EXOTIC PROJECTIVE STRUCTURES AND BOUNDARY OF QUASI-FUCHSIAN SPACE

KENTARO ITO

Abstract. Let $P(S)$ denote the space of projective structures on a closed surface S. It is known that the subset $Q(S) \subset P(S)$ of projective structures with quasi-Fuchsian holonomy has infinitely many connected components. In this paper, we investigate the configuration of these components. In particular, we show that the closure of any *exotic* component of $Q(S)$ intersects the closure of the *standard* component of $Q(S)$. As a consequence, $Q(S)$ has connected closure in $P(S)$. We also mention the complexity of the boundary of the quasi-Fuchsian space.

1. Introduction

Let S be an oriented closed surface of genus $g > 1$. A projective structure on S is a maximal system of local coordinates modeled on the Riemann sphere $\widehat{\mathbf{C}}$, whose transition functions are Möbious transformations. For a given projective structure on S, we have a pair (f, ρ) of a local homeomorphism f from the universal cover $\widetilde{S}$ of S to $\widehat{\mathbf{C}}$, called a developing map, and a group homomorphism ρ of $\pi_1(S)$ into $\mathrm{PSL}_2(\mathbf{C})$, called a holonomy representation. Let $P(S)$ denote the space of all (marked) projective structures on S, and let $V(S)$ denote the space of all conjugacy classes of representations of $\pi_1(S)$ into $\mathrm{PSL}_2(\mathbf{C})$. Holonomy representations give a mapping $\mathrm{hol} : P(S) \to V(S)$, which is called the holonomy mapping. It is known that the map hol is a local homeomorphism ([13]). The quasi-Fuchsian space $QF(S)$ is the subspace of $V(S)$ consisting of faithful representations whose holonomy images are quasi-Fuchsian groups.

In this paper, we investigate the subset $Q(S) = \mathrm{hol}^{-1}(QF(S))$ of $P(S)$. We say an element of $Q(S)$ is *standard* if its developing map is injective; otherwise it is *exotic*. The set of standard projective structures with fixed underlying complex structure is well known as the image of the Teichmüller space under Bers embedding (see [5]). On the other hand, the existence of exotic projective structures was first shown by Maskit [21]. More investigations of exotic projective structures are found in [11], [12], [13], [25], [30] and [32]. As we shall see in Proposition 2.3, each connected component of $Q(S)$ is biholomorphically isomorphic to $QF(S)$. Moreover, as a consequence of the result of Goldman [12], the connected components of $Q(S)$ are in one to one correspondence with the set $\mathcal{ML}_{\mathbf{Z}}(S)$ of integral points of measured laminations (see 2.4 for precise definition). We denote by $\mathcal{Q}_\lambda$ the component of

Date: October 20, 1999.
1991 *Mathematics Subject Classification.* Primary 30F40; Secondary 30F30, 32G15, 57M50.

$Q(S)$ corresponding to $\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)$, where $\mathcal{Q}_0$ is the component consisting of all standard projective structures.

Recently McMullen [25, Appendix A] discovered the next phenomenon.

Theorem 1.1 (McMullen). *There exists a sequence of exotic projective structures which converges to an element of $\partial\mathcal{Q}_0$.*

This phenomenon deeply depends on the following phenomenon in the theory of Kleinian groups: There is a sequence of quasi-Fuchsian groups whose algebraic limit is properly contained in the geometric limit. Such a sequence of quasi-Fuchsian groups used in the proof of Theorem 1.1 is essentially constructed in Anderson and Canary [2]. Related topics can be found in [7], [8] and [18].

Theorem 1.1 brings up naturally the following questions;

(1) Can we characterize the points on $\partial\mathcal{Q}_0$ which are limits of exotic projective structures?
(2) Can we characterize how a sequence of exotic projective structures can converge to a point of $\partial\mathcal{Q}_0$?

As for the first question, we first remark that the holonomy image of the limit projective structure constructed in Theorem 1.1 is a regular b-group. Moreover, we will show that any element of $\partial\mathcal{Q}_0$ whose holonomy image is a degenerate group without accidental parabolics can not be an accumulation point of exotic projective structures (Corollary 3.5). This result has been announced by Matsuzaki already. We discuss this topic in Section 3.

In this paper, we are mainly concerned with the second question. Our first main result shows that there exists a sequence in *any* exotic component $\mathcal{Q}_\lambda$ which converges to a point of $\partial\mathcal{Q}_0$.

Theorem A. *For any $\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)$, we have $\overline{\mathcal{Q}_0} \cap \overline{\mathcal{Q}_\lambda} \neq \emptyset$. Especially, the closure of $Q(S)$ in $P(S)$ is connected.*

The proof depends on the following observation: For a converging sequence of exotic projective structures, what component of $Q(S)$ the sequence is contained is closedly related to how the algebraic limit is contained in the geometric limit of corresponding holonomy representations. In Section 5, we develop a technique to construct some sequences of representations with the same algebraic limit but with mutually distinct geometric limits. Using this technique, we can extend Theorem A to the following form.

Theorem B. *For any finite set $\{\lambda_i\}_{i=1}^m$ of $\mathcal{ML}_{\mathbf{Z}}(S)$ satisfying $i(\lambda_j, \lambda_k) = 0$ for all $j, k \in \{1, \ldots, m\}$, we have $\overline{\mathcal{Q}_0} \cap \overline{\mathcal{Q}_{\lambda_1}} \cap \cdots \cap \overline{\mathcal{Q}_{\lambda_m}} \neq \emptyset$. Where $i(\cdot,\cdot)$ denotes the geometric intersection number.*

Since the holonomy mapping hol : $P(S) \to V(S)$ is a local homeomorphism, the complexity of $Q(S)$ at $\partial\mathcal{Q}_0$ is inherited by the complexity of $\partial QF(S)$. In fact, Theorem 1.1 implies that the closure of $QF(S)$ in $V(S)$ is not a manifold with boundary (Theorem A.1 in [25]). This shows the advantage of consideration of projective structures to investigate the quasi-Fuchsian space. As a consequence of Theorem B, we obtain the following

Theorem C. *For any positive integer $n \in \mathbf{N}$, there exists a point $[\rho]$ of $\partial QF(S)$ such that $U \cap QF(S)$ consists of more than n components for any sufficiently small neighborhood U of $[\rho]$.*

Theorem A and B can be viewed as the projective structure anlogues of the works of Anderson and Canary [2] and Anderson, Canary and McCullough [3] in characterizing when components of the set of discrete faithful representations of finitely generated group G into $\mathrm{PSL}_2(\mathbf{C})$ have intersecting closures. Theorem C also can be viewed as the analogue of their work, while Theorem C describe how the closure of a unique component of the set of discrete faithful representations intersects itself.

This paper is organized as follows: In Section 2, we provide detailed definitions and basic properties of the spaces and maps with which we will be concerned. In Section 3, we investigate the relationship between sequences of exotic projective structures and algebraic and geometric limits of their holonomy representations. Section 4 and Section 5 are devoted to the proofs of Theorem A and Theorem B, respectively.

Acknowledgements. The author would like to express his gratitude to Hiroshige Shiga and Katsuhiko Matsuzaki for their encouragement and useful suggestions. He also appreciates the referee's valuable comments on the previous manuscript of this paper.

2. Notation and basic facts

A *Kleinian group* G is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$, which acts on the hyperbolic space $\mathbf{H}^3$ as isometries, and on the sphere at infinity $S^2_\infty = \widehat{\mathbf{C}}$ as conformal automorphisms. The *region of discontinuity* $\Omega(G)$ is the largest open subset of $\widehat{\mathbf{C}}$ on which G acts properly discontinuously, and the *limit set* $\Lambda(G)$ of G is its compliment $\widehat{\mathbf{C}} - \Omega(G)$. The quatient manifold $N_G = \mathbf{H}^3 \cup \Omega(G)/G$ is called the *Kleinian manifold* of G. A *quasi-Fuchsian group* is a Kleinian group whose limit set is a Jordan curve and which contains no element interchanging the two components of its region of discontinuity. A quasi-Fuchsian group is obtained by a quasi-conformal deformation of a Fuchsian group.

2.1. Beltrami differentials.

For a given Kleinian group G with $\Omega(G) \neq \emptyset$, a measurable function μ on $\widehat{\mathbf{C}}$ is called a *Beltrami differential* for G if

$$\mu(g(z))\overline{g'(z)} = \mu(z)g'(z)$$

holds for a.e. $z \in \widehat{\mathbf{C}}$ and for all $g \in G$. The space of all Beltrami differentials μ for G whose essential sup-norm satisfying $||\mu||_\infty < 1$ is denoted by $\mathrm{Belt}(G)_1$. For a given element $\mu \in \mathrm{Belt}(G)_1$, there exists a unique quasi-conformal map $w : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ satisfying the Beltrami equation $w_{\bar{z}} = \mu w_z$ and fixing $0, 1$ and ∞. Throughout this paper, this normalized quasi-conformal map with the Beltrami coefficient μ will be denoted by w_μ. For more information about quasi-conformal map, see Lehto-Virtanen [20] for example. The quasi-conformal map w_μ induces a group isomorphism Θ_μ of G

into $\mathrm{PSL}_2(\mathbf{C})$ satisfying $w_\mu \circ g = \Theta_\mu(g) \circ w_\mu$ for all $g \in G$. For a G-invariant open set $U \subset \Omega(G)$, we denote by $\mathrm{Belt}(U, G)_1$ the subset of $\mathrm{Belt}(G)_1$ consisting of all elements with support in U.

2.2. **Teichmüller space.** Let S be an oriented closed surface of genus $g > 1$. The Teichmüller space $T(S)$ consists of pairs (f, X), where X is a Riemann surface and $f : S \to X$ is an orientation preserving diffeomorphism. Two pairs (f_1, X_1) and (f_2, X_2) represent the same point in $T(S)$, if there is a holomorphic isomorphism $h : X_1 \to X_2$ such that $h \circ f_1$ is isotopic to f_2. It is known that the space $T(S)$ is a $3g - 3$ dimensional complex manifold, diffeomorphic to a cell.

There is another but equivalent definition of the Teichmüller space. We fix a Fuchsian group Γ acting on the upper half plane $\mathbf{H} = \{z \in \mathbf{C} : \mathrm{Im} z > 0\}$ such that $S = \mathbf{H}/\Gamma$. Two elements $\mu, \nu \in \mathrm{Belt}(\mathbf{H}, \Gamma)_1$ are called equivalent if $w_\mu|\partial\mathbf{H} = w_\nu|\partial\mathbf{H}$, or equivalently, $\Theta_\mu = \Theta_\nu$. The Teichmüller space $T(\Gamma)$ (or $T(S)$) is the space of equivalence classes $[\mu]$ of elements μ in $\mathrm{Belt}(\mathbf{H}, \Gamma)_1$. For each $t = [\mu] \in T(S)$, let Γ_t denote the quasi-Fuchsian group $\Theta_\mu(\Gamma)$, whose region of discontinuity is a union of $\mathbf{H}_t = w_\mu(\mathbf{H})$ and $\mathbf{H}_t^* = w_\mu(\mathbf{H}^*)$, where $\mathbf{H}^* = \{z \in \mathbf{C} : \mathrm{Im} z < 0\}$ is the lower half plane. The quasi-conformal map $w_\mu|\mathbf{H} : \mathbf{H} \to \mathbf{H}_t$ descends to a quasi-conformal map $g_t : S = \mathbf{H}/\Gamma \to S_t = \mathbf{H}_t/\Gamma_t$, such that the pair (g_t, S_t) represents $t \in T(S)$.

2.3. **The space of projective structures.** For a given projective structure on S, we obtain a local homeomorphism $f : \widetilde{S} \to \widehat{\mathbf{C}}$ by lifting the structure to the universal cover $\widetilde{S}$ of S and continuing the coordinates analytically. This map f is called a *developing map* of the projective structure. A developing map f induces a group homomorphism $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$ satisfying $f \circ g = \rho(g) \circ f$ for all $g \in \pi_1(S)$, which is called a *holonomy representation.* This pair (f, ρ) is called a *projective pair.* Note that a projective structure determines a projective pair (f, ρ) uniquely up to the action of $\mathrm{PSL}_2(\mathbf{C})$; the action is defined by

$$(f, \rho) \mapsto (A \circ f, A \circ \rho \circ A^{-1})$$

for $A \in \mathrm{PSL}_2(\mathbf{C})$.

For each $t \in T(S)$, let $B_2(\mathbf{H}_t, \Gamma_t)$ denote the space of holomorphic quadratic differentials for Γ_t on $\mathbf{H}_t$, whose element is a holomorphic function φ on $\mathbf{H}_t$ satisfying

$$\varphi(\gamma(z))\gamma'(z)^2 = \varphi(z)$$

for all $\gamma \in \Gamma_t, z \in \mathbf{H}_t$. The space $B_2(\mathbf{H}_t, \Gamma_t)$ is a $3g - 3$ dimensional complex vector space.

