2012年度
「リーマン面・不連続群論」研究集会

大阪大学

2013年 1月 12日（土）– 1月 14日（月）

プログラム＋アブストラクト＋講演スライド

RIEMANN SURFACES AND DISCONTINUOUS GROUPS 2013

Osaka Univeristy, January 12-14, 2013
Organaizers: Hiroshige Shiga (Tokyo Institute of Technology)
Tamas Kalman (Tokyo Institute of Technology GE)
Kentaro Ito (Nagoya University)
Hideki Miyachi (Osaka University)

This is an annual conference on topics of Riemann surfaces and discontinuous groups, including geometric function theory, potential theory, Teichmuller theory and hyperbolic geometry. This conference is partially supported by the Global Edge Institute at Tokyo Tech and the "Program to Promote the Tenure Track System" (テニュアトラック普及・定着事業) of the Ministry of Education, Culture, Sports, Science & Technology (文部科学省), and Grant-in-Aid for Scientific Research (A) 22244005.

Venue : Graduate school of Science, Building E, Room E301 (Third floor)
Osaka University, Machikaneyama 1-1, Toyonaka, Osaka

Invited speakers

Ege Fujikawa (Chiba University)
Yuki Iguchi (Tokyo Institute of Technology)
Yuichi Kabaya (Osaka University)
David Kalaj (University of Montenegro)
Eiko Kin (Osaka University)
Erina Kinjo (Tokyo Institute of Technology)
Hidetoshi Masai (Tokyo Institute of Technology)
Katsuhiko Matsuzaki (Waseda University)
Toshihiro Nogi (Osaka City University)
Kasra Rafi (University of Toronto)
Ken-ichi Sakan (Osaka City Univeristy)
Masaharu Tanabe (Tokyo Institute of Technology)
Masahiro Yanagishita (Waseda University)

Program

January 12 (Saturday).

13:30–14:20 Ege Fujikawa (Chiba Univeristy)
The order of periodic elements of the asymptotic Teichmuller modular group

14:30–15:20 Kasra Rafi (University of Toronto)
Geometry of Teichmüller space (part I)

15:40–16:30 Toshihiro Nogi (Osaka City University)
On extendibility of a map induced by Bers isomorphism

16:40–17:30 Eiko Kin (Osaka University)
Minimal dilatations of pseudo-Anosovs generated by the magic 3-manifold and their asymptotic behavior

January 13 (Sunday).

10:00–10:50 Yuichi Kabaya (Osaka University)
Parametrization of $\mathrm{PGL}(n, \mathbb{C})$-representations of surface groups

11:00–11:50 Hidetoshi Masai (Tokyo Institute of Technology)
On commensurability of fibrations on a hyperbolic 3-manifold

Lunch Break

13:30–14:20 Ken-ichi Sakan (Osaka City University)
Quasiconformal and Lipschitz harmonic mappings of the unit disk onto bounded convex domains

14:30–15:20 David Kalaj (University of Montenegro)
Energy-minimal diffeomorphisms between doubly connected Riemann surfaces

15:40–16:30 Erina Kinjo (Tokyo Institute of Technology)
On the length spectrum metric in infinite-dimensional Teichmuller spaces

16:40–17:30 Kasra Rafi (University of Toronto)
Geometry of Teichmüller space (part II)

January 14 (Monday).

10:00–10:50 Yuki Iguchi (Tokyo Institute of Technology)
On accumulation points of geodesics in Thurston's boundary of Teichmüller spaces

11:00–11:50 Masaharu Tanabe (Tokyo Institute of Technology)
On the combinatorial Hodge star operator and holomorphic cochains

Lunch Break

13:30–14:20 Masahiro Yanagishita (Waseda University)
Teichmüller distance and Kobayashi distance on subspaces of the universal Teichmüller space

14:30–15:20 Katsuhiko Matsuzaki (Waseda University)
Circle diffeomorphisms and Teichmüller spaces

Abstruct

Ege Fujikawa.

Title. The order of periodic elements of the asymptotic Teichmuller modular group

Abstruct. We give a sufficient condition for an asymptotic Teichmuller modular transformation to be of finite order. Furthermore, we estimate the order by using hyperbolic geometry.

Yuki Iguchi.

Title. On accumulation points of geodesics in Thurston's boundary of Teichmüller spaces

Abstract. In this talk, we detect accumulation points of arbitrary Teichmüller geodesic rays in Thurston's compactification of a Teichmüller space. We showed that accumulation points of a ray are written as a sum of measured foliations supported on the partitions of minimal decomposition of the vertical foliation associated with the ray. We also showed that there exists a boundary point to which no ray accumulates.

Yuichi Kabaya.

Title. Parametrization of $\mathrm{PGL}(n, \mathbb{C})$-representations of surface groups

Abstruct. Let S be a surface of genus g with n boundary components. The interior of S is ideally triangulated into $2(2g - 2 + n)$ ideal triangles. Fock and Goncharov gave a parametrization of (framed) $\mathrm{PGL}(n, \mathbb{C})$-representations of the fundamental group of S by $(n-1)(n-2)/2$ parameters for each ideal triangle and $(n-1)$ parameters for each edge of the triangulation. In this talk, we will give a parametrization of $\mathrm{PGL}(n, \mathbb{C})$-representations as an analogue of the Fenchel-Nielsen coordinates using the Fock-Goncharov coordinates. This is joint work with Xin Nie.

David Kalaj.

Title. Energy-minimal diffeomorphisms between doubly connected Riemann surfaces

Abstruct. Let $N = (\Omega, \sigma)$ and $M = (\Omega^*, \rho)$ be doubly connected Riemann surfaces and assume that ρ is a smooth metric with bounded Gauss curvature $\mathcal{K}$ and finite area. We establishes the existence of homeomorphisms between Ω and Ω^* that minimize the Dirichlet energy. *Among all homeomorphisms $f : \Omega \twoheadrightarrow \Omega^*$ between doubly connected domains such that* $\mathrm{Mod}\Omega \leq \mathrm{Mod}\Omega^*$ *there exists, unique up to conformal authomorphisms of Ω, an energy-minimal diffeomorphism which is a harmonic diffeomorphism. The results improve and extend some recent results of Iwaniec, Koh, Kovalev and Onninen (Inven. Math. (2011)). Further, the case of radial metrics is discussed in details.*

Eiko Kin.

Title. Minimal dilatations of pseudo-Anosovs generated by the magic 3-manifold and their asymptotic behavior

Abstract. Let $\delta_g > 1$ be the minimal dilatation of pseudo-Anosovs defined on a closed surface of genus g. Penner proved that $\log \delta_g$ behaves like $\frac{1}{g}$. We are interested in McMullen's question: Does $\lim\limits_{g \to \infty} g \log \delta_g$ exist? What is its value? We examine his question in the large set $\widehat{\mathcal{M}}$ of pseudo-Anosovs on closed surfaces "generated by" the magic manifold N which is homeomorphic to the 3-chain link exterior. Let $\widehat{\delta}_g$ be the minimum among dilatations of elements in $\widehat{\mathcal{M}}$ defined on a closed surface of genus g. We prove that $\lim\limits_{g \to \infty} g \log \widehat{\delta}_g = \log(\frac{3+\sqrt{5}}{2})$. Moreover for large g, $\widehat{\delta}_g$ is achieved by the monodromy of some $\varSigma_g$-bundle over the circle obtained from either $N(\frac{3}{-2})$ ($\simeq (-2,3,8)$-pretzel link exterior) or $N(\frac{1}{-2})$ ($\simeq 6_2^2$ link exterior) by Dehn filling both cusps, where $N(r)$ is the manifold obtained from N by Dehn filling one cusp along the slope $r \in \mathbb{Q}$. This is a joint work with Sadayoshi Kojima and Mitsuhiko Takasawa.

Erina Kinjo.

Title. On the length spectrum metric in infinite-dimensional Teichmuller spaces

Abstruct. We consider Teichmuller metric and the length spectrum metric in Teichmuller spaces. It is known that these metrics define the same topology in finite-dimensional Teichmuller spaces. In this talk, we study infinite-dimensional Teichmuller spaces where they define the same topology.

Hidetoshi Masai.

Title. On commensurability of fibrations on a hyperbolic 3-manifold

Abstruct. We discuss the fibered commensurability of fibrations on a hyperbolic 3-manifold. The notion of fibered commensurability is defined by Calegari, Sun and Wang (2010). Calegari, Sun and Wang asked in their paper if there is a manifold with a pair of commensurable fibrations whose fiberes are of different topology. In this talk we will construct an infinite sequence of manifolds with such pairs of fibrations. We further show that two fibrations of each pair belong to the same fibered face.

Katsuhiko Matsuzaki.

Title. Circle diffeomorphisms and Teichmüller spaces

Abstruct. By characterizing a diffeomorphism of the circle with Hölder continuous derivative in terms of the quasiconformal Teichmüller theory, we show certain rigidity of groups of circle diffeomorphisms.

Toshihiro Nogi.

Title. On extendibility of a map induced by Bers isomorphism

Abstruct. Let $T(S)$ be the Teichmüller space of a closed Riemann surface S of genus $g(> 1)$. Denote by U the universal covering surface of S, that is, the upper half-plane and denote by $\dot{S}$ the surface obtained by removing a point from S. By the Bers isomorphism theorem, we have a homeomorphism of $T(S) \times U$ onto $T(\dot{S})$. The Bers embedding shows that the spaces $T(S) \times U$ and $T(\dot{S})$ are embedded in $(3g-2)$-dimensional complex vector space. Thus the boundaries of both spaces are naturally defined.

Let A be a subset of the boundary ∂U of U consisting of all points filling S. In this talk, we show that the homeomorphism of $T(S) \times U$ onto $T(\dot{S})$ has a continuous extension to $T(S) \times (U \cup A)$. This is a joint work with Hideki Miyachi (Osaka University).

Kasra Rafi.

Title. Geometry of Teichmüller space (part I and II).

Abstruct. We review recent results about the Teichmüller space equipped with the Teichmüller metric. We give an inductive description of a Teichmüller geodesic using the Teichmüller geodesics of surfaces with lower complexity. We use these results to compare how the Teichmüller space is similar or different from the hyperbolic space.

Ken-ichi Sakan.

Title. Quasiconformal and Lipschitz harmonic mappings of the unit disk onto bounded convex domains

Abstract. For a sense-preserving univalent harmonic self-mapping F of the unit disk, Pavlović showed that F is quasiconformal iff F is bi-Lipschitz. He gave another characterization,too, for the quasiconformality of F by means of some properties of the boundary-valued mapping of F. If the target of F is a bounded convex domain, then this result does not hold in general as it stands. If the Lipschitz property of F is pre-assumed, however, then we could obtain a variant of the result by Pavlović. In other words, in this talk we show some characterizations of a quasiconformal and Lipschitz harmonic mapping of the unit disk onto a bounded convex domain.This is a joint work with Dariusz Partyka.

Masaharu Tanabe.

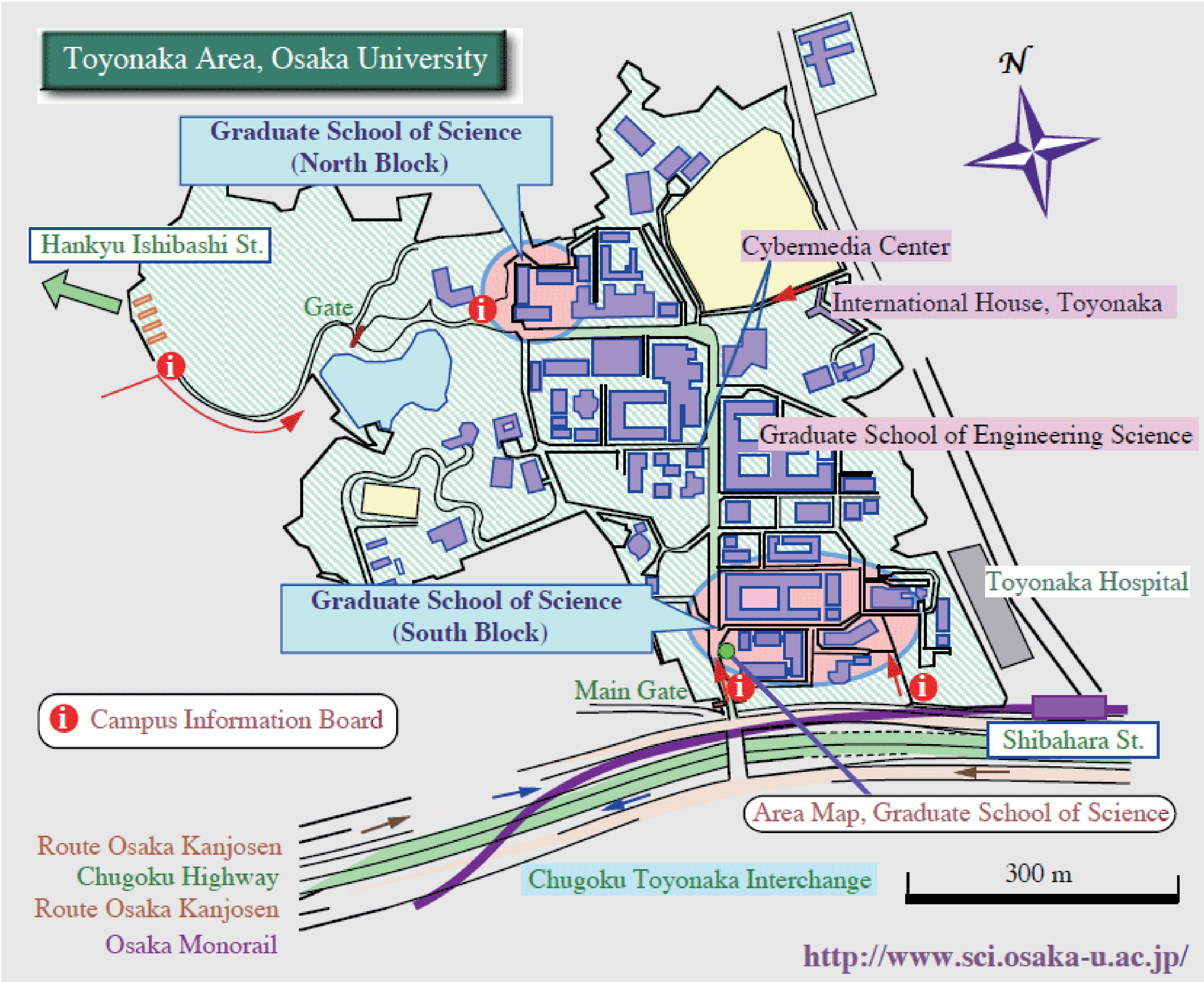
Title. On the combinatorial Hodge star operator and holomorphic cochains

Abstruct. For cochains equipped with an inner product of a triangulated manifold, S.O. Wilson defined the combinatorial Hodge star operator $\bigstar$ in his paper of 2007 and showed that for a certain cochain inner product which he named the Whitney inner product, this operator converges to the smooth Hodge star operator if the manifold is Riemannian. He also stated that $\bigstar\bigstar \neq \pm\mathrm{Id}$ in general and raised a question if $\bigstar\bigstar$ approaches $\pm\mathrm{Id}$ as the mesh of the triangulation tends to zero. In this talk, we solve this problem affirmatively.

Masahiro Yanagishita.

Title. Teichmüller distance and Kobayashi distance on subspaces of the universal Teichmüller space

Abstruct. It is known that the Teichmüller distance on the universal Teichmüller space T coincides with the Kobayashi distance. For a metric subspace of T having a comparable complex structure with that of T, we can similarly consider whether or not the Teichmüller distance on the subspace coincides with the Kobayashi distance. In this talk, we give a sufficient condition for metric subspaces under which the problem above has a affimative answer. Moreover, we introduce an example of such subspaces.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Graduate School of Science (South Block)
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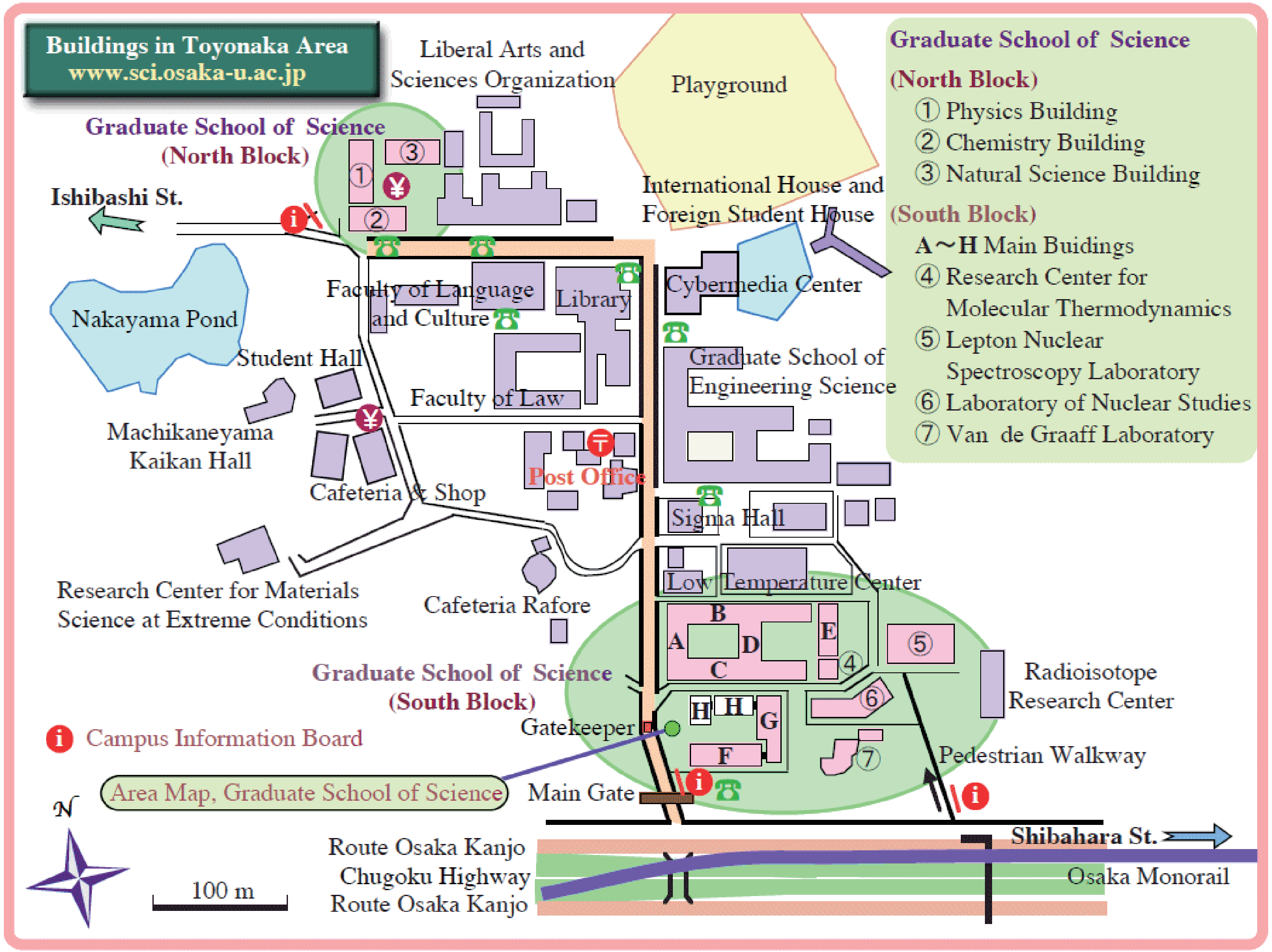

Buildings in Toyonaka Area
www.sci.osaka-u.ac.jp
Liberal Arts and Sciences Organization
Playground
Graduate School of Science
(North Block)
① Physics Building
② Chemistry Building
③ Natural Science Building
(South Block)
A～H Main Buidings
④ Research Center for Molecular Thermodynamics
⑤ Lepton Nuclear Spectroscopy Laboratory
⑥ Laboratory of Nuclear Studies
⑦ Van de Graaff Laboratory
Graduate School of Science (North Block)
Ishibashi St.
International House and Foreign Student House
Nakayama Pond
Faculty of Language and Culture
Library
Cybermedia Center
Student Hall
Graduate School of Engineering Science
Faculty of Law
Machikaneyama Kaikan Hall
Post Office
Cafeteria & Shop
Sigma Hall
Research Center for Materials Science at Extreme Conditions
Cafeteria Rafore
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Pedestrian Walkway
Area Map, Graduate School of Science
Main Gate
Shibahara St.
Route Osaka Kanjo
Chugoku Highway
Route Osaka Kanjo
Osaka Monorail
100 m

On commensurability of fibrations on a hyperbolic 3-manifold

Hidetoshi Masai

Tokyo institute of technology, DC2

13th, January, 2013

Contents

Notations

- surface = compact orientable surface of negative Euler characteristic possibly with boundary.
- hyperbolic manifold = orientable manifold whose interior admits complete hyperbolic metric of finite volume.
- F : surface
- $\phi : F \to F$, automorphism (isotopy class of self-homeomorphisms) which may permute components of ∂F.
- (F, ϕ): pair of surface F and automorphism ϕ.

