Symmetries of Julia sets in higher dimensions

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Introduction

We consider the iteration of a holomorphic map on a complex manifold as complex dynamics. The Fatou set is defined as the largest open set on which the family of iterates is normal, or, roughly speaking, on which the dynamics are stable. The Julia set is defined as the complement of the Fatou set. Recently, the study of one-dimensional complex dynamical systems has developed because of the progress of computer technology. The success of the study of one-dimensional complex dynamics is now promoting the study of higher-dimensional complex dynamics.

We are concerned with symmetries of Julia sets in higher dimensions. We say that a Julia set has symmetries if it is preserved by non-elementary transformations. For example, one can construct holomorphic maps which commute with each action of a polyhedral group acting on the Riemann sphere $\mathbf{P}^1$. Its Julia sets are preserved by each action of the polyhedral group. As another example, there are Julia sets of polynomials on $\mathbf{C}$ that are each preserved by some rotations. We generalize these objects and results of symmetries of Julia sets in one dimension to those in higher dimensions.

This thesis consists of two parts. In chapter 1, we will show that a family of holomorphic maps on complex projective spaces has good dynamical properties. S. Crass constructed a family of holomorphic maps which has the following properties: for each integer k, there exists a holomorphic map that commutes with each element of the $(k+2)$-th symmetry group acting on the k-dimensional complex projective space $\mathbf{P}^k$, and whose critical set coincides with the special hyperplane of the $(k+2)$-th symmetry group action. We prove that the Fatou sets of this family consist of attracting basins and that each map of this family satisfies Axiom A. This result gives the first nontrivial example of holomorphic maps whose Julia sets have the symmetries of finite group actions, and for which the global dynamics are understood. In higher dimensions, there are only few examples whose dynamics are well understood.

In chapter 2, we will investigate symmetries of Julia sets of polynomial skew products on $\mathbf{C}^2$. We define polynomial skew products on $\mathbf{C}^2$ as regular polynomial maps on $\mathbf{C}^2$ whose first component depends only on the first component of the coordinates. The dynamics of polynomial skew products on $\mathbf{C}^2$ are closely related to those of polynomials on $\mathbf{C}$. The second Julia sets of polynomial skew products are analogues to the Julia sets of polynomials. We consider only those symmetries of the second Julia sets of polynomial skew products, which are defined by affine maps whose first component depends only on the first coordinate. First, we investigate the structure of symmetries and give a necessary and sufficient condition for the group of symmetries to be infinite. Next, we show that, except for two types, polynomial skew products with the same second Julia set are essentially the same. As a corollary, except for two types, the first Julia set is determined only by the second Julia set. The first Julia set of a polynomial skew product is defined by the Julia set of its extension to the 2-dimensional projective space.

Notes

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Chapter 1

Dynamics of symmetric holomorphic maps on projective spaces

We consider complex dynamics of a *critically finite* holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$, which has symmetries associated with the symmetric group S_{k+2} acting on $\mathbf{P}^k$, for each $k \geq 1$. The Fatou set of each map of this family consists of attractive basins of superattracting points. Each map of this family satisfies Axiom A.

1.1 Introduction

For a finite group G acting on $\mathbf{P}^k$ as projective transformations, we say that a rational map f on $\mathbf{P}^k$ is *G-equivariant* if f commutes with each element of G. That is, $f \circ r = r \circ f$ for any $r \in G$, where $\circ$ denotes the composition of maps. Doyle and McMullen [4] introduced the notion of *equivariant* functions on $\mathbf{P}^1$ to solve quintic equations. See also [11] for *equivariant* functions on $\mathbf{P}^1$. Crass [2] extended Doyle and McMullen's algorithm to higher dimensions to solve sextic equations. Crass [3] found a good family of finite groups and *equivariant* maps for which one may say something about global dynamics. Crass [3] conjectured that the Fatou set of each map of this family consists of attractive basins of superattracting points. Although I do not know whether this family has relation to solving equations or not, our results will give affirmative answers for the conjectures in [3].

In section 2 we shall explain an action of the symmetric group S_{k+2} on $\mathbf{P}^k$ and properties of our S_{k+2}-equivariant map. In section 3 and 4 we shall show our results about the Fatou sets and hyperbolicity of our maps by using properties of our maps and Kobayashi metrics.

1.2 S_{k+2}-equivariant maps

Crass [3] selected the symmetric group S_{k+2} as a finite group acting on $\mathbf{P}^k$ and found an S_{k+2}-*equivariant* map which is holomorphic and *critically finite* for each $k \geq 1$. We denote by $C = C(f)$ the critical set of f and say that f is *critically finite* if each irreducible component of $C(f)$ is periodic or preperiodic. More precisely, S_{k+2}-*equivariant* map g_{k+3} defined in section 1.2.2 preserves each irreducible component of $C(g_{k+3})$, which is a projective hyperplane. The complement of $C(g_{k+3})$ is Kobayashi hyperbolic. Furthermore restrictions of g_{k+3} to invariant projective subspaces have the same properties as above. See section 1.2.3 for details.

1.2.1 S_{k+2} acts on $\mathbf{P}^k$

An action of the $(k+2)$-th symmetric group S_{k+2} on $\mathbf{P}^k$ is induced by the permutation action of S_{k+2} on $\mathbf{C}^{k+2}$ for each $k \geq 1$. The transposition (i, j) in S_{k+2} corresponds with the transposition "$u_i \leftrightarrow u_j$" on $\mathbf{C}_u^{k+2}$, which pointwise fixes the hyperplane $\{u_i = u_j\} = \{u \in \mathbf{C}_u^{k+2} \mid u_i = u_j\}$. Here $\mathbf{C}^{k+2} = \mathbf{C}_u^{k+2} = \{u = (u_1, u_2, \cdot\cdot, u_{k+2}) \mid u_i \in \mathbf{C} \ \text{ for } \ i = 1, \cdot\cdot, k+2\}$.

The action of S_{k+2} preserves a hyperplane H in $\mathbf{C}_u^{k+2}$, which is identified with $\mathbf{C}_x^{k+1}$ by projection $A : \mathbf{C}_u^{k+2} \to \mathbf{C}_x^{k+1}$,

$$H = \left\{ \sum_{i=1}^{k+2} u_i = 0 \right\} \overset{\mathrm{A}}{\simeq} \mathbf{C}_x^{k+1} \text{ and } A = \begin{pmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \end{pmatrix}.$$

Here $\mathbf{C}^{k+1} = \mathbf{C}_x^{k+1} = \{x = (x_1, x_2, \cdot\cdot, x_{k+1}) \mid x_i \in \mathbf{C} \ \text{ for } \ i = 1, \cdot\cdot, k+1\}$.

Thus the permutation action of S_{k+2} on $\mathbf{C}_u^{k+2}$ induces an action of "S_{k+2}" on $\mathbf{C}_x^{k+1}$. Here "S_{k+2}" is generated by the permutation action S_{k+1} on $\mathbf{C}_x^{k+1}$ and a

$(k+1, k+1)$-matrix T which corresponds to the transposition $(1,\ k+2)$ in S_{k+2},

$$T = \begin{pmatrix} -1 & 0 & \dots & 0 \\ -1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -1 & 0 & \dots & 1 \end{pmatrix}.$$

Hence the hyperplane corresponding to $\{u_i = u_j\}$ is $\{x_i = x_j\}$ for $1 \leq i < j \leq k+1$. The hyperplane corresponding to $\{u_i = u_{k+2}\}$ is $\{x_i = 0\}$ for $1 \leq i \leq k+1$. Each element in "S_{k+2}" which corresponds to some transposition in S_{k+2} pointwise fixes one of these hyperplanes in $\mathbf{C}_x^{k+1}$.