A projective structure determines naturally its *underlying complex structure.* The set of all projective structures with an underlying complex structure $S_t = \mathbf{H}_t/\Gamma_t$ is parametrized by $B_2(\mathbf{H}_t, \Gamma_t)$ as follows. For any projective structure on S_t, let $f : \mathbf{H}_t \to \widehat{\mathbf{C}}$ be its developing map. Then we assign an element $S(f) \in B_2(\mathbf{H}_t, \Gamma_t)$ to this projective structure, where $S(f)$ is the Schwarzian derivative of f defined by $S(f) = (f''/f')' - 1/2(f''/f')^2$. Conversely, for any element $\varphi \in B_2(\mathbf{H}_t, \Gamma_t)$, there is a holomorphic map $f : \mathbf{H}_t \to \widehat{\mathbf{C}}$ satisfying $S(f) = \varphi$, which descends to a projective structure on S_t. Here and hereafter, with this identification, we regard a pair (t, φ) as a marked projective structure, where $t \in T(S)$ and $\varphi \in B_2(\mathbf{H}_t, \Gamma_t)$.

Let $P(S)$ denote the holomorphic cotangent bundle over $T(S)$ with projection $\pi : P(S) \to T(S)$. Then, each fiber $\pi^{-1}(t)$ over $t \in T(S)$ is identified with $B_2(\mathbf{H}_t, \Gamma_t)$. We regard $P(S)$ as the space of marked projective structures. For any projective structure $(t, \varphi) \in P(S)$, let $f_{t,\varphi} : \mathbf{H}_t \to \widehat{\mathbf{C}}$ denote its developing map and $\bar{\rho}_{t,\varphi} : \Gamma_t \to \mathrm{PSL}_2(\mathbf{C})$ its holonomy representation satisfying $f_{t,\varphi} \circ \gamma = \bar{\rho}_{t,\varphi}(\gamma) \circ f_{t,\varphi}$ for any $\gamma \in \Gamma_t$. A representation $\rho_{t,\varphi} = \bar{\rho}_{t,\varphi} \circ \Theta_\mu$ of $\pi_1(S) = \Gamma$ into $\mathrm{PSL}_2(\mathbf{C})$ is also called a holonomy representation, where $\mu \in \mathrm{Belt}(\mathbf{H}, \Gamma)_1$ is a representative of $t \in T(S)$. It is known that, for any projective structure (t, φ), the holonomy image $\rho_{t,\varphi}(\pi_1(S))$ is a non-abelian subgroup of $\mathrm{PSL}_2(\mathbf{C})$.

A sequence $\{\rho_n\}$ of representations of $\pi_1(S)$ into $\mathrm{PSL}_2(\mathbf{C})$ is said to converge *algebraically* to ρ if $\rho_n(g)$ converges to $\rho(g)$ in $\mathrm{PSL}_2(\mathbf{C})$ for any $g \in \pi_1(S)$. Let $V(S)$ denote the space of all conjugacy classes $[\rho]$ of representations $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$ such that $\rho(\pi_1(S))$ is non-abelian. A sequence $\{[\rho_n]\}$ in $V(S)$ converges to $[\rho] \in V(S)$ if there is a sequence of representatives of $\{[\rho_n]\}$ which converges algebraically to a representative of $[\rho]$. It is known that $V(S)$ is $6g-6$ dimensional complex manifold (see, for example, [23, Theorem 4.21]). The *holonomy map*

$$\mathrm{hol} : P(S) \to V(S)$$

is defined by $\mathrm{hol}(t, \varphi) = [\rho_{t,\varphi}]$. The basic fact is that the holonomy map is a holomorphic local homeomorphism ([13], see also [9] and [14]).

2.4. **Grafting.** Let $\mathcal{S}$ denote the set of homotopy classes of non-trivial simple closed curves on S. By abuse of notation, we also denote a representative of $C \in \mathcal{S}$ by C. Let $\mathcal{ML}_{\mathbf{Z}}(S)$ denote the set of integral points of measured laminations on S. Namely, each element $\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)$ is written as a formal summation $\sum n_j C_j$, where $\{n_j\}$ are positive integers and $\{C_j\}$ are mutually distinct disjoint elements in $\mathcal{S}$. We shall contain the "zero" measured lamination in $\mathcal{ML}_{\mathbf{Z}}(S)$.

Let X be a Riemann surface marked by S. A canonical projective structure on S is provideded by the Fuchsian uniformization $X = \mathbf{H}/G$, where $\mathbf{H}$ is the upper half plane and G is a Fuchsian group acting on $\mathbf{H}$. For any element $\lambda = \sum n_j C_j$ of $\mathcal{ML}_{\mathbf{Z}}(S)$, we can construct a new projective structure on S by cutting X along each C_j and "grafting" some projective annulus at each cut locus. More precisely, let $l_X(C_j)$ denote the hyperbolic length of geodesic representative of C_j on X and let

$$A_j = (\mathbf{C} - i\mathbf{R}^+)/\left\langle z \mapsto e^{l_X(C_j)} z \right\rangle$$

be the annulus equipped with natural projective structure. Then new projective structure on S is obtained by cutting X along geodesic representatives of each C_j and inserting A_j for n_j-times. This new projective structure is said to be obtained by grafting along λ. (See [12], [17], [25] and [32] for more information.)

We explain this grafting operation in the contex of complex analysis, for our later use. The following construction is due to Maskit [21] and our explanation is based on that of Gallo [11]. An element $(t, \varphi) \in P(S)$ is called a *Fuchsian projective structure* if $f_{t,\varphi}$ is injective and $\rho_{t,\varphi}(\pi_1(S))$ is a Fuchsian group.

For simplicity, we first explain for the case that $\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)$ is a simple closed curve with weight 1. Take a Fuchsian projective structure (t, φ) and an element

$C \in \mathcal{S}$. We denote the holonomy image $\rho_{t,\varphi}(\pi_1(S))$ by G. We may assume that the image of developing map $f_{t,\varphi}(\mathbf{H}_t)$ coincides with the upper half plane $\mathbf{H}$, and that the imaginary axis $i\mathbf{R}^+$ projects onto C via the covering map $H \to S_t = \mathbf{H}/G$. Choose $\alpha \in (0, \pi/2)$ so that $B_\alpha = \{z \in \mathbf{H} : \pi/2 - \alpha < \arg z < \pi/2 + \alpha\}$ projects onto a collar about C in S_t. Take a C^1 homeomorphism

$$v : [\pi/2 - \alpha, \pi/2 + \alpha] \to [\pi/2 - \alpha, 5\pi/2 + \alpha]$$

satisfying $v(\pi/2 - \alpha) = \pi/2 - \alpha$, $v(\pi/2 + \alpha) = 5\pi/2 + \alpha$, and $v'(\pi/2 - \alpha) = v'(\pi/2 + \alpha) = 1$. We now define a local homeomorphism $W : \mathbf{H} \to \widehat{\mathbf{C}}$ as follows. Let $g \in G$ be a generator for the stabilizer of $i\mathbf{R}^+$ in G and set $D_\alpha = \bigcup_{h \in G/\langle g \rangle} h(B_\alpha)$. Let $W(z) = z$ for $z \in \mathbf{H} - D_\alpha$, and $W(z) = re^{iv(\theta)}$ for $z = re^{i\theta} \in B_\alpha$. For $z \in h(B_\alpha)$ with some $h \in G$, let $W(z) = h \circ W \circ h^{-1}(z)$. One can easily verify that $W \circ h = h \circ W$ for all $h \in G$ and hence that $\mu = W_{\bar{z}}/W_z \in \mathrm{Belt}(\mathbf{H}, G)_1$.

Now let $\hat{\mu} \in \mathrm{Belt}(\mathbf{H}_t, \Gamma_t)_1$ be the pull back $f^*_{t,\varphi}(\mu)$ of μ via developing map $f_{t,\varphi}$, which is defined by

$$f^*_{t,\varphi}(\mu) = (\mu \circ f_{t,\varphi})\overline{f'_{t,\varphi}}/f'_{t,\varphi}.$$

Let $\nu \in \mathrm{Belt}(\mathbf{H}, \Gamma)_1$ be a representative of $t \in T(S)$. Let $t' \in T(S)$ be the equivalent class of the Beltrami coefficient ν' of the quasi-conformal map $w_{\hat{\mu}} \circ w_\nu : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$. Since $W \circ f_{t,\varphi} \circ (w_{\hat{\mu}})^{-1}$ is locally conformal on $\mathbf{H}_{t'} = w_{\hat{\mu}}(\mathbf{H}_t)$, we can take its Schwarzian derivative $\varphi' \in B_2(\mathbf{H}_{t'}, \Gamma_{t'})$. Now we obtain a new projective structure (t', φ') with *surjective* developing map $f_{t',\varphi'}$ which commutes the following diadram;

$$\begin{array}{ccccc}
\mathbf{H} & \xrightarrow{w_\nu} & \mathbf{H}_t & \xrightarrow{f_{t,\varphi}} & \widehat{\mathbf{C}} \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle w_{\hat{\mu}}} & & \downarrow{\scriptstyle W} \\
\mathbf{H} & \xrightarrow{w_{\nu'}} & \mathbf{H}_{t'} & \xrightarrow{f_{t',\varphi'}} & \widehat{\mathbf{C}}.
\end{array}$$

Moreover one can see that $\rho_{t',\varphi'} = \rho_{t,\varphi}$ from the fact that $W \circ h = h \circ W$ for all $h \in G$. It is shown in [11, Lemma 3.1] that the element $(t', \varphi') \in P(S)$ does not depend on the choice of α, v, and ν. The projective structure (t', φ') is said to be obtained from (t, φ) by grafting along C and is denoted by $\mathrm{Gr}_C(t, \varphi)$.

An important remark is that, for $(t, \varphi) = \mathrm{Gr}_C(t, \varphi)$, the subset $f^{-1}_{t',\varphi'}(\Lambda(G))/\Gamma_{t'}$ in $S_{t'}$ consists of two simple closed curves each of which is homotopic to C (see [12]). This can be seen as follows: We first note that the limit set $\Lambda(G)$ of $G = \rho_{t',\varphi'}(\pi_1(S))$ coincides with $\mathbf{R} \cup \{\infty\}$. Since $f^{-1}_{t,\varphi}(B_\alpha)$ projects onto a collar about C in S_t via the covering map $\mathbf{H}_t \to S_t$, $w_{\hat{\mu}}(f^{-1}_{t,\varphi}(B_\alpha))$ is also projected onto a collar about C in $S_{t'}$ via the covering map $\mathbf{H}_{t'} \to S_{t'}$. Since the developing map $f_{t',\varphi'}$ maps $w_{\hat{\mu}}(f^{-1}_{t,\varphi}(B_\alpha))$ onto the multi-sheeted domain $\{z \in \widehat{\mathbf{C}} : \pi/2 - \alpha < \arg z < 5\pi/2 + \alpha\}$, $w_{\hat{\mu}}(f^{-1}_{t,\varphi}(B_\alpha)) \cap f^{-1}_{t',\varphi'}(\Lambda(G))$ consists of two connected components each of which is projected onto a simple closed curve homotopic to C.

The grafting operation can be naturally extended to $\mathcal{ML}_{\mathbf{Z}}(S)$. For example, if $\lambda = nC \in \mathcal{ML}_{\mathbf{Z}}(S)$, we only have to change v into $v : [\pi/2 - \alpha, \pi/2 + \alpha] \to [\pi/2 - \alpha, 2n\pi + \pi/2 + \alpha]$. Let $\mathrm{Gr}_\lambda(t, \varphi)$ denote the projective structure obtaind from (t, φ) by grafting along $\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)$.

As we have observed, the grafting operator does not change holonomy representations. Conversely, Goldman [12] showed that all projective structures with Fuchsian holonomy are obtained by grafting.

Theorem 2.1 (Goldman). *For any Fuchsian projective structure* (t, φ)*,*

$$\mathrm{hol}^{-1}(\mathrm{hol}(t, \varphi)) = \{\mathrm{Gr}_\lambda(t, \varphi)\}_{\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)}.$$

2.5. **Quasi-conformal deformations of projective structures.** Let $AH(S)$ denote the subset of $V(S)$ consisting of discrete faithful representations. The *quasi-Fuchsian space* $QF(S)$ is the subset of $AH(S)$ consisting of faithful representations whose images are quasi-Fuchsian groups. It is known that $AH(S)$ is a closed subset in $V(S)$(see [15, Theorem 1]) and that the interior $\mathrm{int}AH(S)$ of $AH(S)$ coincides with $QF(S)$(see [31, Theorem A]). (It is conjectured that $\overline{QF(S)} = AH(S)$, which is so called Bers-Thurston conjecture.) We denote the subset $\mathrm{hol}^{-1}(AH(S))$ of $P(S)$ by $K(S)$ and $\mathrm{hol}^{-1}(QF(S))$ by $Q(S)$. Then, since the holonomy map $\mathrm{hol} : P(S) \to V(S)$ is a local homeomorphism, one obtains $\mathrm{int}K(S) = Q(S)$.

