $\mathbb{S}^1$ [**Yo2**, Theorem 1.4]. So we have the Herman ring $\mathcal{H}$ such that $\mathbb{S}^1 \subset \mathcal{H}$.

We use proof by contradiction. Assume that $\mathcal{H} \cap \gamma_1 = \emptyset$. Let $\{K_n\}_{n\geq 1}$ be a sequence of closed annuli in the Herman ring $\mathcal{H}$ such that $f(K_n) = K_n$, $K_n \subset \mathrm{Int} K_{n+1}$ and $\bigcup_{n=1}^{+\infty} K_n = \mathcal{H}$. So K_n converges to $\overline{\mathcal{H}}$ in the sense of Hausdorff convergence. Let Ω_n be the unbounded component of $\widehat{\mathbb{C}} - K_n$, let Ω be the unbounded component of $\widehat{\mathbb{C}} - \overline{\mathcal{H}}$. So Ω_n converges to Ω with respect to ∞ in the sense of Carathéodory kernel convergence. We consider the following conformal isomorphisms

$$\Phi_n : \widehat{\mathbb{C}} - \overline{\mathbb{D}} \to \Omega_n, \ \Phi : \widehat{\mathbb{C}} - \overline{\mathbb{D}} \to \Omega$$

so that $\Phi_n(\infty) = \Phi(\infty) = \infty$, $\lim_{z\to\infty} \Phi_n(z)/z > 0$ and $\lim_{z\to\infty} \Phi(z)/z > 0$. So Φ_n converges locally uniformly to Φ by the Carathéodory kernel theorem (see for example [**Po**, Theorem 1.8]).

Since f is univalent on a neighborhood of $\overline{U}$ and $\mathcal{H} \cap \gamma_1 = \emptyset$, there exists $r_0 > 1$ such that $g = \Phi^{-1} \circ f \circ \Phi$ is univalent on $\{z : 1 < |z| < r_0\}$. So $g_n = \Phi_n^{-1} \circ f \circ \Phi_n$ is also univalent on $\{z : 1 < |z| < r_0\}$. By the reflection principle, g_n and g are extended and univalent on $\{z : 1/r_0 < |z| < r_0\}$. We fix r such that $1 < r < r_0$. Since Φ_n converges locally uniformly to Φ, g_n converges uniformly to g on $r\mathbb{S}^1$. So g_n converges uniformly to g on $\mathbb{S}^1/r$. By the maximum principle, g_n converges uniformly to g on $\{z : 1/r \leq |z| \leq r\}$, particularly on $\mathbb{S}^1$.

Let L_n be the outer boundary of K_n, let L be the outer boundary of the Herman ring $\mathcal{H}$. We notice that the dynamics of g_n on $\mathbb{S}^1$ corresponds to the dynamics of f on L_n. Since L_n is a Jordan closed curve in the Herman ring $\mathcal{H}$ such that $f(L_n) = L_n$, the dynamics of f on L_n corresponds to the dynamics of the irrational rotation $z \mapsto e^{2\pi i \mathrm{Rot}(f|_{\mathbb{S}^1})} z$. Therefore, the rotation number $\mathrm{Rot}(g_n|_{\mathbb{S}^1})$ corresponds to $\mathrm{Rot}(f|_{\mathbb{S}^1})$. Then,

$$\mathrm{Rot}(g|_{\mathbb{S}^1}) = \lim_{n\to+\infty} \mathrm{Rot}(g_n|_{\mathbb{S}^1}) = \lim_{n\to+\infty} \mathrm{Rot}(f|_{\mathbb{S}^1}) = \mathrm{Rot}(f|_{\mathbb{S}^1}).$$

Therefore, $\mathrm{Rot}(g|_{\mathbb{S}^1})$ is Diophantine, and thus g is linearizable on $\mathbb{S}^1$. So we can take a Jordan closed curve η in an outer neighborhood of $\mathbb{S}^1$ such that $g(\eta) = \eta$, and thus $\Phi(\eta)$ is a Jordan closed curve such that $f(\Phi(\eta)) = \Phi(\eta)$. Let V be the Jordan annular domain which is surrounded by $\Phi(\eta)$ and $\mathbb{S}^1$ (see Figure 1.5).

We notice $f(V) = V$. Moreover, the dynamics of f on V corresponds to the dynamics of the irrational rotation $z \mapsto e^{2\pi i \mathrm{Rot}(f|_{\mathbb{S}^1})} z$ by the classification theorem of dynamics on hyperbolic surfaces (see for example [**Mi**, Theorem

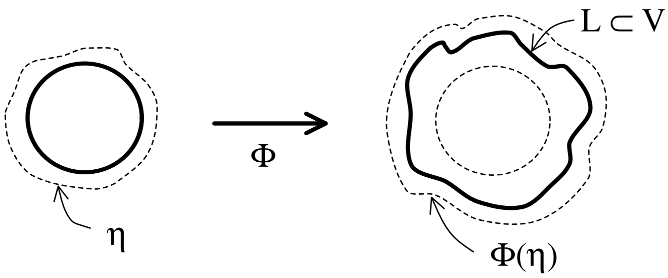

Figure 1.5

5.2]). Then $L \subset V \subset F_f$. This contradicts that L is the outer boundary of the Herman ring $\mathcal{H}$. Therefore, we conclude $\mathcal{H} \cap \gamma_1 \neq \emptyset$. It is possible to see $\mathcal{H} \cap \gamma_2 \neq \emptyset$, as in the above argument.

Proof. (Proof of Proposition 1.2.3) This proof is referred from [**PM**, Section III.2]. Since the rotation number $\mathrm{Rot}(f|_{\mathbb{S}^1})$ is irrational, there exists a sequence $\{\alpha_n\}_{n\geq 1}$ such that $\lim_{n\to+\infty} \alpha_n = 0$ and each $f_n(z) = e^{2\pi i \alpha_n} f(z)$ has the rotation number $\mathrm{Rot}(f_n|_{\mathbb{S}^1})$ which is Diophantine (see also [**MS**, Lemma 4.1]). So f_n is univalent on a neighborhood of $\overline{U}$.

From Lemma 1.2.1, we take the closed annulus H_n in the Herman ring $\mathcal{H}_n$ of f_n with the following properties:

- H_n is compact, connected, and $\widehat{\mathbb{C}} - H_n$ has just two connected components;
- $\mathbb{S}^1 \subset H_n \subset \overline{U}$, $H_n \cap \gamma_1 \neq \emptyset$, $H_n \cap \gamma_2 \neq \emptyset$, and $f_n(H_n) = H_n$.

Every H_n is contained in $\overline{U}$, so there exists a subsequence $\{H_{n_i}\}_{i\geq 1}$ and a set H' such that H_{n_i} converges to H' in the sense of Hausdorff convergence. Then H' has the following properties:

- H' is compact and connected;
- $\mathbb{S}^1 \subset H' \subset \overline{U}$, $H' \cap \gamma_1 \neq \emptyset$ and $H' \cap \gamma_2 \neq \emptyset$.

Since f_{n_i} converges uniformly to f on $\overline{U}$, it follows from [**PM**, Lemma III.1.2] that $f_{n_i}(H_{n_i})$ converges to $f(H')$ in the sense of Hausdorff convergence. Then $f_{n_i}(H_{n_i}) = H_{n_i}$ implies $f(H') = H'$. Let H be the union of H' and all the components of $\widehat{\mathbb{C}} - H'$ contained in U. So $\widehat{\mathbb{C}} - H$ has just two connected components. Since $f(U)$ does not contain the bounded component of $\widehat{\mathbb{C}} - \gamma_2$, it is not difficult to see $f(H) = H$, and thus H satisfies the required properties.

Now, we show the last part of Proposition 1.2.3. If f is linearizable on $\mathbb{S}^1$, it is obvious that $\mathbb{S}^1 \subset \mathrm{Int}H$. Conversely, assume that $\mathbb{S}^1 \subset \mathrm{Int}H$. Let V be the component of $\mathrm{Int}H$ which contains $\mathbb{S}^1$. So V is conformally isomorphic to some annulus, and $f(V) = V$. The dynamics of f on V corresponds to the dynamics of the irrational rotation $z \mapsto e^{2\pi i \mathrm{Rot}(f|_{\mathbb{S}^1})} z$ by the classification theorem of dynamics on hyperbolic surfaces. Therefore, f is linearizable on $\mathbb{S}^1$.

The following lemma corresponds to [**PM**, Theorem 2].

Lemma 1.2.2 *Assuming the hypothesis in Proposition 1.2.3, let H be a Herman compactum for (f, U). Let Ω be the unbounded component of $\widehat{\mathbb{C}} - H$, let $\Phi : \widehat{\mathbb{C}} - \overline{\mathbb{D}} \to \Omega$ be a conformal isomorphism such that $\Phi(\infty) = \infty$. So $g = \Phi^{-1} \circ f \circ \Phi$ is univalent on an outer neighborhood of $\mathbb{S}^1$. Then g is extended and univalent on a neighborhood of $\mathbb{S}^1$ by the reflection principle. Furthermore, the rotation number $\mathrm{Rot}(g|_{\mathbb{S}^1})$ corresponds to the rotation number $\mathrm{Rot}(f|_{\mathbb{S}^1})$.*

Proof. First, we show that there exists a Herman compactum H for (f, U) such that $\mathrm{Rot}(g|_{\mathbb{S}^1}) = \mathrm{Rot}(f|_{\mathbb{S}^1})$. It is referred from the proof of [**PM**, Lemma III.3.3]. Since the rotation number $\mathrm{Rot}(f|_{\mathbb{S}^1})$ is irrational, there exists a sequence $\{\alpha_n\}_{n\geq 1}$ such that $\lim_{n\to+\infty} \alpha_n = 0$ and each $f_n(z) = e^{2\pi i \alpha_n} f(z)$ has the rotation number $\mathrm{Rot}(f_n|_{\mathbb{S}^1})$ which is Diophantine. So f_n is univalent on a neighborhood of $\overline{U}$.

From Lemma 1.2.1, we take the closed annulus H_n in the Herman ring $\mathcal{H}_n$ of f_n as the Herman compactum for (f_n, U). Every H_n is contained in $\overline{U}$, so there exists a subsequence $\{H_{n_i}\}_{i\geq 1}$ and a set H' such that H_{n_i} converges to H' in the sense of Hausdorff convergence.

Since f_{n_i} converges uniformly to f on $\overline{U}$, it follows from [**PM**, Lemma III.1.2] that $f_{n_i}(H_{n_i})$ converges to $f(H')$ in the sense of Hausdorff convergence. Then $f_{n_i}(H_{n_i}) = H_{n_i}$ implies $f(H') = H'$. Let H be the union of H' and all the components of $\widehat{\mathbb{C}} - H'$ contained in U. It is not difficult to see that H is a Herman compactum for (f, U).

Let Ω_{n_i} be the unbounded component of $\widehat{\mathbb{C}} - H_{n_i}$, let $\Phi_{n_i} : \widehat{\mathbb{C}} - \overline{\mathbb{D}} \to \Omega_{n_i}$ be a conformal isomorphism so that $\Phi_{n_i}(\infty) = \infty$. For the sake of convenience, we assume that $\lim_{z\to\infty} \Phi_{n_i}(z)/z > 0$ and $\lim_{z\to\infty} \Phi(z)/z > 0$. We notice that Ω is the unbounded component of $\widehat{\mathbb{C}} - H$, and is the unbounded component of $\widehat{\mathbb{C}} - H'$ as well. So Ω_{n_i} converges to Ω with respect to ∞ in the sense of Carathéodory kernel convergence, and thus Φ_{n_i} converges locally uniformly to Φ by the Carathéodory kernel theorem.

Since f is univalent on a neighborhood of $\overline{U}$, there exists $r_0 > 1$ such that $g = \Phi^{-1} \circ f \circ \Phi$ is univalent on $\{z : 1 < |z| < r_0\}$. So $g_{n_i} = \Phi_{n_i}^{-1} \circ f_{n_i} \circ \Phi_{n_i}$ is also univalent on $\{z : 1 < |z| < r_0\}$. By the reflection principle, g_{n_i} and g are extended and univalent on $\{z : 1/r_0 < |z| < r_0\}$. We fix r such that $1 < r < r_0$. Since Φ_{n_i} converges locally uniformly to Φ, g_{n_i} converges uniformly to g on $r\mathbb{S}^1$. So g_{n_i} converges uniformly to g on $\mathbb{S}^1/r$. By the maximum principle, g_{n_i} converges uniformly to g on $\{z : 1/r \le |z| \le r\}$, particularly on $\mathbb{S}^1$.