The action of "S_{k+2}" on $\mathbf{C}^{k+1}$ projects naturally to the action of "S_{k+2}" on $\mathbf{P}^k$. These hyperplanes on $\mathbf{C}^{k+1}$ projects naturally to projective hyperplanes on $\mathbf{P}^k$. Here $\mathbf{P}^k = \{x = [x_1 : x_2 : \cdot\cdot : x_{k+1}] \mid (x_1, x_2, \cdot\cdot, x_{k+1}) \in \mathbf{C}^{k+1} \setminus \{\mathbf{0}\}\}$. Each element in the action of "S_{k+2}" on $\mathbf{P}^k$ which corresponds to some transposition in S_{k+2} pointwise fixes one of these projective hyperplanes. We denote "S_{k+2}" also by S_{k+2} and call these projective hyperplanes transposition hyperplanes.

1.2.2 Existence of our maps

One way to get S_{k+2}-*equivariant* maps on $\mathbf{P}^k$ which are *critically finite* is to make S_{k+2}-*equivariant* maps whose critical sets coincide with the union of the transposition hyperplanes.

Theorem 1 ([3]). *For each $k \geq 1$, g_{k+3} defined below is the unique S_{k+2}-equivariant holomorphic map of degree $k+3$ which is doubly critical on each transposition hyperplane.*

$$g = g_{k+3} = [g_{k+3,1} : g_{k+3,2} : \cdot\cdot : g_{k+3,k+1}] : \mathbf{P}^k \to \mathbf{P}^k,$$

$$\textit{where}\ \ g_{k+3,l}(x) = x_l^3 \sum_{s=0}^{k} (-1)^s \frac{s+1}{s+3} x_l^s A_{k-s},\ \ A_0 = 1,$$

and A_{k-s} is the elementary symmetric function of degree k-s in $\mathbf{C}^{k+1}$.

Then the critical set of g coincides with the union of the transposition hyperplanes. Since g is S_{k+2}-*equivariant* and each transposition hyperplane is pointwise fixed by some element in S_{k+2}, g preserves each transposition hyperplane.

In particular g is *critically finite*. Although Crass [3] used this explicit formula to prove Theorem 1, we shall only use properties of the S_{k+2}-*equivariant* maps described below.

1.2.3 Properties of our maps

Let us look at properties of the S_{k+2}-*equivariant* map g on $\mathbf{P}^k$ for a fixed k, which is proved in [3] and shall be used to prove our results. Let L^{k-1} denote one of the transposition hyperplanes, which is isomorphic to $\mathbf{P}^{k-1}$. Let L^m denote one of the intersections of $(k-m)$ or more distinct transposition hyperplanes which is isomorphic to $\mathbf{P}^m$ for $m = 0, 1, \cdot\cdot, k-1$.

First, let us look at properties of g itself. The critical set of g consists of the union of the transposition hyperplanes. By S_{k+2}-*equivariance*, g preserves each transposition hyperplane. Furthermore the complement of the critical set of g is Kobayashi hyperbolic.

Next, let us look at properties of g restricted to L^m for $m = 1, 2, \cdot\cdot, k-1$. Let us fix any m. Since g preserves each L^m, we can also consider the dynamics of g restricted to any L^m. Each restricted map has the same properties as above. Let us fix any L^m and denote by $g|_{L^m}$ the restricted map of g to the L^m. The critical set of $g|_{L^m}$ consists of the union of intersections of the L^m and another L^{k-1} which does not include the L^m. We denote it by L^{m-1}, which is an irreducible component of the critical set of $g|_{L^m}$. By S_{k+2}-*equivariance,* $g|_{L^m}$ preserves each irreducible component of the critical set of $g|_{L^m}$. Furthermore the complement of the critical set of $g|_{L^m}$ in L^m is Kobayashi hyperbolic.

Finally, let us look at a property of superattracting fixed points of g. The set of superattracting points, where the derivative of g vanishes for all directions, coincides with the set of L^0's.

Remark 1. *For every $k \geq 1$ and every m, $1 \leq m \leq k$, a restricted map of g_{k+3} to any L^m is not conjugate to g_{m+3}.*

1.2.4 Examples for $k = 1$ and 2

Let us see transposition hyperplanes of the S_3-*equivariant* function g_4 and the S_4-*equivariant* map g_5 to make clear what L^m is. In [3] one can find explicit formulas and figures of dynamics of S_{k+2}-equivariant maps in low-dimensions .

S_3-equivariant function g_4 in $\mathbf{P}^1$

$$g_3([x_1 : x_2]) = [x_1^3(-x_1 + 2x_2) : x_2^3(2x_1 - x_2)] : \mathbf{P}^1 \to \mathbf{P}^1,$$
$$C(g_3) = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_1 = x_2\} = \{0, 1, \infty\} \text{ in } \mathbf{P}^1.$$

In this case ”transposition hyperplanes” are points in $\mathbf{P}^1$ and L^0 denotes one of three superattracting fixed points of g_3.

S_4-equivariant map g_5 in $\mathbf{P}^2$

$$C(g_5) = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\} \cup$$
$$\{x_1 = x_2\} \cup \{x_2 = x_3\} \cup \{x_3 = x_1\} \text{ in } \mathbf{P}^2.$$

In this case L^1 denotes one of six transposition hyperplanes in $\mathbf{P}^2$, which is an irreducible component of $C(g_5)$. For example, let us fix a transposition hyperplane $\{x_1 = 0\}$. Since g_5 preserves each transposition hyperplane, we can also consider the dynamics of g_5 restricted to $\{x_1 = 0\}$. We denote by $g_5|_{\{x_1=0\}}$ the restricted map of g_5 to $\{x_1 = 0\}$. The critical set of $g_5|_{\{x_1=0\}}$ in $\{x_1 = 0\} \simeq \mathbf{P}^1$ is

$$C(g_5|_{\{x_1=0\}}) = \{[0 : 1 : 0], [0 : 0 : 1], [0 : 1 : 1]\}.$$

When we use L^0 after we fix $\{x_1 = 0\}$, L^0 denotes one of intersections of $\{x_1 = 0\}$ and another transposition hyperplane, which is a superattracting fixed point of $g_5|_{\{x_1=0\}}$ in $\mathbf{P}^1$. The set of superattracting fixed points of g_5 in $\mathbf{P}^2$ is

$$\{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1], [1 : 1 : 0], [1 : 0 : 1], [0 : 1 : 1]\}.$$

In general L^0 denotes one of intersections of two or more transposition hyperplanes, which is a superattracting fixed point of g_5 in $\mathbf{P}^2$.