We now introduce the notion of a quasi-conformal deformation of a projective structure with quasi-Fuchsian holonomy, which was developed by Shiga and Tanigawa in [30]. Fix an element $(t, \varphi) \in Q(S)$ and denote its holonomy image $\rho_{t,\varphi}(\pi_1(S))$ by G. For each $\mu \in \mathrm{Belt}(G)_1$, we take the pull back $\hat{\mu} = f_{t,\varphi}^*(\mu) \in \mathrm{Belt}(\mathbf{H}_t, \Gamma_t)_1$ of μ, which determines a new point $t' \in T(S)$ in the same manner described in 2.4. Since $w_\mu \circ f_{t,\varphi} \circ (w_{\hat{\mu}})^{-1}$ is locally conformal on $\mathbf{H}_{t'}$, we can take its Schwarzian derivative $\varphi' \in B_2(\mathbf{H}_{t'}, \Gamma_{t'})$. Now we obtain a new projective structure (t', φ'), a quasi-conformal deformation of (t, φ), satisfying

$$\begin{cases} f_{t',\varphi'} = w_\mu \circ f_{t,\varphi} \circ (w_{\hat{\mu}}|\mathbf{H}_t)^{-1} : \mathbf{H}_{t'} \to \widehat{\mathbf{C}}, \\ \rho_{t',\varphi'} = \Theta_\mu \circ \rho_{t,\varphi} : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C}), \end{cases}$$

where Θ_μ is the group isomorphism of G into $\mathrm{PSL}_2(\mathbf{C})$ induced by w_μ. We define a map

$$\widetilde{\Psi}_{t,\varphi} : \mathrm{Belt}(G)_1 \to P(S)$$

by $\widetilde{\Psi}_{t,\varphi}(\mu) = (t', \varphi')$. Two elements $\mu, \nu \in \mathrm{Belt}(G)_1$ are said to be equivalent if Θ_μ is $\mathrm{PSL}_2(\mathbf{C})$-conjugate to Θ_ν. The quotient space of $\mathrm{Belt}(G)_1$ by this equivalent relation can be naturally identified with the quasi-Fuchsian space $QF(S)$.

Lemma 2.2 (cf. [30]). *The map* $\widetilde{\Psi}_{t,\varphi}$ *descends to a map*

$$\Psi_{t,\varphi} : QF(S) \to P(S).$$

Proof. For any equivalent two elements $\mu, \nu \in \mathrm{Belt}(G)_1$, we will show that $\widetilde{\Psi}_{t,\varphi}(\mu) = \widetilde{\Psi}_{t,\varphi}(\nu)$. The same argument in [10] reveals that there is a path $c_\tau, \tau \in [0, 1]$ in $\mathrm{Belt}(G)_1$ jointing μ and ν and contained in the equivalence class $[\mu]$ of μ. Note that $\mathrm{hol} \circ \widetilde{\Psi}_{t,\varphi}(c_\tau)$ is constant on $\tau \in [0, 1]$. Since the map hol is a local homeomorphism, it implies that $\widetilde{\Psi}_{t,\varphi}(c_\tau)$ is constant on $\tau \in [0, 1]$ and that $\widetilde{\Psi}_{t,\varphi}(\mu) = \widetilde{\Psi}_{t,\varphi}(\nu)$. □

Using the map in Lemma 2.2, we can show the following

Proposition 2.3. *For any connected component $\mathcal{Q}$ of $Q(S)$,*

$$\mathrm{hol}|\mathcal{Q} : \mathcal{Q} \to QF(S)$$

is a biholomorphic map. Moreover, $\Psi_{t,\varphi} = (\mathrm{hol}|\mathcal{Q})^{-1}$ holds for any $(t,\varphi) \in \mathcal{Q}$. Therefore the map $\Psi_{t,\varphi}$ does not depend on the choice of $(t,\varphi) \in \mathcal{Q}$.

Proof. To show that $\mathrm{hol}|\mathcal{Q}$ is biholomorphic, it suffices to show that $\mathrm{hol}|\mathcal{Q}$ is bijective, since the map hol is a local biholomorphism. Fix an element $(t,\varphi) \in \mathcal{Q}$. Note that $\Psi_{t,\varphi}(QF(S)) \subset \mathcal{Q}$, since $QF(S)$ is connected and $\Psi_{t,\varphi}$ is continuous. It can be easily seen by definition that $(\mathrm{hol}|\mathcal{Q}) \circ \Psi_{t,\varphi}$ is the identity map of $QF(S)$. Therefore we only have to show that $\Psi_{t,\varphi} \circ (\mathrm{hol}|\mathcal{Q})$ is the identity map of $\mathcal{Q}$. To this end, we will show that a subset

$$\mathcal{Q}' = \{(s,\psi) \in \mathcal{Q} : (s,\psi) = \Psi_{t,\varphi} \circ \mathrm{hol}(s,\psi)\}$$

of $\mathcal{Q}$ is non-empty, open and closed. Since $\Psi_{t,\varphi} \circ (\mathrm{hol}|\mathcal{Q})$ is cotinuous, $\mathcal{Q}'$ is closed. Moreover, $\Psi_{t,\varphi}([\rho_{t,\varphi}]) = (t,\varphi)$ implies $(t,\varphi) \in \mathcal{Q}'$, and hence $\mathcal{Q}' \neq \emptyset$. Take $(s,\psi) \in \mathcal{Q}'$ and its neighborhood U in $\mathcal{Q}$ such that $\mathrm{hol}|U$ is injective. Let V be a neighborhood of (s,ψ) contained in U and satisfying $\Psi_{t,\varphi} \circ \mathrm{hol}(V) \subset U$. Note that, for any $(s',\psi') \in V$, $\mathrm{hol}(\Psi_{t,\varphi} \circ \mathrm{hol}(s',\psi')) = (\mathrm{hol} \circ \Psi_{t,\varphi}) \circ \mathrm{hol}(s',\psi') = \mathrm{hol}(s',\psi')$. Since both $\Psi_{t,\varphi} \circ \mathrm{hol}(s',\psi')$ and (s',ψ') are contained in U and $\mathrm{hol}|U$ is injective, we have $\Psi_{t,\varphi} \circ \mathrm{hol}(s',\psi') = (s',\psi')$ for all $(s',\psi') \in V$. Therefore $\mathcal{Q}'$ is open. □

2.6. **Components of $Q(S)$.** Take a Fuchsian projective structure (t,φ). By Proposition 2.3, each connected component of $Q(S)$ contains a unique projective structure whose holonomy representation coincides with $[\rho_{t,\varphi}]$. But, from Theorem 2.1, these projective structures are written in the form $\{\mathrm{Gr}_\lambda(t,\varphi)\}_{\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)}$. Therefore we obtain the decomposition of $Q(S)$ into its connected components;

$$Q(S) = \coprod_{\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)} \mathcal{Q}_\lambda,$$

where $\mathcal{Q}_\lambda$ is the component containing $\mathrm{Gr}_\lambda(t,\varphi)$. Note that this suffix does not depend on the choice of Fuchsian projective structure (t,φ). Recall that an element of $Q(S)$ is called standard if its developing map is injective; otherwise exotic. Since any element of $\mathcal{Q}_\lambda$ is obtained from $\mathrm{Gr}_\lambda(t,\varphi)$ by a quasi-conformal deformation, one can easily see that $\mathcal{Q}_0$ is the component consisting of all standard projective structures, and that any element of $\mathcal{Q}_\lambda (\lambda \neq 0)$ is an exotic projective structure. Note that, since $\mathrm{Gr}_\lambda(t,\varphi)(\lambda \neq 0)$ has surjective developing map, any exotic projective structure has surjective developing map. Moreover, we can characterize the component of $Q(S)$ in which an element (t,φ) of $Q(S)$ is contained, as follows.

Lemma 2.4. *Take an element $(t,\varphi) \in Q(S)$ and denote $\rho_{t,\varphi}(\pi_1(S))$ by G. Then (t,φ) is contained in a component $\mathcal{Q}_\lambda$ corresponding to $\lambda = \sum n_j C_j \in \mathcal{ML}_{\mathbf{Z}}(S)$ if and only if the subset $f_{t,\varphi}^{-1}(\Lambda(G))/\Gamma_t$ of S_t consists of disjoint unions of $2n_j$ simple closed curves each of which is homotopic to C_j for all j.* □

3. Exotic projective structures and limits of representations

In this section, we investigate the relationship between sequences of exotic projective structures and algebraic and geometric limits of their holonomy representations. We begin with the definition of geometric convergence of Kleinian groups.

Definition 3.1. Let X be a locally compact Hausdorff space. We denote by $\mathcal{C}(X)$ the set of all closed subset of X. A sequence $\{A_n\}$ of closed subsets of X converges to a closed subset $A \subset X$ in the *Hausdorff topology* on $\mathcal{C}(X)$ if every element $x \in A$ is the limit of a sequence $\{x_n \in A_n\}$ and if every accumulation point of every sequence $\{x_n \in A_n\}$ lies in A. A sequence of Kleinian groups $\{G_n\}$ is said to converge *geometrically* to a group $\widehat{G}$ if $\{G_n\}$ converges to $\widehat{G}$ in the Hausdorff topology on $\mathcal{C}(\mathrm{PSL}_2(\mathbf{C}))$.

We recall some basic facts on the convergence of representations. Let $\{\rho_n\}$ be a sequence of discrete faithful representations of $\pi_1(S)$ into $\mathrm{PSL}_2(\mathbf{C})$ which converges algebraically to ρ_∞. Then ρ_∞ is also a discrete faithful representation (see [15, Theorem 1]). Moreover, there is a subsequence of $\{G_n = \rho_n(\pi_1(S))\}$ converging geometrically to a kleinian group $\widehat{G}$ which contains $G_\infty = \rho_\infty(\pi_1(S))$ (see [16, proposition 3.8]). The following theorem is due to Kerckhoff and Thurston [18, Corollary 2.2].

Theorem 3.2 (Kerckhoff-Thurston). *Let $\{\rho_n : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})\}$ be an algebraically convergent sequence of faithful representations whose images $\{G_n = \rho_n(\pi_1(S))\}$ are quasi-Fuchsian groups. Assume that $\{G_n\}$ converges geometrically to $\widehat{G}$. Then, $\{\Lambda(G_n)\}$ converges to $\Lambda(\widehat{G})$ in the Hausdorff topology on $\mathcal{C}(\widehat{\mathbf{C}})$.*

The following Lemma 3.3 plays an important role in this paper, especially in the proof of Theorem A. Since the situation under which we consider Lemma 3.3 is somewhat complicated, we first describe it:

Let $\{(t_n, \varphi_n)\}$ be a sequence in $Q(S)$ converging to an element (t, φ) in $K(S)$. Take a sequence of projective pairs $\{(f_{t_n,\varphi_n}, \rho_{t_n,\varphi_n})\}$ and a projective pair $(f_{t,\varphi}, \rho_{t,\varphi})$ such that $\{\rho_{t_n,\varphi_n}\}$ converges algebraically to $\rho_{t,\varphi}$. Put $G_n = \rho_{t_n,\varphi_n}(\pi_1(S))$ and $G_\infty = \rho_{t,\varphi}(\pi_1(S))$. Moreover, we assume that $\{G_n\}$ converges geometrically to a Kleinian group $\widehat{G}$. Since $\{t_n\}$ converges to t in $T(S)$, one can take a smooth quasi-conformal map $\omega_n : S_t \to S_{t_n}$ such that $\omega_n \circ g_t$ is homotopic to g_{t_n}, where $g_t : S \to S_t$ and $g_{t_n} : S \to S_{t_n}$ are markings for t and t_n respectively, and that the maximal dilatation

$$\frac{1 + ||(\omega_n)_{\bar{z}}/(\omega_n)_z||_\infty}{1 - ||(\omega_n)_{\bar{z}}/(\omega_n)_z||_\infty}$$

of ω_n tends to 1 as $n \to \infty$.