Fibered Manifolds

Definition

- $[F, \phi] = F \times [0, 1]/((\phi(x), 0) \sim (x, 1))$ is called the mapping torus associated to (F, ϕ).
- A 3-manifold M is called fibered if we can find (F, ϕ) s.t. $[F, \phi] \cong M$.

Mapping tori and classification of automorphisms

- ϕ is periodic $\iff$ $[F, \phi]$ is a Seifert fibered space.
- ϕ is reducible $\iff$ $[F, \phi]$ is a toroidal manifold.
- ϕ is pseudo Anosov $\iff$ $[F, \phi]$ is a hyperbolic manifold.

Thurston norm

- M : fibered hyperbolic 3-manifold.
- $F = F_1 \sqcup F_2 \sqcup \cdots F_n$: (possibly disconnected) compact surface .
- $\chi_-(F) = \sum |\chi(F_i)|$
  (F_i : components with negative Euler characteristic).

Definition (Thursotn)

$\omega \in H^1(M;\mathbb{Z}) \subset H^1(M;\mathbb{R})$.
We define $\|\omega\|$ to be

$$\min\{\chi_-(F) \mid (F, \partial F) \subset (M, \partial M) \text{ embedded, and } [F] \in H_2(M, \partial M; \mathbb{Z}) \text{ is the Poincare dual of } \omega. \}$$

Definition

F is called a minimal representative of $\omega \iff F$ realize the minimum $\chi_-(F)$.

We can extend this norm to $H^1(M;\mathbb{Q})$ by $\|\omega\| = \|r\omega\|/r$.

Theorem (Thursotn)

- $\|\cdot\|$ *extends continuously to* $H^1(M;\mathbb{R})$*,*
- $\|\cdot\|$ *turns out to be semi-norm on* $H^1(M;\mathbb{R})$*, and*
- *The unit ball* $U = \{\omega \in H^1(M;\mathbb{R}) \mid \|\omega\| \leq 1\}$ *is a compact convex polygon*

Definition

$\|\cdot\|$ is called the Thurston norm on $H^1(M;\mathbb{R})$.

Fibered cone

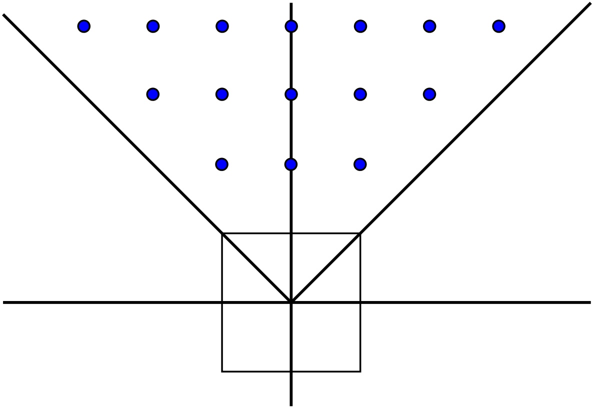

Figure: $H^1(M, \mathbb{R})$

Fibered cone

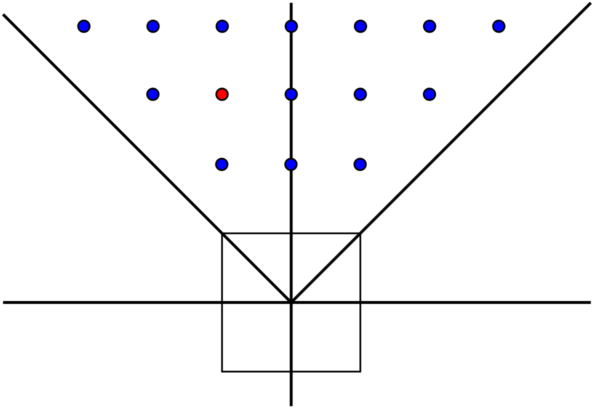

Figure: $H^1(M, \mathbb{R})$

Fibered cone

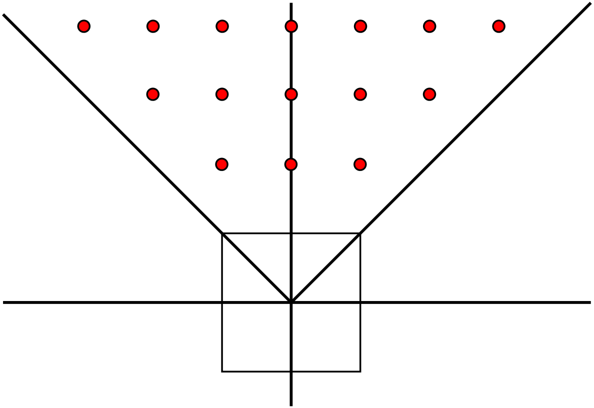

Figure: $H^1(M, \mathbb{R})$

Question.

What is "a relationship" among fibrations on a hyperbolic manifold (or, on the same fibered cone)?

Example.

(Fried) Mapping tori of (un)stable laminations with respect to the pseudo Anosov monodromies on the same fibered cone are isotopic.

Commensurability of Automorphisms

Definition (Calegari-Sun-Wang (2011))

A pair $(\widetilde{F}, \widetilde{\phi})$ covers (F, ϕ) if there is a finite cover $\pi : \widetilde{F} \to F$ and representative homeomorphisms $\tilde{f}$ of $\widetilde{\phi}$ and f of ϕ so that $\pi\widetilde{f} = f\pi$ as maps $\widetilde{F} \to F$.

Definition (CSW)

Two pairs (F_1, ϕ_1) and (F_2, ϕ_2) are said to be commensurable if $\exists$ $(\widetilde{F}, \widetilde{\phi}_i)$, $k_i \in \mathbb{Z} \setminus \{0\}$ $(i = 1, 2)$ such that $\widetilde{\phi_1}^{k_1} = \widetilde{\phi_2}^{k_2}$.

This commensurability generates an equivalence relation.

Commensurability

$$\begin{tikzcd}
(\widetilde{F}_1, \widetilde{\phi_1}^{k_1}) \arrow[d] \arrow[r, phantom, "="] & (\widetilde{F}_2, \widetilde{\phi_2}^{k_2}) \arrow[d] \\
(F_1, \phi_1) & (F_2, \phi_2)
\end{tikzcd}$$

Remark. The above is different from the below.

$$\begin{tikzcd}
(\widetilde{F}_1, \widetilde{\phi_1^{k_1}}) \arrow[d] \arrow[r, phantom, "="] & (\widetilde{F}_2, \widetilde{\phi_2^{k_2}}) \arrow[d] \\
(F_1, \phi_1) & (F_2, \phi_2)
\end{tikzcd}$$

Fibered Commensurability

Definition (CSW)

A fibered pair is a pair $(M, \mathcal{F})$ where

- M is a compact 3-manifold with boundary a union of tori and Klein bottles,
- $\mathcal{F}$ is a foliation by compact surfaces.

Remark. Since $[F, \phi]$ has a foliation whose leaves are homeomorphic to F, fibered pair is a generalization of the pair of type (F, ϕ).

Fibered Commensurability 2

Definition (CSW)

A fibered pair $(\widetilde{M}, \widetilde{\mathcal{F}})$ covers $(M, \mathcal{F})$ if there is a finite covering of manifolds $\pi : \widetilde{M} \to M$ such that $\pi^{-1}(\mathcal{F})$ is isotopic to $\widetilde{\mathcal{F}}$.

Definition (CSW)

Two fibered pairs $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ are commensurable if there is a third fibered pair $(\widetilde{M}, \widetilde{\mathcal{F}})$ that covers both.

Minimal Elements

Proposition. [CSW]

The covering relation on pairs of type (F, ϕ) is transitive.

Definition

An element (F, ϕ) (or $(M, \mathcal{F})$) is called minimal if it does not cover any other elements.

Periodic Case [CSW]

- $\exists$ exactly 2 commensurability classes; with or without boundaries.
- each commensurability class contains ∞-many minimal elements.

(hint: consider elements with maximal period)

Reducible Case

Theorem (CSW)

- $\exists$ *manifold with infinitely many incommensurable fibrations.*
- $\exists$ *manifold with infinitely many fibrations in the same commensurable class.*

Remark. The manifolds in this theorem are graph manifolds.

Pseudo Anosov Case

Theorem (CSW)

Suppose $\partial M = \emptyset$. Then every hyperbolic fibered commensurability class $[(M, \mathcal{F})]$ contains a unique minimal element.

Remark. The assumption $\partial M = \emptyset$ is not explicitly written in their paper.

Result 1

Every hyperbolic fibered commensurability class $[(M, \mathcal{F})]$ contains a unique minimal element.

Corollary (CSW)

M: hyperbolic fibered 3-manifold.
Then number of fibrations on M commensurable to a fibration on M is finite.

Recall that if (the first Betti number of M) > 1, then M admits infinitely many distinct fibrations (Thurston).

Question[CWS].

- When two fibrations on M are commensurable?
- Are there any example of two commensurable fibrations on M with non homeomorphic fiber?

Invariants , pseudo-Anosov case(CSW)

- Commensurability class of dilatations.
- Commensurability class of the vectors of the numbers of n-pronged singular points on $\mathrm{Int}(F)$.

Example. (0,0,1,1,1,0,...) means it has one 3 (4, and 5)-pronged singularity.
Remark. Let $\{p_i\}_{i \in I}$ be the set of singular points and $\{n_i\}_{i \in I}$ their prong number, then

$$\sum_i \frac{2-n_i}{2} = \chi(F).$$

Definition

Two fibrations $\omega_1 \neq \omega_2 \in H^1(M;\mathbb{Z})$ are symmetric if $\exists$ homeomorphism $\varphi : M \to M$ such that $\varphi^*(\omega_1) = \omega_2$ or $\varphi^*(\omega_1) = -\omega_2$.

Definition

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Result 2

Two fibrations on $S^3 \setminus 6_2^2$ or the Magic 3-manifold are either symmetric or non-commensurable.

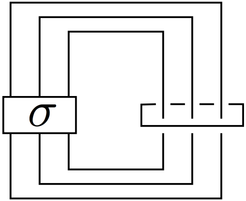

Figure: the fibered link associated to a braid $\sigma \in B_3$

Fibrations on a manifold

Result 3

M : fibered hyperbolic 3-manifold which does not have hidden symmetry.
Then, any two non-symmetric fibrations of M are not fibered commensurable.

Fibrations on a manifold

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Remark

- "Most" hyperbolic 3-manifolds do not have hidden symmetry.
- $S^3 \setminus 6_2^2$ and the Magic 3-manifold have lots of hidden symmetries.

Hidden Symmetries

M: hyperbolic 3-manifold
$\rho : \pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$: a holonomy representation.
$\Gamma := \rho(\pi_1(M))$

Hidden Symmetries

M: hyperbolic 3-manifold
$\rho : \pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$: a holonomy representation.
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Definition

$N(\Gamma) := \{\gamma \in \mathrm{PSL}(2, \mathbb{C}) \mid \gamma\Gamma\gamma^{-1} = \Gamma\}$.
$C(\Gamma) :=$
$\{\gamma \in \mathrm{PSL}(2, \mathbb{C}) \mid \gamma\Gamma\gamma^{-1}$ and Γ are weakly commensurable$\}$

$N(\Gamma)$ and $C(\Gamma)$ are called normalizer and commensurator respectively.
Two groups $\Gamma_i < \mathrm{PSL}(2, \mathbb{C})$ $(i = 1, 2)$ are said to be weakly commensurable if $[\Gamma_i : \Gamma_1 \cap \Gamma_2] < \infty$ for both $i = 1, 2$.

Manifold with (no) Hidden Symmetry

Definition

An elements in $C(\Gamma) \setminus N(\Gamma)$ is called a hidden symmetry.

Definition

A hyperbolic 3-manifold M said to have no hidden symmetry $\iff$ the image $\Gamma := \rho(\pi_1(M))$ of a holonomy representation ρ does not have hidden symmetry.

Remark. By Mostow-Prasad rigidity theorem, this definition does not depend on the choice of a holonomy representation.

Commensurable fibrations on the same fibered cone

Result 4

One can construct an infinite sequence of manifolds $\{M_i\}$ with

- non-symmetric (fiberes are of different topology), and
- commensurable

fibrations whose corresponding elements in $H^1(M_i; \mathbb{Z})$ are on the same fibered cone.

Construction

Lemma

M : fibered hyperbolic 3-manifold.
$\omega_1 \neq \pm\omega_2 \in H^1(M;\mathbb{Z})$: primitive elements correspond to symmetric fibrations.
Then, for all $n >> 1 (n \in \mathbb{N})$, there exists a finite cover $p_n : M_n \to M$ of degree n such that $p_n^*(\omega_1)$ and $p_n^*(\omega_2)$ correspond to commensurable but non-symmetric fibrations.

Idea

Let (F_1, ϕ_1) and (F_2, ϕ_2) be corresponding pair of ω_1 and ω_2, respectively.
Then let $p_n : M_n \longrightarrow M$ be the covering that corresponds to (F_1, ϕ_1^n) (dynamical cover).
Then for large enough n, we see that $p_n^{-1}(F_2)$ is not homeomorphic to F_1.

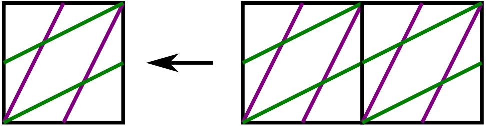

Figure: Schematic picture of the dynamical covering

To be precise

(b is the first Betti number of M)

- $H^1(M;\mathbb{Z}) \cong \mathrm{Hom}(H_1(M)/\mathrm{Tor},\mathbb{Z}) \cong \mathbb{Z}^b$
- $0 \to \pi_1(F_i) \to \pi_1(M) \xrightarrow{\rho_i} \pi_1(S^1) \cong \mathbb{Z} \to 0.$

To be precise

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- $H^1(M;\mathbb{Z}) \cong \mathrm{Hom}(H_1(M)/\mathrm{Tor},\mathbb{Z}) \cong \mathbb{Z}^b$
- $0 \to \pi_1(F_i) \to \pi_1(M) \xrightarrow{\rho_i} \pi_1(S^1) \cong \mathbb{Z} \to 0.$
- $A_i = \mathrm{ab}(\pi_1(F_i))/\mathrm{Tor} \subset H_1(M)$ (ab : abelianization)
- $A_i = \mathrm{Ker}(\omega_i) \cong \mathbb{Z}^{b-1}$

To be precise

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- $H^1(M;\mathbb{Z}) \cong \mathrm{Hom}(H_1(M)/\mathrm{Tor}, \mathbb{Z}) \cong \mathbb{Z}^b$
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- $A_i = \mathrm{ab}(\pi_1(F_i))/\mathrm{Tor} \subset H_1(M)$ $(\mathrm{ab}:$ abelianization)
- $A_i = \mathrm{Ker}(\omega_i) \cong \mathbb{Z}^{b-1}$
- $\rho_1 : \pi_1(M) \xrightarrow{\mathrm{ab}} H_1(M) \xrightarrow{\omega_1} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}.$
- for sufficiently large n, $\exists b \in A_2$ s.t. $\rho_1(b) \neq 0$

$S^3 \setminus 6_2^2$

$S^3 \setminus 6_2^2$ has a symmetry that permutes the components of cusps.

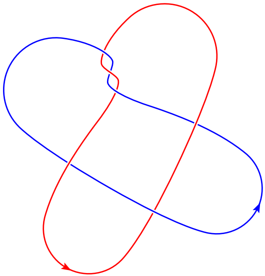


Figure: 6_2^2(Generated by "Kirby Calculator")

Thurston Norm on $H^1(S^3 \setminus 6_2^2, \mathbb{R})$[Hironaka]

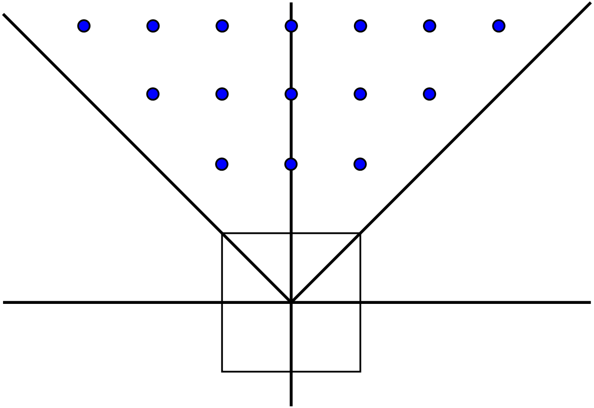
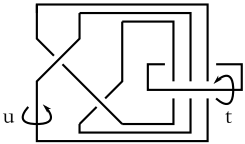
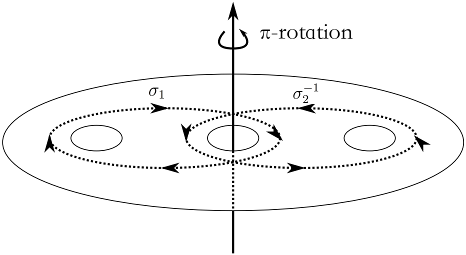


Figure: $H^1(S^3 \setminus 6_2^2, \mathbb{R})$

- By taking conjugate we can prove that $\exists h_1 : M \to M$ s.t. $h_1^*(\omega) = -\omega$.