Let L_{n_i} be the outer boundary of H_{n_i}. We notice that the dynamics of g_{n_i} on $\mathbb{S}^1$ corresponds to the dynamics of f_{n_i} on L_{n_i}. Since L_{n_i} is a Jordan closed curve in the Herman ring $\mathcal{H}_{n_i}$ such that $f_{n_i}(L_{n_i}) = L_{n_i}$, the dynamics of f_{n_i} on L_{n_i} corresponds to the dynamics of the irrational rotation $z \mapsto e^{2\pi i \mathrm{Rot}(f_{n_i}|_{\mathbb{S}^1})}z$. Therefore, the rotation number $\mathrm{Rot}(g_{n_i}|_{\mathbb{S}^1})$ corresponds to the rotation number $\mathrm{Rot}(f_{n_i}|_{\mathbb{S}^1})$. Then,

$$\mathrm{Rot}(g|_{\mathbb{S}^1}) = \lim_{i\to+\infty} \mathrm{Rot}(g_{n_i}|_{\mathbb{S}^1}) = \lim_{i\to+\infty} \mathrm{Rot}(f_{n_i}|_{\mathbb{S}^1}) = \mathrm{Rot}(f|_{\mathbb{S}^1}).$$

Now, we show that such the rotation number $\mathrm{Rot}(g|_{\mathbb{S}^1})$ does not depend on choosing the Herman compactum H for (f, U). It is referred from the proof of [**PM**, Lemma III.3.4]. We fix a Herman compactum H for (f, U). A sequence $\{z_n\}_{n\in\mathbb{Z}}$ is called a *full orbit* of z_0 if $z_{n+1} = f(z_n)$ for all $n \in \mathbb{Z}$, and we denote by $\mathcal{O}(z_0)$. Let H_M be the connected component of the set $\{z \in \overline{U} : \exists \mathcal{O}(z) \subset \overline{U}\}$ which contains $\mathbb{S}^1$. Clearly, $f(H_M) = H_M$ and $H \subset H_M$. It is not difficult to see that H_M is the maximal Herman compactum for (f, U).

Let Ω_M be the unbounded component of $\widehat{\mathbb{C}} - H_M$, let $\Phi_M : \widehat{\mathbb{C}} - \overline{\mathbb{D}} \to \Omega_M$ be a conformal isomorphism such that $\Phi_M(\infty) = \infty$. So $g_M = \Phi_M^{-1} \circ f \circ \Phi_M$ is univalent on an outer neighborhood of $\mathbb{S}^1$. Then g_M is extended and univalent on a neighborhood of $\mathbb{S}^1$ by the reflection principle.

We fix a point $z \in H \cap \gamma_1 \subset H_M \cap \gamma_1$. Since γ_1 is a Jordan closed curve, the point z is accessible from the unbounded component of $\widehat{\mathbb{C}} - \overline{U}$, and is accessible from Ω_M as well. Let $\eta \subset \Omega_M \subset \Omega$ be a path converging to z. Then $\Phi^{-1}(\eta)$ converges to some point $w \in \mathbb{S}^1$ and $\Phi_M^{-1}(\eta)$ converges to some point $w_M \in \mathbb{S}^1$ (see [**Mc**, Corollary 6.4]). Now the conformal isomorphism $\Phi^{-1} \circ \Phi_M$

preserves the cyclic ordering between $\{g^{\circ n}(\Phi^{-1}(\eta))\}_{n\geq 0}$ and $\{g_M^{\circ n}(\Phi_M^{-1}(\eta))\}_{n\geq 0}$ (see Figure 1.6).

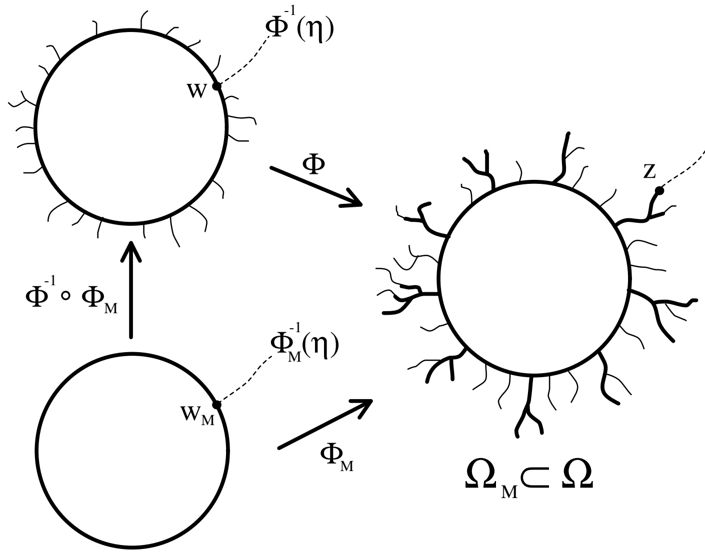

Figure 1.6

Therefore, the cyclic ordering of $\{g^{\circ n}(w)\}_{n\geq 0}$ corresponds to the cyclic ordering of $\{g_M^{\circ n}(w_M)\}_{n\geq 0}$, and thus $\mathrm{Rot}(g|_{\mathbb{S}^1}) = \mathrm{Rot}(g_M|_{\mathbb{S}^1})$.

Proof. (Proof of Proposition 1.2.4) The method of the proof is similar to that of Proposition 1.2.2. We use proof by contradiction.

First, we consider the case where f is linearizable on $\mathbb{S}^1$. Assume that there exists a point z which is biaccessible from the unbounded component Ω of $\widehat{\mathbb{C}} - H$. Let $\Phi : \widehat{\mathbb{C}} - \overline{\mathbb{D}} \to \Omega$ be a conformal isomorphism such that $\Phi(\infty) = \infty$. So $g = \Phi^{-1} \circ f \circ \Phi$ is univalent on an outer neighborhood of $\mathbb{S}^1$. Then g is extended and univalent on a neighborhood of $\mathbb{S}^1$ by the reflection principle. From Lemma 1.2.2, the rotation number $\mathrm{Rot}(g|_{\mathbb{S}^1})$ corresponds to the rotation number $\mathrm{Rot}(f|_{\mathbb{S}^1})$.

Let R_s and R_t be two distinct external rays land at z. Let X be the component of $\mathrm{Int}H$ which contains $\mathbb{S}^1$, let L be the outer boundary of X. Clearly, $f(X) = X$ and $f(L) = L$. Let V be the component of $\mathbb{C} - (R_s \cup \{z\} \cup R_t)$ which does not contain L. We cut off V along an equipotential curve E_r, and thus have the Jordan domain W which is contained in V.

Then $D = \Phi^{-1}(W - H)$ has the interval $I \subset \mathbb{S}^1$ as a part of its boundary (see Figure 1.7).

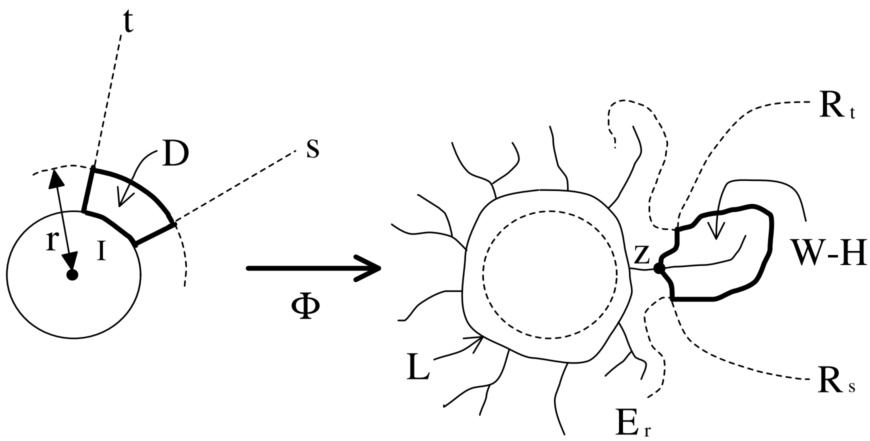

Figure 1.7

Since the rotation number $\mathrm{Rot}(g|_{\mathbb{S}^1})$ is irrational, there exists N such that $\bigcup_{j=0}^{N} g^{\circ j}(I) = \mathbb{S}^1$. We could take a more smaller $r > 1$, so that $g, g^{\circ 2}, \cdots, g^{\circ N}$ are univalent on D, and furthermore, $\bigcup_{j=0}^{N} g^{\circ j}(D)$ is an outer neighborhood of $\mathbb{S}^1$.

Then $f, f^{\circ 2}, \cdots, f^{\circ N}$ are univalent on $W - H$, and thus $\bigcup_{j=0}^{N} f^{\circ j}(W - H)$ is an outer neighborhood of H. So any point of $L \subset \partial\Omega$ can be approximated by some sequence in $\bigcup_{j=0}^{N} f^{\circ j}(W - H)$. Now the injectivity of f implies that each Jordan domain $f^{\circ j}(W)$ does not intersect L, therefore, $\overline{f^{\circ j}(W)} \cap L$ contains at most one point $f^{\circ j}(z)$. This contradicts that L has infinitely many points.

Now, we consider the case where f is not linearizable on $\mathbb{S}^1$. Assume that there exists a point z which is biaccessible from the unbounded component Ω of $\widehat{\mathbb{C}} - H$. Let $\Phi : \widehat{\mathbb{C}} - \overline{\mathbb{D}} \to \Omega$ be a conformal isomorphism such that $\Phi(\infty) = \infty$. So $g = \Phi^{-1} \circ f \circ \Phi$ is univalent on an outer neighborhood of $\mathbb{S}^1$. Then g is extended and univalent on a neighborhood of $\mathbb{S}^1$ by the reflection principle. From Lemma 1.2.2, the rotation number $\mathrm{Rot}(g|_{\mathbb{S}^1})$ corresponds to the rotation number $\mathrm{Rot}(f|_{\mathbb{S}^1})$.

Let R_s and R_t be two distinct external rays land at z. Let V be the component of $\mathbb{C} - (R_s \cup \{z\} \cup R_t)$ which does not contain $\mathbb{S}^1$. We cut off V along an equipotential curve E_r, and thus have the Jordan domain W which is contained in V. Then $D = \Phi^{-1}(W - H)$ has the interval $I \subset \mathbb{S}^1$ as a part of its boundary (see Figure 1.8).

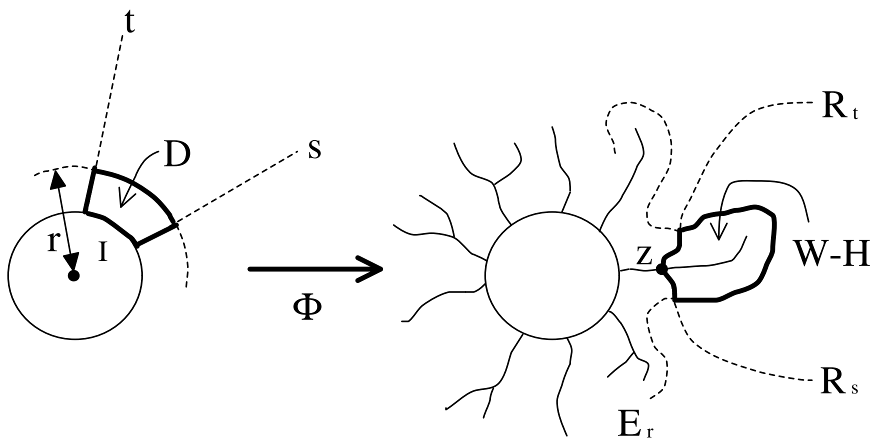

Figure 1.8

Since the rotation number $\mathrm{Rot}(g|_{\mathbb{S}^1})$ is irrational, there exists N such that $\bigcup_{j=0}^{N} g^{\circ j}(I) = \mathbb{S}^1$. We could take a more smaller $r > 1$, so that $g, g^{\circ 2}, \cdots, g^{\circ N}$ are univalent on D, and furthermore, $\bigcup_{j=0}^{N} g^{\circ j}(D)$ is an outer neighborhood of $\mathbb{S}^1$.

Then $f, f^{\circ 2}, \cdots, f^{\circ N}$ are univalent on $W - H$, and thus $\bigcup_{j=0}^{N} f^{\circ j}(W - H)$ is an outer neighborhood of H. So any point of $\mathbb{S}^1 \subset \partial\Omega$ can be approximated by some sequence in $\bigcup_{j=0}^{N} f^{\circ j}(W - H)$. Now the injectivity of f implies that each Jordan domain $f^{\circ j}(W)$ does not intersect $\mathbb{S}^1$, therefore, $\overline{f^{\circ j}(W)} \cap \mathbb{S}^1$ contains at most one point $f^{\circ j}(z)$. This contradicts that $\mathbb{S}^1$ has infinitely many points.