In this situation, we have the following

Lemma 3.3. *The sequence $\{\omega_n^{-1}(f_{t_n,\varphi_n}^{-1}(\Lambda(G_n))/\Gamma_{t_n})\}$ converges to $f_{t,\varphi}^{-1}(\Lambda(\widehat{G}))/\Gamma_t$ in the Hausdorff topology on $\mathcal{C}(S_t)$.*

Proof. We first observe that $\{f_{t_n,\varphi_n}\}$ converges to $f_{t,\varphi}$ locally uniformly in $\mathbf{H}_t$. Since $\{(t_n, \varphi_n)\}$ converges to (t, φ), $\{t_n\}$ converges to t in $T(S)$ and $\{\varphi_n\}$ converges to

φ locally uniformly in $\mathbf{H}_t$. Therefore, one can take a sequence of projective pairs $\{(\check{f}_{t_n,\varphi_n}, \check{\rho}_{t_n,\varphi_n})\}$ and a projective pair $\{(\check{f}_{t,\varphi}, \check{\rho}_{t,\varphi})\}$ such that $\{\check{f}_{t_n,\varphi_n}\}$ converges to $\check{f}_{t,\varphi}$ locally uniformly in $\mathbf{H}_t$. Chose an element $A_n \in \mathrm{PSL}_2(\mathbf{C})$ so that $\check{f}_{t_n,\varphi_n} = A_n \circ f_{t_n,\varphi_n}$ holds. Then, since both $\{\rho_{t_n,\varphi_n}\}$ and $\{\check{\rho}_{t_n,\varphi_n} = A_n \circ \rho_{t_n,\varphi_n} \circ A_n{}^{-1}\}$ are algebraically convergent sequences, $\{A_n\}$ converges to some element $A \in \mathrm{PSL}_2(\mathbf{C})$. Therefore, $\{f_{t_n,\varphi_n}\}$ also converges to $f_{t,\varphi}$ locally uniformly in $\mathbf{H}_t$.

Take a sequence of lifts $\widetilde{\omega}_n : \mathbf{H}_t \to \mathbf{H}_{t_n}$ of ω_n which converges to the identity locally uniformly in $\mathbf{H}_t$. Then $\{f_{t_n,\varphi_n} \circ \widetilde{\omega}_n\}$ also converges to $f_{t,\varphi}$ locally uniformly in $\mathbf{H}_t$. Since, from Theorem 3.2, $\{\Lambda(G_n)\}$ converges to $\Lambda(\widehat{G})$ in the Hausdorff topology on $\mathcal{C}(\widehat{\mathbf{C}})$, one can easily check that $\{\widetilde{\omega}_n^{-1}(f_{t_n,\varphi_n}^{-1}(\Lambda(G_n)))\}$ converges to $f_{t,\varphi}^{-1}(\Lambda(\widehat{G}))$ in the Hausdorff topology on $\mathcal{C}(\mathbf{H}_t)$. This implies that the sequence $\{\omega_n^{-1}(f_{t_n,\varphi_n}^{-1}(\Lambda(G_n))/\Gamma_{t_n})\}$ converges to $f_{t,\varphi}^{-1}(\Lambda(\widehat{G}))/\Gamma_t$ in the Hausdorff topology on $\mathcal{C}(S_t)$. □

Remark. The above lemma implies that the shape of $f_{t,\varphi}^{-1}(\Lambda(\widehat{G}))/\Gamma_t$ restricts the shape of $f_{t_n,\varphi_n}^{-1}(\Lambda(G_n))/\Gamma_{t_n}$, and hence the component of $Q(S)$ in which (t_n, φ_n) is contained by Lemma 2.4. This is the fundamental idea of the proof of Theorem A.

A Kleinian group G is called *geometrically finite* if it has a finite-sided fundamental domain in $\mathbf{H}^3$. A Kleinian group G is said to be a *b-group* if it has the only one simply connected invariant component of $\Omega(G)$, which is denoted by $\Omega_0(G)$. A geometrically finite b-group is said to be *regular*. A *degenerate group* is a b-group with $\Omega_0(G) = \Omega(G)$. For a b-group G, take a Riemann mapping $f : \Omega_0(G) \to \mathbf{H}$, which induce a group isomorphism $\chi_f : G \to fGf^{-1}$. An *accidental parabolic element* g in G is a parabolic element such that $\chi_f(g)$ is a loxodromic element in fGf^{-1}.

Let $U(S)$ denote the subset of $P(S)$ consisting of all projective structures whose developing maps are injective. Then $U(S)$ is closed in $P(S)$, containing $\mathcal{Q}_0$, and contained in $K(S)$. Since $\mathrm{int}AH(S) = QF(S)$, one can easily see that $\mathrm{int}U(S) = \mathcal{Q}_0$, and that $\partial\mathcal{Q}_0 \subset \partial U(S)$. Note that, for an element (t, φ) in $\partial U(S)$, its holonomy image $G = \rho_{t,\varphi}(\pi_1(S))$ is a b-group with an invariant component $f_{t,\varphi}(\mathbf{H}_t) = \Omega_0(G)$ (see [19]).

Using Lemma 3.3, we can characterize a sequence of exotic projective structures converging to an element of $\partial U(S)$ by algebraic and geometric limits of their holonomy representations.

Proposition 3.4. *In the same situation in Lemma 3.3, with additional assumption that $(t, \varphi) \in \partial U(S)$, the followings are equivalent;*

(1) (t_n, φ_n) *are exotic projective structures for large enough* n,

(2) $\Omega_0(G_\infty) \cap \Lambda(\widehat{G}) \neq \emptyset$.

Remark. The "(2) ⇒ (1)" part of Proposition 3.4 is due to McMullen [25]. In fact, he constructs a sequence of representations satisfying (2), and apply the "(2) ⇒ (1)" part to show Theorem 1.1. Later, we will explain his arguments more precisely.

Proof of Proposition 3.4. Recall that any exotic projective structure has surjective developing map. Hence, a projective structure $(t_n, \varphi_n) \in Q(S)$ is exotic if and only if its developing map f_{t_n,φ_n} is surjective. Therefore the condition (1) is equivalent to the condition that $f^{-1}_{t_n,\varphi_n}(\Lambda(G_n))/\Gamma_{t_n} \neq \emptyset$ for large enough n. Using Lemma 3.3, it turns out to be equivalent to the condition $f^{-1}_{t,\varphi}(\Lambda(\widehat{G}))/\Gamma_t \neq \emptyset$. But this condition is equivalent to the condition $f_{t,\varphi}(\mathbf{H}_t) \cap \Lambda(\widehat{G}) \neq \emptyset$. Since $f_{t,\varphi}(\mathbf{H}_t) = \Omega_0(G_\infty)$ holds for $(t,\varphi) \in \partial U(S)$, we have completed the proof. □

As a consequence of the "(1) ⇒ (2)" part of Proposition 3.4, we have the following assertion due to Matsuzaki (oral communication, see also [23, Section 7.4]).

Corollary 3.5 (Matsuzaki). *If a projective structure (t,φ) in $\partial U(S)$ is an accumulation point of exotic projective structures, its holonomy image $\rho_{t,\varphi}(\pi_1(S))$ contains accidental parabolics.*

Proof. Assume that there is a sequence of exotic projective structures $\{(t_n,\varphi_n)\}$ converging to a projective structure $(t,\varphi) \in \partial U(S)$ whose holonomy image contains no accidental parabolics. Then the following theorem due to Thurston (see Ohshika [27, Corollary 6.1]) implies that the geometric limit $\widehat{G}$ of $\{G_n = \rho_{t_n,\varphi_n}(\pi_1(S))\}$ coincides with the algebraic limit $G_\infty = \rho_{t,\varphi}(\pi_1(S))$. This contradicts Proposition 3.4. □

Theorem 3.6 (Thurston). *Let $\{\rho_n\}$ be a sequence of discrete faithful representations of $\pi_1(S)$ into $\mathrm{PSL}_2(\mathbf{C})$ converging algebraically to ρ_∞. Assume that $G_n = \rho_n(\pi_1(S))$ and $G_\infty = \rho_\infty(\pi_1(S))$ contain no accidental parabolics. Then $\{G_n\}$ converges geometrically to G_∞.*

We remark that a b-group with no accidental parabolics is a degenerate group (see [5]). Therefore Corollary 3.5 implies that an element of $\partial U(S)$ whose holonomy image is a degenerate group without accidental parabolics can not be an accumulation point of exotic projective structures. Moreover, we obtain the next corollaries.

Corollary 3.7. *The subset of $\partial\mathcal{Q}_0$ of projective structures which can not be accumulation points of exotic projective structures is dense in $\partial\mathcal{Q}_0$.*

Proof. A similar argument in [5, p.598] and [24, p.221] reveals that the subset of $\partial\mathcal{Q}_0$ of projective structures whose holonomy images contain no accidental parabolics is dense in $\partial\mathcal{Q}_0$. Then the assertion follows immediately from Corollary 3.5. □

Corollary 3.8. *For any $t_0 \in T(S)$, there exists a projective structure (t_0,φ) in $\partial\mathcal{Q}_0$ whose holonomy image $\rho_{t_0,\varphi}(\pi_1(S))$ is a regular b-group, such that there is no sequence of exotic projective structures converging to (t_0,φ).*

Proof. Fix $t_0 \in T(S)$ and consider a subspace $T = \mathcal{Q}_0 \cap B_2(\mathbf{H}_{t_0},\Gamma_{t_0})$ of $B_2(\mathbf{H}_{t_0},\Gamma_{t_0})$. The space T is coincident with the image of, so called, the Bers embedding of the Teichmüller space into $B_2(\mathbf{H}_{t_0},\Gamma_{t_0})$ (see [29]). Note that the boundary ∂T of T in $B_2(\mathbf{H}_{t_0},\Gamma_{t_0})$ is contained in $\partial\mathcal{Q}_0 \cap B_2(\mathbf{H}_{t_0},\Gamma_{t_0})$. It is known by McMullen [24] that the subset $\partial' T \subset \partial T$ of projective structures whose holonomy images are regular b-groups is dense in ∂T. Let (t_0,ψ) be a point in ∂T such that $\rho_{t_0,\psi}(\pi_1(S))$ is a degenerate group and take a sequence $\{(t_0,\varphi_n)\}$ in $\partial' T$ converging to (t_0,ψ). Now suppose that the assertion were false. Then, by the diagonal argument, one can take a sequence of exotic projective structures converging to (t_0, ψ). This contradicts Corollary 3.5. □

4. The proof of Theorem A

We first recall some basic facts of the quasi-Fuchsian space. Take a faithful representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$ whose image $G = \rho(\pi_1(S))$ is a quasi-Fuchsian group. Then the Kleinian manifold $N_G = (\mathbf{H}^3 \cup \Omega(G))/G$ is homeomorphic to $S \times [0,1]$ and $\partial N_G = \Omega(G)/G$ consists of two Riemann surfaces X_1 and X_2. We assume that ∂N_G is equipped with the orientation induced from that of $\widehat{\mathbf{C}}$. Then $\partial N_G = X_1 \cup X_2$, combined with markings induced from ρ, determines a point in $T(S) \times T(\overline{S})$, where $\overline{S}$ denotes S with its orientation reversed. Moreover, this assignment induces a holomorphic bijection (see [4]),

$$\mathrm{qf} : T(S) \times T(\overline{S}) \to QF(S).$$

A subset $B_t = \mathrm{qf}(\{t\} \times T(\overline{S}))$ of $QF(S)$ for some $t \in T(S)$ is called a *vertical Bers slice*. On the other hand, $B^t = \mathrm{qf}(T(S) \times \{t\})$ for some $t \in T(\overline{S})$ is called a *horizontal Bers slice*. Note that the bounbary of any vertical Bers slice is contained in $\mathrm{hol}(\partial \mathcal{Q}_0)$, while the boundary of any horizontal Bers slice is not.

The mapping class group $\mathrm{Mod}(S)$ is the group consisting of isotopy classes of orientation preserving homeomorphism of S. Recall that $\mathrm{Mod}(S)$ acts naturally on $T(S)$ and on $T(\overline{S})$.

We devote the rest of this section to the proof of the following theorem.

Theorem A. *For any $\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)$, we have $\overline{\mathcal{Q}_0} \cap \overline{\mathcal{Q}_\lambda} \neq \emptyset$. Especially, the closure of $Q(S)$ in $P(S)$ is connected.*

The second statement can be easily seen from the first statement. Hence we consentrare our attention to the first statement.

4.1. **Proof for special case.** Here, we prove Theorem A for the case that $\lambda = C \in \mathcal{S}$. Our aim is to show that there exists a sequence in $\mathcal{Q}_C$ which converges to an element of $\partial \mathcal{Q}_0$. This proceeds as follows.

1. First we review the proof of Theorem 1.1:

   Let (u, v) be any pair of Riemann surfaces in $T(S) \times T(\overline{S})$ and $\tau \in \mathrm{Mod}(S)$ be the Dehn twist around C. We will see that the sequence $\{[\rho_n] = \mathrm{qf}(\tau^n u, \tau^{2n} v)\}$ converges algebarically to a point $[\rho_\infty]$ on the boundary of some vertical Bers slice B_t. Let (t, φ) be a point of $\partial \mathcal{Q}_0$ such that $\mathrm{hol}(t, \varphi) = [\rho_\infty]$. Since the holonomy map $\mathrm{hol} : P(S) \to V(S)$ is a local homeomorphism, one can take a sequence $\{(t_n, \varphi_n)\}$ in $Q(S)$ converging to (t, φ) and satisfying $\mathrm{hol}(t_n, \varphi_n) = [\rho_n]$. For large enough n, (t_n, φ_n) turns out to be exotic and the proof of Theorem 1.1 is completed.