1.3 The Fatou sets of the S_{k+2}-equivariant maps

1.3.1 Definitions and preliminaries

Let us recall theorems about *critically finite* holomorphic maps. Let f be a holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$. The Fatou set of f is defined to be the maximal open subset where the iterates $\{f^n\}_{n\geq 0}$ is a normal family. The Julia set of f is defined to be the complement of the Fatou set of f. Each connected component of the Fatou set is called a Fatou component. Let U be a Fatou component of f. A holomorphic map h is said to be a limit map on U if there is a subsequence $\{f^{n_s}|_U\}_{s\geq 0}$ which locally converges to h on U. We say that a point q is a Fatou limit point if there is a limit map h on a Fatou component U such that $q \in h(U)$. The set of all Fatou limit points is called the Fatou limit set. We define the ω-limit set $E(f)$ of the critical points by

$$E(f) = \bigcap_{j=1}^{\infty} \overline{\bigcup_{n=j}^{\infty} f^n(C)}.$$

Theorem 2. *([10, Proposition 5.1]) If f is a critically finite holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$, then the Fatou limit set is contained in the ω-limit set $E(f)$.*

Let us recall the notion of Kobayashi metrics. Let M be a complex manifold and $K_M(x,v)$ the Kobayashi quasimetric on M,

$$\inf\left\{|a| \,\middle|\, \varphi : \mathbf{D} \to M : \text{holomorphic}, \varphi(0) = x, D\varphi\left(a\left(\frac{\partial}{\partial z}\right)_0\right) = v, a \in \mathbf{C}\right\}$$

for $x \in M$, $v \in T_xM$, $z \in \mathbf{D}$, where $\mathbf{D}$ is the unit disk in $\mathbf{C}$. We say that M is Kobayashi hyperbolic if K_M becomes a metric. Theorem 5 is a corollary of Theorem 3 and Theorem 4 for $k = 1$ and 2.

Theorem 3. *(a basic result whose former statement can be found in [8, Corollary 14.5]) If f is a critically finite holomorphic function from $\mathbf{P}^1$ to $\mathbf{P}^1$, then the only Fatou components of f are attractive components of superattracting points. Moreover if the Fatou set is not empty, then the Fatou set has full measure in $\mathbf{P}^1$.*

Theorem 4. *([5, theorem 7.7]) If f is a critically finite holomorphic map from $\mathbf{P}^2$ to $\mathbf{P}^2$ and the complement of $C(f)$ is Kobayashi hyperbolic, then the only Fatou components of f are attractive components of superattracting points.*

1.3.2 Our first result

Let us fix any k and $g = g_{k+3}$. For every m, $2 \leq m \leq k$, we can apply an argument in [5] to a restricted map of g to any L^m because every L^{m-1} is smooth and because every $L^m \backslash C(g|_{L^m})$ is Kobayashi hyperbolic. We shall use this argument in Lemma 2, which is used to prove Proposition 1.

Proposition 1. *For any Fatou component U which is disjoint from $C(g)$, there exists an integer n such that $g^n(U)$ intersects with $C(g)$.*

Proof: We suppose that $g^n(U)$ is disjoint from $C(g)$ for any n and derive a contradiction by using Lemma 2 and Remark 3 below. Take any point $x_0 \in U$. Since $E(g)$ coincides with $C(g)$, $g^n(x_0)$ accumulates to $C(g)$ as n tends to ∞ from Theorem 2. Since $C(g)$ is the union of the transposition hyperplanes, there exists a smallest integer m_1 such that $g^n(x_0)$ accumulates to some L^{m_1}. Let h_1 be a limit map on U such that $h_1(x_0)$ belongs to the L^{m_1}. From Lemma 2 below, the intersection of $h_1(U)$ and the L^{m_1} is an open set in the L^{m_1} and is contained in the Fatou set of $g|_{L^{m_1}}$.

We next consider the dynamics of $g|_{L^{m_1}}$. If there exists an integer n_2 such that $g^{n_2}(h_1(U) \cap L^{m_1})$ intersects with $C(g|_{L^{m_1}})$, then $g^{n_2}(h_1(U) \cap L^{m_1})$ intersects with some L^{m_1-1}. In this case we can consider the dynamics of $g|_{L^{m_1-1}}$. On the other hand, if there does not exist such n_2, then there exists an integer m_2 and a limit map h_2 on $h_1(U) \cap L^{m_1}$ such that the intersection of $h_2(h_1(U) \cap L^{m_1})$ and some L^{m_2} is an open set in the L^{m_2} from Remark 3 below. Thus it is contained in the Fatou set of $g|_{L^{m_2}}$. Here m_2 is smaller than m_1. In this case we can consider the dynamics of $g|_{L^{m_2}}$.

We continue the same argument above. These reductions finally come to some L^1 and we use Theorem 3. One can find a similar reduction argument in the proof of Theorem 5. Consequently $g^n(x_0)$ accumulates to some superattracting point L^0. So there exists an integer s such that g^s sends U to the attractive Fatou component which contains the superattracting point L^0. Thus $g^s(U)$ intersects with $C(g)$, which is a contradiction. □

Remark 2. *Even if a Fatou component U intersects with some L^m and is disjoint from any L^{m-1}, then the similar thing as above holds for the dynamics in the L^m. In this case $U \cap L^m$ is contained in the Fatou set of $g|_{L^m}$ and there exists an integer n such that $g^n(U \cap L^m)$ intersects with $C(g|_{L^m})$.*

Lemma 1. *For any Fatou component U which is disjoint from $C(g)$ and any point $x_0 \in U$, let h be a limit map on U such that $h(x_0)$ belongs to some L^m and does not belong to any L^{m-1}. If $g^n(U)$ is disjoint from $C(g)$ for every $n \geq 1$, then the intersection of $h(U)$ and the L^m is an open set in the L^m.*

Proof: Let B be the complement of $C(g)$. Since B is Kobayashi hyperbolic and B includes $g^{-1}(B)$, $g^{-1}(B)$ is Kobayashi hyperbolic, too. So we can use Kobayashi metrics K_B and $K_{g^{-1}(B)}$. Since B includes $g^{-1}(B)$,

$$K_B(x,v) \leq K_{g^{-1}(B)}(x,v) \text{ for all } x \in g^{-1}(B), v \in T_x\mathbf{P}^k .$$

In addition, since g is an unbranched covering from $g^{-1}(B)$ to B,

$$K_{g^{-1}(B)}(x,v) = K_B(g(x), Dg(v)) \text{ for all } x \in g^{-1}(B), v \in T_x\mathbf{P}^k .$$

From these two inequalities we have the following inequality

$$K_B(x,v) \leq K_B(g(x), Dg(v)) \text{ for all } x \in g^{-1}(B), v \in T_x\mathbf{P}^k .$$

Since the same argument holds for any g^n from $g^{-n}(B)$ to B,

$$K_B(x,v) \leq K_B(g^n(x), Dg^n(v)) \text{ for all } x \in g^{-n}(B), v \in T_x\mathbf{P}^k .$$

Since g^n is an unbranched covering from U to $g^n(U)$ and B includes $g^n(U)$ for every n, a sequence $\{K_B(g^n(x), Dg^n(v))\}_{n\geq 0}$ is bounded for all $x \in U$, $v \in T_x\mathbf{P}^k$. Hence we have the following inequality for any unit vectors v_n in $T_{x_0}U$ with respect to the Fubini-Study metric in $\mathbf{P}^k$,

$$0 < \inf_{|v|=1} K_B(x_0, v) \leq K_B(x_0, v_n) \leq K_B(g^n(x_0), Dg^n(x_0)v_n) < \infty. \tag{1.1}$$

That is, the sequence $\{K_B(g^n(x_0), Dg^n(x_0)v_n)\}_{n\geq 0}$ is bounded away from 0 and ∞ uniformly.

