SCHOTTKY GROUPS AND BERS BOUNDARY OF TEICHMÜLLER SPACE

KENTARO ITO

Abstract. We will show that every Kleinian groups on a Bers boundary of the Teichmüller space is an algebraic limit of a sequence of Schottky groups. To show this, we extend the action of the mapping class group on a Bers slice to that on a class of function groups whose invariant components are covering some fixed Riemann surface. An important observation is that the orbit of every maximal cusp is dense in a Bers boundary.

1. Introduction

In this paper, we extend the action of the mapping class group on a Bers slice to that on a wider class (which will be called an extended Bers slice) of Kleinian groups. Here, we explain the fundamental idea how to extend the action of the mapping class group.

Let S be an oriented compact surface possibly with boundary ∂S, and let $T(S)$ be the Teichmüller space of comprete hyperbolic structures on the interior of S with finite area. Let $V(S)$ be the space of conjugacy classes of representations of $\pi_1(S)$ into $\mathrm{PSL}_2(\mathbf{C})$. The subspace $QF(S)$ of $V(S)$ consisting of discrete faithful representations whose images are quasi-Fuchsian groups is naturally identified with a product of Teichmüller spaces $T(S) \times T(S)$; that is, there exists a holomorphic isomorphism

$$Q : T(S) \times T(S) \to QF(S).$$

The mapping class group $Mod(S)$ of S naturally acts on $T(S)$, and hence on the Bers slice $B_X = Q(\{X\} \times T(S))$ for every $X \in T(S)$;

$$Q(X, Y) \mapsto Q(X, \sigma Y),$$

where $\sigma \in Mod(S)$. The representation $Q(X, \sigma Y)$ has another description as follows;

$$Q(X, \sigma Y) = Q(\sigma^{-1} X, Y) \circ {\sigma_*}^{-1},$$

where σ_* is the group isomorphism of $\pi_1(S)$ induced by σ. The right side of the above equation suggests us a possibility to define the action of $Mod(S)$ even when Y is pinched or degenerated. In this view point, Bers [3] extended the action of $Mod(S)$ to that on the closure of B_X and, in this paper, we extend the action to that on the subset C_X of $V(S)$ consisting of representations whose images are function groups with invariant components covering $X \in T(S)$. The set C_X is called an *extended Bers slice* for X.

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Our main theorem is the following (see Lemma 6.1 and Theorem 6.4):

Main Theorem . The subset S_X of C_X of Schottky groups consists of exactly one orbit under the action of $Mod(S)$ and the set of accumulation points of S_X contains the Bers boundary ∂B_X.

We remark that, in the case of genus 2, it was shown by Gallo [8] that the set of accumulation points of S_X contains the Bers boundary ∂B_X. But our method is different from that of Gallo.

This paper is organized as follows: In section 2, we give a definition of an extended Bers slice C_X on which we define the action of the mapping class group. In section 3, we obtain a sufficient condition so that the action of $Mod(S)$ is continuous at an element of C_X (Corollary 3.2). In section 5, we show that the orbit of every maximal cusp is dense in the Bers boundary ∂B_X (Theorem 5.6). In section 6, we prove our main theorem as a corollary of preceding sections. In section 5 and 6, one of the crucial tool is Thurston's compactness theorem [23], which will be introduced in section 4.

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2. Preliminaries

Let S be a compact oriented surface of negative Euler characteristic possibly with boundary ∂S. The *Teichmüller space* $T(S)$ is the set of equivalence classes of pairs (f, X); where X is a hyperbolic Riemann surface of finite area and $f : int(S) \to X$ is a homeomorphism from the interior of S. Two pairs (f, X) and (g, Y) are equivalent if there is a conformal map $\psi : X \to Y$ such that $\psi \circ f$ is isotopic to g. The *mapping class group* $Mod(S)$ is the group consisting of isotopy classes of orientation preserving homeomorphisms of S. There is a natural action of $\sigma \in Mod(S)$ on $T(S)$ by

$$\sigma(f, X) = (f \circ \sigma^{-1}, X).$$

A Kleinian group G is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$, which acts on the hyperbolic space $\mathbf{H}^3$ as isometries, and on the sphere at infinity $S^2_\infty = \hat{\mathbf{C}}$ by conformal automorphisms. The limit set of G in $\hat{\mathbf{C}}$ is denoted by $\Lambda(G)$ and its compliment $\hat{\mathbf{C}} - \Lambda(G)$, which is called the region of discontinuity of G, is denoted by $\Omega(G)$. A Kleinian group G is called a *function group* if its region of discontinuity $\Omega(G)$ has an invariant component $\Omega_0(G)$. If a function group has exactly two invariant components, it is called a *quasi-Fuchsian group*; otherwise, it has a unique invariant component.

For a given $X \in T(S)$, let , be a Fuchsian group acting on the unit disc $\Delta = \{z \in \hat{\mathbf{C}} : |z| < 1\}$ such that $X = \Delta/,$. We define the space of bounded holomorphic quadratic differentials on Δ for , by

$$B_2(,) = \{\varphi | \varphi \text{ is holomorphic on } \Delta, \varphi \circ \gamma(\gamma')^2 = \varphi \text{ for } \forall \gamma \in , \text{ and } ||\varphi||_\infty < \infty\},$$

where $||\varphi||_\infty$ is the hyperbolic sup-norm of φ defined by $\sup_{z\in\Delta}(1 - |z|^2)^2|\varphi(z)|$. With this norm $||\varphi||_\infty$, $B_2(,)$ is a finite dimensional complex Banach space.

For $\varphi \in B_2(,\ )$, we associate a pair $(f_\varphi, \rho_\varphi)$, called a *normalized projective structure* on X; where,

$$f_\varphi : \Delta \to \hat{\mathbf{C}}$$

is a meromorphic local homeomorphism whose Schwarzian derivative $S(f_\varphi)$ is equal to φ, and

$$\rho_\varphi : ,\ \to \mathrm{PSL}_2(\mathbf{C})$$

is a group homomorphism satisfying $f_\varphi \circ \gamma = \rho_\varphi(\gamma) \circ f_\varphi$ for all $\gamma \in ,\ $. Moreover, f_φ is normalized by the conditions $f_\varphi(0) = 0, f'_\varphi(0) = 1$ and $f''_\varphi(0) = 0$. Then there is a bijective correspondence between normalized projective structures and $B_2(,\ )$.