   We will show that (t_n, φ_n) is contained in $\mathcal{Q}_C$ for large enough n in the following steps.
2. Let $\{(f_{t_n,\varphi_n}, \rho_{t_n,\varphi_n})\}$ and $(f_{t,\varphi}, \rho_{t,\varphi})$ be projective pairs corresponding $\{(t_n, \varphi_n)\}$ and (t, φ), respectively, such that $\{\rho_{t_n,\varphi_n}\}$ converges algebraically to $\rho_{t,\varphi}$. Let $\widehat{G}$ be the geometric rimit of the sequence $\{G_n = \rho_{t_n,\varphi_n}(\pi_1(S))\}$. In Lemma

4.1, we will see that the subset $f_{t,\varphi}^{-1}(\Lambda(\widehat{G}))/\Gamma_t$ in S_t consists of two components and is contained in an annulus whose core is homotopic to C. Ricall that $\{\omega_n^{-1}(f_{t_n,\varphi_n}^{-1}(\Lambda(G_n))/\Gamma_{t_n})\}$ converges to $f_{t,\varphi}^{-1}(\Lambda(\widehat{G}))/\Gamma_t$ in the Hausdorff topology on $\mathcal{C}(S_t)$ by Lemma 3.3. Therefore, one may expect that $f_{t_n,\varphi_n}^{-1}(\Lambda(G_n))/\Gamma_{t_n}$ consists of two simple closed curves each of which is homotopic to C for large enough n. If it were true, (t_n, φ_n) is contained in $\mathcal{Q}_C$ for large enough n by Lemma 2.4, and the proof is completed. We justify the above expectation in the following steps.

3. It is easy to see that all components of $f_{t_n,\varphi_n}^{-1}(\Lambda(G_n))/\Gamma_{t_n}$ are simple close curves homotopic to C for large enough n (Lemma 4.2). Hence, all what we have to show is that any component of $f_{t_n,\varphi_n}^{-1}(\Lambda(G_n))/\Gamma_{t_n}$ does not join with another components as n tends to ∞.
4. In Lemma 4.3, we will see that any two components of $f_{t_n,\varphi_n}^{-1}(\Lambda(G_n))/\Gamma_{t_n}$ are separeted by some annulus whose core is homotopic to C and whose modulus is larger than m_0, where m_0 does not depend on n.
5. We will see that the hyperbolic distance between two boundary components of an annulus in a Riemann surface can be estimated below by using the modulus of the annulus (Lemma 4.4). Hence the hyperbolic distance between any two components of $\omega_n^{-1}(f_{t_n,\varphi_n}^{-1}(\Lambda(G_n))/\Gamma_{t_n})$ is bounded below by some positive constant $L > 0$ which does not depends on n. Therefore, one can see that $f_{t_n,\varphi_n}^{-1}(\Lambda(G_n))/\Gamma_{t_n}$ consists of two simple closed curves each of which is homotopic to C for large enough n, and can complete the proof of Theorem A for the case that λ is a simple closed curve.

We now fill in the details.

Step 1. Theorem 1.1 is due to McMullen [25, Appendix A]. In our sketch of this proof, we make use of a variation of Thurston's hyperbolic Dehn surgery theorem, which is due to Comar [8] (see also [2, Theorem 2.2] and [6]). Related arguments can be found in [7] and [18]. The "adding twist" technique, discovered by Anderson and Canary in [2], also plays an important role in this proof.

Sketch of proof of Theorem 1.1. Let $\widehat{G}$ be a geometrically finite Kleinian group whose Kleinian manifold $N_{\widehat{G}} = (\mathbf{H}^3 \cup \Omega(\widehat{G}))/\widehat{G}$ is homeomorphic to $S \times [0,1] - C \times \{1/2\}$. The existence of such a Kleinian group $\widehat{G}$ is guaranteed by Thurston's geometrization theorem (see [26]). Here, the tubular neighborhood of $C \times \{1/2\}$ corresponds to the rank two cusp end of $N_{\widehat{G}}$. We fix a basis $\langle \gamma, \delta \rangle$ for the fundamental group of the rank two cusp so that γ is homotopic to $C \times \{0\}$ and δ is trivial in $S \times [0,1]$. By performing $(n,1)$ Dehn filling on the cusp ($n \in \mathbf{N}$), we obtain a sequence of representations $\{\beta_n : \widehat{G} \to \mathrm{PSL}_2(\mathbf{C})\}$ which satisfies the following conditions (see [8]);

- $G_n = \beta_n(\widehat{G})$ is a quasi-Fuchsian group,
- The kernel of β_n is normally generated by $\gamma^n \delta$,
- $\{G_n\}$ converges geometrically to $\widehat{G}$, and
- $\{\beta_n\}$ converges algebraically to the identity representation of $\widehat{G}$.

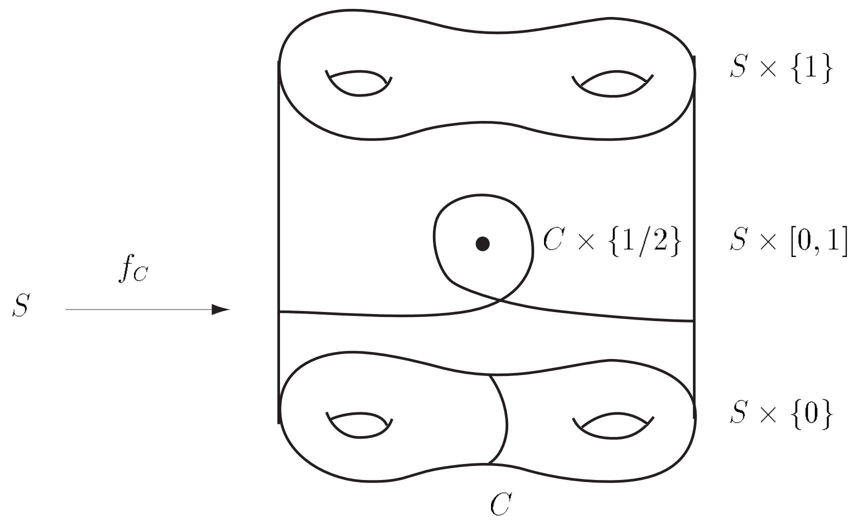

FIGURE 1. The wrapping map f_C.

Let f_0 be the inclusion map $S \to S \times \{1/4\} \subset N_{\widehat{G}}$ and denote by $(f_0)_*$ the induced group homomorphism of $\pi_1(S)$ into $\widehat{G}$. Then we obtain a sequence of faithful representations $\rho'_n = \beta_n \circ (f_0)_*$ of $\pi_1(S)$ onto quasi-Fuchsian groups G_n. By modifying β_n slightly, if necessary, we may assume that ∂N_{G_n} is conformally isomorphic to $\partial N_{\widehat{G}}$ for all n. Then the above representations are expressed as

$$[\rho'_n] = \mathrm{qf}(u, \tau^n v),$$

where $(u, v) \in T(S) \times T(\overline{S})$ is the complex structure on $\partial N_{\widehat{G}}$ combined with trivial markings, and $\tau \in \mathrm{Mod}(S)$ is the Dehn twist around C.

Now we add a twist to f_0. More precisely, we construct an immersion $f_C : S \to N_{\widehat{G}}$ which is homotopic to f_0 in $S \times [0, 1]$ but not in $S \times [0, 1] - C \times \{1/2\}$ in the following way. Let A be a tubular neighborhood of C in S. Then the map $f_C|(S-A)$ is defined by $f_C(x) = (x, 1/4)$, and the map $f_C|A$ is defined so that $f_C(A)$ wraps once around the tubular neighborhood of $C \times \{1/2\}$, see Figure 1. This immersion f_C is called the *wrapping map* associated to C.

Again, we obtain a sequence of faithful representations $\rho_n = \beta_n \circ (f_C)_*$ of $\pi_1(S)$ onto quasi-Fuchsian groups G_n, which can be expressed as

$$[\rho_n] = \mathrm{qf}(\tau^n u, \tau^{2n} v).$$

The sequence $\{\rho_n\}$ converges algebraically to $\rho_\infty = (f_C)_*$. We denote $\rho_\infty(\pi_1(S))$ by G_∞. We can show that G_∞ is a regular b-group and, moreover, that $[\rho_\infty]$ lies on the boundary of some vertical Bers slice B_t (see Lemma 4.1(1)). Therefore, there exists an element $(t, \varphi) \in \partial \mathcal{Q}_0$ with $\mathrm{hol}(t, \varphi) = [\rho_\infty]$. Since the holonomy map $\mathrm{hol} : P(S) \to V(S)$ is a local homeomorphism, one can take a sequence $\{(t_n, \varphi_n)\}$ in $Q(S)$ converging to (t, φ) and satisfying $\mathrm{hol}(t_n, \varphi_n) = [\rho_n]$. Since $f_C : S \to N_{\widehat{G}}$ is

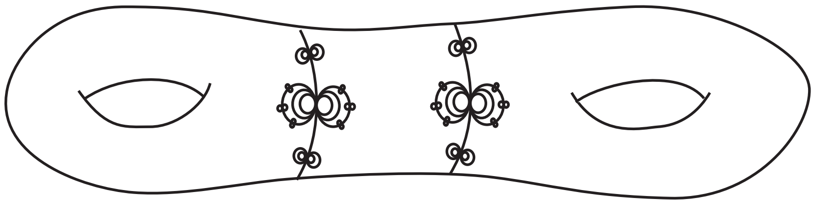

FIGURE 2. The subset $f_{t,\varphi}^{-1}(\Lambda(\widehat{G}))/\Gamma_t$ in S_t.

not homotopic to a map into $\partial N_{\widehat{G}}$, G_∞ does not represent the fundamental group of either component of $\partial N_{\widehat{G}}$ and hence $\Omega_0(G_\infty) \cap \Lambda(\widehat{G}) \neq \emptyset$ (see also Lemma 4.1(2)). Therefore, Proposition 3.4 implies that (t_n, φ_n) are exotic. This completes the proof of Theorem 1.1. □

Step 2. Our aim is to show that the sequence $\{(t_n, \varphi_n)\}$ constructed above is contained in $\mathcal{Q}_C$. To this end, we examine the shape of $f_{t,\varphi}^{-1}(\Lambda(\widehat{G}))/\Gamma_t \subset S_t$ in the following lemma (see Figure 2).

Lemma 4.1. (1) *G_∞ is a regular b-group. Moreover, $[\rho_\infty]$ lies on the boundary of some vertical Bers slice B_t.*

(2) *The subset $f_{t,\varphi}^{-1}(\Lambda(\widehat{G}))/\Gamma_t$ in S_t consists of two connected components and is contained in an annulus whose core is homotopic to C.*

Proof. During this proof, the reader is adviced to refer Figure 3.

(1) Since G_∞ is a finitely generated subgroup of the geometrically finite Kleinian group $\widehat{G}$ with $\Omega(\widehat{G}) \neq \emptyset$, it is also geometrically finite (see, for example, [23, Theorem 3.11]). Moreover, since G_∞ has a parabolic element corresponding to C, it is not a quasi-Fuchsian group. Therefore, to show that G_∞ is a regular b-group, we only have to show that $\Omega(G_\infty)$ has a simply connected invariant component.

Let A be a closed annular neighborhood of C in S. By deforming the wrapping map $f_C : S \to N_{\widehat{G}}$ in its homotopy class, we may assume that f_C maps $S - A$ onto $(S - C) \times \{0\}$ and that $f_C(\mathrm{int} A) \subset \mathrm{int} N_{\widehat{G}}$. We take a lift $\widetilde{f_C} : \widetilde{S} \to \mathbf{H}^3 \cup \Omega(\widehat{G})$ of f_C satisfying

$$\widetilde{f_C} \circ g = \rho_\infty(g) \circ \widetilde{f_C} \quad \text{for all} \quad g \in \pi_1(S),$$

where $\pi_1(S)$ is regarded as the covering transformation group of the universal covering map $p : \widetilde{S} \to S$. Since the map f_C is π_1-injective, we may assume that the map $\widetilde{f_C}$ is an embedding, and hence the image $\widetilde{f_C}(\widetilde{S})$ of $\widetilde{S}$ is simply connected.