We shall choose v_n so that $Dg^n(x_0)v_n$ keeps parallel to the L^m and claim that $Dh(x_0)v \neq \mathbf{0}$ for any accumulation vector v of v_n. Let $h = \lim_{n\to\infty} g^n$ for simplicity. Let V be a neighborhood of $h(x_0)$ and ψ a local coordinate on V so that $\psi(h(x_0)) = \mathbf{0}$ and $\psi(L^m \cap V) \subset \{y = (y_1, y_2, \cdot\cdot, y_k) \mid y_1 = \cdot\cdot = y_{k-m} = 0\}$. In this chart there exists a constant $r > 0$ such that a polydisk $P(\mathbf{0}, 2r)$ does not intersect with any images of transposition hyperplanes which do not include the L^m. Since $\psi(g^n(x_0))$ converges to $\mathbf{0}$ as n tends to ∞, we may assume that $\psi(g^n(x_0))$ belongs to $P(\mathbf{0}, r)$ for large n. Let $\{v_n\}_{n\geq 0}$ be unit vectors in $T_{x_0}\mathbf{P}^k$ and $\{w_n\}_{n\geq 0}$ vectors in $T_{\psi(g^n(x_0))}\mathbf{C}^k$ so that w_n keep parallel to $\psi(L^m)$ with a same direction and

$$Dg^n(x_0)v_n = |Dg^n(x_0)v_n|\, D\psi^{-1}(w_n).$$

So we may assume that the length of w_n is almost unit for large n. We define holomorphic maps φ_n from $\mathbf{D}$ to $P(\mathbf{0}, 2r)$ as

$$\varphi_n(z) = \psi(g^n(x_0)) + rzw_n \;\text{ for } z \in \mathbf{D}$$

and consider holomorphic maps $\psi^{-1} \circ \varphi_n$ from $\mathbf{D}$ to B for large n. Then

$$(\psi^{-1} \circ \varphi_n)(0) = g^n(x_0),$$

$$D(\psi^{-1} \circ \varphi_n)\left(\frac{|Dg^n(x_0)v_n|}{r}\left(\frac{\partial}{\partial z}\right)_0\right) = Dg^n(x_0)v_n.$$

Suppose $Dh(x_0)v = \mathbf{0}$, then $Dg^n(x_0)v$ converges to $\mathbf{0}$ as n tends to ∞ and so does $Dg^n(x_0)v_n$. By the definition of Kobayashi metric we have that

$$K_B(g^n(x_0), Dg^n(x_0)v_n) \leq \frac{|Dg^n(x_0)v_n|}{r} \to 0 \;\text{ as }\; n \to \infty.$$

Since this contradicts (1), we have $Dh(x_0)v \neq \mathbf{0}$. This holds for all directions which are parallel to $\psi(L^m)$. Consequently the intersection of $h(U)$ and the L^m is an open set in L^m. □

Remark 3. *The similar thing as above holds for the dynamics of any restricted map. Thus even if a Fatou component $g^n(U)$ intersects with $C(g)$ for some n, the same result as above holds. Because one can consider the dynamics in the L^m when $g^n(U)$ intersects with some L^m.*

Theorem 5. *For each $k \geq 1$, the Fatou set of the S_{k+2}-equivariant map g consists of attractive basins of superattracting fixed points which are intersections of k or more distinct transposition hyperplanes.*

Proof: This theorem follows from Proposition 1 and Remark 2 immediately. Let us describe details. Take any Fatou component U. From Proposition 1 there exists an integer n_k such that $g^{n_k}(U)$ intersects with $C(g)$. Since $C(g)$ is the union of the transposition hyperplanes, $g^{n_k}(U)$ intersects with some L^{k-1}. By doing the same thing as above for the dynamics of g restricted to the L^{k-1}, there exists an integer n_{k-1} such that $g^{n_k+n_{k-1}}(U)$ intersects with some L^{k-2} from Remark 2. We again do the same thing as above for the dynamics of g restricted to the L^{k-2}.

These reductions finally come to some L^1. That is, there exists integers $n_{k-2}, \cdots, n_2$ such that $g^{n_k+n_{k-1}+\cdots+n_2}(U)$ intersects with some L^1. From Theorem 3 there exists an integer n_1 such that $g^{n_1}(g^{n_k+n_{k-1}+\cdots+n_2}(U))$ contains some L^0. Hence $g^{n_k+n_{k-1}+\cdots+n_1}$ sends U to the attractive Fatou component which contains the superattracting fixed point L^0 in $\mathbf{P}^k$. $\square$

1.4 Axiom A and the S_{k+2}-equivariant maps

1.4.1 Definitions and preliminaries

Let us define hyperbolicity of non-invertible maps and the notion of Axiom A. See [6] for details. Let f be a holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$ and K a compact subset such that $f(K) = K$. Let $\widehat{K}$ be the set of histories in K and $\widehat{f}$ the induced homeomorphism on $\widehat{K}$. We say that f is hyperbolic on K if there exists a continuous decomposition $T_{\widehat{K}} = E^u + E^s$ of the tangent bundle such that $D\widehat{f}(E^{u/s}_{\widehat{x}}) \subset E^{u/s}_{\widehat{f}(\widehat{x})}$ and if there exists constants $c > 0$ and $\lambda > 1$ such that for every $n \geq 1$,

$$|D\widehat{f}^n(v)| \geq c\lambda^n |v| \text{ for all } v \in E^u \text{ and}$$

$$|D\widehat{f}^n(v)| \leq c^{-1}\lambda^{-n} |v| \text{ for all } v \in E^s.$$

Here $|\cdot|$ denotes the Fubini-Study metric on $\mathbf{P}^k$. If a decomposition and inequalities above hold for f and K, then it also holds for $\widehat{f}$ and $\widehat{K}$. In particular we say that f is expanding on K if f is hyperbolic on K with unstable dimension k. Let Ω be the non-wandering set of f, i.e., the set of points for any neighborhood U of which there exists an integer n such that $f^n(U)$ intersects with U. By definition, Ω is compact and $f(\Omega) = \Omega$. We say that f satisfies Axiom A if f is hyperbolic on Ω and periodic points are dense in Ω.

Let us introduce a theorem which deals with repelling part of dynamics. Let f be a holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$. We define the k-th Julia set J_k of f to be the support of the measure with maximal entropy, in which repelling periodic points are dense. It is a fundamental fact that in dimension 1 the 1st Julia set J_1 coincides with the Julia set J. Let K be a compact subset such that $f(K) = K$. We say that K is a repeller if f is expanding on K.

Theorem 6. *([7]) Let f be a holomorphic map on $\mathbf{P}^k$ of degree at least 2 such that the ω-limit set $E(f)$ is pluripolar. Then any repeller for f is contained in J_k. In particular,*

$$J_k = \overline{\{repelling\ periodic\ points\ of\ f\}}$$

If f is critically finite, then $E(f)$ is pluripolar. We need the theorem above to prove our second result.

1.4.2 Our second result

Theorem 7. *For each $k \geq 1$, the S_{k+2}-equivariant map g satisfies Axiom A.*

Proof: We only need to consider the S_{k+2}-*equivariant* map g for a fixed k, because argument for any k is similar as the following one. Let us show the statement above for a fixed k by induction. A restricted map of g to any L^1 satisfies Axiom A by using the theorem of *critically finite* functions (see [8, Theorem 19.1]). We only need to show that a restricted map of g to a fixed L^2 satisfies Axiom A. Then a restricted map of g to any L^2 satisfies Axiom A by symmetry. Argument for a restricted map of g to any L^m, $3 \leq m \leq k$, is similar as for a restricted map of g to the L^2. Let us denote $g|_{L^2}$, $\Omega(g|_{L^2})$, and L^2 by g, Ω, and $\mathbf{P}^2$ for simplicity.