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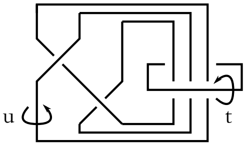

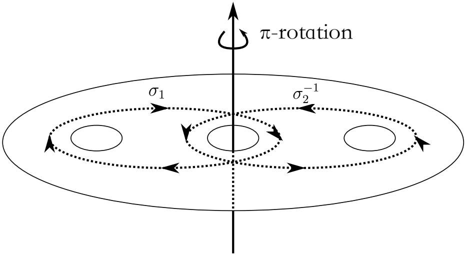

- $+$ the fact that 6_2^2 is amphicheiral, we can find a symmetry $h_2 : M \to M$ s.t.
  - $h_2^*(U) = -U$, and
  - $h_2^*(T) = T$.

$\Rightarrow$ Fibrations $aU + bT$ and $-aU + bT$ are symmetric.

$\Rightarrow$ By taking covers of $S^3 \setminus 6^2_2$, we prove

Result 4

One can construct an infinite sequence of manifolds with non-symmetric but commensurable fibrations (whose corresponding elements in $H^1(M;\mathbb{Z})$ are in the same fibered cone).

Questions

Question 1.

When a manifold has non-symmetric but commensurable fibrations.

- $S^3 \setminus 6^2_2$ and the Magic 3-manifold have many hidden symmetry.

Question 2.

How many commensurable fibrations can a manifold have up to symmetry?

Thank you for your attention

Figure: Marseille

Quasiconformal and Lipschitz harmonic mappings of the unit disk onto bounded convex domains

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$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$
$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$

$$\mathrm{Hom}^{+}(\mathbb{T}) := \left\{ f : \mathbb{T} \to \mathbb{T} \;\middle|\; \begin{array}{l} \text{sense-preserving} \\ \text{homeomorphisms} \end{array} \right\}$$

Radó-Kneser-Choquet Theorem

$f \in \mathrm{Hom}^{+}(\mathbb{T})$

$$\Rightarrow \mathsf{P}[f](z) := \frac{1}{2\pi} \int_{\mathbb{T}} f(u) \,\mathrm{Re}\, \frac{u+z}{u-z} |\,\mathrm{d}\,u| \;, \quad z \in \mathbb{D}$$

is a harmonic and homeomorphic self-mapping of $\mathbb{D}$.

For a continuous mapping $f : \mathbb{T} \to \mathbb{C}$ which gives a rectifiable curve,

$$\begin{aligned}
\dot{f}(z) &:= \frac{df}{d\theta}(e^{i\theta}) \quad \text{a.e.} \quad z = e^{i\theta} \in \mathbb{T}, \\
d_f &:= \operatorname*{ess\,inf}_{z \in \mathbb{T}} |\dot{f}(z)|, \\
e_f &:= \operatorname*{ess\,sup}_{z \in \mathbb{T}} |\dot{f}(z)| \ .
\end{aligned}$$

For $h \in L^1(\mathbb{T})$

$$A(h)(z) := \frac{1}{2\pi} \lim_{\varepsilon \to 0+} \int_{\varepsilon < |t-x| \le \pi} h(e^{it}) \cot \frac{x-t}{2} dt \ ,$$

$$\text{a.e.} \quad z = e^{ix} \in \mathbb{T} \ .$$

Theorem A (Pavlović '02).
Let $f \in \mathrm{Hom}^{+}(\mathbb{T})$ and $F = \mathrm{P}[f]$. Then

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

(i) F is a quasiconformal self-mapping of $\mathbb{D}$,

(ii) F is a bi-Lipschitz self-mapping of $\mathbb{D}$,

(iii) f is absolutely continuous on $\mathbb{T}$,

$$0 < d_f \le e_f < \infty \quad \text{and } A(\dot{f}) \in L^{\infty}(\mathbb{T}).$$

$F = H + \overline{G} : \mathbb{D} \to \Omega$ a sense-preserving univalent harmonic mapping,
(H, G holomorohic mappings, $G(0) = 0$),

Theorem B (Partyka-Sakan '11).
Let $\Omega := F(\mathbb{D})$ be a convex domain. Then the following five conditions are equivalent to each other:

(i) F is a quasiconformal mapping;

(ii) $\exists$ a constant L_1 such that $1 \le L_1 < 2$ and

$$|F(z_2) - F(z_1)| \le L_1 |H(z_2) - H(z_1)| \ , \quad z_1, z_2 \in \mathbb{D} \ ;$$

(iii) $\exists$ a constant l_1 such that $0 \le l_1 < 1$ and

$$|G(z_2) - G(z_1)| \le l_1 |H(z_2) - H(z_1)| \ , \quad z_1, z_2 \in \mathbb{D} \ ;$$

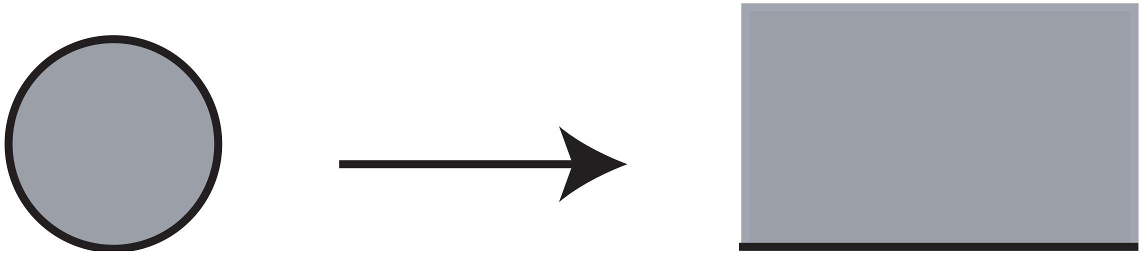
(iv) $\exists$ a constant $L_2 \ge 1$ such that

$$|H(z_2) - H(z_1)| \le L_2 |F(z_2) - F(z_1)| \ , \quad z_1, z_2 \in \mathbb{D} \ ;$$

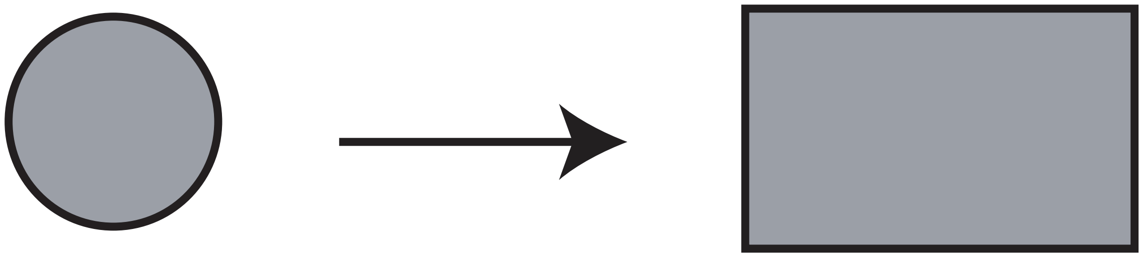
(v) $H \circ F^{-1}$ and $F \circ H^{-1}$ are bi-Lipschtz mappings.

Moreover, (ii) $\Rightarrow \|\mu_{\mathsf{F}}\|_{1,\infty} \le \mathsf{L}_1 - 1$, (iii) $\Rightarrow \|\mu_{\mathsf{F}}\|_{1,\infty} \le l_1$,

(iv) $\Rightarrow \|\mu_{\mathsf{F}}\|_{1,\infty} \le 1 - \frac{1}{L_2}$.



$H : \mathbb{D} \to \{z \in \mathbb{C} |\ \mathrm{Im} z > 0\}, \quad H(z) = \dfrac{i(z+1)}{1-z}$, conformal, not Lipschitz



the Schwarz-Christoffel mapping $H(z)$, conformal, not Lipschitz

$F(z) = H(z) + t\overline{H(z)}, \quad |t| < 1,$ quasiconformal, not Lipschitz

Theorem 1 (a generalization of [PS'07,Th.2.2]). *Let $F = H + \overline{G}$ be a sense-preserving locally univalent harmonic mapping of $\mathbb{D}$ onto a bounded domain $\Omega \subset \mathbb{C}$, where H, G are holomorphic mappings with $G(0) = 0$. Then*

$$\text{(i)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iii)} \Leftrightarrow \text{(iv)}.$$

(i) F *is Lipschitz.*

(ii) $\frac{\partial}{\partial\theta}F,\ S \in h^{\infty}(\mathbb{D})$,

where $S(z) := |z|\frac{\partial}{\partial r}F(z), \qquad z = re^{i\theta} \in \mathbb{D}$,

(iii) $\partial F \in H^{\infty}(\mathbb{D})$,

(iv) F *has a continuous extension* $\tilde{F}$ *to* $\overline{\mathbb{D}}$, *and* $f := \tilde{F}|_{\mathbb{T}}$ *is absolutely continuous, and* $e_f < \infty$, $A(\dot{f}) \in L^{\infty}(\mathbb{T})$.

Theorem 2. *Let* $F = H + \overline{G}$ *be a sense-preserving univalent harmonic mapping of* $\mathbb{D}$ *onto a bounded convex domain* $\Omega \subset \mathbb{C}$*, where* H, G *are holomorphic mappings with* $G(0) = 0$*. Then*

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v)

(i) F *is quasiconformal and Lipschitz.*

(ii) F *is quasiconformal and its boundary valued function* f *is Lipschitz*

(iii) F *is quasiconformal and* H *is bi-Lipschitz*

(iv) F *is bi-Lipschitz*

(v) F *has a continuous extension* $\tilde{F}$ *to* $\overline{\mathbb{D}}$*, and* $f := \tilde{F}|_{\mathbb{T}}$ *is absolutely continuous, and* $0 < d_f \leq e_f < \infty$, $A(\dot{f}) \in L^{\infty}(\mathbb{T})$.

A function $f : \mathbb{T} \to \mathbb{C}$ is called Dini-smooth if f is differentiable on $\mathbb{T}$ and the derivative $\dot{f}$ is not vanishing, and Dini-continuous on $\mathbb{T}$, i.e. its modulus of continuity

$$\omega(\delta) := \sup\{|\dot{f}(\mathrm{e}^{\mathrm{i}\,t}) - \dot{f}(\mathrm{e}^{\mathrm{i}\,s})| : t, s \in \mathbb{R}\,,\ |t-s| \le \delta\}\ , \qquad \delta \in [0; 2\pi]\ ,$$

satisfies the following condition

$$\int_0^{2\pi} \frac{\omega(t)}{t}\,\mathrm{d}\,t < +\infty\ .$$

Corollary 3. *Let $f : \mathbb{T} \to \mathbb{C}$ be a Dini-smooth and injective function. If $f(\mathbb{T})$ is the boundary curve of a convex domain Ω in $\mathbb{C}$, then $F := \mathrm{P}[f]$ is a bi-Lipschitz mapping of $\mathbb{D}$ onto Ω. If additionally $\mathrm{J[F](0)} > 0$ then F is quasiconformal.*

Harmonic Schwarz Lemma

$F : \mathbb{D} \to \mathbb{D}$ (into) harmonic, $F(0) = 0$

$$\Rightarrow \quad |F(z)| \leq \frac{4}{\pi} \arctan |z| \ , \ z \in \mathbb{D}$$

Lemma 4 ([PS'09,Lemma 1.1]). $a, b \in \mathbb{R}$, $-a < b$, $u : \mathbb{D} \to \mathbb{R}$ *harmonic,* $u(0) = 0$, $-a \leq u(z) \leq b$, $z \in \mathbb{D}$

$$\Rightarrow \quad u(z) \leq 2\frac{b+a}{\pi} \arctan \frac{|z| + |p|}{1 + |p||z|} + \frac{b-a}{2} \ , \quad z \in \mathbb{D},$$

where $p = -i \tan \frac{\pi}{4} \frac{b-a}{b+a}$.

Let J[F] stand for the Jacobian of a differentiable mapping $F : \mathbb{D} \to \mathbb{C}$:
$\mathrm{J[F](z)} := |\partial \mathrm{F(z)}|^2 - |\bar{\partial}\mathrm{F(z)}|^2 \ , \quad \mathrm{z} \in \mathbb{D} \ .$
For $R > 0$, $\mathbb{D}(0, R) := \{z \in \mathbb{C} : |z| < R\}$.

Theorem 5. *Let $f : \mathbb{T} \to \mathbb{C}$ be a function of bounded variation and differentiable at a point $z \in \mathbb{T}$ such that $\mathrm{P}[f](0) = 0$, the limit $\lim_{r \to 1^-} \frac{\mathrm{d}}{\mathrm{d}\, r} \mathrm{P}[f](rz)$ exists and the inequality $\liminf_{r \to 1^-} \mathrm{J[P[f]](rz)} \geq 0$. holds. If $f(z)$ is a point linearly accessible from outside of $\Omega := \mathrm{P}[f](\mathbb{D})$, i.e. there exists $\zeta \in \mathbb{T}$ satisfies $\quad \mathrm{Re}(\zeta w) \leq \mathrm{Re}(\zeta f(z)) \ , \quad w \in \Omega$, and $\mathrm{Re}(\zeta\, \mathrm{P(f)(u)})$ is not a constant function ,*
then the following limit exists and
$\lim_{r \to 1^-} \mathrm{J[P[f]](rz)} \geq |\mathrm{f'(z)}| \frac{\mathrm{R_1 + R_2}}{\pi} \tan\left(\frac{\pi}{2} \frac{\mathrm{R_1}}{\mathrm{R_1 + R_2}}\right) \geq |\mathrm{f'(z)}| \frac{\mathrm{R_1}}{2}$
for all $R_1, R_2 > 0$ satisfying the condition
$\mathbb{D}(0, R_1) \subset \mathrm{P}[f](\mathbb{D}) \subset \mathbb{D}(0, R_2)$.

$\Omega \subset \mathbb{C}$ and a function $F : \Omega \to \mathbb{C}$ we denote by $\mathsf{L(F)}$ the Lipschitz constant of F, i.e.

$$\mathsf{L(F)} := \sup\left\{\left|\frac{\mathsf{F(z)-F(w)}}{\mathsf{z-w}}\right| : \mathsf{z}, \mathsf{w} \in \Omega\,,\ \mathsf{z} \neq \mathsf{w}\right\}\ .$$

Note that F is a Lipschitz function iff $\mathsf{L(F)} < +\infty$. If the last condition holds, then F is a L-Lipschitz function for every $L \geq \mathsf{L(F)}$, i.e.

$$|F(w) - F(z)| \leq L|w - z|\ , \quad w, z \in \Omega\ .$$

Theorem 6. $F : \mathbb{D} \to \mathbb{C}$ *harmoic and Lipschitz*
$\Rightarrow$ F *has the continuous extension to* $\bar{\mathbb{D}}$ *and its boundary valued function* f *is absolutely continuous and satisfies*

$$\|A[\dot{f}]\|_\infty \leq \sqrt{2}\mathsf{L(F)}\ , \quad \|\dot{\mathsf{f}}\|_\infty \leq \mathsf{L(F)} \quad \textit{and} \quad \mathsf{L(f)} \leq \mathsf{L(F)}\ .$$

Conversely, if F *has the continuous extension to* $\bar{\mathbb{D}}$ *and its boundary valued function* f *is absolutely continuous,* $A[\dot{f}] \in \mathsf{L}^\infty(\mathbb{T})$ *and* $\dot{f} \in \mathsf{L}^\infty(\mathbb{T})$,
$\Rightarrow$ F *is a Lipschitz mapping with*

$$\mathsf{L(F)} \leq \sqrt{\|A[\dot{f}]\|_\infty^2 + \|\dot{f}\|_\infty^2}\ .$$

A simply connected domain $\Omega \subset \mathbb{C}$ is linearly connected if there exists a constant $M < \infty$ such tat for any points $w_1, w_2 \in \Omega$ are joined by a path $\gamma \subset \Omega$ of length $l(\gamma) \leq M|w_1 - w_2|$.

Lemma 7. *Let $F = H + \overline{G}$ be a sense-preserving univalent harmonic mapping of $\mathbb{D}$ onto a convex domain $\Omega \subset \mathbb{C}$, where H, G are holomorphic mappings with $G(0) = 0$.(Then it is known that H is univalent.) If $H(\mathbb{D})$ is linearly connected, then H is co-Lipschitz. In particular, $H(\mathbb{D})$ is linearly connected and $H'(z)$ is bounded iff H is bi-Lipschitz.*

Set $\mathbb{D}(R) := \{z \in \mathbb{C} : |z| < R\}$ for $R > 0$.
Assume that F is a univalent harmonic mapping of the unit disk $\mathbb{D} := \mathbb{D}(1)$ onto itself and normalized by $F(0) = 0$.
In 1958 E. Heinz proved that

$$|\partial_x F(z)|^2 + |\partial_y F(z)|^2 \geq \frac{2}{\pi^2}\ , \qquad z \in \mathbb{D}\ .$$

Theorem C (Kalaj '03).
If F is a sense-preserving univalent harmonic mapping of $\mathbb{D}$ onto a convex domain Ω satisfying $\Omega \supset \mathbb{D}(R_1)$ and $F(0) = 0$, then

$$|\partial_x F(z)|^2 + |\partial_y F(z)|^2 \geq \frac{1}{8} R_1^2\ , \qquad z \in \mathbb{D}\ , \tag{1}$$

Theorem D (Partyka-Sakan '09).
If F is a sense-preserving univalent harmonic mapping of $\mathbb{D}$ onto a bounded convex domain Ω including 0 and $F(0) = 0$, then for all $R_1, R_2 > 0$ satisfying $\mathbb{D}(R_1) \subset \Omega \subset \mathbb{D}(R_2)$ the following inequalities hold

$$|\partial F(w)| \geq \frac{R_1 + R_2}{2\pi} \tan\left(\frac{\pi}{2}\frac{R_1}{R_1 + R_2}\right) \ , \quad w \in \mathbb{D} \ ,$$

as well as

$$|\partial_x F(w)|^2 + |\partial_y F(w)|^2 \geq 2\left(\frac{R_1 + R_2}{2\pi} \tan\left(\frac{\pi}{2}\frac{R_1}{R_1 + R_2}\right)\right)^2 \ , \quad w \in \mathbb{D} \ .$$

Theorem E (Partyka-Sakan '09).
Let F be a sense-preserving univalent harmonic mapping of $\mathbb{D}$ onto a bounded convex domain Ω including 0 such that $F(0) = 0$, and that F has a continuous extension $\tilde{F}$ to $\overline{\mathbb{D}}$. If $f := \tilde{F}|_{\mathbb{T}}$ is Dini-smooth, both the functions ∂F and $\bar{\partial} F$ have continuous extensions to the closure $\overline{\mathbb{D}}$ and

$$|\partial F(z)|^2 - |\bar{\partial} F(z)|^2) \geq |\dot{f}(z)| \frac{R_1 + R_2}{\pi} \tan\left(\frac{\pi}{2} \frac{R_1}{R_1 + R_2}\right) . \tag{2}$$

for every $z \in \mathbb{T}$ and all $R_1, R_2 > 0$ satisfying $\mathbb{D}(R_1) \subset \Omega \subset \mathbb{D}(R_2)$.