Fix a component $\widetilde{A}_0$ of $\widetilde{A} = p^{-1}(A)$. Let $g_0 \in \pi_1(S)$ be a generator for the stabilizer of $\widetilde{A}_0$ in $\pi_1(S)$. Let $\langle \gamma', \delta' \rangle$ be a rank two parabolic subgroup of $\widehat{G}$ which is conjugate to $\langle \gamma, \delta \rangle$ in $\widehat{G}$ and satisfying $\rho_\infty(g_0) = \gamma'$. Since $\langle \gamma' \rangle$ stabilizes $\widetilde{f_C}(\widetilde{A}_0)$,

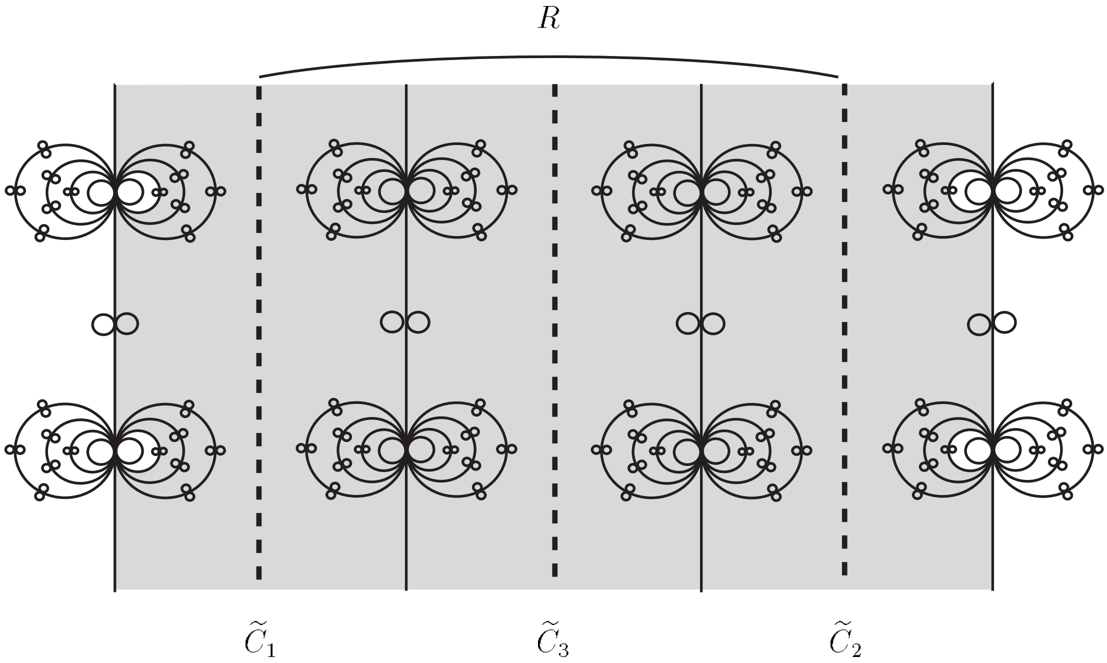

FIGURE 3. $\Lambda(\widehat{G})$ and $\Omega_0(G_\infty)$ (the shaded part).

each of two components $\widetilde{C}_1, \widetilde{C}_2$ of $\widetilde{f}_C(\partial\widetilde{A}_0)$ forms a simple closed curve together with the common fixed point of $\langle\gamma', \delta'\rangle$. Note that $\widetilde{C}_1$ and $\widetilde{C}_2$ projects onto $C \subset S \times \{0\} \subset \partial N_{\widehat{G}}$ via the covering map $\Omega(\widehat{G}) \to \partial N_{\widehat{G}}$. Since $f_C(A)$ wraps once around a tubular neighborhood of $C \times \{1/2\}$ in $N_{\widehat{G}}$, we may assume that $\widetilde{C}_2 = \delta'\widetilde{C}_1$ holds. Let R be the crescent-like domain in $\widehat{\mathbf{C}}$ lying between $\widetilde{C}_1$ and $\widetilde{C}_2$, and let D be the domain which is cut out of $\mathbf{H}^3$ by $\widetilde{f}_C(\widetilde{A}_0)$ and is facing R. Note that $R \cap \widetilde{f}_C(\widetilde{S} - \widetilde{A}) = \emptyset$ and that D is precisely invariant under the subgroup $\langle\gamma'\rangle$ of G_∞, that is, $(\gamma')^l(D) = D$ for any $l \in \mathbf{Z}$ and $D \cap g(D) = \emptyset$ for any $g \in G_\infty - \langle\gamma'\rangle$. Therefore, $\widetilde{f}_C$ is homotopic to an embedding $F : \widetilde{S} \to \widehat{\mathbf{C}}$ satisfying

$$F \circ g = \rho_\infty(g) \circ F \quad \text{for all} \quad g \in \pi_1(S),$$

and the homotopy is constant on $\widetilde{S} - \widetilde{A}$. Since G_∞ acts on $F(\widetilde{S})$ properly discontinuously, there is a component Δ of $\Omega(G_\infty)$ containing $F(\widetilde{S})$. One can easily see that Δ coincides with $F(\widetilde{S})$ because $F : \widetilde{S} \to \widehat{\mathbf{C}}$ descends to an homeomorphism $S \to F(\widetilde{S})/G_\infty \subset \Delta/G_\infty$. Since $F(\widetilde{S})$ is simply connected, G_∞ is a regular b-group with $F(\widetilde{S}) = \Omega_0(G_\infty)$.

A result of Abikoff [1] implies that any representation whose image is a regular b-group lies on the boundary of some (vertical or horizontal) Bers slice. Since $F : \widetilde{S} \to \widehat{\mathbf{C}}$ descends to an orientation preserving homeomorphism $S \to \Omega_0(G_\infty)/G_\infty \subset N_{G_\infty}$

which induces the representation $[\rho_\infty]$, one can see that $[\rho_\infty]$ lies on the boundary of a vertical Bers slice B_t for some $t \in T(S)$.

(2) We first remark that the limit set $\Lambda(\widehat{G})$ is connected since each component of $\Omega(\widehat{G})$ is simply connected, the latter can be seen from the fact that each component of ∂N_G is incompressible in $N_{\widehat{G}}$.

From the above argument, we have $F(\widetilde{A}) = \bigcup_{g \in G_\infty/\langle\gamma'\rangle} g(R)$. Since

$$\Omega_0(G_\infty) - \bigcup_{g \in G_\infty/\langle\gamma'\rangle} g(R) = F(\widetilde{S}) - F(\widetilde{A}) \subset \Omega(\widehat{G}),$$

we have

$$\Omega_0(G_\infty) \cap \Lambda(\widehat{G}) = \left(\bigcup_{g \in G_\infty/\langle\gamma'\rangle} g(R) \right) \cap \Lambda(\widehat{G}) = \bigcup_{g \in G_\infty/\langle\gamma'\rangle} g(R \cap \Lambda(\widehat{G})).$$

Therefore, we concentrate our attention to $R \cap \Lambda(\widehat{G})$. Let G' be a subgroup of $\widehat{G}$ representing the fundamental group of $S \times \{0\} \subset \partial N_{\widehat{G}}$. By conjugating G' in $\widehat{G}$, if necessary, we may assume that $\widetilde{C}_1 \subset \Omega_0(G')$ and that $\widetilde{C}_2 \subset \Omega_0(\delta' G' \delta'^{-1})$. Note that $\Omega_0(G') \cap \Omega_0(\delta' G' \delta'^{-1}) = \emptyset$, since $\delta' \notin G'$. One can easily see that there exists a unique lift $\widetilde{C}_3$ of $C \subset S \times \{1\} \subset \partial N_{\widehat{G}}$ contained in R and terminating in the common fixed point of $\langle \gamma', \delta' \rangle$ at both ends. The curve $\widetilde{C}_3$ divides R into two crescent-like domains R_1 and R_2. Since $\widetilde{C}_1, \widetilde{C}_2$ and $\widetilde{C}_3$ are contained in distinct components of $\Omega(\widehat{G})$, $R_j \cap \Lambda(\widehat{G}) \neq \emptyset$ for $j = 1, 2$. Moreover, since $\Lambda(\widehat{G})$ is connected, $R_j \cap \Lambda(\widehat{G})$, $j = 1, 2$, is also connected. Therefore, the subset $F^{-1}(\Lambda(\widehat{G}))/\pi_1(S)$ in S consists of two connected components and is contained in the annulus A whose core is homotopic to C. Since $F : \widetilde{S} \to \widehat{\mathbf{C}}$ can be regarded as the developing map of the projective structure on S corresponding to $(t, \varphi) \in P(S)$, we obtain the assertion. $\square$

Step 3. Let $\{\lambda_n\}$ be the sequence of $\mathcal{ML}_{\mathbf{Z}}(S)$ satisfying $(t_n, \varphi_n) \in \mathcal{Q}_{\lambda_n}$.

Lemma 4.2. *$\lambda_n \in \{kC : k \in \mathbf{N}\} \subset \mathcal{ML}_{\mathbf{Z}}(S)$ for large enough n.*

Proof. For simplicity, we denote $f_{t,\varphi}^{-1}(\Lambda(\widehat{G}))/\Gamma_t$ by $\widehat{\Lambda}$ and $\omega_n^{-1}(f_{t_n,\varphi_n}^{-1}(\Lambda(G_n))/\Gamma_{t_n})$ by Λ_n. Take an open annulus A in S_t hwose core is homotopic to C and which containing $\widehat{\Lambda}$. Then $K = S_t - A$ is a compact set. Now suppose that there exists a subsequence $\{n_j\}$ such that $\lambda_{n_j} \notin \{kC : k \in \mathbf{N}\}$. Then, from the characterization of Λ_{n_j} in Lemma 2.4, one can easily see that $\Lambda_{n_j} \cap K \neq \emptyset$ for all j. Since K is compact, any sequence $\{z_{n_j} \in \Lambda_{n_j} \cap K\}$ has an accumulation point $z_\infty \in K$. But the Hausdorff convergence $\Lambda_n \to \widehat{\Lambda}$ (Lemma 3.3) implies that $z_\infty \in \widehat{\Lambda}$, which contradicts to $\widehat{\Lambda} \cap K = \emptyset$. $\square$

Step 4. For an annulus A with a conformal structure, the *modulus* $m(A)$ of A is defined by $m(A) = (2\pi)^{-1} \log c$ when A is conformally equivalent to a round annulus $\{z \in \mathbf{C} : 1 < |z| < c\}$. We will make use of the monotonicity of moduli; if an annulus A contains disjoint essential annuli A_1 and A_2, then $m(A_1) + m(A_2) \leq m(A)$ (see [20, Lemma 6.3]).

Lemma 4.3. *There exists a positive constant $m_0 > 0$, independent of n, satisfying the following: Any curve on S_{t_n} joining any two connected components of $f_{t_n,\varphi_n}^{-1}(\Lambda(G_n))/\Gamma_{t_n}$ traverses some annulus whose core is homotopic to C and whose modulus is larger than m_0.*

Proof. Fix a positive integer $k \in \mathbf{N}$. We consider a sequence $\{(s_n, \psi_n)\} = \{\Psi_{kC}([\rho_n])\}$ in $\mathcal{Q}_{kC}$, where

$$\Psi_{kC} = (\mathrm{hol}|\mathcal{Q}_{kC})^{-1} : QF(S) \to \mathcal{Q}_{kC}.$$

We will show that there exists a positive constant m_0, independent of k and n, satisfying the following: Any curve in S_{s_n} joining any two connected components of $f_{s_n,\psi_n}^{-1}(\Lambda(G_n))/\Gamma_{s_n}$ traverses some annulus whose core is homotopic to C and whose modulus is larger than m_0. Once it has shown, since $(t_n, \varphi_n) = \Psi_{k_n C}([\rho_n])$ for $\lambda_n = k_n C$, we obtain the assertion in Lemma 4.3.

The proof depends on the particular form of $[\rho_n] = \mathrm{qf}(\tau^n u, \tau^{2n} v)$. For simplicity, we assume that $[\rho_0] = \mathrm{qf}(u, v)$ is a Fuchsian representation. (The following argument, with a slight modification, works out without this assumption.) Let (p, ϕ) denote the projective structure in $\mathcal{Q}_0$ such that $[\rho_{p,\phi}] = [\rho_0]$. Then $(s_0, \psi_0) = \mathrm{Gr}_{kC}(p, \phi)$. To obtain (s_0, ψ_0) from (p, ϕ), we perform the same construction as described in 2.4. We use the same notation and normalization as in 2.4; for example, $G_0 = \rho_{p,\phi}(\pi_1(S))$ is a Fuchsian group acting on $\mathbf{H}$, $g \in G_0$ is a generator of cyclic subgroup which stabilizes B_α, etc. In addition, we prepare some notations; let $B_\alpha^* = \{z \in \mathbf{H}^* : 3\pi/2 - \alpha < \arg z < 3\pi/2 + \alpha\}$ be the complex conjugation of B_α, put $\widehat{B}_\alpha = B_\alpha \cup B_\alpha^*$ and set $\widehat{D}_\alpha = \bigcup_{h \in G_0/\langle g \rangle} h(\widehat{B}_\alpha)$.