We want to show that $g|_{L^2}$ is hyperbolic on $\Omega(g|_{L^2})$ by using Kobayashi metrics. If g is hyperbolic on Ω, then Ω has a decomposition to S_i,

$$\Omega = S_0 \cup S_1 \cup S_2,$$

where i=0,1,2 indicate the unstable dimensions. Since $C(g)$ attracts all nearby points, S_0 includes all the L^0's and S_1 includes all the Julia sets of $g|_{L^1}$. We denote by $J(g|_{L^1})$ the Julia set of $g|_{L^1}$. Then g is contracting in all directions at

L^0 and is contracting in the normal direction and expanding in an L^1-direction on $J(g|_{L^1})$. Let us consider a compact, completely invariant subset in $\mathbf{P}^2 \setminus C$,

$$S = \{x \in \mathbf{P}^2 \mid \operatorname{dist}(g^n(x), C) \nrightarrow 0 \ \text{ as } \ n \to \infty\}.$$

By definition, we have $J_2 \subset S_2 \subset S$. If g is expanding on S, then it follow that $S_0 = \cup L^0$, $S_1 = \cup J(g|_{L^1})$. Moreover $J_2 = S_2 = S$ holds from Theorem 6 (see Remark 4 below). Since periodic points are dense in $J(g|_{L^1})$ and J_2, expansion of g on S implies Axiom A of g.

Let us show that g is expanding on S. Because f is attracting on C and preserves C, there exists a neighborhood V of C such that V is relatively compact in $g^{-1}(V)$ and the complement of V is connected. We assume one of L^1's to be the line at infinity of $\mathbf{P}^2$. By letting B be $\mathbf{P}^2 \setminus V$ and U one of connected components of $g^{-1}(\mathbf{P}^2 \setminus V)$, we have the following inclusion relations,

$$U \subset g^{-1}(B) \Subset B \subset \mathbf{C}^2 = \mathbf{P}^2 \setminus L^1.$$

Because B and U are in a local chart, there exists a constant $\rho < 1$ such that

$$K_B(x, v) \leq \rho K_U(x, v) \ \text{ for all } x \in U, \ v \in T_x\mathbf{C}^2.$$

In addition, since the map g from U to B is an unbranched covering,

$$K_U(x, v) = K_B(g(x), Dg(v)) \ \text{ for all } x \in U, \ v \in T_x\mathbf{C}^2.$$

From these two inequalities we have the following inequality

$$K_B(x, v) \leq \rho K_B(g(x), Dg(v)) \ \text{ for all } x \in g^{-1}(B), \ v \in T_x\mathbf{C}^2.$$

Since g preserves S, which is contained in $g^{-n}(B)$ for every $n \geq 1$,

$$K_B(x, v) \leq \rho^n K_B(g^n(x), Dg^n(v)) \ \text{ for all } x \in S, \ v \in T_x\mathbf{C}^2.$$

Consequently we have the following inequality for $\lambda = \rho^{-1} > 1$,

$$K_B(g^n(x), Dg^n(v)) \geq \lambda^n K_B(x, v) \ \text{ for all } x \in S, \ v \in T_x\mathbf{C}^2.$$

Since $K_B(x, v)$ is upper semicontinuous and $|v|$ is continuous, $K_B(x, v)$ and $|v|$ may be different only by a constant factor. There exists $c > 0$ such that

$$|Dg^n(x)v| \geq c\lambda^n |v| \ \text{ for all } x \in S, \ v \in T_x\mathbf{C}^2.$$

Thus g is expanding on S and satisfies Axiom A. $\square$

Remark 4. *Unlike the case when $k = 1$, it does not seem obvious that S being a repeller implies $J_k = S$ when $k \geq 2$.*

Remark 5. *From [1, Theorem 4.11] and [9], it follows that the Fatou set of the S_{k+2}-equivariant map g has full measure in $\mathbf{P}^k$ for each $k \geq 1$.*

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Chapter 2

Symmetries of the Julia sets of polynomial skew products on $\mathbf{C}^2$

A polynomial skew product on $\mathbf{C}^2$ is a regular polynomial map of degree at least two such that the first component depends only on the first coordinate. The Julia set of a polynomial skew product can have symmetries, that is, it can be invariant under some linear maps on $\mathbf{C}^2$. We investigate the structure of the group of symmetries and give a necessary and sufficient condition for the group of symmetries to be infinite. We show that, except for two types, polynomial skew products having the same Julia set are essentially the same. As a corollary, except for two types, the first Julia set is determined only by the second Julia set.

2.1 Introduction

The Julia sets of any kind of functions or maps can have symmetries. We say that a Julia set has symmetries if some non-elementary transformations preserve it. Beardon [2] investigated symmetries of the Julia sets of polynomials on $\mathbf{C}$. For a Julia set having symmetries, these symmetries are rotations about some point. The group of symmetries is infinite if and only if the Julia set is a circle. There was a problem: when do polynomials have the same Julia set? Beardon [2] gave an answer to this problem in terms of a functional equation in which symmetries of the Julia set are used. Finally, Schmidt and Steinmetz [5], and Atela and Hu [1] solved the problem independently: polynomials having the same Julia set are essentially the same.

We want to extend these dynamical objects and results in one dimension to those in higher dimensions. As a first step, we extend these dynamical objects and results of polynomials on $\mathbf{C}$ to those of polynomial skew products on $\mathbf{C}^2$. Although the dynamics of polynomial skew products is complicated in higher dimensions, it has many analogies to the dynamics of one-dimensional polynomials.

In section 2.2, we recall the dynamics of a polynomial skew product. In particular, its vertical Böttcher functions are important for the proofs. After providing some basic definitions and a proposition, we will deal with the symmetries of the Julia sets of polynomial skew products. In section 2.3, we show that linear maps which preserve a Julia set are conjugate to rotation-product maps, and give a necessary and sufficient condition for the group of symmetries to be infinite. In section 2.4, we deal with the generalized problem: when do polynomial skew products have the same Julia set? We show that, except for two types, polynomial skew products having the same Julia set are essentially the same. As a corollary, except for two types, the first Julia set is determined only by the second Julia set.

2.2 Dynamics of polynomial skew products

We recall the dynamics of polynomial skew products on $\mathbf{C}^2$, which was studied by Jonsson [4]. A polynomial skew product on $\mathbf{C}^2$ of degree $d \geq 2$ is a map of the form $f(z,w) = (p(z), q(z,w))$, where $p(z)$ and $q(z,w)$ are polynomials of degree d and where $p(z) = az^d + O(z^{d-1})$ and $q(z,w) = bw^d + O_z(w^{d-1})$. This definition is equivalent to that in the abstract. For polynomial skew products f and g, we denote the composition of them by fg , that is, $fg(z,w) = f(g(z,w))$. We also denote the n-th iterate of f by f^n. A polynomial skew product f preserves the set of vertical lines in $\mathbf{C}^2$. In this sense, we often use $q_z(w)$ instead of $q(z,w)$. The restriction of f^n to a line $\{z\} \times \mathbf{C}$ can be viewed as the composition of n polynomials on $\mathbf{C}$, $q_{z_{n-1}} \cdots q_{z_1} q_z(w)$, where $z_n = p^n(z)$. As we will see later, many dynamical objects and results for iterations of a polynomial on $\mathbf{C}$ have vertical counterparts in $\{z\} \times \mathbf{C}$ for a polynomial skew product.