We denote by $C(,\ )$ the set of $\varphi \in B_2(,\ )$ such that the map f_φ is a covering map. For $\varphi \in C(,\ )$, $G = \rho_\varphi(,\ )$ is a function group (possibly with torsion) and $f_\varphi(\Delta)$ is an invariant component of G (see [12] for more information). Furthermore, we denote $C_0(,\ )$ by the set of $\varphi \in C(,\ )$ such that the map $f_\varphi : \Delta \to f_\varphi(\Delta) \subset \hat{\mathbf{C}}$ induces a conformal isomorphism $X = \Delta/,\ \to f_\varphi(\Delta)/\rho_\varphi(,\ )$;

$$C_0(,\ ) = \{\varphi \in C(,\ ) | X = \Delta/,\ \cong f_\varphi(\Delta)/\rho_\varphi(,\ )\}.$$

For $\varphi \in C_0(,\ )$, $G = \rho_\varphi(,\ )$ may have an elliptic element whose fixed points are not contained in the invariant component $\Omega_0(G) = f_\varphi(\Delta)$. In [14], one can find examples of elements of $C(,\ )$ but not of $C_0(,\ )$.

Let $V(S)$ denote the space of conjugacy classes $[\rho]$ of irreducible representations $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$ such that $\rho(g)$ is parabolic for every $g \in \pi_1(\partial S)$. We also use the notation (ρ, G) to represent an element $[\rho] \in V(S)$ whose image is $G = \rho(\pi_1(S))$. The space $V(S)$ is a manifold endowed with the algebraic topology; a sequence of representations $\rho_n : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$ converges *algebraically* to a representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$ if $\rho_n(g) \to \rho(g)$ in $\mathrm{PSL}_2(\mathbf{C})$ for all $g \in \pi_1(S)$.

It is known by Kra [11] that the map

$$hol : B_2(,\ ) \to V(S)$$

assigning the conjugacy class $[\rho_\varphi]$ of the representation $\rho_\varphi : \pi_1(S) \cong ,\ \to \mathrm{PSL}_2(\mathbf{C})$ to $\varphi \in B_2(,\ )$ is a holomorphic embedding. (Here and hereafter, we frequently identify a representation of $\pi_1(S)$ with a representation of , .) For any $X \in T(S)$, we define subsets $\hat{C}_X$ and C_X of $V(S)$ by

$$\hat{C}_X = hol(C(,\ )), \text{ and } C_X = hol(C_0(,\ )).$$

We call C_X an *extended Bers slice*, on which we will define an action of the mapping class group. We can define C_X directly as a subset of $V(S)$ consisting of function groups whose invariant components are covering $X \in T(S)$; more precisely, a representation $(\rho, G) \in V(S)$ is an element of C_X if G is a function group with an invariant component $\Omega_0(G)$ and ρ is induced by the composition of the inclusion map $\iota : \Omega_0(G)/G \hookrightarrow N_G$ into the Kleinian manifold $N_G = \mathbf{H}^3 \cup \Omega(G)/G$ with a conformal isomorphism $g : X \to \Omega_0(G)/G$.

A *Bers slice* B_X is the subset of C_X consisting of faithful representations whose images are quasi-Fuchsian groups. It is known by Bers [2] that a Bers slice B_X is (anti-holomorphically) isomorphic to the Teichmüller space $T(S)$, and that it is relatively compact in $V(S)$. A set $\partial B_X = \bar{B}_X - B_X$ is called a *Bers boundary*, where

$\bar{B}_X$ is the closure of B_X in $V(S)$. Moreover, we denote by $\hat{B}_X$ the subset of C_X consisting of faithful representations. It is conjectured that $\bar{B}_X = \hat{B}_X$ in Bers [2].

The followings are the sets which we want to consider in this paper.

$$B_X \subset \bar{B}_X \subseteq \hat{B}_X \subset C_X \subset \hat{C}_X.$$

Example . In the case that S is a closed surface, a typical example of an element of $C_X - \hat{B}_X$ is a Schottky group. A Kleinian group G is a *Schottky group* if its Kleinian manifold $N_G = \mathbf{H}^3 \cup \Omega(G)/G$ is homeomorphic to a handle body H_g. Let G be a Schottky group which uniformizes X, that is $X = \Omega(G)/G = \partial N_G$, then a representation $\rho : \pi_1(S) \cong \pi_1(\partial N_G) \to G \cong \pi_1(N_G)$ induced by the inclusion map $\partial N_G \hookrightarrow N_G$ is an element of C_X but not of $\hat{B}_X$.

Lemma 2.1. *For any $X \in T(S)$, C_X is a compact subset of $V(S)$.*

Proof. To show that $C_X = hol(C_0(\,,\,))$ is compact, we will show that $C_0(\,,\,)$ is closed and bounded subset of $B_2(\,,\,)$. Since it is known by Kra and Maskit [14] that $C(\,,\,)$ is a closed and bounded subset of $B_2(\,,\,)$ (i.e. $\hat{C}_X$ is compact), we only have to show that $C_0(\,,\,)$ is closed. Let $\varphi_n \in C_0(\,,\,)$ be a sequence converging to $\varphi \in C(\,,\,)$. Let (f_n, ρ_n) and (f, ρ) be normalized projective structures corresponding to φ_n and φ, respectivery. Then f_n converges to f locally uniformly on Δ. Suppose that the map $g : X \to f(\Delta)/\rho(\,,\,)$ induced from $f : \Delta \to f(\Delta)$ is not injective. Then there are two points $x, y \in X (x \neq y)$ such that $g(x) = g(y)$, and hence there are lifts $\tilde{x}, \tilde{y} \in \Delta$ of x and y such that $f(\tilde{x}) = f(\tilde{y})$. Since $f_n(\tilde{x}) \to f(\tilde{x})$ and $f_n(\tilde{y}) \to f(\tilde{y})$, the hyperbolic distances d_n between $f_n(\tilde{x})$ and $f_n(\tilde{y})$ on $f_n(\Delta)$ tend to 0 as $n \to \infty$. On the other hand, since $\varphi_n \in C_0(\,,\,)$, $f_n(\Delta)/\rho_n(\,,\,)$ are conformally equivalent to X. It implies that d_n is equal to the hyperbolic distance between x and y on X for all large n. This is a contradiction. □