Recall that $f_{s_0,\psi_0}^{-1}(\Lambda(G_0))/\Gamma_{s_0} \subset S_{s_0}$ consists of $2k$ simple closed curves each of which is homotopic to C. Moreover, observe that $f_{s_0,\psi_0}^{-1}(\widehat{D}_\alpha)/\Gamma_{s_0}$ contains $2k+1$ annular domains $A_1, \dots A_{2k+1}$ each of whose core is homotopic to C. There exists exactly one connected component of $f_{s_0,\psi_0}^{-1}(\Lambda(G))/\Gamma_{s_0}$ lying between A_j and A_{j+1} for every $j \in \{1, \dots, 2k\}$. Therefore, any curve on S_{s_0} joining any two components of $f_{s_0,\psi_0}^{-1}(\Lambda(G_0))/\Gamma_{s_0}$ traverses some A_j. Note that the developing map f_{s_0,ψ_0} induce a natural conformal isomorphism $\xi_j : A_j \to B_\alpha^{(*)}/\langle g \rangle$ for all j, where $B_\alpha^{(*)}$ is B_α or B_α^*.

One can take an element $\mu_n \in \mathrm{Belt}(G_0)_1$ with $\mathrm{supp}(\mu_n) \subset \widehat{D}_{\alpha/2}$ such that the quasi-conformal map $w_{\mu_n} : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ induces the quasi-conformal deformation $[\rho_n] = \mathrm{qf}(\tau^n u, \tau^{2n} v)$ of $[\rho_0] = \mathrm{qf}(u, v)$. (See [33] for an explicit description of μ_n.) This element $\mu_n \in \mathrm{Belt}(G_0)_1$ also induces the quasi-conformal deformation (s_n, ψ_n) of (s_0, ψ_0), as described in 2.5. Recall that the quasi-conformal map $w_{\hat{\mu}_n}|\mathbf{H}_{s_0} : \mathbf{H}_{s_0} \to \mathbf{H}_{s_n}$ with the Beltrami coefficient $\hat{\mu}_n = f_{s_0,\psi_0}^*(\mu_n)$ descends to a quasi-conformal map $g_{\hat{\mu}_n} : S_{s_0} \to S_{s_n}$.

The two components of $\xi_j^{-1}((B_\alpha^{(*)} - B_{\alpha/2}^{(*)})/\langle g \rangle) \subset A_j$ are denoted by A_j' and A_j'', each of which is essential sub-annulus in A_j. An easy calculation reveals that $m(A_j') = m(A_j'') = \alpha/2l$ for all j, where l is the exponent of g, that is, $g(z) = e^l z$. Since $g_{\hat{\mu}_n}$ is conformal on $A_j' \cup A_j''$, combining with the monotonicity of moduli, we have

$$\begin{aligned} m(g_{\hat{\mu}_n}(A_j)) &\geq m(g_{\hat{\mu}_n}(A_j')) + m(g_{\hat{\mu}_n}(A_j'')) \\ &= m(A_j') + m(A_j'') \\ &= \alpha/l. \end{aligned}$$

Put $m_0 = \alpha/l$. Then any curve on $S_{s_n} = g_{\hat{\mu}_n}(S_{s_0})$ joining any two connected components of $f^{-1}_{s_n,\psi_n}(\Lambda(G_n))/\Gamma_{s_n} = g_{\hat{\mu}_n}(f^{-1}_{s_0,\psi_0}(\Lambda(G_0))/\Gamma_{s_0})$ traverses some $g_{\hat{\mu}_n}(A_j)$ whose core is homotopic to C and whose modulus is larger than m_0. We have completed the proof of Lemma 4.3. □

Step 5. The following lemma implies that the hyperbolic distance between two boundary components of an annulus in a Riemann surface can be estimated below by using the modulus of the annulus. The essencial tool in this proof is Grötzsch's module theorem (see [20]). For $t \in T(S)$, we denote the hyperbolic distance of $z_1, z_2 \in S_t$ by $d_t(z_1, z_2)$ and the hyperbolic length of the closed geodesic representing $C \in \mathcal{S}$ by $l_t(C)$.

Lemma 4.4. *Let $t \in T(S)$ and $C \in \mathcal{S}$. Let $A \subset S_t$ be an annular domain such that ∂A consists of two simple closed curves C_1, C_2 each of which is homotopic to C. Then there is a positive constant $I = I(l_t(C), m(A)) > 0$ which depends only on $l_t(C)$ and $m(A)$ such that*

$$d_t(C_1, C_2) > I(l_t(C), m(A)),$$

where $d_t(C_1, C_2) = \inf\{d_t(z_1, z_2) : z_j \in C_j (j = 1, 2)\}$.

Proof. Let $\mathbf{H} = \{z \in \mathbf{C} : \mathrm{Im} z > 0\}$ denote the upper half plane. Take a holomorphic covering map $p_1 : \mathbf{H} \to S_t$ so that the imaginary axis $i\mathbf{R}^+$ projects onto a simple closed curve homotopic to C. Let $\widetilde{A}$ denote the connected component of ${p_1}^{-1}(A)$ which connect 0 and ∞. Then $\widetilde{A}$ is stabilized by a cyclic group $\langle g \rangle$ which is generated by $g(z) = z\exp(l_t(C))$. Since $m(\mathbf{H}/\langle g \rangle) = \pi/l_t(C)$, there is a holomorphic covering map $p_2 : \mathbf{H} \to R = \{w \in \mathbf{C} : 1 < |w| < \exp(2\pi^2/l_t(C))\}$ whose covering transformation group is $\langle g \rangle$. Let $\lambda_R(w)|dw|$ denote the complete hyperbolic metric on R, and $d_R(\cdot,\cdot)$ the distance with respect to $\lambda_R(w)|dw|$. Note that A is conformally equivalent to $A' = p_2(\widetilde{A})$ and that $d_t(C_1, C_2) = d_R(C_1', C_2')$, where C_1' and C_2' are components of $\partial A'$. One can easily see that there is a positive constant $I_1 = I_1(l_t(C))$ which depends only on $l_t(C)$ such that $\lambda_R(w)|dw| > I_1(l_t(C))|dw|$. Therefore we obtain

$$d_R(C_1', C_2') > I_1(l_t(C)) d_e(C_1', C_2'), \tag{4.1}$$

where $d_e(\cdot,\cdot)$ is the distance with respect to the Euclidean metric $|dw|$.

We can assume that C_1' (resp. C_2') is the inner (resp. outer) component of $\partial A'$. Take $w_1 \in C_1'$ and $w_2 \in C_2'$ satisfying $d_e(w_1, w_2) = d_e(C_1', C_2')$. Let $D \subset \mathbf{C}$ be a connected component of $\mathbf{C} - C_2'$ which contains 0. Take a Riemann mapping $h : \Delta \to D$ such that $h(0) = 0$ and $h^{-1}(w_1) = r > 0$. Now, Koebe's one-quarter theorem and Koebe's distorsion theorem (see, for example, [28, p.9]) implies that there is a positive constant $I_2 = I_2(1-r)$ such that

$$d_e(C_1', C_2') > I_2(1-r). \tag{4.2}$$

Let $\mu(r)$ denote the modulus of the domain $\{z \in \mathbf{C} : |z| < 1, z \notin [0, r]\}$ for $0 < r < 1$. Then Grötzsch's module theorem asserts that $m(\mathcal{A}) \le \mu(r)$ for any annular domain $\mathcal{A} \subset \Delta = \{z \in \mathbf{C} : |z| < 1\}$ which separates 0 and r from $\partial\Delta$ (see [20, Chapter II] for more informations). Since $h^{-1}(A')$ separates 0 and r from $\partial\Delta$, Grötzsch's theorem implies that $m(A') \le \mu(r)$. Since μ is a monotone decreasing function, there is a positive constant $I_3 = I_3(m(A'))$ such that

$$1 - r > I_3(m(A')). \tag{4.3}$$

From the inequalities (4.1)-(4.3), we obtain

$$d_R(C_1', C_2') > I_1(l_t(C)) I_2(I_3(m(A'))).$$

Since $d_R(C_1', C_2') = d_t(C_1, C_2)$ and $m(A') = m(A)$, we obtain the assertion. □

Using Lemma 4.4, we can finally prove the next lemma, and can complete the proof of Theorem A for the special case.

Lemma 4.5. *$(t_n, \varphi_n) \in \mathcal{Q}_C$ for large enough n.*

Proof. Since maximal dilatations of quasi-conformal maps $\omega_n : S_t \to S_{t_n}$ tend to 1, there exists a positive constant $m_1 > 0$ such that moduli of ${\omega_n}^{-1} \circ g_{\hat{\mu}_n}(A_j)$ exseed m_1 for sufficiently large n. Since any curve in S_t joining any two components of $\Lambda_n = {\omega_n}^{-1}(f_{t_n,\varphi_n}^{-1}(\Lambda(G_n))/\Gamma_{t_n})$ traverses some ${\omega_n}^{-1} \circ g_{\hat{\mu}_n}(A_j)$, Lemma 4.4 implies that there exists a positive constant $L > 0$ such that the hyperbolic distance of any two components of Λ_n is bounded below by L for sufficiently large n. Then, from Lemma 3.3 and Lemma 4.1(2), one can easily see that Λ_n consists of two connected components each of which is homotopic to C. Therefore, from Lemma 2.4, we obtain the assertion. □

4.2. **Proof for general case.** Take an arbitrary element $\lambda = \sum_{j=1}^{l} n_j C_j \in \mathcal{ML}_{\mathbf{Z}}(S)$ and let $\widehat{G}$ be a geometrically finite Kleinian group whose Kleinian manifold $N_{\widehat{G}}$ is homeomorphic to $S \times [0,1] - \cup_{j=1}^{l} C_j \times \{1/2\}$. Let $f_\lambda : S \to N_{\widehat{G}}$ be an immersion such that S minus every annular neighborhoods of C_j is mapped into $S \times \{0\}$ by inclusion and the image of each annular neighborhood of C_j is wrapping n_j times around $C_j \times \{1/2\}$. This immersion is called the wrapping map associated to λ.

Now we perform simultaneous $(n, 1)$ Dehn filling on all the cusps to obtain a sequence of quasi-Fuchsian representations $\{\beta_n : \widehat{G} \to G_n\}$ as before. Moreover, we obtain a sequence $\{\rho_n = \beta_n \circ (f_\lambda)_*\}$ of representations converging algebraically to $\rho_\infty = (f_\lambda)_*$, and the corresponding sequence $\{(t_n, \varphi_n)\}$ of projective structures converging to $(t, \varphi) \in \partial\mathcal{Q}_0$. Then we can show, in the same manner in the proof of Lemma 4.1, that the subset $f_{t,\varphi}^{-1}(\Lambda(\widehat{G}))/\Gamma_t$ in S_t consists of disjoint unions of $2n_j$ connected components contained in annuli whose core are homotopic to C_j for all j. Again, we can show that $(t_n, \varphi_n) \in \mathcal{Q}_\lambda$ for large enough n. Finally, we have completed the proof of Theorem A.

4.3. **A remark.** Although it is difficult to understand exactly the shape of the closed set $\overline{\mathcal{Q}_0} \cap \overline{\mathcal{Q}_\lambda}$, one can see that it is not a compact subset of $P(S)$.

Proposition 4.6. *For any $\lambda \in \mathcal{ML}_{\mathbf{Z}}(S)$, $\overline{\mathcal{Q}_0} \cap \overline{\mathcal{Q}_\lambda}$ is not compact in $P(S)$.*

Proof. We proof this, again, for the case that $\lambda = C \in \mathcal{S}$. Let $\widehat{G}$ be the same as in the proof of Theorem A for special case. Let A be an annular domain in a component $S \times \{1\}$ of conformal boundary $\partial N_{\widehat{G}}$, whose core is homotopic to C. Observe that A can be conformally embedded in S_t, where $t \in T(S)$ is the underlying complex structure of the limit point $(t, \varphi) \in \partial\mathcal{Q}_0$ constructed in the proof of Theorem A. If we deform the conformal structure on $S \times \{1\}$ so that $m(A) \to \infty$, then the hyperbolic length of C on $S \times \{1\}$ tends to 0. Then t diverges in $T(S)$, and hence $(t, \varphi) \in \partial\mathcal{Q}_0$ diverges in $P(S)$. □

5. The proofs of Theorems B and C

For $C_1, C_2 \in \mathcal{S}$, the geometric intersection number $i(C_1, C_2)$ is the minimum number of points in which the representations of C_1 and C_2 must intersects. This can be naturally extended to $\mathcal{ML}_{\mathbf{Z}}(S)$. The main aim of this section is to prove the next theorem.

Theorem B. *For any finite set $\{\lambda_i\}_{i=1}^m \subset \mathcal{ML}_{\mathbf{Z}}(S)$ satisfying $i(\lambda_j, \lambda_k) = 0$ for all $j, k \in \{1, \dots, m\}$, we have $\overline{\mathcal{Q}_0} \cap \overline{\mathcal{Q}_{\lambda_1}} \cap \cdots \cap \overline{\mathcal{Q}_{\lambda_m}} \neq \emptyset$.*

To prove this theorem, we first prepair the following Lemma 5.1, which provides us a method to construct some sequences of representations with the same algebraic limit but with mutually distinct geometric limits.