1.3 Preliminaries for proofs

In this section, we shall see preparations for the proofs of the theorems. The following notion will be often used later.

Definition 1.3.1 Let $\Omega \subset \widehat{\mathbb{C}}$ be a simply connected domain which contains ∞. Assume that the boundary $\partial\Omega$ contains at least two points. Let $\Phi : \widehat{\mathbb{C}} - \overline{\mathbb{D}} \to \Omega$ be a conformal isomorphism such that $\Phi(\infty) = \infty$. Let R_s and

R_t be two distinct external rays land at z. Let U_1 and U_2 be two distinct components of $\mathbb{C} - (R_s \cup \{z\} \cup R_t)$. Then for each $l = 1, 2$, *angle* of U_l is defined as

$$A(U_l) = \frac{\text{length}(\Phi^{-1}(U_l \cap E_r))}{2\pi r}.$$

It does not depend on $r > 1$, so it is well defined. Clearly, $0 < A(U_1), A(U_2) < 1$ and $A(U_1) + A(U_2) = 1$. The *angle* between R_s and R_t is defined as $A(R_s, R_t) = \min\{A(U_1), A(U_2)\}$. Clearly, $A(R_s, R_t) \le 1/2$ (see Figure 1.9).

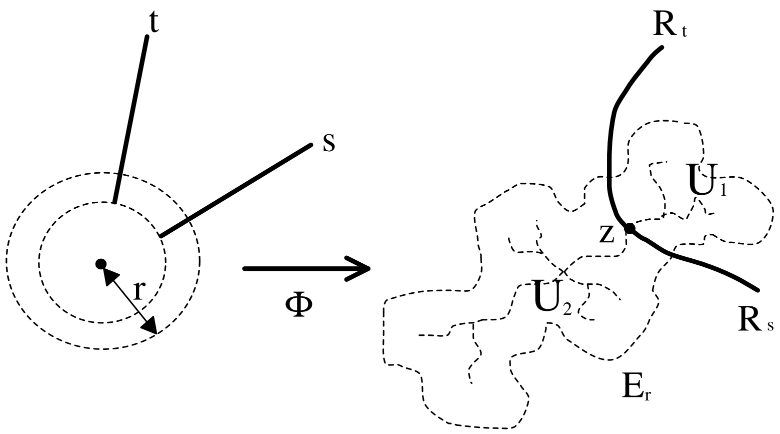

Figure 1.9

The following two lemmas will be used for the proofs of the theorems.

Lemma 1.3.1 *Let K be a compact subset of the complex plane $\mathbb{C}$. Assume that f is analytic on a neighborhood of K, there are no critical points of f in K and f is injective on K. Then there exists $\varepsilon > 0$ such that f is univalent on $N_\varepsilon(K)$, where $N_\varepsilon(K) = \{z \in \mathbb{C} : \min_{w \in K} |z - w| < \varepsilon\}$.*

Proof. Assume that f is not univalent on $N_{1/n}(K)$ for all $n \in \mathbb{N}$. Then there exist $x_n \in N_{1/n}(K)$ and $y_n \in N_{1/n}(K)$ such that $x_n \neq y_n$ and $f(x_n) = f(y_n)$. Since $\{x_n\}_{n \ge 1}$ is contained in $N_1(K)$, we take a subsequence $\{x_{n_i}\}_{i \ge 1}$ and a point x_0 such that $\lim_{i \to +\infty} x_{n_i} = x_0$. Similarly, we take a subsequence $\{y_{n_{i_j}}\}_{j \ge 1}$ of $\{y_{n_i}\}_{i \ge 1}$ and a point y_0 such that $\lim_{j \to +\infty} y_{n_{i_j}} = y_0$. Then both x_0 and y_0 are belong to K, and

$$f(x_0) = \lim_{j \to +\infty} f(x_{n_{i_j}}) = \lim_{j \to +\infty} f(y_{n_{i_j}}) = f(y_0).$$

Now f is injective on K, and thus $x_0 = y_0$. So f is not univalent on any neighborhood of x_0, and thus x_0 is a critical point of f. This contradicts that there are no critical points of f in K.

Lemma 1.3.2 *Let Ω be a bounded domain by a cycle $\gamma \subset \mathbb{C}$ which consists of finite Jordan closed curves. Let f be a complex-valued function defined on a neighborhood of $\overline{\Omega}$. Assume that f is analytic on $\overline{\Omega}$ and injective on $\partial\Omega$. Assume that f preserves the orientation on each Jordan closed curve which constructs a part of $\partial\Omega$. Then Ω' is well defined as the bounded domain by the cycle $f(\partial\Omega) \subset \mathbb{C}$, and f maps Ω conformally onto Ω'.*

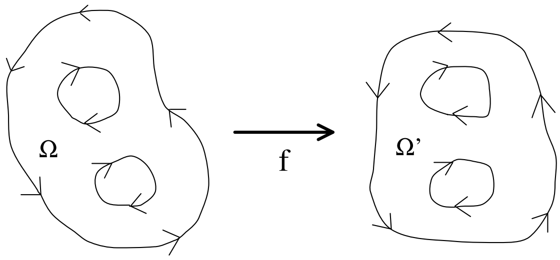

Figure 1.10

Proof. From the open mapping theorem, it is easy to see that Ω' is well defined as the bounded domain by the cycle $f(\partial\Omega) \subset \mathbb{C}$ (see Figure 1.10).

Let w_0 be a point in Ω'. Let $\Gamma(z) = f(z) - w_0 = w - w_0$. Then $\Gamma(z)$ is analytic on $\overline{\Omega}$ and does not take the zeros on $\partial\Omega$. From the argument principle,

$$\frac{1}{2\pi}\int_{\partial\Omega} d\arg\Gamma(z) = \frac{1}{2\pi}\int_{f(\partial\Omega)} d\arg(w - w_0) = N,$$

where N is the number of the zeros in Ω. We obtain $N = 1$, so there exists the zero z_0 of Γ in Ω. Therefore, z_0 is the point in Ω satisfies $f(z_0) = w_0$.

Similarly, we can see that there are no points $z \in \Omega$ such that $f(z) = w_0$ when $w_0 \notin \overline{\Omega'}$.

1.4 Proof of Theorem 1.1.1

In this section, we consider a polynomial $f_c(z) = z^d + c$ of degree $d \geq 2$. For each $0 \leq j \leq d-1$, let $\sigma_j(z) = e^{2\pi i j/d} z$ be a j/d-rotation. Then $f_c \circ \sigma_j = f_c$

implies $\sigma_j(J_{f_c}) = J_{f_c}$. The origin is only one critical point of f_c in $\mathbb{C}$.

Assume that α is an irrationally indifferent fixed point of f_c. Then the origin is recurrent (see [**Ma**]), so the superattracting fixed point ∞ is the only critical point in the immediate basin $\mathcal{A}_\infty$. Therefore, there exists a conformal isomorphism $\Phi : \widehat{\mathbb{C}} - \overline{\mathbb{D}} \to \mathcal{A}_\infty$ such that $\Phi(\infty) = \infty$ and $\Phi^{-1} \circ f_c \circ \Phi(w) = w^d$.

We consider the dynamics of external rays and the equipotential curves in the immediate basin $\mathcal{A}_\infty$. It is easy to see that $f_c(R_t) = R_{dt}$, $f_c^{-1}(R_t) = \bigcup_{j=0}^{d-1} R_{(t+j)/d}$, $f_c(E_r) = E_{r^d}$ and $f_c^{-1}(E_r) = E_{\sqrt[d]{r}}$. Moreover, $\sigma_j(\mathcal{A}_\infty) = \mathcal{A}_\infty$ implies $\sigma_j \circ \Phi = \Phi \circ \sigma_j$, so that $\sigma_j(R_t) = R_{t+j/d}$ and $\sigma_j(E_r) = E_r$.

Lemma 1.4.1 *Let R_s and R_t be two distinct external rays land at $z \neq 0$. Let U be the component of $\mathbb{C} - (R_s \cup \{z\} \cup R_t)$ such that $A(U) = A(R_s, R_t)$. Then $A(U) < 1/d$ and $\overline{\sigma_j(U)} \cap \overline{\sigma_k(U)} = \emptyset$ for $j \neq k$. Therefore, U does not contain both two σ_j-symmetric points and $\mathbb{C} - \overline{U}$ contains the origin.*

Proof. Assume that $A(U) \geq 1/d$. Then $A(\mathbb{C} - \overline{U}) \geq A(R_s, R_t) = A(U) \geq 1/d$, so we may suppose that

$$s < s + \frac{1}{d} \leq t < t + \frac{1}{d} \leq s + 1,$$

and furthermore, $\sigma_1(R_s) \subset \overline{U}$ and $\sigma_1(R_t) \subset \overline{\mathbb{C} - U}$ (see Figure 1.11).

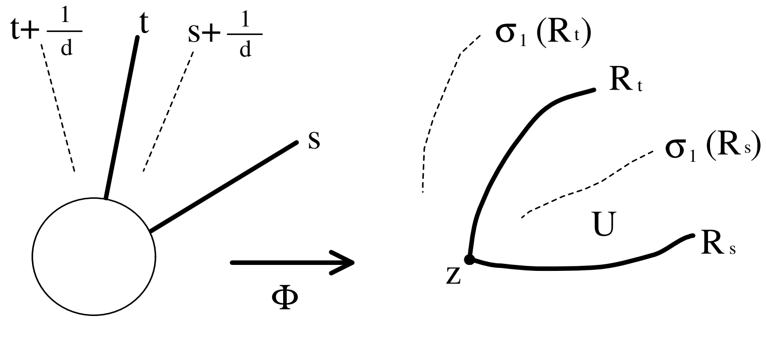

Figure 1.11

Then both $R_{s+1/d}$ and $R_{t+1/d}$ land at $\sigma_1(z)$, so $\sigma_1(z) \in \overline{U} \cap \overline{\mathbb{C} - U} = \partial U$, and thus $\sigma_1(z) = z$. This implies $z = 0$, which contradicts the assumption $z \neq 0$.

Now assume that there are two distinct numbers j and k such that $\overline{\sigma_j(U)} \cap \overline{\sigma_k(U)} \neq \emptyset$. We have $A(U) < 1/d$, so we may suppose

$$s + \frac{j}{d} < t + \frac{j}{d} < s + \frac{k}{d} < t + \frac{k}{d} < s + \frac{j}{d} + 1.$$

Two distinct external rays does not intersect, so we conclude that $\sigma_j(z) = \sigma_k(z)$. This implies $z = 0$, which contradicts the assumption $z \neq 0$.

Lemma 1.4.2 *Let R_s and R_t be two distinct external rays land at $z \neq 0$. Let U be a component of $\mathbb{C} - (R_s \cup \{z\} \cup R_t)$. Then the following three conditions are equivalent to each other:*

(a) $A(U) < 1/d$;

(b) f_c is univalent on U;

(c) U does not contain the origin.

Proof. (a)$\Rightarrow$(b): Assume that $A(U) < 1/d$. So we cut off U along an equipotential curve E_r, and thus have the Jordan domain V which is contained in U. Then f_c is injective on ∂V and preserves the orientation, so Lemma 1.3.2 implies that f_c is univalent on V. We could take a more bigger $r > 1$, so that f_c is univalent on U. Moreover, $f_c(U)$ is the component of $\mathbb{C} - f_c(R_s \cup \{z\} \cup R_t)$ such that $A(f_c(U)) = dA(U)$.

(b)$\Rightarrow$(c): It is obvious.

(c)$\Rightarrow$(a): Assume that U does not contain the origin. If $A(\mathbb{C} - \overline{U}) = A(R_s, R_t)$, then Lemma 1.4.1 implies that U contains the origin. This contradicts the assumption, and thus $A(\mathbb{C} - \overline{U}) \neq A(R_s, R_t)$. Therefore, $A(U) = A(R_s, R_t)$ and thus Lemma 1.4.1 implies $A(U) < 1/d$.

Lemma 1.4.3 *Assume that z is biaccessible from the immediate basin $\mathcal{A}_\infty$ such that $\alpha \notin \{f_c^{\circ n}(z)\}_{n \geq 0}$ and $0 \notin \{f_c^{\circ n}(z)\}_{n \geq 0}$. Then there exist two distinct external rays R_u and R_v with a common landing point w such that $R_u \cup \{w\} \cup R_v$ separates α from the origin.*

Proof. Let R_s and R_t be two distinct external rays landing at z. Let U be the component of $\mathbb{C} - (R_s \cup \{z\} \cup R_t)$ which does not contain the origin. Then Lemma 1.4.2 implies that f_c is univalent on U and thus $A(f_c(U)) = dA(U)$.