Theorem F.
Let $\Omega \subset \mathbb{C}$ be a Jordan domain with a $C^{1,\mu}$, $(0 < \mu \leq 1)$, parametrization of $\partial\Omega$, and $F : \mathbb{D} \to \Omega$ be a sense-preserving univalent (onto) harmonic mapping.

(i) (Kalaj '08) If Ω is convex, then

F is quasiconformal $\iff \quad F$ is bi-Lipschitz

(ii) (Božin and Mateljević, to appear)

F is quasiconformal $\iff \quad F$ is bi-Lipschitz

Lemma 8. *Let V be a linearly connected Jordan domain (resp. a quasi-disk) and let $S : V \to S(V) \subset \mathbb{C}$ be a bi-Lipschitz diffeomorphism. Then $S(V)$ is a linearly connected Jordan domain (resp. a quasi-disk).*

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Energy-minimal diff. between doubly conn. Riemann surfaces

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Osaka, January, 2013

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Harmonic mappings between Riemann surfaces

Let (M, σ) and (N, ρ) be Riemann surfaces with metrics σ and ρ, respectively. If a mapping $f : (M, \sigma) \to (N, \rho)$ is C^2, then f is said to be harmonic (to avoid the confusion we will sometimes sayρ-harmonic) if

$$f_{z\overline{z}} + (\log \rho^2)_w \circ f f_z \, f_{\overline{z}} = 0, \quad (1)$$

where z and w are the local parameters on M and N respectively.

Also f satisfies (1) if and only if its Hopf differential $\Psi = \rho^2 \circ f f_z \overline{f_{\bar z}}$ is a holomorphic quadratic differential on M. For $g : M \mapsto N$ the energy integral is defined by

$E_\rho[g] = \int_M (|\partial g|^2 + |\bar\partial g|^2) dV_\sigma.$

Then f is harmonic if and only if f is a critical point of the corresponding functional where the homotopy class of f is the range of this functional.

Mappings of finite distortion

A homeomorphism $w = f(z)$ between planar domains Ω and D has finite distortion if a) f lies in the Sobolev space $W^{1,1}_{loc}(\Omega, D)$ of functions whose first derivatives are locally integrable, and b) f satisfies the distortion inequality $|f_{\bar{z}}| \le \mu(z)|f_z|$, $0 \le \mu(z) < 1$ almost everywhere in Ω. Such mappings are generalizations of quasiconformal homeomorphisms.

The Nitsche conjecture

The conjecture in question concerns the existence of a har. homeo. between circular annuli $A(r, 1)$ and $A(\tau, \sigma)$, and is motivated in part by the existence problem for doub.-conn. min. surf. with prescribed boundary. In 1962 J. C. C. Nitsche observed that the image annulus cannot be too thin, but it can be arbitrarily thick (even a punctured disk).

Indeed Nitsche observed that a radial harmonic mapping $f : A(r,1) \to A(\tau,\sigma)$, i.e. satisfy. the condition $f(se^{it}) = f(s)e^{it}$ ($f(z) = az + b/\bar{z}$), is a homeomorphism if and only if $\frac{\sigma}{\tau} \geq \frac{1}{2}\left(\frac{1}{r} + r\right)$ (Nitsche bound).

Then he conjectured that for arbitrary harmonic homeomorphism between annuli we have Nitsche bound. Some partial solutions are presented by Weitsman, Lyzzaik, Kalaj (2001,2003).

The Nitsche was right

The Nitsche conjecture for Euclidean harmonic mappings is settled recently by Iwaniec, Kovalev and Onninen (2010, JAMS), showing that, only radial harmonic mappings $h(\zeta) = C\left(\zeta - \frac{\omega}{\bar{\zeta}}\right)$, $C \in \mathbb{C}$, $\omega \in \mathbb{R}$, $|C|(1-\omega) = \sigma$, **which inspired the Nitsche conjecture, make the extremal distortion of rounded annuli.**

Radial $\rho-$harmonic mappings

We state a similar conjecture (see Kalaj, arXiv:1005.5269) with respect to $\rho-$ harmonic mappings. In order to do this, we find all examples of radial ρ-harmonic maps between annuli. We put $w(z) = g(s)e^{it}, \quad z = se^{it}$ in harm. eq. $h_{z\bar{z}} + (\log \rho^2)_w \circ h h_z\, h_{\bar{z}} = 0$ where g is an increasing or a decreasing function.

The resulting functions are $w(z) = g(s)e^{it}$, $z = se^{it}$, **where** g **is the inverse of** $h(\gamma) = \exp\left(\int_\sigma^\gamma \frac{dy}{\sqrt{y^2+c\varrho^2}}\right), \ \tau \le \gamma \le \sigma,$ $\varrho = 1/\rho$. **Moreover they are homeomorphisms iff we have** $\rho-$**Nitsche bound:**

$$r \ge \exp\left(\int_\sigma^\tau \frac{\rho(y)dy}{\sqrt{y^2\rho^2(y) - \tau^2\rho^2(\tau)}}\right).$$

Generalization of Nitsche conjecture

Let ρ be a radial metric. If $r < 1$, and there exists a $\rho-$ harmonic mapping of the annulus $A' = A(r, 1)$ onto the annulus $A = A(\tau, \sigma)$, then there hold the $\rho-$Nitsche bound. Notice that if $\rho = 1$, then this conjecture coincides with standard Nitsche conjecture. For some partial solution see Kalaj (2011, Israel J. of Math)

Another justification of the previous conjecture

Assume that a homeomorphism $f : A(1, r) \to A(\tau, \sigma)$ **has a finite distortion and minimize the integral means** $\mathcal{K}_\rho[f] = \int_\Omega \mathbb{K}(z, f)\rho^2(z)dxdy$. **Here** $\mathbb{K}(z, f) = (|f_z|^2 + |f_{\bar z}|^2)/(|f_z|^2 - |f_{\bar z}|^2)$ **is Distortion function. If the annuli** $A(1, r)$ **and** $A(\tau, \sigma)$ **are conformaly equivalent then the absolute minimum is achieved by a conformal mapping.**

If the annuli $A(1, r)$ and $A(\tau, \sigma)$ are not conformaly equivalent then the absolute minimum is achieved by a homeomorphism f whose inverse h is a ρ harmonic mapping between $A(\tau, \sigma)$ and $A(1, r)$ if and only if the annuli satisfy the $\rho-$Nitsche bound.

The previous result is generalization of a result of Astala, Iwaniec and Martin (2010, ARMA) and has been obtained by Kalaj (2010, arXiv:1005.5269).
If the annuli $A(1, r)$ and $A(\tau, \sigma)$ do not satisfies the $\rho-$Nitsche condition, then no such homeomorphisms of finite distortion exists between them.

Energy minimizers, radial metric, (Kalaj, arXiv:1005.5269)

Let ρ be a regular metric. Within the Nitsche rang, for the annuli A and A', the absolute minimum of the energy integral $h \to E_\rho[h], \quad h \in W^{1,2}(A, A')$ is attained by a $\rho-$Nitsche map

$$h^c(z) = q^{-1}(s)e^{i(t+\beta)}, \; z = se^{it}, \; \beta \in [0, 2\pi),$$

where

$$q(s) = \exp\left(\int_\sigma^s \frac{dy}{\sqrt{y^2 + c\varrho^2}}\right), \tau < s < \sigma.$$

Energy minimizers, Euclidean metric

Let A **and** A' **be doub. conn. domains in compl. plane such that** $\mathrm{Mod}(A) \leq \mathrm{Mod}(A')$**. The absolute minimum of the energy integral** $h \to E[h], \quad h \in W^{1,2}(A, A')$ **is attained by an Euclidean harmonic homeomorphism between** A **and** A'**. (Iwaniec, Kovalev, Onninen, Inventiones, (2011))**

Energy minimizers, Arbitrary metric with bounded Gauss curvature

Let A and A' be doub. conn. domains in Riemann surfaces (M, σ) and (N, ρ), such that $\mathrm{Mod}(A) \le \mathrm{Mod}(A')$. The absolute minimum of the energy integral $h \to E_\rho[h], \quad h \in W^{1,2}(A, A')$ is attained by an $\rho-$ harmonic homeomorphism between A and A'. (See Kalaj, arXiv:1108.0773)

The proof uses

- **The so called deformations $D^{\rho}(\Omega, \Omega^{*})$ which make a compact family.**
- **A modification of Choquet-Rado-Kneser theorem for Riemann surfaces with bounded Gauss curvature**
- **The fact that energy integral is weak lower semicontinuous.**

And the key lemma

Let $\Omega = A(r, R)$ **be a circular annulus,** $0 < r < R < \infty$**, and** Ω^* **a doubly connected domain. If** $h \in D^\rho(\Omega, \Omega^*)$ **is a stationary deformation, then the Hopf differential of the mapping** h**:** $\rho^2(h(z))h_z\overline{h_{\bar z}} \equiv \frac{c}{z^2}$ **in** Ω **where** $c \in \mathbb{R}$ **is a constant. This lemma follows by a result of Jost.**

Conjecture a)

If f is a minimizer of the energy E^ρ between two doubly connected domains Ω, ($\tau = \mathrm{Mod}(\Omega)$) and Ω^*, then f is harmonic and $K(\tau)$-quasiconformal if and only if τ is smaller that the modulus $\tau_\diamond$ of critical Nitsche domain $A(\tau_\diamond)$.

Conjecture b)

b) Under condition of a) we conjecture that $c_\diamond < 0$ and

$$K(\tau) = \max\{\sqrt{\frac{c_\diamond - c}{c_\diamond}}, \sqrt{\frac{c_\diamond}{c_\diamond - c}}\}. \tag{1}$$

Notice that, this is true if the image domain is a circular annulus with a radial metric.

Quasiconformal harmonic mappings (HQC)

The first author who studied the class HQC is Olli Martio on 1968 (AASF). The class of HQC has been studied later by: Hengartner, Schober, Mateljević, Pavlović, Partyka, Sakan, Kalaj, Vuorinen, Manojlović, Nesi, Alessandrini, Božin, Marković, Wan, Onninen, Iwaniec, Kovalev etc.

Lipschitz and co-Lipschitz mappings

Let $w : \Omega \to D$ be a mapping between two domains Ω and D. Then w is called Lipschitz (co-Lipschitz) continuous if there exists a constant $C > 1$ ($c > 0$) such that

$$|w(z) - w(z')| \leq C|z - z'|,$$

$$(c|z - z'| \leq |w(z) - w(z')|) \quad z, z' \in \Omega.$$

Kellogg theorem

Let $w : \Omega \to D$ **be a conformal map between two Jordan d.** $\Omega, D \in C^{1,\alpha}$**. Then** $w \in C^{1,\alpha}(\overline{\Omega})$**,** $w^{-1} \in C^{1,\alpha}(\overline{D})$ **.**

Corollary

Under the conditions of Kellogg theorem w is bi-Lipschitz continuous.

There exists a conf. mapp. of the unit disk onto a C^1 **Jordan domain which is not Lips. cont. (Lesley & Warschawski, 1978, Math Z).**

Motivation

Let $f(e^{it}) = e^{i(t+\alpha \sin t)}$**,** $0 \le \alpha \le 1$ **and let** $w = P[f](z)$ **be harmonic extension of** f **in the unit disk. In my master thesis (1997), I posed the following question, is** w **q.c. At that time I was not aware of the paper of Martio (1969, AASF). It follows by results of Martio that,** w **is q.c. if and only if** $\alpha < 1$**, or what is the same iff** f **is bi-Lipschitz.**

Kellogg type results, Lipschitz continuity

Let $w : \Omega \to D$ be a q.c and $\mathbb{U}$ be the unit disk. Then w is Lipschitz provided that

- $\Omega = D = \mathbb{U}$ **and** $\Delta u = 0$**: (Pavlovic-AASF 2002)**
- $\Omega = D = \mathbb{U}$ **and** $\Delta u = 0$**: (Partyka & Sakan quant. estim. - AASF, 2005, 2007)**
- $\Omega = D =$**Half-plane and** $\Delta u = 0$**: (Pavlovic & Kalaj, 2005, AASF)**

Kellogg type results, Lipschitz continuity

- $\Omega = D =$**Half-plane and** $\Delta u = 0$**, explicit constants (Mateljević& Knežević, 2007, JMAA)**
- **If** $\partial\Omega, \partial D \in C^{2,\alpha}$ **and** $|\Delta w| \leq A|\nabla w|^2 + B$**, (Mateljević & Kalaj, J. D. Analyse, 2006 & Potential Analysis 2010)**
- **If** $\partial\Omega, \partial D \in C^{1,\alpha}$**, and** $\Delta w = 0$ **(Kalaj, Math Z, 2008)**

Kellogg type results, bi-Lipschitz continuity with respect to quasihyperbolic metric

- **If w is hyp. harm. q.c. mapping of the unit disk onto itself, then w is bi-Lipschitz w. r. to hyp. metric (Wan, J. Dif. Geom. 1992).**
- **If $f : \Omega \to D$ is a quasiconformal and harmonic mapping, then it is bi-Lipschitz with respect to quasihyperbolic metric on D and D', (Manojlović, 2009, Filomat)**

Higher dimensional, Lipschitz continuity

- **If** $\partial\Omega, \partial D \in C^{2,\alpha}$ **and** $|\Delta w| \le A|\nabla w|^2 + B$, **(Kalaj, to appean in J. d' Analyse)**
- **Some other High. dim. gen. (Mateljevic & Vuorinen, JIA-2010)**

- **Hyperbolic harmonic (Tam & Wan, Pac. J. math, 1998)**
- **If** $\Omega = D$**= Unit ball and** $\Delta w = g$**, then** u **is** $C(K)$ **Lipschitz with**

Kellogg type results, co-Lipschitz continuity

Let $w : \Omega \to D$ be a q.c. Then w is co-Lipschitz provided that

- $\Omega = D = \mathbb{U}$ **and** $\Delta u = 0$**: (Follows by Heinz theorem)**
- **If** $\partial\Omega \in C^{1,\alpha}$**,** D **is convex and** $\Delta w = 0$ **(Kalaj-2002,)**
- **If** $\partial\Omega, \partial D \in C^{2,\alpha}$ **and** $\Delta w = 0$**, (Kalaj, AASF-2009)**
- **If** $\partial\Omega, \partial D \in C^2$ **and** $\Delta w = 0$**, (Kalaj, Annali SNSP, 2011.)**

Main tools of the proofs of results for HQC

- **Poisson integral formula**
- **Lewy's theorem**
- **Choquet-Rado-Kneser theorem**
- **Mori's theorem**
- **Kellogg theorem**
- **Isoperimetric inequality**
- **Hopf's Boundary Point lemma**
- **Heinz inequality**
- **Max. principle for (sub)harmonic**

- **Distance function from the boundary of the domain Ω: $d(x) = \mathrm{dist}(x, \partial\Omega)$.**
- **Möbius transformations of the unit disk**

$$w = e^{it}\frac{z-a}{1-z\bar{a}}, \quad |z| < 1, |a| < 1$$

or of the unit ball B^n.
- **The Carleman-Hartman-Wintner lemma**

Composing by conformal mappings

If w is a harmonic and g a conformal mapping, then the function $w \circ g$ is harmonic, but the function $g \circ w$ need not be harmonic. Thus the results, for example for the half plane and for the unit disk are not equivalent.

The Poisson integral formula

Let $P(r,x) = \frac{1-r^2}{2\pi(1-2r\cos x + r^2)}$ **denote the Poisson kernel. Then every bounded harm. func.** w **defined on the unit disk** $\mathbf{U} := \{z : |z| < 1\}$ **has the foll. rep.** $w(z) = P[f](z) = \int_0^{2\pi} P(r, x-\varphi) f(e^{ix}) dx$, **where** $z = re^{i\varphi}$ **and** f **is a bounded integr. func. def. on the unit circle T. There exist Poisson integral formula for the halp-plane (space) and for the unit ball.**

The Mori's theorem (Fehlmann & Vuorinen (AASF))

If u is a K quasi-conformal self-mapping of the unit ball B^n with $u(0) = 0$, then there exists a constant $M_1(n, K)$, satisfying the condition $M_1(n, K) \to 1$ as $K \to 1$, such that

$$|u(x) - u(y)| \leq M_1(n, K)|x - y|^{K^{1/(1-n)}}. \quad (2)$$

The Lewy theorem

If $f = g + \overline{h}$ **is a univalent harmonic mapping between two plane domain, then** $J_f := |g'|^2 - |h'|^2 > 0$. **Some extensions of this theorem have been done by: Schulz and Berg.**

The Choquet-Rado-Kneser theorem

If $f = g + \overline{h}$ is a harmonic mapping between two Jordan domains Ω and D such that D is convex, and $f|_{\partial\Omega} : \partial\Omega \to \partial D$ is a homeomorphism (or more general if $f|_{\partial\Omega}$ is a pointwise limit of homeomorphisms) then f is univalent.

Some extensions are given by: Alessandrini and Nessi, Kalaj, Duren, Schober, Jost, Yau and Schoen.

The Heinz theorem, Pac. J. Math, 1959

If $f = g + \overline{h}$ is a univalent harmonic mapping of the unit disk onto itself with $f(0) = 0$, then there holds the inequality

$$|g'(z)|^2 + |h'(z)|^2 \geq \frac{2}{\pi^2}, \quad z \in \mathbf{U}.$$

This theorem has been generalized by Kalaj on 2002 (Comp. Variabl.)

The Hilbert transformation

of a function χ is defined by the formula

$$H(\chi)(\varphi) = -\frac{1}{\pi}\int_{0+}^{\pi} \frac{\chi(\varphi+t)-\chi(\varphi-t)}{2\tan(t/2)}\mathrm{d}t$$

for a.e. φ and $\chi \in L^1(S^1)$. Assume that $w = P[F](z)$, $z = re^{it} \in \mathbb{U}$, $F' \in L^1(S^1)$. Then, if w_t and rw_r are bounded, there hold

$$w_t = P[F'] \text{ and } rw_r = P[H(F')]. \qquad (3)$$

Characterization theorem for HQC

Let $F : S^1 \to \gamma$ **be a sense pr. homeo. of the unit circle onto the Jordan curve** $\gamma = \partial D \in C^2$**. Then** $w = P[F]$ **is a q.c. mapping of the unit disk onto** D **if and only if** F **is abs. cont. and**

$$0 < l(F) := \operatorname{ess\,inf} l(\nabla w(e^{i\tau})), \tag{4}$$

$$||F'||_\infty := \operatorname{ess\,sup} |F'(\tau)| < \infty \tag{5}$$

$$||H(F')||_\infty := \operatorname{ess\,sup} |H(F')(\tau)| < \infty. \tag{6}$$

If F satisfies the conditions (4), (5) and (6), then $w = P[F]$ is K quasiconformal, where

$$K := \frac{\sqrt{||F'||_\infty^2 + ||H(F')||_\infty^2 - I(F)^2}}{I(F)}. \quad (7)$$

The constant K is the best possible in the following sense, if w is the identity or it is a mapping close to the identity, then $K = 1$ or $K$is close to 1 (respectively).