For a given Kleinian group G with $\Omega(G) \neq \emptyset$, we denote by $B(U, G)_1$ the space of measurable Beltrami differentials μ for G satisfying $||\mu||_\infty = \text{ess.sup}|\mu| < 1$ with support in an open set $U \subset \hat{\mathbf{C}}$. For $\mu \in B(U, G)_1$, there is a unique quasiconformal homeomorphism

$$w_\mu : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$$

satisfying $(w_\mu)_{\bar{z}} = \mu \times (w_\mu)_z$ (a.e.) and fixing $0, 1$ and ∞. Two elements $\mu, \nu \in B(U, G)_1$ are *equivalent* (denoted by $\mu \sim \nu$) if w_μ and w_ν induce the same group isomorphism; that is, $w_\mu \circ g \circ (w_\mu)^{-1} = w_\nu \circ g \circ (w_\nu)^{-1}$ for all $g \in G$.

Let $\,$, be a Fuchsian group acting on the unit disc Δ such that $X = \Delta/,$. Let $(\rho, G) \in C_X$ and let $(f : \Delta \to \Omega_0(G), \rho : \pi_1(S) \to G)$ be the corresponding projective structure. For $\mu \in B(\Delta,\,)_1$, the push-forward $f_*\mu$ is an element of $B(\Omega_0(G), G)_1$ which is defined locally by the pull-back

$$(f^{-1})^*\mu = \mu \circ f^{-1} \times \overline{(f^{-1})'}/(f^{-1})'$$

of μ via some branch of f^{-1}. Then, the map

$$f_* : B(\Delta,\,)_1 \to B(\Omega_0(G), G)_1,$$

assigning $f_*\mu$ to μ is an isomorphism. The quotient space $B(\Delta, ,)_1/\sim$ is naturally identified with the Teichmüller space $T(S)$. We define a subspace $QC_0(\rho)$ of $V(S)$ by

$$QC_0(\rho) = \{[\rho'] \in V(S) | \rho'(\gamma) = w_\mu \circ \rho(\gamma) \circ (w_\mu)^{-1}, \gamma \in \pi_1(S), \mu \in B(\Omega_0(G), G)_1\}.$$

This space $QC_0(\rho)$ is also identified with the quotient space $B(\Omega_0(G), G)_1/\sim$. Then, it was shown by Maskit [17] (see also Kra [13]) that the map $f_* : B(\Delta, ,)_1 \to B(\Omega_0(G), G)_1$ descends to an unbranched covering map

$$\Psi_\rho : T(S) \to QC_0(\rho)$$

with $\Psi_\rho(X) = [\rho]$. One can easily see that $\Psi_\rho(Y) \in C_Y$ for all $Y \in T(S)$ and that if $[\rho] \in C_X$ then $[\rho] \circ \sigma_*^{-1} \in C_{\sigma X}$ for all $\sigma \in Mod(S)$, here σ_* is the group isomorphism of $\pi_1(S)$ induced by σ. Now we define the action of $\sigma \in Mod(S)$ on C_X by

$$[\rho] \mapsto [\rho]^\sigma = \Psi_\rho(\sigma^{-1}X) \circ \sigma_*^{-1}.$$

One can easily check that $[\rho]^{\sigma_1 \circ \sigma_2} = ([\rho]^{\sigma_2})^{\sigma_1}$ holds for any $\sigma_1, \sigma_2 \in Mod(S)$. This action of $Mod(S)$ coincides with the natural action on $B_X \cong T(S)$ described in Introduction. We also remark that the action restricted to $\hat{B}_X$ is the same with that defined by Bers in [3].

3. Continuity of the action of the mapping class group

In this section, we obtain a sufficient condition for $[\rho] \in C_X$ so that the action of $Mod(S)$ at $[\rho]$ is continuous. The same result, for the case that C_X is replaced by $\hat{B}_X$, was obtained by Bers [3].

We first show the continuity under base change.

Proposition 3.1. *Let $[\rho] = (\rho, G)$ be an element of C_X such that all components of $\Omega(G)/G$ except for $X = \Omega_0(G)/G$ have (if there exist) no moduli of deformation. Then the following holds:*

If $[\rho_n] \to [\rho]$ in C_X then $\Psi_{\rho_n}(Y) \to \Psi_\rho(Y)$ in C_Y for all $Y \in T(S)$.

Proof. Let φ_n and φ be elements in $C_0(,)$ such that $hol(\varphi_n) = [\rho_n]$ and $hol(\varphi) = [\rho]$, respectively. Since $C_0(,)$ is compact and the map $hol : B_2(,) \to V(S)$ is injective, $\varphi_n \to \varphi$ in $C_0(,)$. Let (f_n, ρ_n) and (f, ρ) be normarized projective structures for φ_n and φ, respectively. Then f_n converges to f locally uniformly on Δ. Let $\mu \in B(\Delta, ,)_1$ be a representative of $Y \in T(S)$. We may assume that μ is continuous. Put $\hat{\mu}_n = (f_n)_*\mu \in B(\Omega_0(G_n), G_n)_1$ and $\hat{\mu} = f_*\mu \in B(\Omega_0(G), G)_1$, where $G_n = \rho_n(\pi_1(S))$ and $G = \rho(\pi_1(S))$. Since $\{w_{\hat{\mu}_n}\}$ fix $0, 1$ and ∞ and their dilatations are uniformly bounded, it has a subsequence (which we denote by the same symbol) $\{w_{\hat{\mu}_n}\}$ converging uniformly to some quasiconformal homeomorphism w_∞ of $\hat{\mathbf{C}}$. Since the representatives of $\Psi_{\rho_n}(Y)$ are induced by $w_{\hat{\mu}_n} \circ f_n$, $\{\Psi_{\rho_n}(Y)\}$ converges algebraically to the conjugacy class of a representation induced by $w_\infty \circ f$. Therefore, we only have to show that w_∞ and $w_{\hat{\mu}}$ induce the same group isomorphism from G into $\mathrm{PSL}_2(\mathbf{C})$.