If $f_c(U)$ does not contain the origin, then we have that f_c is univalent on $f_c(U)$ and thus $A(f_c^{\circ 2}(U)) = d^2 A(U)$ as the above argument. Otherwise, $f_c(U)$ contains the origin.

By repeating the above step, we see that there exists $N \geq 0$ such that $f_c^{\circ N}(U)$ does not contain the origin and $f_c^{\circ N+1}(U)$ contains the origin. Then f_c is univalent on $f_c^{\circ N}(U)$ and thus $A(f_c^{\circ N+1}(U)) = d^{N+1}A(U)$.

If $\alpha \in f_c^{\circ N}(U)$, then put $R_u \cup \{w\} \cup R_v = f_c^{\circ N}(R_s \cup \{z\} \cup R_t)$.

Otherwise, if $\alpha \notin f_c^{\circ N}(U)$, then we may consider the following two cases:

(1) $f_c^{\circ N}(U)$ contains some $\sigma_{j_0}(\alpha)$;

(2) $f_c^{\circ N}(U)$ does not contain any $\sigma_j(\alpha)$.

In the case (1), put $R_u \cup \{w\} \cup R_v = \sigma_{d-j_0}(f_c^{\circ N}(R_s \cup \{z\} \cup R_t))$.

In the case (2), if $f_c^{\circ N+1}(U)$ contains α, then $f_c^{\circ N}(U)$ contains one point of inverse image of α. Since $f_c^{-1}(\alpha) = \{\sigma_j(\alpha) | 0 \leq j \leq d-1\}$, it follows that $f_c^{\circ N}(U)$ contains some $\sigma_{j_0}(\alpha)$. However, this contradicts that $f_c^{\circ N}(U)$ does not contain any $\sigma_j(\alpha)$. Therefore, $f_c^{\circ N+1}(U)$ does not contain α, and thus we put $R_u \cup \{w\} \cup R_v = f_c^{\circ N+1}(R_s \cup \{z\} \cup R_t)$.

Proof. (Proof of Theorem 1.1.1) We use proof by contradiction. If α is a Siegel point, assume that $0 \notin \{f_c^{\circ n}(z_0)\}_{n\geq 0}$. If α is a Cremer point, assume that $\alpha \notin \{f_c^{\circ n}(z_0)\}_{n\geq 0}$ and $0 \notin \{f_c^{\circ n}(z_0)\}_{n\geq 0}$. In both cases, it follows that z_0 is biaccessible from $\mathcal{A}_\infty$ such that $\alpha \notin \{f_c^{\circ n}(z_0)\}_{n\geq 0}$ and $0 \notin \{f_c^{\circ n}(z_0)\}_{n\geq 0}$.

Lemma 1.4.3 implies that there exist two distinct external rays R_u and R_v with a common landing point w such that $R_u \cup \{w\} \cup R_v$ separates α from the origin. Let U be the component of $\mathbb{C} - (R_u \cup \{w\} \cup R_v)$ which contains α. Then f_c is injective on $\overline{U}$. We cut off U along an equipotential curve E_r, and thus have the Jordan domain V which contains α.

Since $\overline{V}$ contains no critical points of f_c, it follows from Lemma 1.3.1 that there exists a Jordan domain W such that $\overline{V} \subset W$ and f_c is univalent on a neighborhood of $\overline{W}$ (see Figure 1.12).

Now we take a Siegel compactum S for (f_c, W) by Proposition 1.2.1. Then S meets the boundary ∂W but not $\partial V - \{w\}$, so S must contain w. Furthermore, $\partial(\widehat{\mathbb{C}} - S) - \{w\}$ is disconnected, and thus the point w is biaccessible from $\widehat{\mathbb{C}} - S$. However, the biaccessibility of w contradicts Proposition 1.2.2.

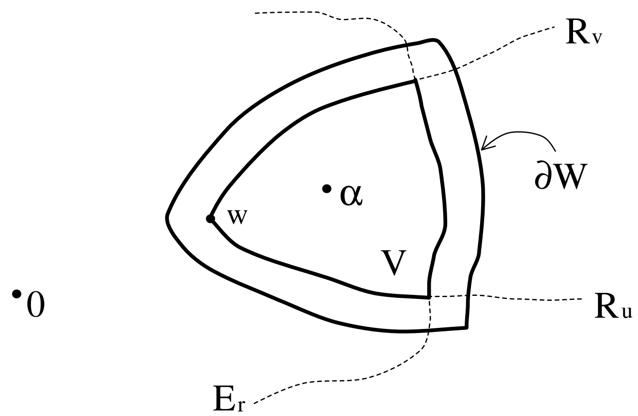

Figure 1.12

1.5 Proof of Theorem 1.1.2

In this section, we consider a polynomial $g_\theta(z) = e^{2\pi i\theta}z + z^d$ of degree $d \geq 2$. Actually, we may consider the cases of $d \geq 3$ and thus assume that $d \geq 3$ in the following arguments. For each $0 \leq j \leq d-2$, let $\tau_j(z) = e^{2\pi ij/(d-1)}z$ be a $j/(d-1)$-rotation. Then $g_\theta \circ \tau_j = \tau_j \circ g_\theta$ implies $\tau_j(J_{g_\theta}) = J_{g_\theta}$. So g_θ has $d-1$ symmetric critical points $c_j = \tau_j(c)$, where c is one of the solutions of $e^{2\pi i\theta} + dz^{d-1} = 0$.

Assume that the origin is an irrationally indifferent fixed point of g_θ. Then some critical point c_{j_0} is recurrent (see [**Ma**]), so $g_\theta \circ \tau_j = \tau_j \circ g_\theta$ implies that every critical point c_j is recurrent. Therefore, the superattracting fixed point ∞ is the only critical point in the immediate basin $\mathcal{A}_\infty$. Then there exists a conformal isomorphism $\Phi : \widehat{\mathbb{C}} - \overline{\mathbb{D}} \to \mathcal{A}_\infty$ such that $\Phi(\infty) = \infty$ and $\Phi^{-1} \circ g_\theta \circ \Phi(w) = w^d$.

We consider the dynamics of external rays and the equipotential curves in the immediate basin $\mathcal{A}_\infty$. It is easy to see that $g_\theta(R_t) = R_{dt}$, $g_\theta^{-1}(R_t) = \bigcup_{j=0}^{d-1} R_{(t+j)/d}$, $g_\theta(E_r) = E_{r^d}$ and $g_\theta^{-1}(E_r) = E_{\sqrt[d]{r}}$. Moreover, $\tau_j(\mathcal{A}_\infty) = \mathcal{A}_\infty$ implies $\tau_j \circ \Phi = \Phi \circ \tau_j$, so that $\tau_j(R_t) = R_{t+j/(d-1)}$ and $\tau_j(E_r) = E_r$.

Lemma 1.5.1 *Let R_s and R_t be two distinct external rays land at $z \neq 0$. Let U be the component of $\mathbb{C} - (R_s \cup \{z\} \cup R_t)$ such that $A(U) = A(R_s, R_t)$. Then $A(U) < 1/(d-1)$ and $\overline{\tau_j(U)} \cap \overline{\tau_k(U)} = \emptyset$ for $j \neq k$. Therefore, U does not contain both two τ_j-symmetric points and $\mathbb{C} - \overline{U}$ contains the origin.*

The method of the proof is similar to that of Lemma 1.4.1.

Lemma 1.5.2 *Let R_s and R_t be two distinct external rays land at $z \neq 0$. Let U be a component of $\mathbb{C} - (R_s \cup \{z\} \cup R_t)$. Then the following three conditions are equivalent to each other:*

(a) $A(U) < 1/d$;

(b) g_θ *is univalent on* U;

(c) U *does not contain any* c_j.

Proof. The method of the proof of (a)⇒(b) is similar to that of Lemma 1.4.2. The proof of (b)⇒(c) is obvious. We give the proof of (c)⇒(a) here.

Assume that U does not contain any c_j. If $A(\mathbb{C} - \overline{U}) = A(R_s, R_t)$, then Lemma 1.5.1 implies that $\mathbb{C} - \overline{U}$ does not contain both two τ_j-symmetric points. Therefore, U contains at least one point of c_j. This contradicts the assumption, and thus we have $A(\mathbb{C} - \overline{U}) \neq A(R_s, R_t)$. Therefore, $A(U) = A(R_s, R_t)$ and we see from Lemma 1.5.1 that $\overline{\tau_j(U)} \cap \overline{\tau_k(U)} = \emptyset$ for $j \neq k$.

If $A(U) \geq 1/d$, then $A(\tau_j(U)) \geq 1/d$ and thus $g_\theta(\overline{\tau_j(U)}) = \mathbb{C}$. Then each $\overline{\tau_j(U)}$ contains at least one point of inverse image of some critical value v_{j_0}, where $v_{j_0} = g_\theta(c_{j_0})$. Therefore, $\bigcup_{j=0}^{d-2} \overline{\tau_j(U)}$ contains at least $d-1$ points of inverse image of v_{j_0}. However, this contradicts that $\mathbb{C} - \bigcup_{j=0}^{d-2} \overline{\tau_j(U)}$ contains the critical point c_{j_0}. Therefore, we conclude $A(U) < 1/d$.

Lemma 1.5.3 *Assume that z is biaccessible from the immediate basin $\mathcal{A}_\infty$ such that $0 \notin \{g_\theta^{\circ n}(z)\}_{n \geq 0}$ and $c_j \notin \{g_\theta^{\circ n}(z)\}_{n \geq 0}$ for all j. Then for each j, there exist two distinct external rays R_{u_j} and R_{v_j} with a common landing point w_j such that $R_{u_j} \cup \{w_j\} \cup R_{v_j}$ separates c_j from the origin.*

Proof. By τ_j-symmetry, it is enough to show Lemma 1.5.3 for some j_0. Now let R_s and R_t be two distinct external rays landing at z. Let U be the component of $\mathbb{C} - (R_s \cup \{z\} \cup R_t)$ which does not contain the origin. If U contains some c_{j_0}, put $R_{u_{j_0}} \cup \{w_{j_0}\} \cup R_{v_{j_0}} = R_s \cup \{z\} \cup R_t$.

On the other hand, assume that U does not contain any c_j. Then Lemma 1.5.2 implies g_θ is univalent on U and thus $A(g_\theta(U)) = dA(U)$.

If $g_\theta(U)$ does not contain any c_j, then we have that g_θ is univalent on $g_\theta(U)$ and thus $A(g_\theta^{\circ 2}(U)) = d^2 A(U)$ as the above argument. Otherwise, $g_\theta(U)$ contains some c_{j_0}.

By repeating the above step, we see that there exists $N \geq 0$ such that $g_\theta^{\circ N}(U)$ does not contain any c_j and $g_\theta^{\circ N+1}(U)$ contains some c_{j_0}. Then g_θ is univalent on $g_\theta^{\circ N}(U)$ and thus $A(g_\theta^{\circ N+1}(U)) = d^{N+1}A(U)$.

So we may consider the following three cases:

(1) $g_\theta^{\circ N+1}(U)$ contains only some one of c_j;

(2) $\mathbb{C} - \overline{g_\theta^{\circ N+1}(U)}$ contains only some one of c_j;

(3) $g_\theta^{\circ N+1}(U)$ contains all c_j.

In the case (1) and case (2), put $R_{u_{j_0}} \cup \{w_{j_0}\} \cup R_{v_{j_0}} = g_\theta^{\circ N+1}(R_s \cup \{z\} \cup R_t)$.

Now, we consider the case (3). To simplify the notation, we set as the following:

$$L = g_\theta^{\circ N}(R_s \cup \{z\} \cup R_t), \quad V = g_\theta^{\circ N}(U),$$

$$W = \mathbb{C} - \bigcup_{j=0}^{d-2} \overline{\tau_j(V)}, \quad W' = \bigcap_{j=0}^{d-2} g_\theta(\tau_j(V)) = \bigcap_{j=0}^{d-2} \tau_j(g_\theta(V)).$$

Then both W and W' are τ_j-symmetrical domains, which contain the origin as well as all c_j (see Figure 1.13).