For a Kleinian group G, let $M_G = \mathbf{H}^3/G$ denote the interior of the Kleinian manifold $N_G = (\mathbf{H}^3 \cup \Omega(G))/G$.

Take an element $\lambda = \sum_{j=1}^l n_j C_j \in \mathcal{ML}_{\mathbf{Z}}(S)$ and an ordered set of positive integers $\mathbf{k} = \{k_1, \dots, k_l\}$. For this pair $(\lambda, \mathbf{k})$, we define a new element $\lambda/\mathbf{k}$ of $\mathcal{ML}_{\mathbf{Z}}(S)$ by

$$\lambda/\mathbf{k} = \sum_{j=1}^{l} [n_j/k_j] C_j,$$

where $[n_j/k_j]$ is the largest integer which does not exceed n_j/k_j.

Lemma 5.1. *Let $\lambda = \sum_{j=1}^l n_j C_j$ be an element of $\mathcal{ML}_{\mathbf{Z}}(S)$, and let $\widehat{G}$ be a geometrically finite Kleinian group such that $N_{\widehat{G}}$ is homeomorphic to $S \times [0,1] - \bigcup_{j=1}^l C_j \times \{1/2\}$. Then, for any ordered set of positive integers $\mathbf{k} = \{k_1, \dots, k_l\}$, there exists a geometrically finite Kleinian group $\widehat{G}'$ such that*

- *$N_{\widehat{G}'}$ is also homeomorphic to $S \times [0,1] - \bigcup_{j=1}^l C_j \times \{1/2\}$, and*
- *the wrapping map $f_\lambda : S \to M_{\widehat{G}}$ associated to λ and the wrapping map $f_{\lambda'} : S \to M_{\widehat{G}'}$ associated to $\lambda' = \lambda/\mathbf{k}$ induce the same representation up to conjugation in $\mathrm{PSL}_2(\mathbf{C})$, that is, $[(f_\lambda)_*] = [(f_{\lambda'})_*]$.*

Proof. Take an element $\bar{\lambda} = \sum_{j=1}^l m_j C_j \in \mathcal{ML}_{\mathbf{Z}}(S)$ satisfying $m_j = n_j - [n_j/k_j]k_j$ for all j. Note that $0 \le m_j < k_j$ hold for all j. We consider the wrapping map

$f_{\bar\lambda} : S \to M_{\widehat{G}}$ and denote $(f_{\bar\lambda})_*(\pi_1(S))$ by G. Recall that G is a regular b-group, and that M_G is homeomorphic to the interior of $S \times [0,1]$. Via the canonical projection $p_1 : M_G \to M_{\widehat{G}}$, the map $f_{\bar\lambda}$ is lifted to an embedding $\widetilde{f_{\bar\lambda}} : S \to M_G$ which is a homotopy equivalence.

We now choose a basis $\langle \gamma_j, \delta_j \rangle \subset \widehat{G}$ for the fundamental group of each rank two cusp in $M_{\widehat{G}}$ corresponding to $C_j \times \{1/2\}$ as before, but with an additional assumption that γ_j is contained in G. We take a subgroup

$$\widehat{G'} = \langle G, \delta_1^{k_1}, \dots, \delta_l^{k_l} \rangle$$

of $\widehat{G}$. Since $m_j < k_j$ hold for all j, one can take Jordan domains $\{D_j, D'_j\}_{j=1}^l$ in $\Omega(G)$ such that

- both D_j and D'_j are stabilized by $\langle \gamma_j \rangle$,
- ${\delta_j}^{k_j}$ maps ∂D_j onto $\partial D'_j$ and the interior of D_j onto the exterior of D'_j, and
- $\{D_j, D'_j\}_{j=1}^l$ are projected onto $2l$ disjoint cusp neighborhoods in ∂N_G via the covering map $\Omega(G) \to \partial N_G$.

Now one can apply the Klein-Maskit combination theorem II [22, Theorem E.5 (xi)] inductively to show that $N_{\widehat{G'}}$ is homeomorphic to $S \times [0,1] - \bigcup_{j=1}^l C_j \times \{1/2\}$. Moreover, note that the canonical projection $p_2 : M_G \to M_{\widehat{G'}}$ maps the end of M_G corresponding to $S \times \{0\}$ homeomorphically onto the end of $M_{\widehat{G'}}$ corresponding to $S \times \{0\}$. Therefore $p_2 \circ \widetilde{f_{\bar\lambda}}$ is homotopic to the inclusion map $f_0 : S \to M_{\widehat{G'}}$ onto $S \times \{1/4\}$.

We now consider the canonical projection $p_3 : M_{\widehat{G'}} \to M_{\widehat{G}}$. Note that $p_3 \circ f_0$ is homotopic to $f_{\bar\lambda}$. Observe that the restriction of p_3 on the tubular neighborhood of $C_j \times \{1/2\}$ in $M_{\widehat{G'}}$ is a k_j-fold covering map onto the tubular neighborhood of $C_j \times \{1/2\}$ in $M_{\widehat{G}}$. We now perform a surgery to obtain the wrapping map $f_{\lambda'} : S \to M_{\widehat{G'}}$ from $f_0 : S \to M_{\widehat{G'}}$ as before. Then $p_3 \circ f_{\lambda'}$ is homotopic to the wrapping map $f_\lambda : S \to M_{\widehat{G}}$ associated to λ. Therefore $[(p_3 \circ f_{\lambda'})_*] = [(f_\lambda)_*]$, and since $(p_3)_* : \widehat{G'} \to \widehat{G}$ is the inclusion map, we have $[(f_{\lambda'})_*] = [(f_\lambda)_*]$. □

Now Theorem B follows immediately from Lemma 5.1.

Proof of Theorem B. For any finite set $\{\lambda_i\}_{i=1}^m \subset \mathcal{ML}_{\mathbf{Z}}(S)$ satisfying $i(\lambda_j, \lambda_k) = 0$ for all $j, k \in \{1, \dots, m\}$, one can easily find an element $\lambda = \sum_{j=1}^l n_j C_j \in \mathcal{ML}_{\mathbf{Z}}(S)$ and ordered sets of positive integers $\mathbf{k}_i = \{k_1^{(i)}, \dots, k_l^{(i)}\}$ satisfying $\lambda_i = \lambda / \mathbf{k}_i$ for all i. By Lemma 5.1, there exist Kleinian groups $\widehat{G}$ and $\widehat{G}_i (i = 1, \dots, m)$ such that the wrapping maps $f_\lambda : S \to M_{\widehat{G}}$ and $f_{\lambda_i} : S \to M_{\widehat{G}_i}$ induce the same representations $[(f_\lambda)_*] = [(f_{\lambda_i})_*]$ for all i. Take the element $(t, \varphi) \in \partial \mathcal{Q}_0$ such that $\mathrm{hol}(t, \varphi) = [(f_\lambda)_*]$. By performing simultaneous $(n, 1)$ Dehn filling on all the cusps in $M_{\widehat{G}_i}$, we obtain a sequence $\{(t_n^{(i)}, \varphi_n^{(i)})\}$ in $\mathcal{Q}_{\lambda_i}$ converging to (t, φ). We have completed the the proof. □

As a consequence of Theorem B, we obtain the following

Theorem C. *For any positive integer $n \in \mathbf{N}$, there exists a point $[\rho] \in \partial QF(S)$ such that $U \cap QF(S)$ consists of more than n components for any sufficiently small neighborhood U of $[\rho]$.*

Proof. For any positive integer $n \in \mathbf{N}$, a finite set $\{kC\}_{k=1}^{n}$ in $\mathcal{ML}_{\mathbf{Z}}(S)$ satisfies the condition in Theorem B. Since the holonomy map hol : $P(S) \to V(S)$ is local homeomorphism, we obtain the assertion. □

References

[1] W. Abikoff, *On boundaries of Teichmüller spaces and on kleinian groups:III*, Acta Math. **134** (1975), 211–237.

[2] J. W. Anderson and R. D. Canary, *Algebraic limits of Kleinian groups which rearrange the pages of a book*, Invent. Math. **126** (1996), 205–214.

[3] J. W. Anderson, R. D. Canary and D. McCullough, *On the topology of deformation spaces of Kleinian groups*, preprint.

[4] L. Bers, *Simultaneous uniformization*, Bull. Amer. Math. Soc. **66** (1960), 94–97.

[5] ———, *On boundaries of Teichmüller spaces and on kleinian groups:I*, Ann. of Math. **91** (1970), 570–600.

[6] F. Bonahon and J. P. Otal, *Variétés hyperboliques à géodésiques arbitrairement courtes*, Bull. London Math. Soc. **20** (1988), 255–261.

[7] J. F. Brock, *Iteration of mapping classes on a Bers slice: examples of algebraic and geometric limits of hyperbolic 3-manifolds*, Contemporary Math. **211** (1997), 81–106.

[8] T. D. Comar, *Hyperbolic Dehn surgery and convergence of Kleinian groups*, Ph. D. Thesis, University of Michigan, 1996.

[9] C. J. Earle, *On variation of projective structures*, Riemann surfaces and related topics, Ann. Math. Studies **97** (1981), 87–99.

[10] C. J. Earle and C. T. McMullen, *Quasiconformal isotopes*, Holomorphic functions and Moduli II, Springer-Verlag, 1987, 143–154.

[11] D. M. Gallo, *Deforming real projective structures*, Ann. Acad. Sci. Fenn. **22** (1997), 3–14.

[12] W. M. Goldman, *Projective structures with Fuchsian holonomy*, J. Diff. Geom. **25** (1987), 297–326.

[13] D. A. Hejhal, *Monodromy groups and linearly polymorphic functions*, Acta Math. **135** (1975), 1–55.

[14] J. H. Hubbard, *The monodromy of projective structures*, Riemann surfaces and related topics, Ann. Math. Studies **97** (1981), 257–275.

[15] T. Jørgensen, *On discrete groups of Möbius transformations*, Amer. J. Math. **98** (1986), 739–749.

[16] T. Jørgensen and A. Marden, *Algebraic and geometric convergence of Kleinian groups*, Math. Scand. **66** (1990), 47–72.

[17] Y. Kamishima and S. P. Tan, *Deforming spaces on geometric structures*, Aspects of Low Dimensional Manifolds, advanced Studies in Pure Mathematics 20, Kinokuniya Co., 1992, 263–299.

[18] S. P. Kerckhoff and W. P. Thurston, *Non-continuity of the action of the modular group at Bers' boundary of Teichmuller space*, Invent. Math. **100** (1990), 25–47.

[19] I. Kra, *Deformations of Fuchsian groups*, Duke Math. J. **36** (1969), 537–546.

[20] O. Lehto and K. I. Virtanen, *Quasiconformal mappings in the plane*, Springer-Verlag, 1973.

[21] B. Maskit, *On a class of Kleinian groups*, Ann. Acad. Sci. Fenn. Ser. A I **442** (1969), 1–8.

[22] ———, *Kleinian groups*, Springer-Verlag, 1988.

[23] K. Matsuzaki and M. Taniguchi, *Hyperbolic manifolds and Kleinian groups*, Oxford University Press, 1998.

[24] C. T. McMullen, *Cusps are dense*, Ann. of Math. **133** (1991), 217–247.

[25] ———, *Complex earthquakes and Teichmuller theory*, J. Amer. Math. Soc. **11** (1998), 283–320.
[26] J. W. Morgan, *On Thurston's uniformization theorem for three-dimensional manifolds*, The Smith conjecture, Academic Press, 1984, 37–125.
[27] K. Ohshika, *Divergent sequences of Kleinian groups*, Geometry & Topology Monographs Volume 1: The Epstein Birthday Schrift, 419–450.
[28] C. Pommerenke, *Boundary behavior of conformal maps*, Springer-Verlag, 1992.
[29] H. Shiga, *Projective structures on Riemann surfaces and Kleinian groups*, J. Math. Kyoto Univ. **27** (1987), 433–438.
[30] H. Shiga and H. Tanigawa, *Projective structures with discrete holonomy representations*, Trans. Amer. Math. Soc. **351** (1999), 813–823.
[31] D. P. Sullivan, *Quasiconformal homeomorphisms and dynamics II: structural stability implies hyperbolicity of Kleinian groups*, Acta Math. **155** (1985), 243–260.
[32] H. Tanigawa, *Grafting, harmonic maps and projective structures on surfaces*, J. Diff. Geom. **47** (1997), 399–419.
[33] S. Worpert, *The Fenchel-Nielsen deformation*, Ann. of Math. **115** (1982), 501–528.

Department of Mathematics, Tokyo Institute of Technology, Tokyo 152-8551 Japan
E-mail address: `itoken@@math.titech.ac.jp`