Remark

a) This theorem has been proved by Pavlović in (AASF, 2002) for $D = \mathbf{U}$.
b) If the image domain D is convex, then the condition (4) is equivalent (and can be replaced) with

$$0 < I(F) := \operatorname{ess\,inf} |F'(\tau)|, \qquad (8)$$

(Kalaj, Math z. 2008)
c) In this form by Kalaj (Arxiv). It is a general. of a theorem of Choquet.

Elliptic operator

Let $A(z) = \{a^{ij}(z)\}_{i,j=1}^2$ **be a symm. matrix function,** $z \in \Omega \subset \mathbf{C}$**. Assume that**

- $\Lambda^{-1} \leq \langle A(z)h, h\rangle \leq \Lambda$ **for** $|h| = 1$, $\Lambda \geq 1$
- $|A(z) - A(\zeta)| \leq L|\zeta - z|$ $z, \zeta \in D$**.**

For $L[u] := \sum_{i,j=1}^2 a^{ij}(z) D_{ij} u(z)$ **under previous cond. consider the diff. ineq.** $|L[u]| \leq \mathcal{B}|\nabla u|^2 + \Gamma$, **(***ell. part. diff. ineq.***),** $\mathcal{B} > 0$ **and** $\Gamma > 0$**. If** $A = Id$**, then** $L = \Delta$

Theorem 1

(To appear in Proceedings A of The Royal Society of Edinburgh)
If $w : \mathbf{U} \to \mathbf{U}$ is a q.c. solution of the elliptic partial differential inequality

$$|L[w]| \leq a|\nabla w|^2 + b,$$

then ∇w is bounded and w is Lipschitz continuous.

The main steps of the proofs

The proofs are differs from the proof of corresponding results for the class HQC. Some methods of the proof are borrowed from the paper of Nagumo. Two main steps in the proof are the following two lemmas

Global behaviour

Let D **be a complex domain with diameter** d **satisfying exterior sphere cond. Let** $u(z)$ **be a twice diff. mapping satisfying the ell. diff. ineq. in** D, $u = 0$ $(z \in \partial D)$. **Assume also that** $|u(z)| \leq M$, $z \in D$, $64\mathcal{B}\Gamma M < \pi$ **and** $u \in C(\overline{D})$. **Then** $|\nabla u| \leq \gamma, \quad z \in D$, **where** γ **is a constant depending only on** M, $\mathcal{B}$, Γ, $\mathfrak{L}$, Λ **and** d.

Interior estimate of gradient

Let D be a bounded domain, whose diameter is d. Let $u(z)$ be any C^2 solution of elliptic partial differential inequality such that $|u(z)| \leq M$. Then there exist constants $C^{(0)}$ and $C^{(1)}$, depending on modulus of continuity of u, Λ, L, B, Γ, M and d such that

$$|\nabla u(z)| < C^{(0)}\rho(z)^{-1}\max_{|\zeta - z|\leq\rho(z)}\{|u(\zeta)|\} + C^{(1)}$$

where $\rho(z) = \mathrm{dist}(z, \partial D)$.

The idea of the proof of Theorem 1

By using the interior estimates of gradient and estimates near the boundary we show global and a priory bondedness of gradient of a q.c. mapping w. It is previously showed that the function $u = |w|$ satisfies a certain elliptic differential inequality near the boundary of the unit disk. By using the quasiconformality, we prove that ∇w is bounded by a constant not depending on w.

Some open problems

- **Is every q.c. harmonic between smooth domains on the space bi-Lipschitz continuous?**
- **Is every q.c. solution of elliptic PDE between smooth domains on plane or on the space bi-Lipschitz continuous?**
- **Do some q.c. harmonic mappings on the space have critical points?**

Thanks for attention

On the length spectrum metric in infinite-dimensional Teichmüller spaces

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Riemann surfaces and Discontinuous groups 2013

Abstract.
We study when the length spectrum metric and Teichmüller metric define the same topology on Teichmüller space.

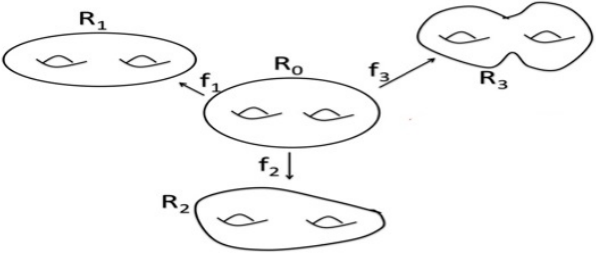
1. Introduction

R_0: a Riemann surface ($\neq \hat{\mathbb{C}}, \mathbb{C}, \mathbb{C} - \{0\}$, torus)
$T(R_0)$: the Teichmüller space of R_0

$$:= \{(R, f) \mid f : R_0 \to R \text{ a quasiconformal map } \}/ \overset{\text{Teich.}}{\sim}$$

$$(R, f) \overset{\text{Teich.}}{\sim} (S, g) \overset{\text{def.}}{\Leftrightarrow} \exists\, h : R \to S \text{ conformal s.t. } h \cong g \circ f^{-1}.$$

We denote the equivalence class of (R, f) by $[R, f]$.

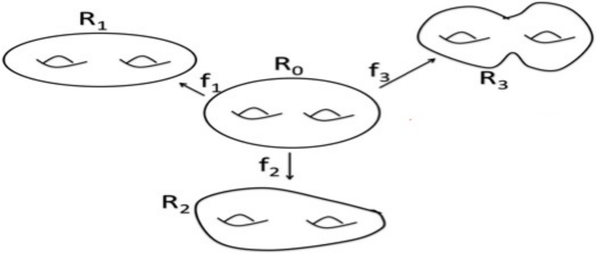


We define Teichmüller metric and the length spectrum metric on $T(R_0)$.

- Teichmüller metric d_T

$$d_T([R_1, f_1], [R_2, f_2]) = \inf_{f \cong f_2 \circ f_1{}^{-1}} \log K(f),$$

where the infimum is taken over all quasiconformal maps which are homotopic to $f_2 \circ f_1{}^{-1}$ and $K(f)$ is the maximal dilatation of f.

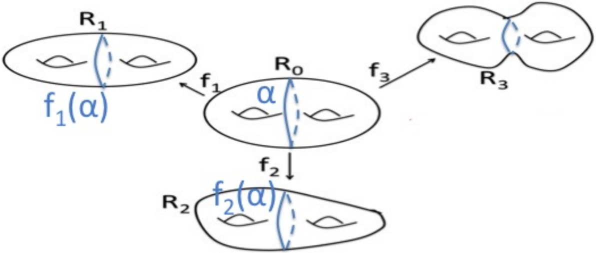


$d_T([R_1, f_1], [R_2, f_2]) = 0 \Leftrightarrow f_2 \circ f_1{}^{-1}$ is homotopic to some conformal map.

- the length spectrum metric d_L

$$d_L([R_1, f_1], [R_2, f_2]) = \sup_{\alpha \in \Sigma_{R_0}} \left| \log \frac{\ell_{R_2}(f_2(\alpha))}{\ell_{R_1}(f_1(\alpha))} \right|,$$

where $\Sigma_{R_0} := \{\alpha \mid \alpha \text{ is a closed curve in } R_0\}$ and $\ell_{R_i}(\alpha)$ is the hyperbolic length of a geodesic which is freely homotopic to α.



$d_L([R_1, f_1], [R_2, f_2]) = 0 \Leftrightarrow \ell_{R_1}(f_1(\alpha)) = \ell_{R_2}(f_2(\alpha))$ for any $\alpha \in \Sigma_{R_0}$.

Lemma (Sorvali, 1972)

For any $p_1, p_2 \in T(R_0)$*,*

$$d_L(p_1, p_2) \leq d_T(p_1, p_2)$$

holds.

Question:

When do d_T and d_L define the same topology on Teichmüller space?

d_T and d_L define the same topology on $T(R_0)$.

$\Leftrightarrow$ For sequence $\{p_n\}_{n=0}^{\infty} \subset T(R_0)$,

$$\lim_{n\to\infty} d_L(p_n, p_0) = 0 \Leftrightarrow \lim_{n\to\infty} d_T(p_n, p_0) = 0$$

$\overset{\text{Lemma}}{\Leftrightarrow}$ For sequence $\{p_n\}_{n=0}^{\infty} \subset T(R_0)$,

$$\lim_{n\to\infty} d_L(p_n, p_0) = 0 \Rightarrow \lim_{n\to\infty} d_T(p_n, p_0) = 0$$

We write $d_T \sim d_L$ on $T(R_0)$ if both metrics define the same topology.

⋆ **History**

1972: T. Sorvali conjectures that $d_T \sim d_L$ on $T(R_0)$ if R_0 is a topologically finite Riemann surface (i.e., a compact surface from which at most finitely many points have been removed).

1999: Z. Li and L. Liu prove that Sorvali's conjecture is true.

We consider Teichmüller spaces of topologically infinite Riemann surfaces below.

2003: H. Shiga shows that there exists a topologically infinite Riemann surface R_0 s.t. $d_T \not\approx d_L$ on $T(R_0)$. Also he gives a sufficient condition for d_T and d_L to define the same topology.

2008: Liu-Sun-Wei give a sufficient condition for d_T and d_L to define different topologies.

Our results

- We extend a theorem of Liu-Sun-Wei.
- We show that the converse of Shiga's theorem is not true.
- We extend Shiga's theorem.

Liu-Sun-Wei gave a sufficient condition for d_T and d_L to define different topologies on the Teichmüller space.

Theorem (Liu-Sun-Wei)

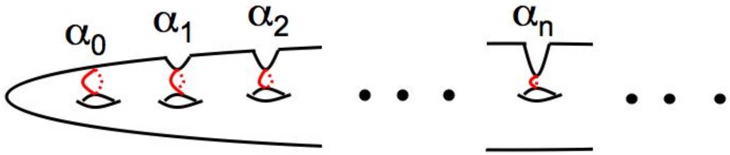
R_0: a Riemann surface,
$\exists\{\alpha_n\}_{n=1}^{\infty} \subset \Sigma_{R_0}$ *s.t.* $\ell_{R_0}(\alpha_n) \to 0$ $(n \to \infty)$.
$\Rightarrow d_T \nsim d_L$ *on* $T(R_0)$.

Liu-Sun-Wei gave a sufficient condition for d_T and d_L to define different topologies on the Teichmüller space.

Theorem (Liu-Sun-Wei)

R_0: a Riemann surface,
$\exists\{\alpha_n\}_{n=1}^{\infty} \subset \Sigma_{R_0}$ *s.t.* $\ell_{R_0}(\alpha_n) \to 0$ $(n \to \infty)$.
$\Rightarrow d_T \not\sim d_L$ *on* $T(R_0)$.

Example.



We extend a theorem of Liu-Sun-Wei as follows.

Theorem 1

R_0: a Riemann surface.
$\exists\{\alpha_n\}_{n=1}^{\infty} \subset \Sigma_{R_0}$ *s.t. for* $\forall\{\beta_n\}_{n=1}^{\infty} \subset \Sigma_{R_0}$ *with* $\alpha_n \cap \beta_n \neq \emptyset$ $(n = 1, 2, \dots)$,

$$\frac{\sharp(\alpha_n \cap \beta_n)\ell_{R_0}(\alpha_n)}{\ell_{R_0}(\beta_n)} \to 0 \ (n \to \infty).$$

$\Rightarrow d_T \not\sim d_L$ *on* $T(R_0)$.

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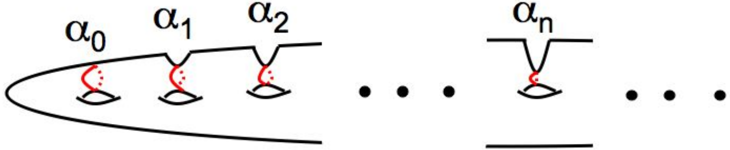
$$\frac{\sharp(\alpha_n \cap \beta_n)\ell_{R_0}(\alpha_n)}{\ell_{R_0}(\beta_n)} \to 0 \ (n \to \infty).$$

$\Rightarrow d_T \not\sim d_L$ *on* $T(R_0)$.

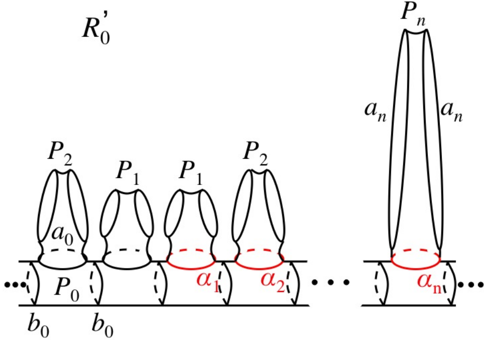
Example 1.
Any Riemann surface R_0 satisfying the assumption of Theorem of Liu-Sun-Wei satisfies the assumption of Theorem 1 by the collar lemma.



Example 2.
The Riemann surface R_0 constructed by Shiga (the first example of R_0 s.t. $d_T \not\approx d_L$ on $T(R_0)$) satisfies the assumption of Theorem 1. (This does not satisfy the assumption of Theorem of Liu-Sun-Wei.)

Example 3.
Except for Examples 1 and 2, we can construct a Riemann surface R_0 satisfying the assumption of Theorem 1 as follows:

P_0: a pair of pants with boundary lengths (a_0, b_0, b_0).
Make countable copies of P_0 and glue them as in the below Figure.
$\{a_n\}_{n=1}^{\infty}$: a monotone divergent sequence of positive numbers.
P_n: a pair of pants with boundary lengths (a_0, a_n, a_n).
Make two copies of P_n and glue each copy with the union of the copies of P_0 as in the below Figure.
Let R_0' denote a Riemann surface with boundary we have obtained.



Take some pants $\{P_m'\}_{m=1}^{\infty}$ and define $R_0 := R_0' \cup \bigcup_{m=1}^{\infty} P_m'$.

We prove Theorem 1. (We write it again.)

Theorem 1

R_0: a Riemann surface.
$\exists\{\alpha_n\}_{n=1}^{\infty} \subset \Sigma_{R_0}$ *s.t. for* $\forall\{\beta_n\}_{n=1}^{\infty} \subset \Sigma_{R_0}$ *with* $\alpha_n \cap \beta_n \neq \emptyset$ $(n = 1, 2, \ldots)$,

$$\frac{\sharp(\alpha_n \cap \beta_n)\ell_{R_0}(\alpha_n)}{\ell_{R_0}(\beta_n)} \to 0 \ (n \to \infty).$$

$\Rightarrow d_T \not\approx d_L$ *on* $T(R_0)$.

The proof is short.

We prove Theorem 1. (We write it again.)

Theorem 1

R_0: a Riemann surface.
$\exists\{\alpha_n\}_{n=1}^{\infty} \subset \Sigma_{R_0}$ s.t. for $\forall\{\beta_n\}_{n=1}^{\infty} \subset \Sigma_{R_0}$ with $\alpha_n \cap \beta_n \neq \emptyset$ $(n = 1, 2, \dots)$,

$$\frac{\sharp(\alpha_n \cap \beta_n)\ell_{R_0}(\alpha_n)}{\ell_{R_0}(\beta_n)} \to 0 \ (n \to \infty).$$

$\Rightarrow d_T \not\asymp d_L$ on $T(R_0)$.

The proof is short. We use the following lemma.

Matsuzaki's Lemma

α: a simple closed geodesic on a Riemann surface R_0,
$f : R_0 \to R_0$: the n-times Dehn twist along α.
$\Rightarrow$ The maximal dilatation $K(f)$ of an extremal quasiconformal map of f satisfies

$$K(f) \geq \left\{ \left(\frac{(2|n|-1)\ell_{R_0}(\alpha)}{\pi} \right)^2 + 1 \right\}^{1/2}.$$

3. A counterexample to the converse of Shiga's theorem

Shiga gave a sufficient condition for d_T and d_L to define the same topology on the Teichmüller space.

Theorem (Shiga)

R_0: a Riemann surface
$\exists$ a pants decomposition $R_0 = \cup_{k=1}^{\infty} P_k$ satisfying following conditions:
(1)Each connected component of ∂P_k is either a puncture or a simple closed geodesic of $R_0.(k = 1, 2, \dots)$
(2)$\exists M > 0$ s.t. if α is a boundary curve of some P_k then

$$0 < M^{-1} < \ell_{R_0}(\alpha) < M.$$

$\Rightarrow d_T \sim d_L$ on $T(R_0)$.

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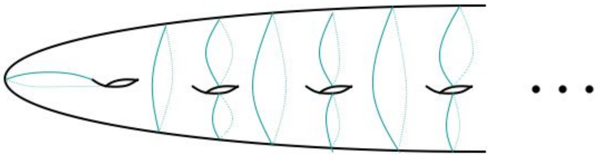
(2)$\exists M > 0$ s.t. if α is a boundary curve of some P_k then

$$0 < M^{-1} < \ell_{R_0}(\alpha) < M.$$

$\Rightarrow d_T \sim d_L$ on $T(R_0)$.

Example 1.
All Riemann surfaces of finite topological type satisfy Shiga's condition.

Example 2.
Some Riemann surfaces of infinite topological type satisfy Shiga's condition.



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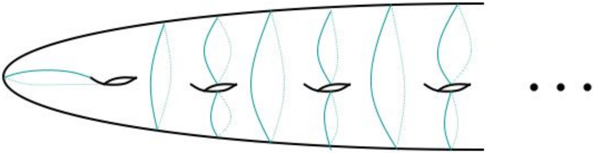
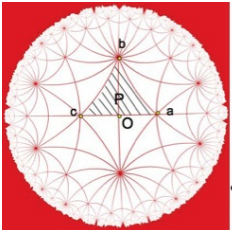
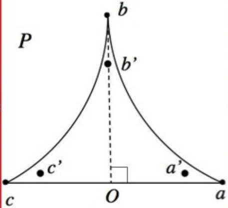
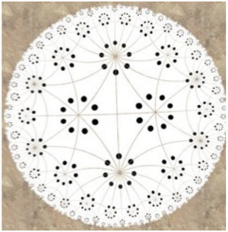


Now, we show that the converse of Shiga's theorem is not true by giving a counterexample.