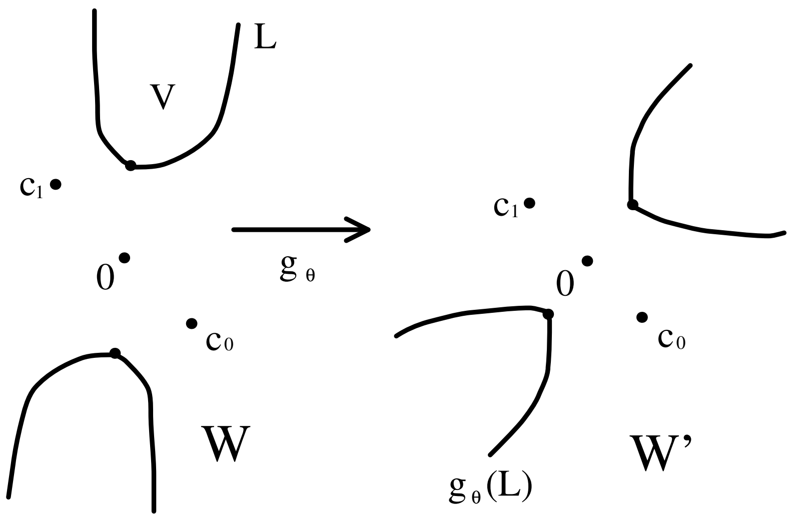

Figure 1.13

If $\overline{W'}$ contains some critical value $v_{j_0} = g_\theta(c_{j_0})$, then each $\overline{\tau_j(V)}$ contains one point of inverse image of v_{j_0}, and thus $\bigcup_{j=0}^{d-2} \overline{\tau_j(V)}$ contains $d-1$ points of inverse image of v_{j_0}. However, this contradicts that $W = \mathbb{C} - \bigcup_{j=0}^{d-2} \overline{\tau_j(V)}$ contains the critical point c_{j_0}. Therefore, $\overline{W'}$ does not contain any $v_j = g_\theta(c_j)$.

Now we may suppose that $\mathbb{C} - \overline{g_\theta(V)}$ contains some $v_{j_0} = g_\theta(c_{j_0})$. For each $0 \leq j \leq d-2$, we consider the following bijection:

$$g_\theta|_{\overline{\tau_j(V)}} : \overline{\tau_j(V)} \to g_\theta(\overline{\tau_j(V)}).$$

Then the image $g_\theta(\overline{\tau_j(V)})$ contains $g_\theta(L)$. Therefore, $\deg g_\theta = d$ implies $W \cap g_\theta^{-1}(g_\theta(L)) \neq \emptyset$. Now we set $L' = W \cap g_\theta^{-1}(g_\theta(L))$.

If L' does not separate c_{j_0} from the origin, then there exists a continuous curve γ in $W - L'$ between c_{j_0} and the origin. Then $g_\theta(\gamma)$ is a continuous curve between v_{j_0} and the origin. So $g_\theta(\gamma) \cap g_\theta(L) \neq \emptyset$ and thus $\gamma \cap g_\theta^{-1}(g_\theta(L)) \neq \emptyset$. However, this contradicts $\gamma \subset W - L'$.

Therefore, it is concluded that L' separates c_{j_0} from the origin, and thus we put $R_{u_{j_0}} \cup \{w_{j_0}\} \cup R_{v_{j_0}} = L'$.

Proof. (Proof of Theorem 1.1.2) We use proof by contradiction. If the origin is a Siegel point, assume that $c_j \notin \{g_\theta^{\circ n}(z_0)\}_{n \geq 0}$ for all j. If the origin is a Cremer point, assume that $0 \notin \{g_\theta^{\circ n}(z_0)\}_{n \geq 0}$ and $c_j \notin \{g_\theta^{\circ n}(z_0)\}_{n \geq 0}$ for all j.

In both cases, it follows that z_0 is biaccessible from $\mathcal{A}_\infty$ so that $0 \notin \{g_\theta^{\circ n}(z_0)\}_{n \geq 0}$ and $c_j \notin \{g_\theta^{\circ n}(z_0)\}_{n \geq 0}$ for all j. Lemma 1.5.3 implies that for each j, there exist two distinct external rays R_{u_j} and R_{v_j} with a common landing point w_j such that $R_{u_j} \cup \{w_j\} \cup R_{v_j}$ separates c_j from the origin. Then we may suppose that all $R_{u_j} \cup \{w_j\} \cup R_{v_j}$ are τ_j-symmetrical.

Let U be the component of $\mathbb{C} - \bigcup_{j=0}^{d-2} (R_{u_j} \cup \{w_j\} \cup R_{v_j})$ which contains the origin. We cut off U along an equipotential curve E_r and thus have the τ_j-symmetric Jordan domain V which contains the origin. Then g_θ is injective on ∂V and preserves the orientation, so Lemma 1.3.2 implies that g_θ is injective on $\overline{V}$.

Since $\overline{V}$ contains no critical points of g_θ, it follows from Lemma 1.3.1 that there exists a Jordan domain W such that $\overline{V} \subset W$ and g_θ is univalent on a neighborhood of $\overline{W}$ (see Figure 1.14).

Now we take a Siegel compactum S for (g_θ, W) by Proposition 1.2.1. Then S meets the boundary ∂W but not $\partial V - \bigcup_{j=0}^{d-2} \{w_j\}$, so S must contain some w_{j_0}. Furthermore, $\partial(\widehat{\mathbb{C}} - S) - \{w_{j_0}\}$ is disconnected, and thus the point w_{j_0} is biaccessible from $\widehat{\mathbb{C}} - S$. However, the biaccessibility of w_{j_0} contradicts Proposition 1.2.2.

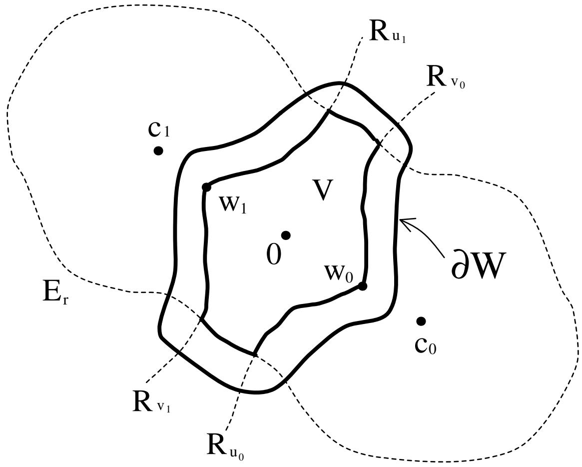

Figure 1.14

1.6 Proof of Theorem 1.1.3

In this section, we consider a rational function $h(z) = e^{2\pi i\theta} z^2 (z-a)/(1-\bar{a}z)$. Let $\upsilon(z) = 1/\bar{z}$ be an inversion. Then $h \circ \upsilon = \upsilon \circ h$ implies $\upsilon(J_h) = J_h$. The zeros are the origin and a, and the poles are infinity and $\upsilon(a)$. We suppose $|a| > 3$ such that $h|_{\mathbb{S}^1}$ is an analytic circle diffeomorphism. Then both of infinity and the origin are superattracting fixed points with local degree 2, and thus $h \circ \upsilon = \upsilon \circ h$ implies $\upsilon(\mathcal{A}_\infty) = \mathcal{A}_0$. Let c be the critical point of h such that $|c| > 1$, and thus $\upsilon(c)$ is also a critical point of h.

Assume that the rotation number $\mathrm{Rot}(h|_{\mathbb{S}^1})$ is irrational. If h is linearizable on $\mathbb{S}^1$, then there exists a Herman ring $\mathcal{H}$ and thus $\mathbb{S}^1 \subset \mathcal{H} \subset F_h$. On the other hand, if h is not linearizable on $\mathbb{S}^1$, then $\mathbb{S}^1 \subset J_h$. In either case, some critical point is recurrent (see [**Ma**]), so that both c and $\upsilon(c)$ are recurrent by $h \circ \upsilon = \upsilon \circ h$. Therefore, each of superattracting fixed points infinity and the origin is the only critical point in each immediate basin. We may consider only the immediate basin $\mathcal{A}_\infty$. So there exists a conformal isomorphism $\Phi : \widehat{\mathbb{C}} - \overline{\mathbb{D}} \to \mathcal{A}_\infty$ such that $\Phi(\infty) = \infty$ and $\Phi^{-1} \circ h \circ \Phi(w) = w^2$.

We consider the dynamics of external rays and the equipotential curves in the immediate basin $\mathcal{A}_\infty$. It is easy to see that $h(R_t) = R_{2t}$, $h^{-1}(R_t) \cap \mathcal{A}_\infty = R_{t/2} \cup R_{(t+1)/2}$, $h(E_r) = E_{r^2}$ and $h^{-1}(E_r) \cap \mathcal{A}_\infty = E_{\sqrt{r}}$.

Lemma 1.6.1 *There are no points in $\mathbb{S}^1$ which are biaccessible from $\mathcal{A}_\infty$.*

Proof. This proof is referred from the last part of the proof of [**Za**, Theorem 5]. We use proof by contradiction. Assume that there exists a point $z_0 \in \mathbb{S}^1$ which is biaccessible from $\mathcal{A}_\infty$. Let R_s and R_t be two distinct external rays landing at z_0, let U_0 be the component of $\mathbb{C} - (R_s \cup \{z_0\} \cup R_t)$ which does not contain $\mathbb{S}^1$. Let $z_n = h^{\circ n}(z_0)$ and U_n be the component of $\mathbb{C} - h^{\circ n}(R_s \cup \{z_0\} \cup R_t)$ which does not contain $\mathbb{S}^1$ (see Figure 1.15).

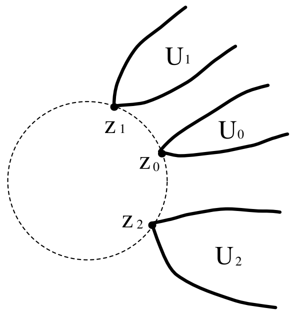

Figure 1.15

There are no critical points in $\mathbb{S}^1$, and so we notice that $A(U_n) \neq 1/2$ for all $n \geq 0$. First, we show that $A(U_n) > 1/2$ for some U_n. Assume that $A(U_0) < 1/2$. By the similar method of the proof of (a)$\Rightarrow$(b) in Lemma 1.4.2, we see h is injective on $\overline{U}_0$. Since z_0 is not a critical point, $\mathbb{S}^1 \not\subset h(\overline{U}_0)$, therefore, $h(\overline{U}_0) = \overline{U}_1$ and $A(U_1) = 2A(U_0)$. If $A(U_1) < 1/2$, then we similarly have that $h(\overline{U}_1) = \overline{U}_2$ and $A(U_2) = 2A(U_1)$. By repeating the above step, we conclude there exists U_N such that $A(U_N) > 1/2$.

We shall see contradiction. Let $V = \mathbb{C} - \overline{U}_N$. Then $A(V) < 1/2$ by $A(U_N) > 1/2$. Since the rotation number $\mathrm{Rot}(h|_{\mathbb{S}^1})$ is irrational, the orbit $\{z_n\}_{n \geq 0}$ is infinite. So $U_n \subset V$ for all $n \geq N+1$ (see Figure 1.16).

By the above argument, we obtain that $h(\overline{U}_n) = \overline{U}_{n+1}$ and $A(U_{n+1}) = 2A(U_n)$ for all $n \geq N+1$. This monotonous increasing contradicts $A(U_n) < A(V) < 1/2$ for all $n \geq N+1$.

In the rest of this section, we shall use the above lemma without any explanation.

Lemma 1.6.2 *Let R_s and R_t be two distinct external rays land at $z \neq c$. Let U be a component of $\mathbb{C} - (R_s \cup \{z\} \cup R_t)$. Then the following two conditions are equivalent to each other:*

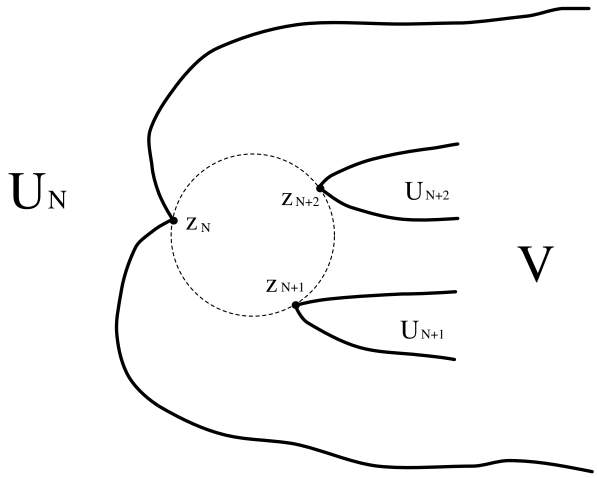

Figure 1.16

(a) $A(U) < 1/2$*;*

(b) U *does not contain* c*.*

Proof. (a)⇒(b): Assume that $A(U) < 1/2$. Then we cut off U along an equipotential curve E_r, and thus have the Jordan domain V which is contained in U. Then h is injective on ∂V and preserves the orientation. We may consider the following two cases:

(1) $\mathbb{S}^1 \cap \overline{V} = \emptyset$;

(2) $\mathbb{S}^1 \subset V$.

In the case (1), Lemma 1.3.2 implies that h is injective on $\overline{V}$. We could take a more bigger $r > 1$, so that h is univalent on U. Therefore, U does not contain c.