Since injectivity radii (with respect to the Poincaré metric on Δ) of f_n are uniformly bounded below (see [14], Lemma 5.1), for any $z \in \Omega_0(G)$, there is an open neighborhood U of z and suitable branches of inverse maps ${f_n}^{-1}$ and f^{-1} on U such that ${f_n}^{-1}$ converges to f^{-1} uniformly on U. Hence one can see that $\hat{\mu}_n$ converges to $\hat{\mu}$ locally uniformly on $\Omega_0(G)$. Therefore, $w_{\hat{\mu}_n} \circ (w_{\hat{\mu}})^{-1}$ converges to a conformal map $w_\infty \circ (w_{\hat{\mu}})^{-1}$ locally uniformly on $w_{\hat{\mu}}(\Omega_0(G))$, and hence, the Beltrami coefficient of w_∞ is equal to $\hat{\mu}$ almost everywhere on $\Omega_0(G)$. Since there is no essential deformation on $\Omega(G) - \Omega_0(G)$ by assumption and on $\Lambda(G)$ by Sullivan's rigidity theorem [22], w_∞ and $w_{\hat{\mu}}$ induce the same group isomorphism. □

Corollary 3.2. *Let $[\rho]$ be an element of C_X satisfying the same condition as in Proposition 3.1. Then the action of $Mod(S)$ is continuous at $[\rho]$; that is, if $[\rho_n] \to [\rho]$ in C_X then $[\rho_n]^\sigma \to [\rho]^\sigma$ for all $\sigma \in Mod(S)$.*

Proof. By Proposition, $\Psi_{\rho_n}(\sigma^{-1}X) \to \Psi_\rho(\sigma^{-1}X)$ for all $\sigma \in Mod(S)$. Therefore, $[\rho_n]^\sigma = \Psi_{\rho_n}(\sigma^{-1}X) \circ {\sigma_*}^{-1}$ converges algebraically to $[\rho]^\sigma = \Psi_\rho(\sigma^{-1}X) \circ {\sigma_*}^{-1}$. □

Remark . In [10], Kerckhoff and Thurston showed that there is a Bers slice B_X and a point $[\rho] \in \partial B_X$ at which the action of $Mod(S)$ is not continuous.

4. Thurston's compactness theorem

In this section, we introduce Thurston's compactness theorem [23], which will play an important roll in the following sections.

Let M be a compact 3-manifold with boundary ∂M. A closed curve γ on ∂M is said to be *compressible* if it is null homotopic in M but not in ∂M; otherwise it is *incompressible*. A proper map $f : (A, \partial A) \to (M, \partial M)$ of an annulus A into M is said to be *essential* if $f_* : \pi_1(A) \to \pi_1(M)$ is an injection and f is not homotopic (as a map of pairs) to a map into ∂M.

Definition 4.1. Let M be a compact 3-manifold with boundary ∂M. Let λ be a system of non-trivial homotopically distinct simple closed curves on ∂M. Then a pair (M, λ) is *doubly incompressible* if

(1) every compressible simple closed curve on ∂M intersects λ at least three times,
(2) there are no essential annuli with boundary in $\partial M - \lambda$, and
(3) every maximal abelian subgroup of $\pi_1(\partial M - \lambda)$ is mapped to a maximal abelian subgroup of $\pi_1(M)$.

Let M be a compact 3-manifold with boundary ∂M. We denote by $AH(M)$ the space of conjugacy classes $[\rho] = (\rho, G)$ of discrete faithful representations $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$, where $G = \rho(\pi_1(M))$. The space $AH(M)$ is equipped with the algebraic topology. Let γ be an incompressible closed curve on ∂M. For $(\rho, G) \in AH(M)$, $l_\rho(\gamma)$ denotes the length of the geodesic representative of γ in the hyperbolic manifold $\mathbf{H}^3/G$, or $l_\rho(\gamma) = 0$ if $\rho(\gamma)$ is parabolic. We define $AH(M, \lambda, K)$ to be the set of $(\rho, G) \in AH(M)$ such that $l_\rho(\lambda) \le K$, where $l_\rho(\lambda)$ is the total sum of the lengths of every component of λ.

Theorem 4.2 (Thurston [23]). *If (M, λ) is doubly incompressible, then $AH(M, \lambda, K)$ is compact for all K.*

A curve system $\lambda = \{\alpha_j\}_{j=1}^N$ on S is called *homotopically independent* if it has the following properties: (1) each α_j is a simple closed curve on S and for $i \neq j$, $\alpha_i \cap \alpha_j = \emptyset$, (2) each α_j is non-trivial and not freely homotopic to a component of ∂S, and (3) for $i \neq j$, α_i is not freely homotopic to α_j. A homotopically independent curve system $\lambda = \{\alpha_j\}_{j=1}^N$ on S is *maximal* if it divides S into a union of pairs of pants. (If S is a surface of type (g, n); that is, S is a closed surface of genus g with n open disc removed, then $N = 3g - 3 + n$.) A pair (λ, λ') of maximal curve systems on S is called *binding* S if they have no curves in common and if (after suitable deformation of λ and λ' by homotopy) each component of $S - (\lambda \cup \lambda')$ is a simply connected domain or an annulus containing a component of ∂S in its boundary.

The following lemma is discussed in more general setting in Ohshika [21].

Lemma 4.3. *Let (λ', λ'') be a pair of maximal curve systems which binds S. For this pair, we define a maximal curve system λ on $\partial(S \times I)$, where $I = [0, 1]$ is a closed interval, as*

$$\lambda = (\lambda' \times \{0\}) \cup (\lambda'' \times \{1\}) \cup (\partial S \times \{1/2\}).$$

Then $(S \times I, \lambda)$ is doubly incompressible.