Counterexample (to the converse of Shiga's theorem)
Γ: a hyperbolic triangle group of signature (2,4,8) acting on $\mathbb{D}$
P: a fundamental domain for Γ with angles $(\pi, \pi/4, \pi/4, \pi/4)$.
O, a, b, c: the vertices of P, where the angle at O is π.
$\varepsilon > 0$: a sufficiently small number
b': the point on the segment $[Ob]$ whose hyperbolic distance from b is ε.
Similarly, we take a' and c' in P.
We define a Riemann surface R_0 by removing the Γ-orbits of a', b', c' from the unit disk $\mathbb{D}$; $R_0 := \mathbb{D} - \{\gamma(a'), \gamma(b'), \gamma(c') \mid \gamma \in \Gamma\}$.

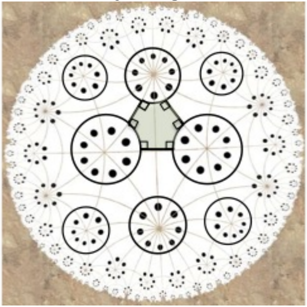






It is not difficult to show that R_0 does not satisfy the assumption of Shiga's Theorem.
It is difficult to show that $d_T \sim d_L$ on $T(R_0)$.

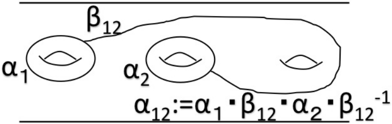
Outline of the proof (that $d_T \sim d_L$ on $T(R_0)$).
We show that for the sequence $\{p_n\}_{n=o}^{\infty} \subset T(R_0)$ s.t. $d_L(p_n, p_0) \to 0$ $(n \to \infty)$, $d_T(p_n, p_0)$ converges to 0 as $n \to \infty$.
We may assume that $p_0 = [R_0, id]$. Put $p_n = [R_n, f_n]$.
Step 1: We divide R_0 into punctured-disks and hyperbolic right-hexagons. Also, we divide R_n for a sufficiently large n similarly.



In Step 1, we note the following lemma.

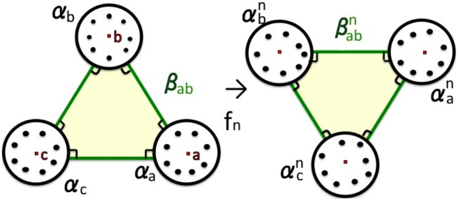
Lemma

R_0: a hyperbolic Riemann surface.
α_1, α_2: disjoint simple closed geodesics in R_0.
β_{12}: a simple arc connecting α_1 and α_2.
Assume that a closed curve $\alpha_{12} := \alpha_1 \cdot \beta_{12} \cdot \alpha_2 \cdot \beta_{12}^{-1}$ is non-peripheral.
$\Rightarrow \exists\ \beta_{12}^{\star}$: a geodesic connecting α_1 and α_2 s.t.
(1) β_{12} and $\beta_{12}^{\star}$ are homotopic, where the homotopy map moves each endpoint on each closed geodesic;
(2) $\beta_{12}^{\star}$ is orthogonal to α_1 and α_2;
(3) the length of $\beta_{12}^{\star}$ is determined by $\ell_{R_0}(\alpha_1)$, $\ell_{R_0}(\alpha_2)$ and $\ell_{R_0}(\alpha_{12})$.



Proof. This follows from properties of pants and right-hexagons. □

Step 2: We construct $(1+C_n)$-qc maps from right-hexagons in R_0 to right-hexagons in R_n, where $C_n \to 0$ $(n \to \infty)$.



Step 3: We construct $(1+C_n)$-qc maps from punctured-disks in R_0 to punctured-disks in R_n, where $C_n \to 0$ $(n \to \infty)$. Consequently, we obtain a quasiconformal map g_n of the whole of R_0 such that g_n is homotopic to f_n and $K(g_n) \to 1$ $(n \to \infty)$. Thus $d_T(p_n, p_0) \to 0$ $(n \to \infty)$.

In Steps 2 and 3, we use Bishop's Lemma.

Bishop's Lemma

$T_1, T_2 \subset \mathbb{D}$: two hyperbolic triangles with sides (a_1, b_1, c_1) and (a_2, b_2, c_2). Suppose all their angles are bounded below by $\theta > 0$ and

$$\varepsilon := \max(|\log \frac{a_1}{a_2}|, |\log \frac{b_1}{b_2}|, |\log \frac{c_1}{c_2}|) \leq A.$$

$\Rightarrow \exists\ C = C(\theta, A) > 0$ and $\exists$ a $(1 + C\varepsilon)$-quasiconformal map $\varphi : T_1 \to T_2$ such that φ maps each vertex to the corresponding vertex and φ is affine on the edge of T_1.

$\square$

4. Extension of Shiga's theorem

We extend Shiga's theorem as follows.

Theorem 2.

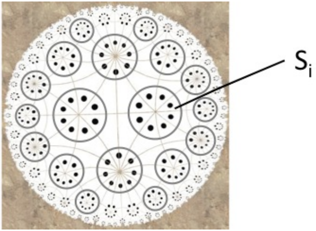
R_0: a Riemann surface.
$\exists\ M > 0$ and $\exists$ a decomposition

$$R_0 = S \cup (R_0 - S)$$

s.t.
(1) S is an open subset of R_0 whose relative boundary consists of simple closed geodesics and each connected component of S has a pants decomposition satisfying the same condition as that of Shiga's theorem for M
(2) $R_0 - S$ is of genus 0 and $d_{R_0}(x, S) < M$ for any $x \in R_0 - S$.
$\Rightarrow d_T \sim d_L$ on $T(R_0)$.

Examples 1.

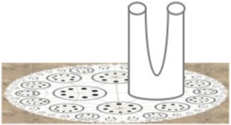
The above counterexample to the converse of Shiga's theorem satisfies Theorem 2. Let S_i $(i = 1, 2, ...)$ be a punctured disk, then $S = \cup_{i=1}^{\infty} S_i$.

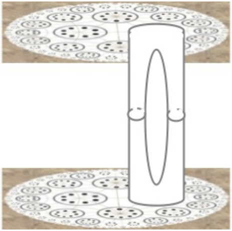


Also, we can construct a Riemann surface R_0 satisfying Theorem 2 by replacing a hyperbolic triangle group Γ with an arbitrary Fuchsian group with a compact fundamental region.

Examples 2.

In a Riemann surface of Example 1, we replace a punctured disk S_i with a Riemann surface satisfying Shiga's condition. We regard it as a block and construct a Riemann surface R_0 with two or more holes. (See figure.) Then R_0 satisfies Theorem 2.





Corollary 3.

Let R_0 be a Riemann surface with bounded geometry. Also, assume that R_0 has finite genus.
$\Rightarrow d_T \sim d_L$ on $T(R_0)$.

We say that a Riemann surface R_0 has *bounded geometry* if it satisfies the following condition:
There exists a constant $M > 0$ such that any closed geodesic has the length greater than $1/M$ and for any $x \in R_0$, there exists a closed curve based on x with the length less than M.

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On accumulation points of geodesics in Thurston's boundary of Teichmüller spaces

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January 14, 2013

Teichmüller spaces

- X, Y : Riemann surfaces of genus $g \geq 2$
- $f : X \to Y$: a quasi-conformal mapping (q.c.)
- (Y, f) : a marked Riemann surface of genus g

$(Y_1, f_1) \sim (Y_2, f_2) \overset{\text{def}}{\Leftrightarrow} \exists h \colon Y_1 \to Y_2$: biholo.
s.t. the diagram is commutative.

$$\begin{tikzcd} & Y_1 \arrow[d, "h"] \\ X \arrow[ur, "f_1"] \arrow[r, "f_2"'] & Y_2 \end{tikzcd}$$

Definition (Teichmüller spaces)

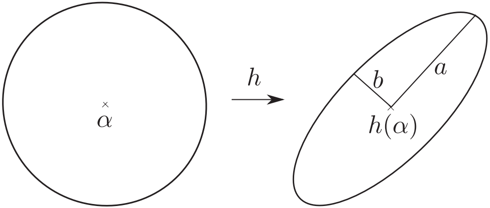
$$T_g := \{\text{marked Riemann surfaces of genus } g\} / \sim$$

Teichmüller distance

Definition (Teichmüller distance)

$$d([Y_1, f_1], [Y_2, f_2]) := \log \inf_h K_h,$$

where $h : Y_1 \to Y_2$ moves over all q.c. homotopic to $f_2 \circ f_1^{-1}$, and where K_h is the maximal dilatation of h.

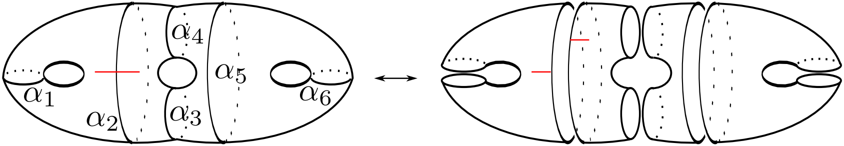


$$K_h := \sup_{\alpha \in X} \frac{a}{b}$$

$\rightsquigarrow$ (T_g, d) is a complete, geodesic metric space.

Fenchel-Nielsen cordinates of T_g

$$\boldsymbol{T_g} \cong \mathbb{R}^{\boldsymbol{6g-6}} \quad \text{(homeomorphic)}$$



- $\mathcal{P} := \{\alpha_i\}_{i=1}^{3g-3}$: a pants curve system of X
- ℓ_{α_i} : lengh parameter of α_i, t_{α_i} : twist parameter of α_i
- $(\ell_{\alpha_i}, t_{\alpha_i})_{i=1}^{3g-3}$: a Fenchel-Nielsen coordinate of T_g

Problem

Formalize the boundary behavior of geodesics in (T_g, d).

1. Determine a condition that geodesics converge.
2. Find a divergent geodesic.
3. Find a boundary point to which no geodesic accumulates.
4. Determine the limit sets (the set of all accumulation points in the boundary) of geodesics.

From the uniformization theorem,

$$T_g \cong \{\text{hyperbolic metrics on } X\}/\sim.$$

- $\mathcal{S} := \{\text{non-trivial simple closed curves on } X\}/\text{free homotopy}$
- $\ell_\rho(\alpha) := \inf_{\alpha' \simeq \alpha} \text{length}_\rho(\alpha') \quad (\alpha \in \mathcal{S},\ \rho \in T_g)$

The map $\ell_\rho : \alpha \mapsto \ell_\rho(\alpha)$ is an element of the space $\mathbb{R}^{\mathcal{S}}_{\geq 0}$. So we define the map ℓ as

$$\ell : T_g \ni \rho \mapsto \ell_\rho \in \mathbb{R}^{\mathcal{S}}_{\geq 0}.$$

We consider the map

$$\tilde{\ell}\colon T_g \xrightarrow{\ell} \mathbb{R}_{\geq 0}^{\mathcal{S}} \xrightarrow{\text{proj.}} \left(\mathbb{R}_{\geq 0}^{\mathcal{S}} \setminus \{0\}\right) \Big/ \mathbb{R}_+.$$

Theorem (Thurston)

1. *$\tilde{\ell}$ is an embedding and $\tilde{\ell}(T_g)$ is relatively compact.*
2. $\overline{\tilde{\ell}(T_g)} \cong \mathbb{B}^{6g-6} \cup \mathbb{S}^{6g-7}$ *(closed ball)*
3. $\partial\tilde{\ell}(T_g) \cong \mathcal{PMF} \cong \mathbb{S}^{6g-7}$ *(sphere)*
   (Projective Measured Foliations)

$\diamondsuit$ The action of the mapping class group on T_g extends continuously to the boundary.

Teichmüller geodesics

Around a regular point of a holomorphic quadratic differential $\varphi = \varphi(z)dz^2$ on X, the local coordinates

$$w = \int \sqrt{\varphi(z)}dz$$

determine a (singular) flat structure on X. Letting X_t be the Riemann surface with the local coordinates

$$w_t = e^{t/2}u + ie^{-t/2}v \quad (w = u + iv),$$

we can get the map

$$\mathbb{R} \ni t \mapsto X_t \in T_g.$$

This map is an isometric embedding (Teichmüller geodesic).

Definition (convergence in $\mathcal{PMF}$)

A sequence ρ_n on T_g converges to $[G] \in \mathcal{PMF}$ if $\exists c_n \to \infty$ s.t.

$$\frac{\ell_{\rho_n}(\alpha)}{c_n} \to i(G, \alpha) \quad (\alpha \in \mathcal{S}),$$

where $i(\cdot, \cdot) : \mathcal{MF} \times \mathcal{MF} \to \mathbb{R}_{\geq 0}$ (the geometric intersection number function).

- F : the vertical foliation of φ
- $\mathcal{G}_t$: the hyperbolic metric uniformizing the surface X_t
- $\mathcal{G}_{F,X} = \{\mathcal{G}_t\}_{t \geq 0}$: the Teichmüller geodesic ray from X

$\diamondsuit$ The limit set $L(\mathcal{G}_{F,X})$ is a non-empty, connected, closed subset of $\mathcal{PMF}$.

Problem

Formalize the asymptotic behavior of geodesics in Thurston's compactification of T_g.

1. Determine a condition for $\sharp L(\mathcal{G}_{F,X}) = 1$.
2. Find a geodesic with $\sharp L(\mathcal{G}_{F,X}) \geq 2$.
3. For all $X \in T_g$, show
$$\mathcal{PMF} \neq \bigcup_{F \in \mathcal{MF}} L(\mathcal{G}_{F,X}).$$
4. Examine a relation between two foliations F and G representing an accumuration point of $\mathcal{G}_{F,X}$.

Problem 1 : On the convergence of geodesics

Theorem (Masur, 1982)

1. *If $F = \sum_{i=1}^{N} a_i \alpha_i$ is rational, namely, F has only closed leaves, then*
$$\lim_{t \to \infty} \mathcal{G}_t = [\alpha_1 + \cdots + \alpha_N] \in \mathcal{PMF}.$$
2. *If F is uniquely ergodic, namely, F has only one transverse measure up to multiplication, then*
$$\lim_{t \to \infty} \mathcal{G}_t = [F] \in \mathcal{PMF}.$$

Problem 2 : On the existance of diverging geodesics

Remark

- $\{F \in \mathcal{MF} \mid F$ is uniquely ergodic $\}$ is full measure and
- $\{F \in \mathcal{MF} \mid F$ is rational $\}$ is dense.

Corollary

Limit sets $L(\mathcal{G}_{F,X})$ are null sets, and they have no interior point.

Theorem (Lenzhen, 2008)

There exists a Teichmüller geodesic that do not have a limit in $\mathcal{PMF}$.

Theorem (I)

Let $G = \sum_{i=1}^{N} b_i \alpha_i$ be a rational measured foliation. Then the following holds.

1. *If $b_i \neq b_j$ for some $i \neq j$, then there is no Teichmüller geodesic which accumulates to $[G]$.*
2. *If $b_1 = \cdots = b_N$, then the following three conditions are equivalent.*
   (a) $[G] \in L(\mathcal{G}_{F,X})$.
   (b) $F = \sum_{i=1}^{N} a_i \alpha_i$ *for some* $a_i > 0$.
   (c) $L(\mathcal{G}_{F,X}) = \{[\sum_{i=1}^{N} \alpha_i]\}$.

Sketch of the proof

Proof by contradiction

Suppose that $[G]$ is an accumulation point of some geodesic $\mathcal{G}_{F,X}$.

- Since $i(F,G)=0$, we see $i(F,\alpha_i)=0$ for all i.
- We show that $i(F,\beta)=0$ for any curve $\beta\in\mathcal{S}$ with $i(\beta,\alpha_i)=0$ for all i.

These imply that $F=\sum a_i\alpha_i$ where $a_i\geq 0$. It follows from Masur's theorem that

$$\mathcal{G}_{F,X}\to[\alpha_1+\cdots+\alpha_N]\neq[G].$$

This is a contradiction.

Minimal decompositions of foliations

$\diamondsuit$ Each leaf of a measured foliation F either is closed or is dense in a subsurface Ω (called a *minimal domain*). We write F as the sum

$$F = \sum_{\Omega} F_{\Omega} + \sum_{i=1}^{N} a_i \alpha_i \ \text{(minimal decomposition)},$$

where F_{Ω} is a minimal foliation on Ω, and where α_i is a closed curve and $a_i \geq 0$.

Remark If $a_i = 0$, then α_i is homotopic to a boundary component of a minimal domain.

Problem 4 : On accumulation points of geodesics

Theorem (I)

$$F = \sum_{\Omega} F_\Omega + \sum_{i=1}^{N} a_i \alpha_i \text{ (\textit{minimal decomposition})}$$

If $\sum_\Omega F_\Omega \neq 0$*, then* $[G] \in L(\mathcal{G}_{F,X})$ *is written as the sum*

$$G = \sum_{\Omega} G_\Omega + \sum_{i=1}^{N} b_i \alpha_i$$

satisfying the following properties.

1. $\sum_\Omega G_\Omega \neq 0$.
2. G_Ω *and* F_Ω *are topologically equivalent unless* $G_\Omega = 0$.
3. *If* $b_1 + \cdots + b_N > 0$*, then* $G_\Omega \neq 0$ *for all* Ω.
4. $a_i = 0$ *implies* $b_i = 0$.

Problem 4 : On accumulation points of geodesics

We say a sequence $\rho_n \in T_g$ is *thick along a curve* $\alpha \in \mathcal{S}$ if

$$\inf_n \ell_{\rho_n}(\alpha_n) \neq 0,$$

where $\alpha_n \in \mathcal{S}$ denotes the ρ_n-shortest curve intersecting α essentially.

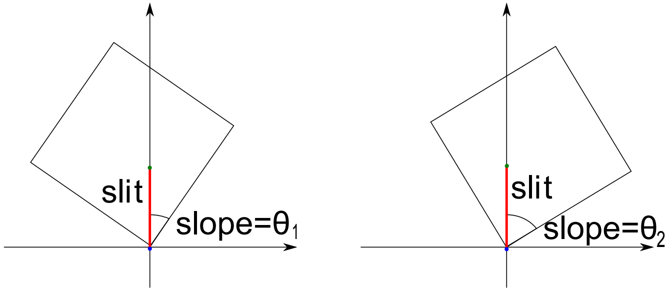
Theorem (I)

Under the same condition for the previous theorem, we write $\mathcal{G}_{t_n} \to [G]$. If there exist a minimal domain Ω_0 and a non-peripheral curve $\alpha_0 \subset \Omega_0$ such that $\mathcal{G}_{t_n}$ is thick along α_0, then

$$G = \sum_{\Omega} G_{\Omega},$$

where $G_{\Omega} \cong F_{\Omega}$ unless $G_{\Omega} = 0$ and $G_{\Omega_0} \neq 0$.

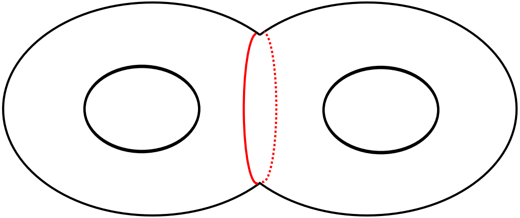
Example(Lenzehn's construction)



1. Take two square tori X_1 and X_2.
2. Cut along the slits (=red lines).
3. Glue together along the slits crosswise.

Example(Lenzehn's construction)

The resulting Riemann surface X is of genus two.