Let $f(z,w) = (p(z), q(z,w))$ be a polynomial skew product on $\mathbf{C}^2$. The first component p defines a dynamics on the base space $\mathbf{C}$. A useful tool in the study of the dynamical of p is the Green function G_p of p, defined by

$$G_p(z) = \lim_{n\to\infty} d^{-n} \log^+ |p^n(z)|.$$

Let $K_p = \{G_p = 0\}$ and $J_p = \partial K_p$. In this paper we call J_p the base Julia set of f. We also have the Green function G_f of f on $\mathbf{C}^2$, defined by

$$G_f(z) = \lim_{n\to\infty} d^{-n} \log^+ |f^n(z,w)|,$$

where $|(z,w)| = \max\{|z|,|w|\}$ is a norm on $\mathbf{C}^2$. Define $G_z(w) = G_f(z,w) - G_p(z)$. Then G_z is a nonnegative, continuous, and subharmonic function on $\mathbf{C}$. Let $K_z = K_z(f) = \{G_z = 0\}$ and $J_z = J_z(f) = \partial K_z(f)$. Then K_z and J_z are compact. The function G_z coincides with the Green function for K_z with a pole at the infinity. For z in K_p, w belongs to K_z if and only if the orbit $\{q_{z_{n-1}} \cdots q_{z_1} q_z(w)\}_{n\geq 1}$ is bounded. In this paper we call J_z the vertical Julia set of f.

We define three completely invariant sets of the polynomial skew product f:

$$J_f = \bigcup_{z\in J_p} \{z\} \times J_z, \quad J_2(f) = \overline{\bigcup_{z\in J_p} \{z\} \times J_z},$$

$$\text{and} \;\; J_1(f) = \overline{\bigcup_{z\in \mathbf{C}} \{z\} \times J_z \cup \bigcup_{z\in J_p} \{z\} \times K_z},$$

where the closure is taken in the 2-dimensional projective space $\mathbf{P}^2$. Hence these sets have the inclusion relation $J_f \subset \overline{J_f} = J_2(f) \subset J_1(f)$. Heinemann [3] called J_f the pre-Julia set of f. In this paper we call J_f the Julia set of f. In general, J_f is not compact because J_z is not continuous in z with respect to the Hausdorff metric. By definition, f extends to a holomorphic map on $\mathbf{P}^2$. It is known that $J_1(f)$ coincides with the support of the Green current T of the extension of f to $\mathbf{P}^2$, and that $J_2(f)$ coincides with the support of the Green measure $T \wedge T$. In the study of the dynamics of holomorphic maps on projective spaces, these sets are called the first and the second Julia set of f.

For a polynomial P of degree d, the dynamics of P near infinity is conjugate to $z \to z^d$ by a Böttcher function. For a polynomial skew product, similar Böttcher functions exist on vertical lines in $\mathbf{C}^2$. The following proposition is a modified version of Jonsson's result.

Proposition 2 ([4]). *For any polynomial skew product f on $\mathbf{C}^2$, where $f(z,w) = (p(z), q(z,w))$ and $q(z,w) = bw^d + O_z(w^{d-1})$, and for a constant $c = b^{\frac{1}{d-1}}$ there exists $R > 0$ and, for every z in K_p, a unique conformal map φ_z of $\{w : G_z(w) > R\}$ onto $\{W : |W| > e^R\}$ such that*

(i) $\varphi_z(w) = c(w + c_z + o(1))$ *as* $w \to \infty$,

(ii) $\log|\varphi_z(w)| = G_z(w)$,

(iii) $\varphi_{p(z)}(q_z(w)) = (\varphi_z(w))^d$,

where a constant c_z depends on z.

We call φ_z the vertical Böttcher function of f at z. Now we are ready to investigate the structure of the groups of symmetries of the completely invariant sets for a polynomial skew product.

2.3 Symmetries of a Julia set

First, let us recall objects and results of symmetries of the Julia sets of polynomials on $\mathbf{C}$, which was investigated by Beardon [2]. Let $P(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $d \geq 2$ and J_P its Julia set. The group of symmetries of a polynomial P is defined by

$$\Sigma = \Sigma(P) = \{\, \sigma \in E : \ \sigma(J_f) = J_f \,\},$$

where E is the set of conformal Euclidean isometries, that is, $E = \{\, \sigma(z) = \mu z + c : |\mu| = 1 \,\}$. The centroid of P is defined by

$$\zeta = \frac{-a_{d-1}}{d a_d}.$$

If the solutions of $P(z) = Z$ are $z_1, z_2, \cdots, z_d$, then

$$P(z) = a_d(z - z_1)(z - z_2)\cdots(z - z_d) + Z$$

and so the center of gravity of the points z_j coincides with ζ. Each symmetry σ is a rotation about the centroid of P, that is, $\sigma(z) = \mu(z - \zeta) + \zeta$ for some μ in the unit circle S^1. We can normalize P by $z \to z - \zeta$ so that the centroid is at the origin. We say that a polynomial is in normal form if its centroid is at the origin. For a normalized polynomial P, the group $\Sigma(P)$ can be identified with a subgroup of S^1.

Let us generalize these dynamical objects and results to those of polynomial skew products. We restrict the symmetries of the Julia sets of polynomial skew products to affine maps whose first component depends only on the first coordinate. Hence the group Γ of symmetries of the Julia set of a polynomial skew product $f(z,w) = (p(z), q(z,w))$ is defined by

$$\Gamma = \Gamma(J_f) = \{ \gamma \in S : \gamma(J_f) = J_f \},$$

$$\text{where } S = \left\{ \gamma \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} c_1 z + c_2 \\ c_3 z + c_4 w + c_5 \end{pmatrix} : |c_1| = |c_4| = 1 \right\}.$$

Each element of S is a linear map which preserves the set of vertical lines in $\mathbf{C}^2$. In addition, it preserves the metrics on the base space and on vertical lines. In the same manner, we can define $\Gamma(J_2(f))$ and $\Gamma(J_1(f) \cap \mathbf{C}^2)$. Because the orbits of points in $J_2(f)$ are bounded, it follows that γ in S preserves J_f if and only if it preserves $J_2(f)$. We will see later that γ in S preserves J_f if and only if it preserves both $J_1(f) \cap \mathbf{C}^2$ and $J_p \times \mathbf{C}$.

Let $f(z,w) = (p(z), q(z,w))$ be a polynomial skew product such that

$$\begin{pmatrix} p(z) \\ q(z,w) \end{pmatrix} = \begin{pmatrix} a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0 \\ b_d w^d + b_{d-1}(z) w^{d-1} + \cdots + b_1(z) w + b_0(z) \end{pmatrix}.$$

Note that $b_{d-l}(z)$ is a polynomial of degree at most l in z. As in the one-dimensional case, we define the centroid of q_z by

$$\zeta_z = \frac{-b_{d-1}(z)}{d b_d}.$$

If the solutions of $q_z(w) = W$ are $w_1, w_2, \cdots, w_d$, then the center of gravity of the points w_j coincides with ζ_z. We can normalize f by the conjugation map $(z,w) \to (z - \zeta, w - \zeta_z)$ so that all centroids ζ and ζ_z are at the origin. Before normalizing the polynomial skew product, we express Γ using the centroids ζ and ζ_z.

Proposition 3. *For a polynomial skew product* $f(z,w) = (p(z), q(z,w))$, *any linear map* γ *in* Γ *can be written as*

$$\gamma\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \mu(z-\zeta)+\zeta \\ \nu(w-\zeta_z)+\zeta_{\sigma(z)} \end{pmatrix},$$

for some μ, ν *in* S^1, *where* $\sigma(z) = \mu(z-\zeta)+\zeta$ *belongs to* $\Sigma(p)$.

proof. Let us denote $\gamma(z,w)$ in Γ by $(\sigma(z), \gamma_z(w))$. From the one-dimensional result, it follows that $\sigma(z) = \mu(z-\zeta)+\zeta$ for some μ in S^1. Instead of using the one-dimensional argument in [2], one can prove the above by a similar argument as below.