Proof. We only consider the case of $\partial S \neq \emptyset$, since the case of $\partial S = \emptyset$ is easier. If $\partial S \neq \emptyset$, then $S \times I$ is homeomorphic to a handle body H_g of some genus g. We identify $S \times I$ and H_g via this homeomorphism. We first check the condition (1) in the Definition 4.1. Let γ be a compressible simple closed curve on ∂H_g. Since $S \times \{0\}$ and $S \times \{1\}$ contain no compressible curves, γ must intersect a component of $\partial S \times \{1/2\}$. If $i(\gamma, \lambda) \leq 2$ (here $i(\cdot, \cdot)$ denotes the geometric intersection number), one can easily see that the only possible case is the following one: there exists a component δ of $\partial S \times \{1/2\}$ and a component W of $\partial H_g - ((\lambda' \times \{0\}) \cup (\lambda'' \times \{1\}))$ homeomorphic to a four-time-punctured sphere such that $\gamma \cup \delta \subset W$ and $i(\gamma, \delta) = 2$. Let α and β be components of ∂W such that α, β and γ bound a pair of pants T. Take a base point x in T. By abuse of notation, α, β and γ also denote the elements of $\pi_1(\partial H_g, x)$ contained in T. Since (λ', λ'') is binding, the curves α and β form a rank 2 free subgroup $\langle \alpha, \beta \rangle$ of $\pi_1(\partial H_g, x)$ which is mapped into $\pi_1(H_g, x)$ injectively. Hence $\gamma = \alpha \cdot \beta \in \pi_1(\partial H_g, x)$ is incompressible, which is a contradiction.

Now we check the condition (2). Suppose that there exists an essential annulus $f : (A, \partial A) \to (H_g, \partial H_g)$ with boundary in $\partial H_g - \lambda$. let γ and γ' be images of ∂A on ∂H_g. Then, since $i(\gamma \cup \gamma', \partial S \times \{1/2\}) = 0$, γ and γ' may be assumed to be contained in $S \times \{0, 1\}$. Let $p : S \times I \to S$ be the canonical retraction. Then, $p(\gamma)$ is homotopic to $p(\gamma')$ in S. Since (λ', λ'') is binding S, γ and γ' must be contained in the same component of $S \times \{0, 1\}$. Now the retraction above gives a homotopy between f and a map into ∂H_g. This is a contradiction.

Finally, we check the condition (3). Since all abelian subgroups of $\pi_1(\partial H_g - \lambda)$ or $\pi_1(H_g)$ are isomorphic to $\mathbf{Z}$, we only have to show that all primitive element of $\pi_1(\partial H_g - \lambda)$ is also primitive in $\pi_1(H_g)$. But this is trivial since H_g is homotopic to S. □

5. Orbit density for maximal cusps

Let $[\rho] \in \hat{B}_X$. The *accidental parabolic locus* of $[\rho]$ is a homotopically independent curve system $\lambda = \{\alpha_j\}$ on S such that $\rho(\alpha_j)$ is (a conjugacy class of) a parabolic element of $G = \rho(\pi_1(S))$ for every j, and no simple closed curve which is not homotopic to a component of λ have this property. For $[\rho] \in \hat{B}_X$, its accidental parabolic locus is uniquely determined up to homotopy. An element $[\rho] \in \hat{B}_X$ is called *maximal cusp* if its accidental parabolic locus is maximal. It is a well known facts that every maximal cusp is contained in ∂B_X and that, for any maximal curve system λ on S, there exists a unique maximal cusp whose accidental parabolic locus is λ (see Abikoff [1] and Maskit [16]).

Lemma 5.1. *Let $[\rho] \in C_X$ and let $\lambda = \{\alpha_j\}$ be a maximal curve system on S. If $\rho(\alpha_j)$ are parabolic for all j, $[\rho]$ is the maximal cusp in ∂B_X whose accidental parabolic locus is λ.*

Proof. We only have to show that $[\rho]$ is a faithful representation. Suppose that $\rho : \pi_1(S) \to G$ is not faithful. Then the covering map $p : \Omega_0(G) \to X = \Omega_0(G)/G$ is not universal, where $\Omega_0(G)$ is a unique invariant component of G. Then, by the planarity theorem (see [18], X.A.4), there exists a non-trivial simple closed curve δ on X and $\tilde{\delta}$ on $\Omega_0(G)$ such that $p|\tilde{\delta} : \tilde{\delta} \to \delta$ is a finite-sheeted covering map; say k-sheeted. Let $g \in G$ be a generator of the subgroup of G stabilizing $\tilde{\delta}$. Since $\lambda \subset X$ is maximal and δ is not parallel to a component of λ, δ must intersect some component of λ, say α_1. We may assume that the number of intersection of δ and α_1 is equal to $i(\delta, \alpha_1)$. Let $\tilde{\alpha}_1$ be a lift of α_1 on $\Omega_0(G)$ which intersects $\tilde{\delta}$. Let h be a parabolic element which conjugates to $\rho(\alpha_1)$ in G and stabilizing $\tilde{\alpha}_1$. By adjoining the fixed point of h to $\tilde{\alpha}_1$, we obtain a simple closed curve, which divides $\hat{\mathbf{C}}$ into two domains; let D be one of the two domains. If $k > 1$, we require that D satisfies $D \cap g(D) = \emptyset$. Let η_1 be a component of $D \cap \tilde{\delta}$ and β be an arc in $\tilde{\alpha}_1$ which connects end points of η_1. Let $\tilde{\delta}_1 = \eta_1 \cup \beta$ and let $\tilde{\delta}_2$ be the closed curve $\tilde{\delta}$ with $\eta_1, g(\eta_1), \dots, g^{k-1}(\eta_1)$ replaced by $\beta, g(\beta), \dots, g^{k-1}(\beta)$. Then, for $j = 1, 2$, $\tilde{\delta}_j$ projects to a simple closed curve δ_j on X such that $p|\tilde{\delta}_j : \tilde{\delta}_j \to \delta_j$ is a finite-sheeted covering map. Moreover, note that $i(\delta_j, \lambda)$ is strictly less than $i(\delta, \lambda)$ for $j = 1, 2$. Since δ is non-trivial and $\delta = \delta_1 \cdot \delta_2$, either δ_1 or δ_2 are non-trivial. After a finite number of steps as above, we obtain a non-trivial simple closed curve δ' such that $i(\delta', \lambda) = 0$ and that, for a lift $\tilde{\delta}'$ of δ', $p|\tilde{\delta}' : \tilde{\delta}' \to \delta'$ is a finite-sheeted covering map. This is a contradiction. □

For a simple closed curve α on S, let $D_\alpha \in Mod(S)$ denote the Dehn twist (once) around α.