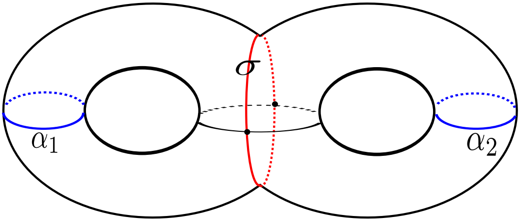


- θ_1, θ_2 : the slopes of slits
- σ : the curve in X corresponding to the slits
- F_{θ_i} : the vertical foliation on X_i ($i = 1, 2$)
- F : the vertical foliation on X

Then

$$F = F_{\theta_1} + 0 \cdot \sigma + F_{\theta_2}.$$

Case 1 : θ_1 and θ_2 are rational



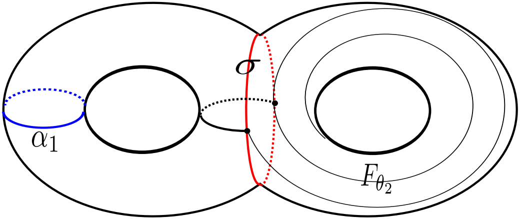
$$\exists \boldsymbol{\alpha_i} \in \boldsymbol{\mathcal{S}} \text{ s.t. } \boldsymbol{F_{\theta_i} = a_i \alpha_i} \quad \boldsymbol{(a_i > 0)}$$

So $\boldsymbol{F = a_1\alpha_1 + a_2\alpha_2}$. Since $\boldsymbol{F}$ is rational,

$$\boldsymbol{L(\mathcal{G}_{F,X}) = \{[\alpha_1 + \alpha_2]\}}$$

from the theorem of Masur.

Case 2 : θ_1 is irrational and θ_2 is rational



$$\boldsymbol{F = a_1\alpha_1 + 0 \cdot \sigma + F_{\theta_2}} \quad \text{(minimal decomposition)},$$

where $\boldsymbol{\alpha_1} \in \boldsymbol{\mathcal{S}}$ and $\boldsymbol{F_{\theta_2}}$ is minimal and uniquely ergodic in $\boldsymbol{X_2}$. Then

$$\boldsymbol{L(\mathcal{G}_{F,X}) = \{[F_{\theta_2}]\}}.$$

Case 3 : θ_1 and θ_2 are irrational

$$F = F_{\theta_1} + 0 \cdot \sigma + F_{\theta_2} \quad \text{(minimal decomposition)}.$$

So

$$L(\mathcal{G}_{F,X}) \subset \{[a_1 F_{\theta_1} + a_2 F_{\theta_2}] \mid a_1 + a_2 = 1\}.$$

Theorem (Lenzhen, 2008)

Under the above notation, suppose that θ_1 is of bounded type as a continued fraction and that θ_2 is of unbounded type. Then

$$\sharp L(\mathcal{G}_{F,X}) \geq 2 \quad \textit{and} \quad [F_{\theta_2}] \in L(\mathcal{G}_{F,X}).$$

Continued fractions

Every real number θ has a continued fraction expantion of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

where $a_0 \in \mathbb{Z}, a_i \in \mathbb{N} \cup \{0\}$.

- θ is irrational $\Leftrightarrow$ the number of $a_i \neq 0$ is infinite
- θ is of bounded type $\Leftrightarrow$ $\{a_i\}_{i\in\mathbb{N}}$ is bounded
- θ is unbounded type $\Leftrightarrow$ $\{a_i\}_{i\in\mathbb{N}}$ is unbounded

The meaning of Lenzehn's condition

「θ_1, θ_2 : irrational」 $\cdots$ They determine the minimal ergodic foliation on the torus X_i induced by the parallel line field of slope θ_i.

「θ_1 : bounded type」 $\cdots$ There is a curve in X_1 along which $\mathcal{G}_t$ is thick.

「θ_2 : unbounded type」 $\cdots$ There are sequences s_n, t_n of time such that $\mathcal{G}_{s_n}$ is thick along a curve in X_2 but $\mathcal{G}_{t_n}$ is *thin along a curve* in X_2.

A sufficient condition for diverging

Theorem (I)

Suppose that F has at least two minimal domains and that there exist two minimal domains Ω_1, Ω_2 which satisfy the following three conditions.

(a) *There is a sequence s_n such that $\mathcal{G}_{s_n}$ is thick along a curve in Ω_1.*

(b) *There is a sequence t_n such that $\mathcal{G}_{t_n}$ is thick along a curve in Ω_2.*

(c) *There is a pants curve system $\mathcal{P}_{t_n}(\Omega_2)$ of Ω_2 such that*

$$\max_{\gamma \in \mathcal{P}_{t_n}(\Omega_2)} \ell_{\mathcal{G}_{t_n}}(\gamma) \to 0 \quad as \quad n \to \infty.$$

Then

$$\sharp L(\mathcal{G}_{F,X}) \geq 2.$$

Future task

Suppose that F is minimal, namely, every leaf of F is dense in the surface X, and that F is not uniquely ergodic.

It is known that there are finitely many ergodic measures $\{\mu_i\}_{i=1}^p$ such that any transverse measures of the foliation F are written as the sum $a_1\mu_1 + a_2\mu_2 + \cdots + a_p\mu_p$ where $a_i \geq 0$. Hence

$$L(\mathcal{G}_{F,X}) \subset \left\{ \sum_{i=1}^p a_i\mu_i \middle| a_i \geq 0,\ \sum_{i=1}^p a_i = 1 \right\}.$$

Question is whether the above inclusion is equal or not.

On the combinatorial Hodge star operator and holomorphic cochains

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January 14. 2013

S.O. Wilson, 2007

$\bigstar$; *combinatorial Hodge star operator defined on cochains equipped with an inner product of a triangulated manifold.*

Wilson showed that for a certain cochain inner product which he named the Whitney inner product, this operator converges to the smooth Hodge star operator as the mesh of the triangulation tends to zero if the manifold is Riemannian.

He also stated that $\bigstar\bigstar \neq \pm\mathrm{Id}$ *in general and raised a question if* $\bigstar\bigstar$ *approaches* $\pm\mathrm{Id}$.

In this talk, we solve this problem affirmatively.

Remark; The (smooth) Hodge star $\star$ on forms satisfies

$$\star\star = (-1)^{j(n-j)}\mathrm{Id},$$

where j is the degree of differential forms and n is the dimension of the manifold.

2. Preliminaries

Notation 2.1

- M; a closed oriented C^∞ manifold of dimension n.
- K; a fixed C^∞ triangulation of M.
- $C^j(K)$; the simplicial j-cochains of K with values in $\mathbb{R}$, where $j = 0, 1, \cdots, n$.
- $\Lambda^j(M)$; C^∞ differential forms of degree j on M.
- $\Lambda(M) := \bigoplus \Lambda^j(M)$.

Coboundary operator δ and its adjoint δ^*

Giving an ordering of the verticies, we have a coboundary operator

$$\delta : C^j \to C^{j+1}.$$

If the cochains $C(K) = \bigoplus_j C^j(K)$ are equipped with an inner product $\langle\ ,\ \rangle$, then we can define the adjoint of δ denoted by δ^* as

$$\langle \delta^* \sigma, \tau \rangle = \langle \sigma, \delta \tau \rangle.$$

Definition 2.2

The combinatorial Laplacian is defined to be $\blacktriangle = \delta^*\delta + \delta\delta^*$, and the space of harmonic j-cochains of K is defined to be

$$\mathcal{H}C^j(K) = \{a \in C^j \,|\, \blacktriangle a = 0\}.$$

Recall that for forms the Laplacian is $\Delta = d^*d + dd^*$. The following theorem is formally analogous to the Hodge decomposition of forms.

Theorem 2.3 (Eckmann)

Let (C, δ) be a finite dimensional complex with inner product and induced adjoint δ^*. There is an orthogonal direct sum decomposition

$$C^j(K) = \delta C^{j-1}(K) \oplus \mathcal{H}C^j(K) \oplus \delta^* C^{j+1}(K)$$

and $\mathcal{H}C^j(K) \cong H^j(K)$, the cohomology of (K, δ) in degree j.

Suppose M is a Riemannian manifold. The Riemannian metric provides the space $\Lambda(M) = \bigoplus \Lambda^j(M)$ of smooth differential forms on M with an inner product

$$\langle \omega, \psi \rangle = \int_M \omega \wedge \star \psi, \quad \omega, \psi \in \Lambda(M),$$

where $\star$ is the Hodge star operator.

We denote by $L^2\Lambda^j$ the completion of $\Lambda^j(M)$ with respect to this inner product.

Whitney map

We now define the Whitney map W of $C^j(K)$ into $L^2\Lambda^j$.
To do so

- identify K with M and fix some ordering of the set of vertices of K.
- Denote by μ_k the barycentric coordinate corresponding to the k-th vertex p_k in K.
- Write every cochain $c \in C^j(K)$ as the sum $c = \sum c_\tau \cdot \tau$ with $c_\tau \in \mathbb{R}$ and τ running through all j-simplexes $[p_0, p_1, \cdots, p_j]$ whose vertices form an increasing sequence with respect to the ordering of K.

Now we define $W\tau$ for such simplexes τ.

Whitney map

Definition 2.4

Let $\tau = [p_0, p_1, \cdots, p_j]$, where $p_0, p_1, \cdots, p_j$ is an increasing sequence of vertices of K. Define $W\tau \in L_2\Lambda^j$ by the formula

$$W\tau = j! \sum_{i=0}^{j} (-1)^i \mu_i d\mu_0 \wedge d\mu_1 \wedge \cdots \wedge \widehat{d\mu_i} \wedge \cdots \wedge d\mu_j$$

where $\widehat{\ }$ over a symbol means deletion.

We extend W by linearity to all of $C^j(K)$ and call it the Whitney map.

Examples

Example 2.5

If $\tau = [p_0] \in C^0(K)$, then

$$W\tau = \mu_0.$$

If $\tau = [p_0, p_1] \in C^1(K)$, then

$$W\tau = \mu_0 d\mu_1 - \mu_1 d\mu_0.$$

Some properties of W

Lemma 2.6

The following hold:
(1) $W\tau = 0$ on $M \setminus \overline{St(\tau)}$,
(2) $dW = W\delta$,
where St denotes the open star and the bar denotes closure.

de Rham map

There is a map of converse direction, the de Rham map R from differential forms to $C^j(K)$ which is given by integration, i.e.

Definition 2.7

for any differential form ω and chain c we have:

$$R\omega(c) = \int_c \omega.$$

One can check that $RW = Id$. In general $WR \neq Id$, but Dodziuk and Patodi showed that WR is approximately equal to the identity.

Definition 2.8

K: a triangulation of a Riemaniann manifold M.
The mesh $\eta = \eta(K)$ of a triangulation is:

$$\eta = \sup r(p, q),$$

where r means the geodesic distance in M and the supremum is taken over all the pair of vertices p, q of a 1-simplex in K.

Theorem 2.9 (Dodziuk and Patodi)

Let M be a Riemannian manifold with triangulation K of mesh η. There exist a positive constant C and a positive integer m, independent of K, such that

$$\|\omega - WR\omega\| \leq C \cdot \|(Id + \Delta)^m \omega\| \cdot \eta$$

for all C^∞ differential forms ω on M.

▸ Theorem 2.14

Whitney inner product

Using W, we can define an inner product on $C(K)$ which is induced by the inner product on differential forms, namely,

Definition 2.10

For $\sigma, \tau \in C(K)$

$$\langle \sigma, \tau \rangle = \int_M W\sigma \wedge \star W\tau = \langle W\sigma, W\tau \rangle,$$

where we use the same notation $\langle\ ,\ \rangle$ for the inner product on $C(K)$ and for the inner product on Λ. We call this inner product the Whitney inner product.

Product operation

Whitney also defined a product operation.

Definition 2.11

We define $\cup : C^j(K) \otimes C^k(K) \to C^{j+k}(K)$ by

$$\sigma \cup \tau = R(W\sigma \wedge W\tau).$$

combinatorial Hodge star operator

Wilson defined the combinatorial Hodge star operator and showed that for the Whitney inner product, this operator converges to the smooth Hodge star operator as the mesh of the triangulation tends to zero.

Definition 2.12

Let K be a triangulation of a closed oriented manifold M, with simplicial cochains $C = \bigoplus_j C^j$. Let $\langle\ ,\ \rangle$ be an inner product on C such that C^i is orthogonal to C^j for $i \neq j$. For $\sigma \in C^j$ we define $\bigstar\sigma \in C^{n-j}$ by:

$$\langle \bigstar\sigma, \tau \rangle = (\sigma \cup \tau)[M],$$

where $[M]$ denotes the fundamental class of M.

Remark; $(\sigma \cup \tau)[M] = R(W\sigma \wedge W\tau)[M] = \int_M W\sigma \wedge W\tau$. app of Lemma 3.1

Several properties of $\bigstar$

Lemma 2.13

The following hold:
(1) $\bigstar\delta = (-1)^{j+1}\delta^*\bigstar$, i.e. $\bigstar$ is a chain map.
(2) For $\sigma \in C^j$ and $\tau \in C^{n-j}$, $\langle \bigstar\sigma, \tau \rangle = (-1)^{j(n-j)}\langle \sigma, \bigstar\tau \rangle$, i.e. $\bigstar$ is (graded) skew-adjoint.
(3) $\bigstar$ induces isomorphisms $\mathcal{H}C^j(K) \to \mathcal{H}C^{n-j}(K)$ on harmonic cochains.

$\bigstar$ converges to $\star$

Theorem 2.14 (Wilson)

Let M be a Riemannian manifold with triangulation K of mesh η. There exist a positive constant C' and a positive integer m, independent of K, such that

$$\| \star \omega - W \bigstar R\omega \| \leq C' \cdot \| (Id + \Delta)^m \omega \| \cdot \eta$$

for all C^∞ differential forms ω on M.

Theorem 2.9

Proof of the thm

Useful lemma

Lemma 2.15

For $a \in C^j$ and $b \in C^{n-j}$,

$$\langle W\bigstar a,\ Wb\rangle = \langle \star Wa,\ Wb\rangle,$$

where $\langle\ ,\ \rangle$ is the inner product on $L^2\Lambda^{n-j}$.

In general, $\langle W\bigstar a, \omega\rangle \neq \langle \star Wa, \omega\rangle$ for a form $\omega \in L^2\Lambda^{n-j}$.
If $\omega \in W(C^{n-j})$, $=$ holds.

3. Main theorem

For the remainder, we fix the inner product on $C^j(K)$ the Whitney inner product unless otherwise mentioned. (Thus M must be Riemannian.)

Lemma 3.1

For an arbitrary $a \in C^j(K)$,

$$\langle a, a \rangle \geq \langle \bigstar a, \bigstar a \rangle.$$

This means

$$\|Wa\| \geq \|W \bigstar a\|.$$

▸ pf of the theorem

Proof.

We put $V = W(C^j(K))$, the image of $C^j(K)$ in $L^2\Lambda^j$ via the Whitney map W. Since V is a finite dimensional subspace of $L^2\Lambda^j$, we have an orthogonal direct sum decomposition

$$L^2\Lambda^j = V \oplus V^\perp.$$

We write $\star Wa = \alpha_1 + \alpha_2$, where $\alpha_1 \in V$ and $\alpha_2 \in V^\perp$. Then

$$\langle a,\, a\rangle = \langle Wa,\, Wa\rangle = \langle \star Wa,\, \star Wa\rangle = \langle \alpha_1,\, \alpha_1\rangle + \langle \alpha_2,\, \alpha_2\rangle.$$

On the other hand, by Lemma 2.15,

$$\begin{aligned}\langle \bigstar a,\, \bigstar a\rangle &= \langle W\bigstar a,\, W\bigstar a\rangle = \langle \star Wa,\, W\bigstar a\rangle = \langle \alpha_1,\, W\bigstar a\rangle \\ &= \langle \alpha_1,\, \star Wa\rangle = \langle \alpha_1,\, \alpha_1\rangle.\end{aligned}$$

$\square$

Application of Lemma 3.1 to holomorphic cochains

To define holomorphic 1-cochains we need to extend some of our definitions to the case of complex valued cochains.

- $\langle\, ,\, \rangle$; any hermitian inner product on the complex valued simplicial 1-cochains of a triangulated topological surface K.

Definition 3.2

We define the associated combinatorial star operator $\bigstar$ by

$$\langle \bigstar\sigma, \tau\rangle = (\sigma \cup \overline{\tau})[M],$$

where the bar denotes complex conjugation and $\cup$ is as Definition 2.11, extended over $\mathbb{C}$ linearly. Definition 2.12

Just as with real coefficients, we have the Hodge decomposition

$$C^1(K) = \delta C^0(K) \oplus H^1(K) \oplus \delta^* C^2(K)$$

where H^1 is the space of complex valued harmonic 1-cochains.

Definition 3.3

The space of holomorphic 1-cochains $\mathcal{H}^{1,0}(K)$ is defined to be

$$\mathcal{H}^{1,0}(K) = \{\omega \in H^1(K) | \bigstar\omega = -i\lambda\omega, \ (\lambda > 0)\}.$$

The space of anti-holomorphic 1-cochains $\mathcal{H}^{0,1}(K)$ is defined to be

$$\mathcal{H}^{0,1}(K) = \{\omega \in H^1(K) | \bigstar\omega = i\lambda\omega, \ (\lambda > 0)\}.$$

Lemma 3.4

Let K be a triangulation of a surface M of genus g. A hermitian inner product on the simplicial 1-cochains of K gives an orthogonal direct sum decomposition

$$H^1(K) = \mathcal{H}^{1,0}(K) \oplus \mathcal{H}^{0,1}(K).$$

Each summand on the right has complex dimension g and complex conjugation maps $\mathcal{H}^{1,0}(K)$ to $\mathcal{H}^{0,1}(K)$ and vice versa.

Riemann's bi-linear relations

Definition 3.5

Let $\{a_1, \cdots, a_g, b_1, \cdots, b_g\}$ be a canonical homology basis for M. For $h \in \mathcal{H}^{1,0}(K)$, the A-periods and B-periods of h are the following complex numbers:

$$A_j = h(a_j), \quad B_j = h(b_j) \quad \textit{for} \quad 1 \leq j \leq g.$$

Theorem 3.6 (Riemann's bi-linear relations)

If σ, $\sigma' \in \mathcal{H}^{1,0}(K)$ have A-periods A_j, A'_j and B-periods B_j, B'_j respectively, then

$$\sum_{j=1}^{g} (A_j B'_j - B_j A'_j) = 0.$$

Corollary to Lemma 3.1

Corollary 3.7

Let K be a triangulation of a compact Riemann surface M. Let $\bigstar$ be the combinatorial Hodge star induced by the Whitney inner product.
Then for any $\omega \in \mathcal{H}^{1,0}(K)$, the negative imaginary eigenvalue $-i\lambda$ of $\bigstar$ satisfies $\lambda \leq 1$, that is,

$$\bigstar\omega = -i\lambda\omega, \qquad \lambda \in (0, 1].$$

Yamaki-kun found this fact independently, by a different method.