In the case (2), we set $W = V - \overline{\mathbb{D}}$ (see Figure 1.17).

Then h is injective on ∂W and preserves the orientation. So Lemma 1.3.2 implies that h is injective on $\overline{W}$, and thus $c \notin W$. Since $c \notin \overline{\mathbb{D}}$, the domain V does not contain c. We could take a more bigger $r > 1$, so that U does not contain c.

(b)⇒(a): Assume that U does not contain c. Then $\mathbb{C} - \overline{U}$ contains c. It follows from the contraposition of (a)⇒(b) that $A(\mathbb{C} - \overline{U}) > 1/2$, and thus $A(U) < 1/2$.

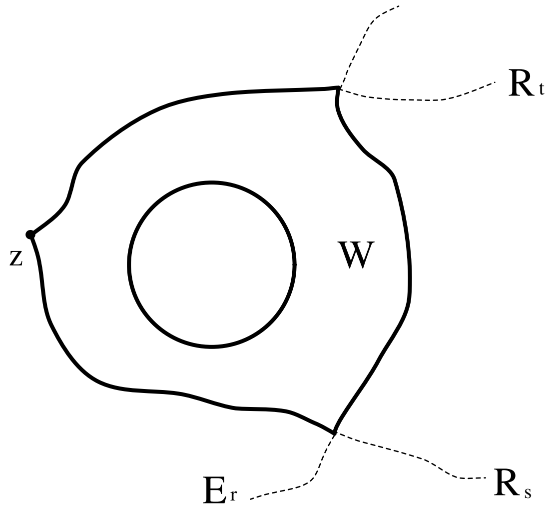

Figure 1.17

Lemma 1.6.3 *Assume that z is biaccessible from the immediate basin $\mathcal{A}_\infty$ such that $c \notin \{h^{\circ n}(z)\}_{n\geq 0}$. Then there exist two distinct external rays R_u and R_v with a common landing point w such that $R_u \cup \{w\} \cup R_v$ separates $\mathbb{S}^1$ from c.*

Proof. Let R_s and R_t be two distinct external rays landing at z. Let U be the component of $\mathbb{C} - (R_s \cup \{z\} \cup R_t)$ which does not contain c. Then U satisfies $A(U) < 1/2$ by Lemma 1.6.2. If $\mathbb{S}^1 \subset U$, we put $R_u \cup \{w\} \cup R_v = R_s \cup \{z\} \cup R_t$. On the other hand, if $\mathbb{S}^1 \cap \overline{U} = \emptyset$, then we see h is univalent on U and thus $A(h(U)) = 2A(U)$ by the similar method of the proof of (a)$\Rightarrow$(b) in Lemma 1.4.2.

We consider $h(U)$ instead of U. If $h(U)$ contains neither c nor $\mathbb{S}^1$, then we similarly have that h is univalent on $h(U)$ and thus $A(h^{\circ 2}(U)) = 2^2 A(U)$. Otherwise, $h(U)$ contains c or $\mathbb{S}^1$.

By repeating the above step, we see that there exists $N \geq 0$ such that $h^{\circ N}(U)$ does not contain c nor $\mathbb{S}^1$ and $h^{\circ N+1}(U)$ contains c or $\mathbb{S}^1$. Then h is univalent on $h^{\circ N}(U)$ and thus $A(h^{\circ N+1}(U)) = 2^{N+1} A(U)$. So we may consider the following three cases:

(1) $h^{\circ N+1}(U)$ contains $\mathbb{S}^1$ but not c;

(2) $h^{\circ N+1}(U)$ contains c but not $\mathbb{S}^1$;

(3) $h^{\circ N+1}(U)$ contains both c and $\mathbb{S}^1$.

In the case (1) and case (2), put $R_u \cup \{w\} \cup R_v = h^{\circ N+1}(R_s \cup \{z\} \cup R_t)$.

Now, we consider the case (3). Since $h|_{\overline{h^{\circ N}(U)}} : \overline{h^{\circ N}(U)} \to \overline{h^{\circ N+1}(U)}$ is bijective, $\overline{h^{\circ N}(U)}$ contains the Jordan closed curve γ such that $h(\gamma) = \mathbb{S}^1$. So $h \circ \upsilon = \upsilon \circ h$ implies that $h^{-1}(\mathbb{S}^1) = \mathbb{S}^1 \cup \gamma \cup \upsilon(\gamma)$ (see Figure 1.18).

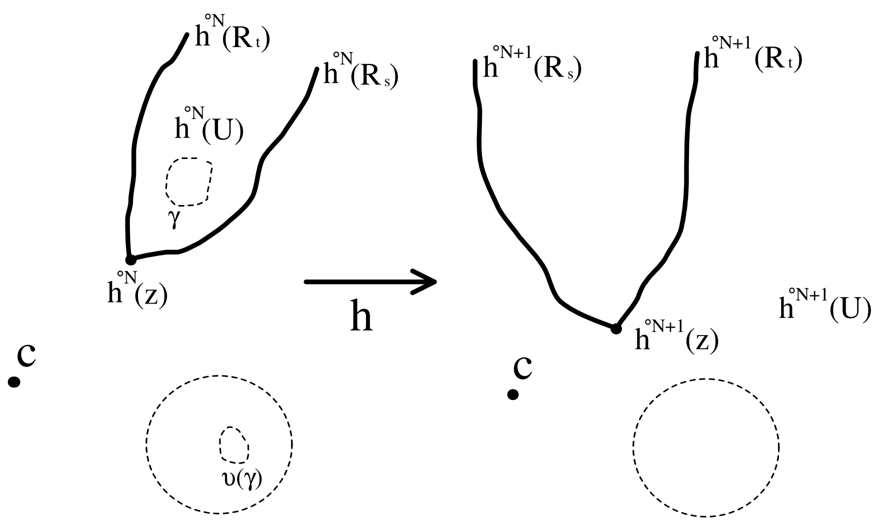

Figure 1.18

To simplify the notation, we set $h^{\circ N}(R_s) = R_{s'}$ and $h^{\circ N}(R_t) = R_{t'}$. Let $R_u = R_{s'+1/2}$, $R_v = R_{t'+1/2}$, and w be their landing point. Then $h(R_u \cup \{w\} \cup R_v) = h^{\circ N+1}(R_s \cup \{z\} \cup R_t)$. We shall see that $R_u \cup \{w\} \cup R_v$ separates $\mathbb{S}^1$ from c as following.

Assume that $R_u \cup \{w\} \cup R_v$ does not separate $\mathbb{S}^1$ from c. Let V be the component of $\mathbb{C} - (R_u \cup \{w\} \cup R_v)$ which does not contain c, and thus it does not contain $\mathbb{S}^1$. Then $A(V) = A(h^{\circ N}(U))$ by $A(V) < 1/2$. So h is univalent on V and thus $A(h(V)) = 2A(V)$. Then $A(h(V)) = 2A(V) = 2A(h^{\circ N}(U)) = A(h^{\circ N+1}(U))$ implies that $h(V) = h^{\circ N+1}(U) \supset \mathbb{S}^1$. So V contains a preimage of $\mathbb{S}^1$. This is impossible, for $h^{-1}(\mathbb{S}^1) = \mathbb{S}^1 \cup \gamma \cup \upsilon(\gamma)$.

Proof. **(Proof of Theorem 1.1.3)** We use proof by contradiction, and thus assume that $c \notin \{h^{\circ n}(z_0)\}_{n \geq 0}$. Then there exist two distinct external rays R_u and R_v with a common landing point w such that $R_u \cup \{w\} \cup R_v$ separates $\mathbb{S}^1$ from c by Lemma 1.6.3. Let U be the component of $\mathbb{C} - (R_u \cup \{w\} \cup R_v)$ which contains $\mathbb{S}^1$. We cut off U along an equipotential curve E_r, and thus have the Jordan closed curve $\gamma \subset \mathbb{C} - \overline{\mathbb{D}}$. Then h is injective on γ and preserves the orientation. Let V' be the Jordan annular domain which is surrounded by γ and $\mathbb{S}^1$. Since $\overline{V'}$ does not contain the pole $\upsilon(a)$, it follows from Lemma 1.3.2 that h is injective on $\overline{V'}$. Then $\overline{h(V')} \subset \mathbb{C} - \mathbb{D}$ implies that $\overline{V'}$ does not contain the zero a.

We put $V = V' \cup \mathbb{S}^1 \cup \upsilon(V')$. So $\overline{V}$ does not contain any of the pole $\upsilon(a)$, the zero a, two critical points c and $\upsilon(c)$ (see Figure 1.19).

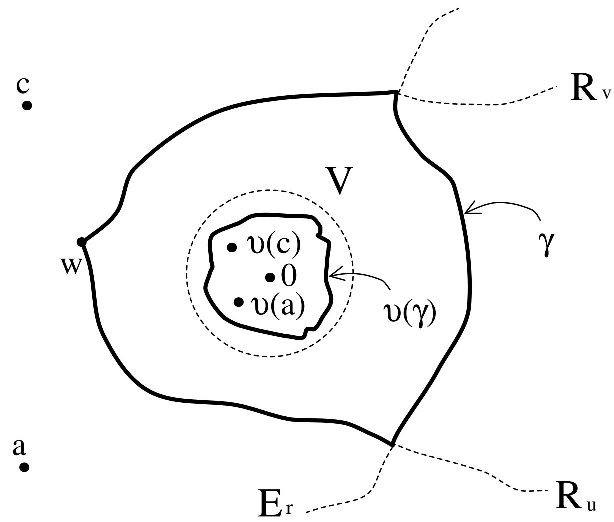

Figure 1.19

Moreover, h is injective on $\overline{V}$ by $h \circ \upsilon = \upsilon \circ h$. It follows from Lemma 1.3.1 that there exists a Jordan annular domain W such that $\overline{V} \subset W$ and h is univalent on a neighborhood of $\overline{W}$. We may suppose that both W and $h(W)$ do not contain the origin.

Now we take a Herman compactum H for (h, W) by Proposition 1.2.3. Then H meets the outer component of the boundary ∂W but not $\gamma - \{w\}$, so H must contain w. Let Ω be the unbounded component of $\widehat{\mathbb{C}} - H$. Then $\partial\Omega - \{w\}$ is disconnected, and thus the point w is biaccessible from Ω. However, the biaccessibility of w contradicts Proposition 1.2.4.

Chapter 2

Periodic points on the boundaries of rotation domains of some rational functions

2.1 Introduction and the main theorem

The dynamics on a periodic Fatou component is well understood, actually there are three possibilities. They are the attracting case, the parabolic case or the irrational rotation case. However, it is difficult to see the dynamics on the boundary of a periodic Fatou component. A positive answer to the question of local connectivity of the boundary sometimes gives a model of the dynamics. Even when the boundary fails to be locally connected, we are interested in the dynamics of the boundary. Especially, we may ask can the boundary have a dense orbit or a periodic orbit?

It is interesting that the periodic points on the boundary $\partial\Omega$ of an immediate attracting or parabolic basin Ω are dense in $\partial\Omega$ [**PrZ**, Theorem A]. According to [**RY**, Theorem 1], if Ω is a bounded Fatou component of a polynomial that is not eventually a Siegel disk, then the boundary $\partial\Omega$ is a Jordan curve. For a geometrically finite rational function with connected Julia set, the Julia set is locally connected [**TY**, Theorem A], and thus every Fatou component is locally connected.

We are interested in the topological structures of the boundaries of rotation domains and the dynamics on the boundaries. There are some results about the Julia sets which contain the boundaries of Siegel disks (see for example [**ABC**, **He**, **Pe**, **PZ**, **R**, **Ro**]).

If the boundary $\partial\Omega$ of a Siegel disk Ω is locally connected, then it follow from the Carathéodory's theorem in conformal mapping theory that $\partial\Omega$ is a Jordan closed curve and the dynamics on $\partial\Omega$ is topologically conjugate to an irrational rotation. In particular, there are no periodic points on the boundary $\partial\Omega$.

According to R. Pérez-Marco, the injectivity on a simply connected neighborhood of the closure of a Siegel disk implies that no periodic points on the boundary of the Siegel disk. More precisely, we have the following proposition [**PM**, Theorem IV.4.2].

Proposition 2.1.1 *Let Ω be an invariant Siegel disk of a rational function R, and let U be a neighborhood of $\overline{\Omega}$ so that the boundary ∂U consists of a Jordan closed curve γ. If R is injective on a neighborhood of $\overline{U}$, and both of γ and $R(\gamma)$ are contained in a component of $\widehat{\mathbb{C}} - \overline{\Omega}$, then the boundary $\partial\Omega$ contains no periodic points.*

In general, it may be hard to find a Jordan domain where the function is injective. The following theorem implies that there are still no periodic points except for the Cremer points on the boundary of invariant rotation domains even when the injective neighborhood is not a Jordan domain.