Proposition 5.2. *Let (λ', λ'') be a binding pair of maximal curve systems on S. Let $[\rho] \in \partial B_X$ be a maximal cusp whose accidental parabolic locus is λ''. Put $\sigma = D_{\alpha_1} \circ \cdots \circ D_{\alpha_N} \in Mod(S)$, where $\lambda' = \{\alpha_j\}_{j=1}^N$. Then the sequence $\{[\rho]^{\sigma^n}\}_{n \in \mathbf{Z}}$ converges to the maximal cusp $[\rho_\infty]$ whose accidental parabolic locus is λ' as $|n| \to \infty$.*

In the proof of Proposition 5.2, we will make use of the following two lemmas; the first is due to Canary [5] and the second is a well known technical lemma.

Lemma 5.3 (Canary [5]). *Given $A > 0$, there exists a constant R such that if G is a non-elementary, torsion-free Kleinian group such that every incompressible closed geodesic on $\Sigma = \Omega(G)/G$ has hyperbolic length at least A, then for any closed curve γ on Σ,*

$$l_N(\gamma) \leq R l_\Sigma(\gamma),$$

where $l_N(\gamma)$ and $l_\Sigma(\gamma)$ are hyperbolic length of geodesic representatives of γ in $N = \mathbf{H}^3/G$ and in Σ, respectively.

Lemma 5.4. *Let F_2 be a rank 2 free group and let $\{\chi_n : F_2 \to \mathrm{PSL}_2(\mathbf{C})\}$ be a sequence of discrete faithful representations which converges algebraically to χ_∞. If $\{\chi'_n = \psi_n \cdot \chi_n \cdot \psi_n^{-1}\}$ also converges algebraically to χ'_∞ for a sequence $\{\psi_n\}$ of $\mathrm{PSL}_2(\mathbf{C})$, then ψ_n converges to some element ψ_∞ in $\mathrm{PSL}_2(\mathbf{C})$.* $\square$

Proof of 5.2. Our argument is almost parallel to that of Kerckhoff and Thurston [10] (see also [4]). We denote by $AH_{\partial S}(S \times I)$ the set of $[\chi] \in AH(S \times I)$ such that $\chi(\gamma)$ are parabolic for all $\gamma \in \pi_1(\partial S \times I)$. Then, for a given maximal cusp $[\rho] \in \partial B_X$, $AH_{\partial S}(S \times I)$ is properly embedded into $V(S)$ so that $[\rho] \in AH_{\partial S}(S \times I)$ and hence $QC_0(\rho) \subset AH_{\partial S}(S \times I)$. Let λ be a maximal curve system on $\partial(S \times I)$ as in Lemma 4.3, so that $(S \times I, \lambda)$ is doubly incompressible. Then the sequence $\{[\bar{\rho}_n] = \Psi_\rho(\sigma^{-n} X)\}_{n \in \mathbf{Z}}$ in $QC_0(\rho)$ is contained in

$$AH_{\partial S}(S \times I, \lambda, K) = AH_{\partial S}(S \times I) \cap AH(S \times I, \lambda, K)$$

for some K, since

$$l_{\bar{\rho}_n}(\lambda'' \times \{1\}) = l_{\bar{\rho}_n}(\partial S \times \{1/2\}) = 0$$

and

$$l_{\bar{\rho}_n}(\lambda' \times \{0\}) \leq R l_{\sigma^{-n} X}(\lambda' \times \{0\}) = R l_X(\lambda' \times \{0\}),$$

where R is a constant in Lemma 5.3. Since $AH(S \times I, \lambda, K)$ is compact by Theorem 4.2, and $AH_{\partial S}(S \times I)$ is closed in $AH(S \times I)$, $AH_{\partial S}(S \times I, \lambda, K)$ is compact. Therefore $\{[\bar{\rho}_n]\}_{n \in \mathbf{Z}}$ has a convergent subsequence. On the other hand, since C_X is compact (Lemma 2.1), $\{[\rho]^{\sigma^n}\}_{n \in \mathbf{Z}}$ also has a convergent subsequence. We also denote this subsequence by the same symbol; in fact, in the following argument, we will show that this subsequence converges to a unique maximal cusp, therefore $\{[\rho]^{\sigma^n}\}_{n \in \mathbf{Z}}$ converges without passing to a subsequence. Take representatives ρ_n of $[\rho]^{\sigma^n}$ converging to a representation ρ_∞. Then $\bar{\rho}_n = \rho_n \circ \sigma^n$ are representatives of $[\bar{\rho}_n]$. Therefore, there are elements $\psi_n \in \mathrm{PSL}_2(\mathbf{C})$ such that $\psi_n \cdot \bar{\rho}_n \cdot {\psi_n}^{-1}$ converges to a representation $\bar{\rho}_\infty$.

Take a component α of λ' and let T be a component of $S - \lambda'$ containing α in its boundary. Let $\alpha' (\neq \alpha)$ be a component of λ' or a component of ∂S contained in the boundary of T. Choose a base point x in T and regard $\pi_1(S) = \pi_1(S, x)$. By abuse of notation, α and α' also denote the elements of $\pi_1(S, x)$ contained in T. Since $\rho_n(\alpha) = \bar{\rho}_n(\alpha)$ and $\rho_n(\alpha') = \bar{\rho}_n(\alpha')$ and since ρ_n converges on α and α', by Lemma 5.4, the elements $\psi_n \in \mathrm{PSL}_2(\mathbf{C})$ may be taken to be the identity.