Let us show that there exist ν in S^1 such that $\gamma_z(w) = \nu(w-\zeta_z)+\zeta_{\sigma(z)}$ holds for any z in J_p. Note that the Böttcher function φ_z has a relationship with the centroid ζ_z of q_z. By combining (i) and (iii) in proposition 2, we have the following equation

$$c(b_d w^d + b_{d-1}(z)w^{d-1} + \cdots) = c^d(w^d + dc_z w^{d-1} + \cdots).$$

By comparing the second terms, c_z coincides with $-\zeta_z$ and so $\varphi_z(w) = c(w-\zeta_z+o(w))$. Next, let us show that $\gamma(J_f) = J_f$ induces the equation $\gamma_z(w) = \nu(w-\zeta_z)+\zeta_{\sigma(z)}$ for any z in J_p. Since $J_{\sigma(z)}$ coincides with $\gamma_z(J_z)$, $K_{\sigma(z)}$ coincides with $\gamma_z(K_z)$. Thus G_z and $G_{\sigma(z)}\gamma_z$ are the Green functions for K_z for any z in J_p. From the uniqueness property of Green functions, G_z coincides with $G_{\sigma(z)}\gamma_z$. Thus there exists ν in S^1 such that $\nu\varphi_z(w) = \varphi_{\sigma(z)}\gamma_z(w)$. Comparing the regular terms on this equation, it follows that $\gamma_z(w) = \nu(w-\zeta_z)+\zeta_{\sigma(z)}$ for any z in J_p. By the uniqueness theorem of holomorphic functions on horizontal lines, the equation above holds on $\mathbf{C}^2$. □

Therefore we can identify $\Gamma = \{\gamma_{\mu,\nu}(z,w) = (\mu z, \nu w) : \gamma_{\mu,\nu}(J_f) = J_f\}$ with $\{(\mu,\nu) \in S^1 \times S^1 : \gamma_{\mu,\nu} \in \Gamma\}$ for a normalized polynomial skew product f. The following lemma helps us to investigate the structure of Γ. The proof is similar to the proof in the one dimensional case, see [2]. In the w-direction, we use vertical Böttcher functions instead of a Böttcher function. Such an argument was used in the proof of proposition 3.

Lemma 2. *Let* $f(z,w) = (p(z), q(z,w))$ *be a polynomial skew product of degree* d. *Then, for* γ *in* S, γ *belongs to* Γ *if and only if* $f\gamma = \gamma^d f$ *holds on* $\mathbf{C}^2$.

Corollary 1. *Let $f(z,w) = (p(z), q(z,w))$ be a polynomial skew product of degree d. Then,* $\Gamma(J_f) = \Gamma(J_2(f)) = \Gamma(J_1(f) \cap \mathbf{C}^2) \cap \Gamma(J_p \times \mathbf{C})$.

proof. We have already shown that $\Gamma(J_f) = \Gamma(J_2(f))$. We only have to show that $\Gamma(J_f) \subset \Gamma(J_1(f) \cap \mathbf{C}^2) \cap \Gamma(J_p \times \mathbf{C})$. Let γ be an element of S that preserves J_f. Then $f\gamma = \gamma^d f$ holds on $\mathbf{C}^2$. For z in K_p, J_z coincides with the boundary of the set of points whose second coordinates are bounded under the iterations. For z in $\mathbf{C} \setminus K_p$, J_z coincides with the boundary of the set of points whose ratio of the second coordinates to the first are bounded under the iterations. Hence the equation $f\gamma = \gamma^d f$ implies that γ preserves $J_1(f) \cap \mathbf{C}^2$. □

Let us give three examples of symmetries of the Julia sets of polynomial skew products. All of these are in normal form.

Example 1 (polynomial product). *Let $f(z,w) = (p(z), q(w)) = (z^3 + c, w^3 + dw)$ be a polynomial product with $c, d \neq 0$. Then it follows that $\Gamma = \Sigma(p) \times \Sigma(q) = \{(\mu, \nu) : \mu^3 = \nu^2 = 1\}$.*

Example 2 (polynomial skew product with finite group of symmetries). *Let $f(z,w) = (z^3, w^3 + czw + dz^3)$, c, $d \neq 0$. Then lemma 2 implies that $\Gamma = \{(\mu, \nu) : \mu^3 = \mu\nu = 1\} = \{(1,1), (\rho, \rho^2), (\rho^2, \rho)$ for $\rho^3 = 1\}$.*

Example 3 (polynomial skew product with infinite group of symmetries). *Let $f(z,w) = (z^2, w^2 + cz)$, $c \neq 0$. Then lemma 2 implies that $\Gamma = \{(\mu, \nu) : \mu = \nu^2 \in S^1\}$. It will be proved that f is semi-conjugate to $(z,w) \to (z^2, w^2 + c)$ by $\pi(z,w) = (z^2, zw)$ in proposition 4.*

Now, let us consider when the group of symmetries is infinite. A polynomial skew product is conjugate to a map that is in normal form. In addition, it is conjugate to a map for which the leading terms of p and q_z are 1. Hence we may assume that the polynomial skew product is in normal form and that the leading terms of p and q_z are both 1 without loss of generality.

Theorem 8. *Let $f(z,w) = (p(z), q(z,w))$ be a normalized polynomial skew product of degree d with leading terms of $p(z)$ and $q_z(w)$ being 1. Then Γ is infinite if and only if one of the following holds:*

(i) J_f *is a product of the unit circle and a Julia set* J*,*

(ii) J_f *is a product of a Julia set* J_p *and the unit circle,*

(iii) for some integers n, m, and a Julia set J on $\mathbf{C}$*,*

$$J_f = \bigcup_{z \in S^1} \{z\} \times z^{\frac{n}{m}} J.$$

Moreover, Γ *is infinite if and only if one of the following holds:*

(i) f *is a polynomial product and* $p(z) = z^d$*,*

(ii) f *is a polynomial product and* $q(w) = w^d$*,*

(iii) f *is semi-conjugate to a polynomial product by* $\pi(z, w) = (z^n, z^m w)$ *for some integers n and m, and* $p(z) = z^d$*.*

proof. Each condition implies that Γ is infinite. We prove the converse. Let Γ be infinite. We identify $\Gamma = \{\gamma_{\mu,\nu}(z, w) = (\mu z, \nu w) : \gamma_{\mu,\nu}(J_f) = J_f\}$ with $\{(\mu, \nu) \in S^1 \times S^1 : \gamma_{\mu,\nu} \in \Gamma\}$.

If Γ has only finitely many indifferent μ's, then it must have infinitely many indifferent ν's. Since Γ is compact, each vertical Julia set J_z is a circle. By using vertical Böttcher functions, it follows that $q_z(w) = c_z w^d$ for some $c_z \neq 0$. By assumption, c_z is equal to 1. Hence f is a product, $q(w) = w^d$, and J_q is the unit circle.

Assume that Γ has infinitely many indifferent μ. Since Γ is compact, J_p is a circle, which is equivalent to p being conjugate to $z \to z^d$. By assumption, $p(z) = z^d$ and J_p is the unit circle. Finally, proposition 4 completes the proof. □

Proposition 4. *Let* $f(z, w) = (z^d, q(z, w))$ *be a polynomial skew product of degree d. Then the following are equivalent:*

(i) there exist integers n and m such that

$$q(z^n, z^m w) = z^{md} q(1, w),$$

(ii) f *is semi-conjugate to a polynomial product given by* $\pi(z, w) = (z^n, z^m w)$ *for some integers n and m,*

(iii) Γ *is infinite,*

(iv) $f\tau = \tau^d f$ *holds for some* $\tau(z,w) = (\epsilon z, \delta w)$ *with* $|\epsilon| \neq 1$*,*

(v) there exists integers n, m and a Julia set J on $\mathbf{C}$ *such that*

$$J_f = \bigcup_{z\in S^1} \{z\} \times z^{\frac{m}{n}} J.$$

proof. Polynomial products satisfy all of these conditions. So we assume that f is not a polynomial product.