Theorem 2.1.1 *Let Ω be an invariant rotation domain of a rational function R, and let U be a neighborhood of $\overline{\Omega}$. If R is injective on U, then the boundary $\partial\Omega$ contains no periodic points except Cremer points.*

In the last section, we will discuss some related topics.

2.2 Basic definitions

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, and let $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function of degree at least two. We define the *Fatou set* of R as the union of all open sets $U \subset \widehat{\mathbb{C}}$ such that the family of iterates $\{R^n\}$ is equicontinuous on U, and the *Julia set* of R as the complement of the Fatou set of R. We denote the Julia set of R by $J(R)$ and the Fatou set of R by $F(R)$. The Fatou set $F(R)$ is a completely invariant open set and the Julia set $J(R)$ is a completely invariant compact set. Their fundamental properties can be found in [**Be**, **Mi**].

For each periodic point z_0 with period k, the *multiplier* is defined as $(R^k)'(z_0)$ and we denote by λ. A connected component of the Fatou set $F(R)$ is called a *Fatou component.*

A periodic point z_0 with period k is called *attracting* if $|\lambda| < 1$. Then the point z_0 is contained in the Fatou set $F(R)$. The Fatou component Ω containing the point z_0 is called the *immediate attracting basin* of z_0. Then $\{(R^k)^n\}$ converges locally uniformly to z_0 on Ω.

A periodic point z_0 with period k is called *parabolic* if λ is a root of unity, or equivalently there exists an rational number p/q such that $\lambda = e^{2\pi i p/q}$. Then the point z_0 is contained in the Julia set $J(R)$. A Fatou component Ω whose boundary contains the point z_0 is called an *immediate parabolic basin* of z_0 if $\{(R^{kq})^n\}$ converges locally uniformly to z_0 on Ω.

A periodic point z_0 with period k is called *irrationally indifferent* if $|\lambda| = 1$ but λ is not a root of unity, or equivalently there exists an irrational number θ such that $\lambda = e^{2\pi i\theta}$. Then we distinguish between two possibilities. If the point z_0 lies in the Fatou set $F(R)$, we say that a *Siegel point.* The Fatou component Ω containing the Siegel point z_0 is called the *Siegel disk* with *center* z_0. Then Ω is conformally isomorphic to the unit disk $\mathbb{D}$, and the dynamics of R^k on Ω corresponds to the dynamics of the irrational rotation λz on $\mathbb{D}$. Otherwise, if the point z_0 belongs to the Julia set $J(R)$, we say that a *Cremer point.*

A periodic point z_0 is called *weakly repelling* if $\lambda = 1$ or $|\lambda| > 1$, in particular, is called *repelling* if $|\lambda| > 1$. It well known that the repelling periodic points are dense in the Julia set $J(R)$ and the non-repelling periodic points are finite.

A periodic Fatou component Ω with period k is called a *Herman ring* if Ω is conformally isomorphic to some annulus $\mathbb{A}_r = \{z : 1/r < |z| < r\}$. Then the dynamics of R^k on Ω corresponds to the dynamics of an irrational rotation on $\mathbb{A}_r$. We say that a Siegel disk or a Herman ring is a *rotation domain.* It well known that every Fatou component is eventually periodic, and a periodic Fatou component is either an immediate attracting basin or an immediate parabolic basin or a Siegel disk or a Herman ring.

2.3 Local surjectivity

In this section, we shall see local surjectivity of rational function R of degree at least two. The notion of local surjectivity is referred from [**Sch**].

Definition 2.3.1 Let Ω be a Fatou component, and let $z_0 \in \partial\Omega$. We say R is *locally surjective* for (z_0, Ω), if there exists $\epsilon > 0$ such that $R(N \cap \Omega) = R(N) \cap R(\Omega)$ for any neighborhood $N \subset B_\epsilon(z_0) = \{z : d(z, z_0) < \epsilon\}$ of z_0.

Lemma 2.3.1 *Let Ω be a Fatou component, and let $z_0 \in \partial\Omega$. Assume that R is locally surjective for $(z_0, \Omega), (R(z_0), R(\Omega)), \cdots, (R^{n-1}(z_0), R^{n-1}(\Omega))$. Then R^n is locally surjective for (z_0, Ω).*

Proof. It follows from the assumption that there exists $\epsilon > 0$ such that

$$R(N \cap \Omega) = R(N) \cap R(\Omega),$$

$$R(R(N) \cap R(\Omega)) = R(R(N)) \cap R(R(\Omega)),$$

$$\vdots$$

$$R(R^{n-1}(N) \cap R^{n-1}(\Omega)) = R(R^{n-1}(N)) \cap R(R^{n-1(}\Omega)),$$

for any neighborhood $N \subset B_\epsilon(z_0)$ of z_0. So $R^n(N \cap \Omega) = R^n(N) \cap R^n(\Omega)$.

The following two propositions are described in [**Sch**]. Since the proofs are not given in [**Sch**], we will give proofs for the sake of completeness.

Proposition 2.3.1 *Let Ω be a Fatou component, and let $z_0 \in \partial\Omega$. Assume that z_0 is not a critical point, and there exists a Fatou component $\Omega' \neq \Omega$ such that $z_0 \in \partial\Omega'$ and $R(\Omega') = R(\Omega)$. Then R is not locally surjective for (z_0, Ω).*

Proof. Since z_0 is not a critical point, for any $\epsilon > 0$ there is a sufficiently small neighborhood $N \subset B_\epsilon(z_0)$ of z_0 such that $R|_N : N \to R(N)$ is a homeomorphism. Then $R(N \cap \Omega) \cap R(N \cap \Omega') = \emptyset$ and $R(N \cap \Omega') \subset R(N) \cap R(\Omega') = R(N) \cap R(\Omega)$. Therefore, $R(N \cap \Omega) \subset R(N) \cap R(\Omega) - R(N \cap \Omega') \subsetneq R(N) \cap R(\Omega)$.

Proposition 2.3.2 *Let Ω be a Fatou component, and let $z_0 \in \partial\Omega$. Assume that R is not locally surjective for (z_0, Ω). Then there exists a Fatou component $\Omega' \neq \Omega$ such that $z_0 \in \partial\Omega'$ and $R(\Omega') = R(\Omega)$.*

Proof. From the assumption, for each $n \in \mathbb{N}$ there exists a neighborhood $N_n \subset B_{1/n}(z_0)$ of z_0 such that $R(N_n \cap \Omega) \subsetneq R(N_n) \cap R(\Omega)$. Hence, there is a point $z_n \in N_n - \Omega$ so that $R(z_n) \in R(N_n) \cap R(\Omega) - R(N_k \cap \Omega)$. Let Ω_n be the

Fatou component contains z_n. Then, $\Omega_n \neq \Omega$ and $R(\Omega_n) = R(\Omega)$. Thus, we can set $\Omega' = \Omega_{n_i}$ for a subsequence $\{n_i\}$. Then $z_{n_i} \in \Omega'$ and $\lim_{i \to +\infty} z_{n_i} = z_0$, therefore, $z_0 \in \partial\Omega'$.

As it has been pointed out in [**Sch**], the above proposition implies that if Ω is a completely invariant Fatou component and $z_0 \in \partial\Omega$, then R is locally surjective for (z_0, Ω).

Lemma 2.3.2 *Let Ω be a Fatou component, and let $z_0 \in \partial\Omega$. If R is injective on a neighborhood V of the boundary $\partial\Omega$, then R is locally surjective for (z_0, Ω).*

Proof. Since R is injective on the neighborhood V of $\partial\Omega$, there are no Fatou components of $R^{-1}(R(\Omega))$ which contain z_0 on their boundaries. By the contraposition of Proposition 2.3.2, the proof is finished.

For a Fatou component whose boundary contains no critical point, the injectivity on the closure implies local surjectivity.

Theorem 2.3.1 *Let Ω be a Fatou component. Assume that R is injective on Ω and the boundary $\partial\Omega$ contains no critical points. Then, either R is injective on the boundary $\partial\Omega$ or there exists $z_0 \in \partial\Omega$ such that R is not locally surjective for (z_0, Ω).*

Proof. Suppose that R is injective on $\partial\Omega$ and let $z_0 \in \partial\Omega$. Then, R is injective on a neighborhood V of the boundary $\partial\Omega$ (see also [**Im1**, Lemma 3.1]). Therefore, R is locally surjective for (z_0, Ω) by Lemma 2.3.2.

Now suppose that R is not injective on $\partial\Omega$. Then, there are two distinct points $z_0 \in \partial\Omega$ and $w_0 \in \partial\Omega$ such that $R(z_0) = R(w_0)$. Since the boundary $\partial\Omega$ contains no critical points, there exists $\epsilon > 0$ such that $B_\epsilon(z_0) \cap B_\epsilon(w_0) = \emptyset$ and $R|_{B_\epsilon(z_0)} : B_\epsilon(z_0) \to R(B_\epsilon(z_0))$ is a homeomorphism. Let $w_n \in \Omega$ be a sequence so that $\lim_{n \to +\infty} w_n = w_0$. For any neighborhood $N \subset B_\epsilon(z_0)$ of z_0, the image $R(N)$ is a neighborhood of $R(z_0)$. Since $\lim_{n \to +\infty} R(w_n) = R(w_0) = R(z_0)$, there is some point $R(w_n)$ in $R(N)$. From the injectivity of $R|_\Omega$, there is no point in $N \cap \Omega$ whose image is equal to the point $R(w_n)$. Then, $R(w_n) \in R(N) \cap R(\Omega) - R(N \cap \Omega)$, and thus $R(N \cap \Omega) \subsetneq R(N) \cap R(\Omega)$. Therefore, R is not locally surjective for (z_0, Ω).

Since R is injective on a rotation domain, the following corollary argues that the injectivity on the boundary implies local surjectivity.

Corollary 2.3.1 *Let Ω be an invariant rotation domain. Assume that the boundary $\partial\Omega$ contains no critical points. Then, either R is injective on the boundary $\partial\Omega$ or there exists $z_0 \in \partial\Omega$ such that R is not locally surjective for (z_0, Ω).*

2.4 The proof of the main theorem

Definition 2.4.1 Let $\Omega \subset \widehat{\mathbb{C}}$ be a Fatou component. A point $z \in \partial\Omega$ is called *accessible* from Ω if there exists a continuous curve $\gamma : [0, 1) \to \Omega$ such that $\lim_{s \nearrow 1} \gamma(s) = z$. We say that such a curve γ is a *periodic curve* if $R^k(\gamma) \subset \gamma$ or $R^k(\gamma) \supset \gamma$ for some k.

We show Theorem 2.1.1 by using the following key proposition [**Sch**, Theorem 1].

Proposition 2.4.1 *Let Ω be an invariant Fatou component, and let $z_0 \in \partial\Omega$ be a weakly repelling fixed point. If R is locally surjective for (z_0, Ω), then z_0 is accessible from Ω by a periodic curve.*

So we have the following lemma.

Lemma 2.4.1 *Let Ω be an invariant Fatou component, and let $z_0 \in \partial\Omega$ be a parabolic fixed point. If R is locally surjective for (z_0, Ω), then z_0 is accessible from Ω by a periodic curve.*

Proof. Let $\lambda = e^{2\pi i p/q}$ be the multiplier at z_0. It is clear that Ω is an invariant Fatou component for R^q. So $(R^q)'(z_0) = \lambda^q = 1$ and thus z_0 is a weakly repelling fixed point of R^q. Since $R^n(z_0) = z_0$ and $R^n(\Omega) = \Omega$ for $0 \leq n \leq q$, Lemma 2.3.1 implies that R^q is locally surjective for (z_0, Ω). From Proposition 2.4.1, z_0 is accessible from Ω by a periodic curve for R^q. This curve is periodic for R.

Proof. (Proof of Theorem 2.1.1) We give the proof by contradiction. Suppose that the boundary $\partial\Omega$ contains a periodic point z_0 with period k which is not a Cremer point. So the point z_0 is a parabolic or repelling fixed point of R^k. It is clear that $R^n(\Omega) = \Omega$ and $R^n(z_0) \in \partial\Omega$ for $0 \leq n \leq k$, and thus Ω is an invariant Fatou component for R^k. Since R is injective on U, it follows from Lemma 2.3.2 that R is locally surjective for $(z_0, \Omega), (R(z_0), \Omega), \cdots, (R^{k-1}(z_0), \Omega)$. Lemma 2.3.1 implies that R^k is locally surjective for (z_0, Ω). By Proposition 2.4.1 and Lemma 2.4.1, the point z_0 is accessible from Ω by a periodic curve for R^k. This contradicts that Ω is a rotation domain.