One can find non-trivial elements $\gamma_1, \gamma_2 \in \pi_1(S, x)$ each of which intersects α twice in the opposite direction and does not intersect any other components of λ', and that $\langle\gamma_1,\gamma_2\rangle$ is a rank 2 free subgroup of $\pi_1(S,x)$. Then

$$\bar{\rho}_n(\gamma_1)=\rho_n(\alpha^n)\cdot\rho_n(\gamma_1)\cdot\rho_n(\alpha^{-n})$$

and

$$\bar{\rho}_n(\gamma_2)=\rho_n(\alpha^n)\cdot\rho_n(\gamma_2)\cdot\rho_n(\alpha^{-n})$$

holds. Since both ρ_n and $\bar{\rho}_n$ converge on γ_1 and γ_2, Lemma 5.4 again implies that $\rho_n(\alpha^n)$ converges to an element $\hat{\alpha}$ in $\mathrm{PSL}_2(\mathbf{C})$. Since $\rho_n(\alpha)$ commutes with $\rho_n(\alpha^n)$ for all n, $\rho_\infty(\alpha)$ commutes with $\hat{\alpha}$. If $\langle\rho_\infty(\alpha),\hat{\alpha}\rangle$ were isomorphic to $\mathbf{Z}$, then $\rho_\infty(\alpha^k)=\hat{\alpha}^l$ for some integers k and l, and thus $\rho_n(\alpha^{nl-k})\to id$. This contradicts the fact that $[\rho_n]$ are discrete faithful representations. Therefore we conclude that $\langle\rho_\infty(\alpha),\hat{\alpha}\rangle$ is a rank 2 parabolic subgroup in $\mathrm{PSL}_2(\mathbf{C})$. Hence, $\rho_\infty(\alpha)$ is parabolic. The same argument works well for all components of λ'. Therefore, by Lemma 5.1, we can conclude that $[\rho_\infty]$ is a maximal cusp whose accidental parabolic locus is λ'. □

Lemma 5.5. *For any two maximal curve systems $\lambda=\{\alpha_j\}_{j=1}^N$ and $\lambda'=\{\beta_j\}_{j=1}^N$ on S, there exists a maximal curve system $\nu=\{\gamma_j\}_{j=1}^N$ such that the pairs (λ,ν) and (ν,λ') are binding S.*

Proof. There exists a simple closed curve δ on S such that $i(\delta,\alpha_j)>0$ for all j (see [6]). Put $\sigma=D_{\alpha_1}\circ\cdots\circ D_{\alpha_N}$ and put $\delta_n=\sigma^n(\delta)$. If $i(\beta_j,\lambda)=0$ then $\beta_j=\alpha_i$ for some i and hence $i(\beta_j,\delta_n)>0$ for all n. If $i(\beta_j,\lambda)>0$ then $i(\beta_j,\alpha_i)>0$ for some i. In this case, $i(\beta_j,\delta_n)>0$ for all but finitely many n. Therefore, for sufficiently large n, $i(\beta_j,\delta_n)>0$ holds for all j. Fix such n and let $\gamma_1=\delta_n$. Choose simple closed curves $\gamma_2,\ldots,\gamma_N$ so that $\nu=\{\gamma_j\}_{j=1}^N$ is a maximal curve system. This ν satisfies the desired condition. □

It was shown by McMullen [20] that the set of maximal cusps is dense in ∂B_X. Since the manner of decomposition of S into pairs of pants up to $Mod(S)$ is finite, the set of maximal cusps in ∂B_X decomposes into finitely many orbits under the action of $Mod(S)$. The next theorem shows that *each* orbit is dense in ∂B_X.

Theorem 5.6. *For any maximal cusp $[\rho]\in\partial B_X$, its orbit $\{[\rho]^\sigma\}_{\sigma\in Mod(S)}$ under the action of $Mod(S)$ is dense in ∂B_X.*

Proof. Since the set of maximal cusps is dense in ∂B_X, we only have to show that, for arbitrary fixed two maximal cusps $[\rho]$ and $[\rho']$ in ∂B_X, the orbit $\{[\rho]^\sigma\}_{\sigma\in Mod(S)}$ of $[\rho]$ contains a sequence converging to $[\rho']$. Let λ and λ' are accidental parabolic loci for $[\rho]$ and $[\rho']$, respectively. Then we can find a maximal curve system $\nu=\{\gamma_j\}_{j=1}^N$ such that the pairs (λ,ν) and (ν,λ') are binding (Lemma 5.5). Put $\sigma=D_{\gamma_1}\circ\cdots\circ D_{\gamma_N}$ and $\tau=D_{\beta_1}\circ\cdots\circ D_{\beta_N}$, where $\lambda'=\{\beta_j\}_{j=1}^N$. Then $[\rho]^{\sigma^n}$ converges to an maximal cusp $[\rho'']\in\partial B_X$ whose accidental parabolic locus is ν by Proposition 5.2. Similarly $[\rho'']^{\tau^n}$ converges to $[\rho']$. Since the action of $Mod(S)$ is continuous at maximal cusps (Corollary 3.2), we can find a desired sequence by a diagonal method. □

6. Orbits of Schottky groups and Bers boundary

In this section, we assume that S is a closed surface of genus g. We denote by S_X the set of $[\rho] = (\rho, G) \in C_X$ such that G is a Schottky group.

Lemma 6.1. *The set S_X consists of one orbit under the action of $Mod(S)$; that is, $S_X = \{[\rho]^\sigma\}_{\sigma \in Mod(S)}$ for any $[\rho] \in S_X$.*

Proof. Let (ρ_1, G_1) and (ρ_2, G_2) be arbitrary two elements of S_X. Then there exists a homeomorphism $N_{G_1} \to N_{G_2}$ such that the restriction of this map to the boundaries is a quasiconformal map $\Omega_0(G_1)/G_1 \to \Omega_0(G_2)/G_2$. Now one can see that $[\rho_2] = [\rho_1]^\sigma$, where $\sigma \in Mod(S)$ is an isotopy class of a homeomorphism of S induced by the quasiconformal map. □

A Kleinian group is called *geometrically finite* if it has a finite sided convex fundamental polyhedron in $\mathbf{H}^3$.