Proof.

$$\langle \omega, \omega \rangle \geq \langle \bigstar\omega, \bigstar\omega \rangle = \lambda^2 \langle \omega, \omega \rangle.$$

Thus $\lambda \leq 1$. □

Main theorem

Theorem 3.8

Let M be a Riemannian manifold with triangulation K of mesh η. There exist a positive constant C and a positive integer m, independent of K, such that

$$\| \star\star\omega - W\bigstar\bigstar R\omega\| \leq C \cdot \|(Id + \Delta)^m\omega\| \cdot \eta$$

for all C^∞ differential forms ω on M.

Remark;

$$\star\star\,\omega = (-1)^{j(n-j)}\omega,$$

where j is the degree of the form ω.

Proof of the theorem

$$
\begin{aligned}
\|\star\star\omega - W\bigstar\bigstar R\omega\| &\leq \|\star\star\omega - \star W\bigstar R\omega\| + \|\star W\bigstar R\omega - W\bigstar\bigstar R\omega\| \\
&= \|\star\,\omega - W\bigstar R\omega\| + \|\star W\bigstar R\omega - W\bigstar\bigstar R\omega\|
\end{aligned}
$$

The first term satisfies

$$
\|\star\,\omega - W\bigstar R\omega\| \leq C' \cdot \|(Id + \Delta)^m \omega\| \cdot \eta \tag{1}
$$

by Theorem 2.14. Theorem 2.14

For the second term, we calculate

$$
\begin{aligned}
& \| \star W\bigstar R\omega - W\bigstar\bigstar R\omega\| \\
\leq \quad & \| \star W\bigstar R\omega - \star\star\, \omega\| + \| \star\star\omega - W\bigstar R \star \omega\| \\
& + \|W\bigstar R \star \omega - W\bigstar\bigstar R\omega\| \\
\leq \quad & C'\|(Id + \Delta)^m\omega\| \cdot \eta + C'\|(Id + \Delta)^m \star \omega\| \cdot \eta \\
& + \|W\bigstar R \star \omega - W\bigstar\bigstar R\omega\| \\
= \quad & 2C'\|(Id + \Delta)^m\omega\| \cdot \eta + \|W\bigstar R \star \omega - W\bigstar\bigstar R\omega\|. \qquad (2)
\end{aligned}
$$

Using Lemma 3.1, we have Lemma 3.1

$$\begin{aligned} & \|W\bigstar R\star\omega - W\bigstar\bigstar R\omega\| = \|W\bigstar(R\star\omega - \bigstar R\omega)\| \\ \leq\ & \|W(R\star\omega - \bigstar R\omega)\| = \|WR\star\omega - W\bigstar R\omega\| \\ \leq\ & \|WR\star\omega - \star\omega\| + \|\star\omega - W\bigstar R\omega\| \\ \leq\ & C\|(Id+\Delta)^m\star\omega\|\cdot\eta + C'\|(Id+\Delta)^m\omega\|\cdot\eta \\ =\ & (C+C')\|(Id+\Delta)^m\omega\|\cdot\eta. \end{aligned} \tag{3}$$

Combining (2) with (3), we have

$$\|\star W\bigstar R\omega - W\bigstar\bigstar R\omega\| \leq (C+3C')\|(Id+\Delta)^m\omega\|\cdot\eta.$$

With the inequality (1), we see that

$$\|\star\star\omega - W\bigstar\bigstar R\omega\| \leq (C+4C')\|(Id+\Delta)^m\omega\|\cdot\eta.$$

□

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Teichmüller distance and Kobayashi distance on subspaces of the universal Teichmüller space

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Purpose

(1) Giving a sufficient condition for metric subspaces on which the Teichmüller distance coincides with the Kobayashi distance;

(2) Introducing examples of such subspaces.

The universal Teichmüller space

$$\Delta = \{|z| < 1\} \ : \ \text{the unit disk in the complex plane,}$$
$$QC = \{f : \Delta \to \Delta \text{ quasiconformal mapping } | f \text{ fixes } 1, i, -1\}.$$

We introduce the following equivalence relation: For $f, g \in QC$,

$$f \sim g \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad f|_{\partial\Delta} = g|_{\partial\Delta}.$$

This relation is called Teichmüller equivalence.

$T = QC/\sim$: the universal Teichmüller space;

$[f]$: the Teichmüller equivalence class represented by $f \in QC$.

Especially, $[id|_{\Delta}]$ is called the base point of T and denoted by 0.

Another representation of the universal Teichmüller space

$$\mu = \frac{\bar{\partial} f}{\partial f} \text{ : the Beltrami coefficient of } f \in QC,$$

B : the open unit ball of measurable functions in Δ.

Then

$$B \ni \mu \longleftrightarrow f^{\mu} \in QC \text{ i.e. } QC = \{f^{\mu} | \mu \in B\}.$$

We introduce the following equivalence relation: For $\mu, \nu \in B$,

$$\mu \sim_B \nu \quad \overset{\text{def}}{\Longleftrightarrow} \quad f^{\mu}|_{\partial\Delta} = f^{\nu}|_{\partial\Delta}.$$

Then it follows that $T = B/\sim_B = \{[\mu] | \mu \in B\}$.

$[\mu]$: the Teichmüller equivalence class represented by $\mu \in B$

Two distances on the universal Teichmüller space

The Teichmüller distance

For $p, q \in T$,

$$d_T(p,q) = \frac{1}{2} \inf\{\log K(g \circ f^{-1}) | f \in p, g \in q\},$$

where $K(f) = \frac{1+\|\mu_f\|_\infty}{1-\|\mu_f\|_\infty}$ is the maximal dilatation of $f \in QC$.

This function d_T is called the Teichmüller distance on T.

The Teichmüller distance on T is complete and T is contractible in d_T.

The Kobayashi distance

This distance is defined for complex manifolds.

(Contractility of the Kobayashi distance)
M, N : two complex manifolds,
$d_{K,M}, d_{K,N}$: the Kobayashi distances on M and N.
Then for any holomorphic map F of M into N and for any $p, q \in M$,

$$d_{K,N}(F(p), F(q)) \leq d_{K,M}(p, q).$$

Especially, if F is biholomorphic of M onto N, then F becomes a isomorphism between M and N.

$$\mathcal{B} = \{\phi : \text{holomorphic on } \Delta | \sup_{z \in \Delta} |\phi(z)|\rho(z)^{-2} < \infty\},$$

where ρ is the Poincaré metric on Δ.Then there exists a homeomorphism β of T into $\mathcal{B}$ (the Bers embedding). By identifying T as $\beta(T)$, T becomes a complex manifold modeled on $\mathcal{B}$.

Theorem (Gardiner, Earle-Kra-Krushkal). The Teichmüller distance d_T on T coincides with the Kobayashi distance d_K.

Main Theorem. Let T' be a complex manifold with a holomorphic embedding ι of T' into T, and identify T' with $\iota(T')$. If T' satisfies the following three conditions, then the Teichmüller distance on T' coincides with the Kobayashi distance.

(1) The set $T' \setminus \{0\}$ is contained in the set of Strebel points of T;

(2) For any $\tau \in T'$, the right translation map for τ maps T' onto itself;

(3) For every $\tau \in T' \setminus \{0\}$, there exists a representative $\mu \in \tau$ corresponding to a frame mapping such that, for every $\mu' \in \tau$ that coincides with μ outside some compact subset of Δ and for every $t \in \Delta$, $[t\mu']$ is in T'.

Strebel's Frame Mapping Theorem

Definition. For $\tau \in T$, let f_0 be a extremal mapping for τ. If $f_1 \in \tau$ satisfies the following condition, then f_1 is called a frame mapping for τ:
(Condition) There exists a compact subset E of Δ such that

$$K(f_1|_{\Delta \setminus E}) < K(f_0).$$

If a point of T has a frame mapping, then it is called a Strebel point.

The set of Strebel points is open and dense in T.

Theorem (Strebel's frame mapping theorem, Teichmüller's uniqueness theorem). If a point $\tau \in T$ is a Strebel point, then it has the unique extremal mapping with Beltrami coefficient of the form $k\frac{\overline{\phi}}{|\phi|}$, where $0 < k < 1$ and ϕ is a holomorphic function with $\iint_\Delta |\phi(z)| dxdy = 1$.

Sketch of the main theorem's proof

$d_{T'} = d_T|_{T'}$, $d_{K'}$: the Kobayashi distance on T'.

We show that $d_{T'} = d_{K'}$.

(About $d_{T'} \leq d_{K'}$)

From $d_T = d_K$ and the contractility of the Kobayashi pseudo distance, for $p, q \in T'$,

$$d_{K'}(p,q) \geq d_K(p,q) = d_T(p,q) = d_{T'}(p,q).$$

This implies that $d_{T'} \leq d_{K'}$.

(About $d_{T'} \geq d_{K'}$)

Let $[g] \in T$ and define

$$\alpha([f]) = [f \circ g^{-1}] : \text{the right translation map for } [g].$$

Then by condition (2), α becomes a biholomorphic self map of T and an isomorphism in d_K. Moreover, α also becomes an isomorphism in d_T.

Hence it is sufficient to show that for any $\tau \in T' \setminus \{0\}$,

$$d_{K'}(0, \tau) \leq d_{T'}(0, \tau) = \frac{1}{2} \log K_0,$$

where $K_0 = \frac{1+k_0}{1-k_0}$ is the extremal maximal dilatation of τ.

(Process)

I. Composing a sequence $\{K_n\}$ satisfying $d_{K'}(0, \tau) \leq \frac{1}{2} \log K_n$;

II. Showing the existence of a subsequence of $\{K_n\}$ converging to K_0.

(About I)

By condition (1), τ has a frame mapping f with Beltrami coefficint μ. It follows from Strebel's frame mapping theorem that τ has the unique extremal mapping with Beltrami coefficient of the form $k_0\frac{\overline{\phi_0}}{|\phi_0|}$, where ϕ_0 is a holomorphic function with $\iint_\Delta |\phi_0(z)|dxdy = 1$.

Let $D_n = \{|z| < 1 - \frac{1}{n}\}$. It follows that $f|_{D_n}$ becomes a frame mapping for $[f|_{D_n}] \in T(D_n)$ ($T(D_n)$ is the Teichmüller space of D_n).

By Strebel's frame mapping theorem, $[f|_{D_n}]$ has the unique extremal mapping $\tilde{f}_n : D_n \to f(D_n)$ with Beltrami coefficient of the form $k_n\frac{\overline{\phi_n}}{|\phi_n|}$, where $0 < k_n < 1$ and ϕ_n is a holomorphic function with $\iint_{D_n} |\phi_n(z)|dxdy = 1$.

Set

$$f_n(z) = \begin{cases} \tilde{f}_n(z) & (z \in D_n) \\ \\ f(z) & (z \in \Delta \setminus D_n). \end{cases}$$

Let μ_n be the Beltrami coefficient f_n and $K_n = K(f_n)$. Then for each n,

$$(a) \quad K_n > K_0$$
$$(b) \quad [\mu_n] = \tau.$$

Let $g(t) = [(t\mu_n)/k_n]$ for $t \in \Delta$. It follows from condition (3) that g is a holomorphic map of Δ into T'. By condition (b),

$$d_{K'}(0, \tau) \leq \frac{1}{2} \log \frac{1 + k_n}{1 - k_n} = \frac{1}{2} \log K_n,$$

(About II)

We show there exists a subsequence of $\{K_n\}$ converging to K_0.

There exists a subsequence of $\{\phi_n\}$ converging locally uniformly in Δ to a holomorphic function ϕ^* satisfying $\iint_\Delta |\phi^*| dxdy \leq 1$. It follows that $\iint_\Delta |\phi^*| dxdy > 0$.

From $0 < k_n < 1$ for any n, there exists a convergent subsequence of $\{k_n\}$. Let k^* be the limit and $\mu^* = k^* \frac{\overline{\phi^*}}{|\phi^*|}$. Then it follows that $[\mu^*] = \tau$. By Teichmüller's uniqueness theorem, we obtain $k^* = k_0$. Hence there exists a subsequence of $\{K_n\}$ converging to K_0.

Therefore it follows that $d_{T'} = d_{K'}$. $\square$

Examples of the main theorem

Example 1. asymptotically conformal classes

Definition. A quasiconformal mapping f of Δ is asymptotically conformal if

$$\forall \epsilon > 0,\ \exists E \subset \Delta : \text{a compact subset s.t. } K(f|_{\Delta \setminus E}) < 1 + \epsilon.$$

If a point of T has an asymptotically conformal map, then we call it an asymptotically conformal class. Let

$$T_0 = \{\tau \in T | \tau \text{ is a asymptotically conformal class}\}.$$

The subset T_0 becomes a closed submanifold of T. Indeed, let

$$\mathcal{B}_0 = \{\phi : \text{holomorphic on } \Delta | \limsup_{|z| \to 1} |\phi(z)| \rho(z)^{-2} = 0\}.$$

Then $\mathcal{B}_0$ becomes a closed subspace of $\mathcal{B}$ and $\beta(T_0) = \beta(T) \cap \mathcal{B}_0$.

Theorem (Earle-Gardiner-Lakic, Hu-Jiang-Wang). The Teichmüller distance on T_0 coincides with the Kobayashi distance.

(Proof)

(1) $\forall \tau \in T_0, f \in \tau$: an asymptotically conformal map.
If we take a sufficient large compact subset E of Δ, then it follows that

$$K(f|_{\Delta \setminus E}) < K(\tau).$$

(2) $\forall \tau_i \in T_0$ $(i = 1, 2)$ $f_i \in \tau_i$: two asymptotically conformal maps.
$\forall \epsilon > 0$, $\exists E \subset \Delta$: a compact subset s.t.

$$K((f_1 \circ f_2^{-1})|_{\Delta \setminus E}) \le K(f_1|_{\Delta \setminus E}) K(f_2|_{\Delta \setminus E}) < (1 + \epsilon)^2.$$

(3) $\forall \tau \in T_0, f \in \tau$: an asymptotically conformal map.
This is equivalent to that $\|\mu_f|_{\Delta \setminus D_n}\|_\infty \to 0$ $(n \to \infty)$. Then for any $t \in \Delta$ and any $\mu' \in \tau$ satisfying $\mu' = \mu_f$ on $\Delta \setminus E$ ($E \subset \Delta$: compact), by taking n sufficiently largely,

$$\|t\mu'|_{\Delta \setminus D_n}\|_\infty = |t| \|\mu_f|_{\Delta \setminus D_n}\|_\infty \to 0 \ (n \to \infty).$$

Example 2. p-integrably asymptotic affine classes

Definition. Let $p \geq 2$. If the Beltrami coefficient μ of Δ satisfies

$$\iint_{\Delta} |\mu(z)|^p \rho(z)^2 dxdy < \infty,$$

then we call μ p-integrably asymptotic affine. If a point of T has a p-integrably asymptotic affine Beltrami coefficient, then we call it a p-integrably asymptotic affine class.

Let

$$T^p = \{\tau \in T \mid \tau \text{ is a } p\text{-integrably asymptotic affine class}\}.$$

If $p = 2$, then T^2 becomes contractible in the Weil-Petersson distance of T.

$$\text{Diff} = \{C^\infty\text{-deffeomorphism on } \partial\Delta \text{ fixing -1, i, 1}\}.$$

The subspace T^2 becomes a completion of Diff in the Weil-Petersson distance of T (these results are proved by Cui).

Let

$$A^p = \{\phi : \text{holomorphic on } \Delta \mid \iint_\Delta |\phi(z)|^p \rho(z)^{2-2p} dxdy < \infty\}.$$

Theorem. $\beta(T^p) \subset \beta(T) \cap A^p$.

Then T^p becomes a complex manifold modeled on A^p.

Theorem. The Teichmüller distance on T^p coincides with the Kobayashi distance.

Remark. Let $[f] \in T$ and h be a boundary function on Δ corresponding to $[f]$. Then there exists a quasiconformal extension $E([f])$ of h to Δ such that for any two Möbius transformations γ, δ preserving Δ and fixing $1, i, -1$,

$$E([\gamma \circ f \circ \delta]) = \gamma \circ E([f]) \circ \delta.$$

This map $E([f])$ is called the Douady-Earle extension for $[f]$.

(Proof)

(1) It follows that $A^p \subset \mathcal{B}_0$. Then $\beta(T^p) \subset \beta(T_0)$ i.e. $T^p \subset T_0$.

(3) The following theorem follows:

Theorem. For any $\tau \in T^p$, the Beltrami coefficient μ of $E(\tau)$ becomes p-integrably asymptotic affine.

Moreover, the Douady-Earle extension $E(\tau)$ for each $\tau \in T_0$ is asymptotically conformal (Earle-Markovic-Saric).

For any $t \in \Delta$ and any $\mu' \in \tau$ satisfying $\mu' = \mu$ on $\Delta \setminus E$ ($E \subset \Delta$: compact), it follows that

$$\iint_{\Delta} |t\mu'(z)|^p \rho(z)^2 dxdy < \iint_{\Delta} |\mu(z)|^p \rho(z)^2 dxdy + \iint_{E} |\mu'(z)|^p \rho(z)^2 dxdy < \infty$$

Then we have $[t\mu'] \in T^p$.

(2) The following theorem follows:

Theorem. For any Beltrami coefficient μ of $E(\tau)$. Then there exist constants $C_1, C_2 > 0$ such that

$$C_1\rho(z)^2 \leq J_{f^\mu}(z)\rho(f^\mu(z))^2 \leq C_2\rho(z)^2.$$

For any $\tau_1, \tau_2 \in T^p$, let μ_i be the Beltrami coefficient of $E(\tau_i)$ $(i = 1, 2)$ and η be the Beltrami coefficient of $f^{\mu_1} \circ (f^{\mu_2})^{-1}$. Hence

$$\begin{aligned}\iint_\triangle |\eta(z)|^p \rho(z)^2 dxdy &= \iint_\triangle |\eta(f^{\mu_2}(\zeta))|^p \rho(f^{\mu_2}(\zeta))^2 J_{f^{\mu_2}}(\zeta) d\xi d\eta \\ &\leq C \iint_\triangle \left|\frac{\mu_1(\zeta) - \mu_2(\zeta)}{1 - \mu_1(\zeta)\overline{\mu_2(\zeta)}}\right|^p \rho(\zeta)^2 d\xi d\eta \\ &\leq C \iint_\triangle \frac{2^{p-1}(|\mu_1(\zeta)|^p + |\mu_2(\zeta)|^p)}{(1 - \|\mu_1\|_\infty \|\mu_2\|_\infty)^p} \rho(\zeta)^2 d\xi d\eta < \infty.\end{aligned}$$

Therefore it follows that $\tau = [\eta] \in T^p$.

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