2.5 Some related topics

In this section, we shall give some results on related topics. First, similarly to Proposition 2.1.1, we formulate the following proposition related to Herman rings and give the proof.

Proposition 2.5.1 *Let Ω be an invariant Herman ring of a rational function R, and let U be a neighborhood of $\overline{\Omega}$ so that the boundary ∂U consists of two Jordan closed curves γ and γ' which are separated by invariant curves in the Herman ring Ω. If R is injective on a neighborhood of $\overline{U}$, and both of γ and $R(\gamma)$ are contained in a component V of $\widehat{\mathbb{C}} - \overline{\Omega}$, and both of γ' and $R(\gamma')$ are contained in a component V' of $\widehat{\mathbb{C}} - \overline{\Omega}$, then the boundary $\partial\Omega$ contains no periodic points.*

Proof. This proof is referred from the proof of [**PM**, Theorem IV.4.2]. We give the proof by contradiction. Suppose that the boundary $\partial\Omega$ contains a periodic point with period k. Then, the periodic orbit $O = \{z_1, z_2, \cdots, z_k\}$ is contained in a component L of the boundary $\partial\Omega$. Let $\{K_n\}$ be a sequence of invariant closed annuli in the Herman ring Ω such that $K_n \subset \mathrm{Int} K_{n+1}$ and $\bigcup_{n=1}^{+\infty} K_n = \Omega$. Then $\{K_n\}$ converges to $\overline{\Omega}$ in the sense of Hausdorff convergence. Let $\widetilde{\Omega}$ be the filled set of $\overline{\Omega}$ such that $\widetilde{\Omega} = \widehat{\mathbb{C}} - (V \cup V')$. By the assumption, we note that $R|_{\widetilde{\Omega}} : \widetilde{\Omega} \to \widetilde{\Omega}$ is a homeomorphism.

The component L contains either ∂V or $\partial V'$. For the sake of convenience, we may assume that L contains ∂V, and furthermore, V contains infinity ∞. Let V_n be the component of $\widehat{\mathbb{C}} - K_n$ which contains ∞. Since $\{K_n\}$ converges to $\overline{\Omega}$ in the sense of Hausdorff convergence, $\{V_n\}$ converges to V with respect to ∞ in the sense of Carathéodory kernel convergence. We consider the following conformal isomorphisms

$$\Phi_n : \widehat{\mathbb{C}} - \overline{\mathbb{D}} \to V_n,\ \Phi : \widehat{\mathbb{C}} - \overline{\mathbb{D}} \to V$$

so that $\Phi_n(\infty) = \Phi(\infty) = \infty$, $\lim_{z\to\infty} \Phi_n(z)/z > 0$ and $\lim_{z\to\infty} \Phi(z)/z > 0$. So $\{\Phi_n\}$ converges locally uniformly to Φ by the Carathéodory kernel theorem (see for example [**Po**, Theorem 1.8]). There exists $r > 1$ such that

$g_n = \Phi_n^{-1} \circ R \circ \Phi_n$ and $g = \Phi^{-1} \circ R \circ \Phi$ are injective on $\{z : 1 < |z| < r\}$. By the reflection principle, g_n and g are extended and injective on $\mathbb{A}_r$. We fix r' such that $1 < r' < r$. Since $\{\Phi_n\}$ converges locally uniformly to Φ, $\{g_n\}$ converges uniformly to g on $r'\mathbb{S}^1$. Thus, $\{g_n\}$ converges uniformly to g on $(1/r')\mathbb{S}^1$. By the maximum principle, $\{g_n\}$ converges uniformly to g on $\overline{\mathbb{A}_{r'}}$, particularly on the unit circle $\mathbb{S}^1$.

Let L_n be the component of ∂K_n which is close to L. We notice that the dynamics of g_n on $\mathbb{S}^1$ corresponds to the dynamics of R on L_n. Since L_n is an invariant curve in the Herman ring Ω, the dynamics of R on L_n corresponds to the dynamics of an irrational rotation $z \mapsto e^{2\pi i\theta} z$. Therefore, the rotation number $\mathrm{Rot}(g|_{\mathbb{S}^1})$ is calculated as follows:

$$\mathrm{Rot}(g|_{\mathbb{S}^1}) = \lim_{n\to+\infty} \mathrm{Rot}(g_n|_{\mathbb{S}^1}) = \lim_{n\to+\infty} \theta = \theta.$$

Now let $O'_n = \Phi_n^{-1}(O)$, so O'_n is a periodic orbit of g_n with period k. Since $\{K_n\}$ converges to $\overline{\Omega}$ in the sense of Hausdorff convergence, we see that O'_n get close to $\mathbb{S}^1$ as $n \to +\infty$. More precisely, there are subsequence $\{O'_{n_i}\}$ and a set $O' \subset \mathbb{S}^1$ so that $\{O'_{n_i}\}$ converges to O' in the sense of Hausdorff convergence. Since $O'_{n_i} = \Phi_{n_i}^{-1}(O)$ are finite sets, so the limit set O' is a finite set. Moreover, $g_{n_i}(O'_{n_i}) = O'_{n_i}$ implies that $g(O') = O'$ (see also [**PM**, Lemma III.1.2]), and thus g has a periodic point on $\mathbb{S}^1$. This contradicts that the rotation number $\mathrm{Rot}(g|_{\mathbb{S}^1}) = \theta$ is irrational.

We consider the topology of the boundary of a Siegel disk.

Definition 2.5.1 Let $K \subset \widehat{\mathbb{C}}$ be a non-degenerate continuum. We say $z_0 \in K$ is a *cut point* of K if $K - \{z_0\}$ is disconnected.

Theorem 2.1.1 implies the following corollary, which asserts that the finiteness of cut points on the boundary of a Siegel disk follows from the injectivity of a neighborhood of the boundary.

Corollary 2.5.1 *Let Ω be an invariant Siegel disk of a rational function R, and let U be a neighborhood of $\overline{\Omega}$. If R is injective on U, then there are at most finitely many cut points of the boundary $\partial\Omega$.*

Proof. Assume that $z_0 \in \partial\Omega$ is a cut point of the boundary $\partial\Omega$. Then, z_0 is biaccessible from Ω, and thus z_0 is a periodic point (see [**Im1**, Definition 1.1 and Proposition 1.1]). It follows from Theorem 2.1.1 that z_0 must be a Cremer point. Since there are at most finitely many Cremer points, the proof is finished.

Now we consider the following two functions. Let $P(z) = e^{2\pi i\theta}z + z^2$ be a quadratic polynomial with $\theta \in \mathbb{R} - \mathbb{Q}$. Let $B(z) = e^{2\pi i\tau(\theta)}z^2(z-a)/(1-\bar{a}z)$ be a cubic Blaschke product so that $|a| > 3$ and the rotation number $\mathrm{Rot}(B|_{\mathbb{S}^1}) = \theta \in \mathbb{R} - \mathbb{Q}$. We compare the dynamics of P and the Julia set $J(P)$ with the dynamics of B and the Julia set $J(B)$.

Definition 2.5.2 If there exists a local holomorphic change of coordinate $z = \Phi(w)$, with $\Phi(0) = 0$, such that $\Phi^{-1} \circ P \circ \Phi$ is the irrational rotation $w \mapsto e^{2\pi i\theta}w$ near the origin, then we say that P is *linearizable* at the origin.

The origin is either a Siegel point or a Cremer point, according to whether P is linearizable at the origin or not.

Definition 2.5.3 If there exists an analytic circle diffeomorphism $\Phi : \mathbb{S}^1 \to \mathbb{S}^1$ such that $\Phi^{-1} \circ B \circ \Phi$ is the irrational rotation $w \mapsto e^{2\pi i\theta}w$, then we say that B is *linearizable* on the unit circle.

The unit circle is contained in either the Fatou set $F(B)$ or the Julia set $J(B)$, according to whether B is linearizable on the unit circle or not.

Suppose that P is not linearizable at the origin and B is not linearizable on the unit circle. It follows from [**PM**, Theorem 1 and Theorem V.1.1] that there are Siegel compacta in $J(P)$ and Herman compacta in $J(B)$. There is a recurrent critical point $c_P \in J(P)$ whose forward orbit $\{P^n(c_P)\}_{n\geq 0}$ accumulates the origin, and there is a recurrent critical point $c_B \in J(B)$ whose forward orbit $\{B^n(c_B)\}_{n\geq 0}$ accumulates the unit circle (see [**Ma**, Theorem I]).

Let Ω_P be the immediate attracting basin of infinity with respect to the dynamics of P, and let Ω_B be the immediate attracting basin of infinity with respect to the dynamics of B. A. Douady and D. Sullivan [**Su**, Theorem 8] has shown that $\partial\Omega_P = J(P)$ is not locally connected (see also [**Mi**, Corollary 18.6]). It follows from [**R**, Lemma 1.7 and Proposition 1.6] that the unit circle is contained in the boundary $\partial\Omega_B$, and the boundary $\partial\Omega_B$ is not locally connected. In particularly, the Julia set $J(B)$ is not locally connected. Therefore, we conclude that both of the Julia sets $J(P)$ and $J(B)$ are connected but not locally connected.

It is well known that every repelling periodic point on the boundary $\partial\Omega_P = J(P)$ is accessible from Ω_P by a periodic curve. Furthermore, we have the following proposition.

Proposition 2.5.2 *Let $B(z) = e^{2\pi i\tau(\theta)}z^2(z-a)/(1-\bar{a}z)$ be a cubic Blaschke product so that $|a| > 3$ and the rotation number $\mathrm{Rot}(B|_{\mathbb{S}^1}) = \theta$, let Ω_B be the immediate attracting basin of infinity. Assume that θ is irrational and B is not linearizable on the unit circle. Then, every repelling periodic point on the boundary $\partial\Omega_B$ is accessible from Ω_B by a periodic curve.*

Proof. Let z_0 be a repelling periodic point on the boundary $\partial\Omega_B$ with period k. It is clear that $B^n(\Omega_B) = \Omega_B$ and $B^n(z_0) \in \partial\Omega_B$ for $0 \le n \le k$, and thus Ω_B is an invariant Fatou component for B^k. Let Ω' be the Fatou component containing the pole $1/\bar{a}$. Then, $B^{-1}(\Omega_B) = \Omega' \cup \Omega_B$. Since the unit circle $\mathbb{S}^1$ is contained in the Julia set $J(B)$, the Fatou component Ω' is contained in the unit disk $\mathbb{D}$ and Ω_B is contained in $\widehat{\mathbb{C}} - \overline{\mathbb{D}}$. Therefore, injectivity of $B|_{\mathbb{S}^1}$ implies $\partial\Omega' \cap \partial\Omega_B = \emptyset$.

It follows from the contraposition of Proposition 2.3.2 that B is locally surjective for $(z_0, \Omega_B), (B(z_0), \Omega_B), \cdots, (B^{k-1}(z_0), \Omega_B)$. Lemma 2.3.1 implies that B^k is locally surjective for (z_0, Ω_B). By Proposition 2.4.1, the point z_0 is accessible from Ω by a periodic curve for R^k.

From the results [**SZ**, Theorem 3] and [**Im1**, Theorem 1.3] of biaccessibility, we note that each of the repelling periodic points on $\partial\Omega_P = J(P)$ or $\partial\Omega_B$ has only one external ray landing at the point.

Finally, we consider buried points in the Julia sets. It follows from $\partial\Omega_P = J(P)$ that the Julia set $J(P)$ has no buried points, however, we see that the Julia set $J(B)$ has buried points.

Definition 2.5.4 Let $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function of degree at least two. A point z in the Julia set $J(R)$ is called *buried* if z is not lying in the boundary of any Fatou component.

Interestingly, we have the following (see [**CMTT**, Proposition 1.4] and [**CMMR**, Lemma 1]).

Proposition 2.5.3 *Let $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function of degree at least two. Then there exists a buried point iff there is no periodic Fatou component U such that $\partial U = J(R)$.*

So we have the following proposition.

Proposition 2.5.4 *Let $B(z) = e^{2\pi i \tau(\theta)} z^2(z-a)/(1-\bar{a}z)$ be a cubic Blaschke product so that $|a| > 3$ and the rotation number $\mathrm{Rot}(B|_{\mathbb{S}^1}) = \theta$. Assume that θ is irrational and B is not linearizable on the unit circle. Then there exists a buried point.*

Proof. The unit circle $\mathbb{S}^1$ is contained in the Julia set $J(B)$. There exist two points in $J(B)$ which are separated by $\mathbb{S}^1$ (for example, the recurrent critical points c_B and $1/\bar{c}_B$). Consequently, there is no periodic Fatou component U such that $\partial U = J(B)$, and there exists a buried point by Proposition 2.5.3.

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