$(i) \Rightarrow (ii), (iv)$. The condition (i) implies the following commutative diagram for $\pi(z,w) = (z^n, z^m w)$:

$$\begin{array}{ccc} \mathbf{C}^2 & \xrightarrow{(z^d,\, q(1,w))} & \mathbf{C}^2 \\ \pi \big\downarrow & & \big\downarrow \pi \\ \mathbf{C}^2 & \xrightarrow{(z^d,\, q(z,w))} & \mathbf{C}^2. \end{array}$$

$(ii) \Rightarrow (iii), (iv)$. Let $f_0(z,w) = (p_0(z), q_0(w))$ be a polynomial product such that $\pi f_0 = f\pi$. Since f_0 is product and J_{p_0} is the unit circle, $\gamma_0(z,w) = (\mu z, w)$ belongs to $\Gamma(f_0)$ for any μ in S^1. A rotation-product map γ_0 projects to $\gamma(z,w) = (\mu^n z, \mu^m w)$ by the semi-conjugacy π. The equation $f_0\gamma_0 = {\gamma_0}^d f_0$ implies $f\gamma = \gamma^d f$. By lemma 2, γ belongs to $\Gamma(f_0)$. Similarly (iii) implies (iv) because $\tau(z,w) = (\mu z, w)$ satisfies $f_0\tau_0 = {\tau_0}^d f_0$ for any μ in $\mathbf{C}$.

$(iii) \Rightarrow (i)$. We identify Γ with $\{(\mu,\nu) \in S^1 \times S^1 : \gamma_{\mu,\nu}(J_f) = (J_f)\}$. Note that Γ has infinitely many indifferent μ's. Otherwise Γ has infinitely many indifferent ν's. Thus f is a product from the argument above, which contradicts the assumption. Since Γ is compact, it has all μ's in S^1. Fix μ in S^1 such that $\mu^n \neq 1$ for any integer $n \neq 0$. Lemma 2 implies that $q(\mu z, \nu w) = \nu^l q(z,w)$. Therefore, if q contains the term $z^{m_i} w^{l_i}$ with a non-zero coefficient, then μ and ν are related by $\nu^l = \mu^{m_i}\nu^{l_i}$. The relations $\mu^{m_i}\nu^{d-l_i} = 1$ and $\mu^{m_j}\nu^{d-l_j} = 1$ imply $\mu^{m_i(d-l_j)-m_j(d-l_i)} = 1$. By the property of μ, $m_i(d-l_j) - m_j(d-l_i)$ must be 0. Hence the ratios of m_i and $d - l_i$ are independent of i, and so

$$\frac{m_i}{d-l_i} = \frac{m_j}{d-l_j} =: \frac{m}{d-l}.$$

The integers $n = d - l$ and m satisfy (i). Similarly (ii) implies (iv) because $\epsilon^n \neq 1$ for any integer $n \neq 0$.

$(i) \Rightarrow (v) \Rightarrow (iii)$. Let J be the Julia set of a polynomial $q(1, w)$ on $\mathbf{C}$. Then (i) implies that $J_z = z^{\frac{m}{n}} J$ for z in S^1. On the other hand, (v) implies that the linear maps $(z, w) \to (\mu^n z, \mu^m w)$ preserve J_f for any μ in S^1. Thus Γ is infinite. □

2.4 Polynomial skew products with same Julia set

In this section we consider when polynomial skew products have the same Julia set. We give partial answers to this question, which come from one-dimensional arguments.

Remark 6. *Polynomial skew products have the same Julia set if and only if they have the same second Julia set, because the Julia set or the second Julia set is determined by each other respectively.*

Let us recall Beardon's answer to the problem: when polynomials have the same Julia set? We assume that the degrees of the polynomials are at least two.

Theorem 9 ([2]). *Let P and Q be polynomials. Then $J_P = J_Q$ if and only if $PQ = \sigma QP$ holds for some σ in Σ.*

We can generalize the theorem above to polynomial skew products. The proof is similar to that of the one-dimensional case. In the w-direction, we use vertical Böttcher functions instead of a Böttcher function. Such an argument will appear in the proof of theorem 12 below.

Theorem 10. *Let f and g be polynomial skew products. Then $J_f = J_g$ if and only if $fg = \gamma g f$ holds for some γ in Γ.*

Let us recall the answer of Schmidt and Steinmetz [5], and Atela and Hu [1] to the problem above, which will be used to prove our theorem below.

Theorem 11 ([1] , [5]). *For any Julia set J of a polynomial which is not a circle or a straight line segment, there exists a polynomial R such that any polynomial with the Julia set J can be written in the form σR^k for some integer k and σ in Σ.*

A polynomial P is conjugate to $z \to z^d$ if and only if J_P is a circle. A polynomial P is conjugate to a Chebyshev polynomial if and only if J_P is a straight line segment. By combining these results, it follows that polynomials having the same Julia set are essentially the same.

We generalize the theorem above to that of polynomial skew products. The proof is similar to the proof in the one-dimensional case.

Theorem 12. *Let f and g be polynomial skew products. If J_f coincides with J_g and if its base Julia set is not a circle or a straight line segment, then $f^n = \gamma g^m$ holds for some integers n, m and γ in Γ*

proof. We may assume that $f(z,w) = (p(z), q(z,w))$ and $g(z,w) = (r(z), s(z,w))$ are in normal form. First, we show that if $\deg f = \deg g$, then $f = \gamma g$ holds for some γ in Γ. From theorem 11, it follows that $p = \sigma r$ holds for some $\sigma(z) = \mu z$ in Σ. Let us denote the vertical Böttcher function of f at z by φ_z^f. Since $K_z(f)$ coincides with $K_z(g)$, the Green function of f for $K_z(f)$ coincides with that of g. Thus $\varphi_z^f = s\varphi_z^g$ holds for some s in S^1. Proposition 2 implies that

$$\varphi_{p(z)}^f(q_z(w)) = (\varphi_z^f(w))^d \text{ and } \varphi_{r(z)}^g(s_z(w)) = (\varphi_z^g(w))^d.$$

Thus it follows that $\varphi_{p(z)} q_z = s^d \varphi_{r(z)} s_z$, that is,

$$c(q_z(w) + o(1)) = s^d e(s_z(w) + o(1)),$$

where the constants c and e are determined by the leading terms of q_z and s_z. Since $s = \frac{c}{e}$, it follows that $\frac{e}{c} s^d = s^{d-1}$. Hence $q_z = \nu s_z$ holds for any z in J_p, where $\nu = s^{d-1}$ belongs to S^1. By the uniqueness theorem of holomorphic functions on horizontal lines, $q(z,w) = \nu s(z,w)$ holds on $\mathbf{C}^2$. Hence $g = \gamma f$ holds for $\gamma(z,w) = (\mu z, bw)$ in Γ.

Next, from theorem 11, there exists a polynomial R such that $p = \sigma_1 R^m$ and $r = \sigma_2 R^n$ holds for some integers m, n and σ_1, σ_2 in Σ. Hence $\deg f^n = \deg p^n = \deg R^{nm}$ and $\deg g^m = \deg r^m = \deg R^{nm}$. The argument above then completes the proof. $\square$

Corollary 2. *For polynomial skew products whose base Julia sets are not a circle or a straight line segment, the first Julia set J_1 is uniquely determined by the second Julia set J_2.*

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