Lemma 6.2 (Hejhal [9], Matsuzaki [19])**.** *Each element $[\rho] \in S_X$ is an isolated point in C_X. On the other hand, if torsion-free, geometrically finite element $[\rho] \in C_X$ is isolated in C_X, then $[\rho] \in S_X$.*

Proof. The first statement is due to Hejhal [9], who showed that each Schottky group $[\rho] \in \hat{C}_X$ is isolated in $\hat{C}_X$. Conversely, let $[\rho] \in C_X$ be an isolated point of C_X. Since the same argument of Lemma 2.1 reveals that $\hat{C}_X - C_X$ is closed, $[\rho]$ is also isolated in $\hat{C}_X$. It was shown by Matsuzaki ([19], Theorem 3) that, if a torsion-free, geometrically finite element $[\rho] \in \hat{C}_X$ is isolated in $\hat{C}_X$, then $[\rho]$ is a Schottky group. Thus, the second statement is proved. □

Remark . In Matsuzaki [19], it is obtained the necessary and sufficient condition for a (not necessarily torsion-free) geometrically finite element of $\hat{C}_X$ to be isolated in $\hat{C}_X$.

For $[\rho] \in S_X$, the following lemma gives a characterization of the elements of $Mod(S)$ which stabilize $[\rho]$.

Lemma 6.3. *Let $[\rho] = (\rho, G) \in S_X$ and $\sigma \in Mod(S)$. Then the followings are equivalent:*

1. $[\rho]^\sigma = [\rho]$,
2. $\sigma_*(\ker \rho) = \ker \rho$, *and*
3. *σ can be extended to a homeomorphism of the Kleinian manifold N_G, where σ is regarded as a homeomorphism of $X = \partial N_G$.*

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (2) are trivial. (2) $\Rightarrow$ (1) can be seen from [19, Theorem 2]. We will show that (2) $\Rightarrow$ (3). Let (f, ρ) be the projective structure corresponding to $[\rho]$. We may assume that $\sigma : X \to X$ is a quasiconformal map. Let $\tilde{\sigma} : \Delta \to \Delta$ be a lift of $\sigma : X \to X$. If $\sigma_*(\ker \rho) = \ker \rho$, then $\tilde{\sigma}$ descends to a quasiconformal map $\hat{\sigma} : f(\Delta) \to f(\Delta)$, because the covering group $f : \Delta \to f(\Delta)$ is $\ker \rho$. Since $G = \rho(\pi_1(S))$ is geometrically finite and $\Omega(G) = f(\Delta)$, Marden's isomorphism theorem [15] implies that $\hat{\sigma}$ can be extended to a G-compatible quasiconformal automorphism of $\hat{\mathbf{C}}$. This quasiconformal map can be extended to a G-compatible homeomorphism of $\mathbf{H}^3 \cup \hat{\mathbf{C}}$, which descends to a homeomorphism of N_G. □

Theorem 6.4. *The set of accumulation points of S_X contains ∂B_X.*

Remark . It is known by Gallo [7] that there is an accumulation point of S_X which is not contained in ∂B_X. This can be seen also from a slight modification of the following argument.

Proof of 6.4. Let $[\rho] = (\rho, G) \in S_X$. We claim that, for some $\sigma \in Mod(S)$, the sequence $\{[\rho]^{\sigma^n}\}_{n\in\mathbf{Z}}$ converges to some maximal cusp $[\rho_\infty] \in \partial B_X$ as $|n| \to \infty$. If it has shown, the similar argument in Theorem 5.6 reveals that the claim of the theorem holds. In fact, for any element $[\rho'] \in \partial B_X$, there exists a sequence $\{\tau_n\}$ in $Mod(S)$ such that $[\rho_\infty]^{\tau_n}$ converges to $[\rho']$ by Theorem 5.6. Since the action of $Mod(S)$ is continuous at maximal cusps (Corollary 3.2), we can find a sequence in S_X which converges to $[\rho']$ by a diagonal method.

Now we will show that, for some $\sigma \in Mod(S)$, the sequence $\{[\rho]^{\sigma^n}\}_{n\in\mathbf{Z}}$ converges to some maximal cusp $[\rho_\infty] \in \partial B_X$ as $|n| \to \infty$. Note that the Kleinian manifold N_G is homeomorphic to a handle body H_g. With the identification $G = \pi_1(H_g)$, $AH(H_g)$ is properly embedded in $V(S)$ so that $[\rho] \in AH(H_g)$ and hence $QC_0(\rho) \subset AH(H_g)$. Let Σ be a surface with boundary $\partial\Sigma$ such that $\Sigma \times I$ is homeomorphic to H_g. (For example, let Σ be a surface of type $(1, g-1)$.) We can find a pair (λ', λ'') of maximal curve systems on Σ which binds Σ (cf. Lemma 5.3). For this pair, we define a maximal curve system λ on $S = \partial(\Sigma \times I) = \partial H_g$, as

$$\lambda = (\lambda' \times \{0\}) \cup (\lambda'' \times \{1\}) \cup (\partial\Sigma \times \{1/2\}).$$

Then, by Lemma 4.3, (H_g, λ) is doubly incompressible and hence, by Theorem 4.2, $AH(H_g, \lambda, K)$ is compact. Put $\sigma = D_{\alpha_1} \circ \cdots \circ D_{\alpha_N} \in Mod(S)$, where $\lambda = \{\alpha_j\}_{j=1}^N$. Then $\{[\bar\rho_n] = \Psi_\rho(\sigma^{-n}X)\}_{n\in\mathbf{Z}}$ is contained in a compact set $AH(H_g, \lambda, K)$ of $V(S)$ for some K, since

$$l_{\bar\rho_n}(\lambda) \le R l_{\sigma^{-n}X}(\lambda) = R l_X(\lambda),$$

where R is a constant in Lemma 5.3. Hence, $\{[\bar\rho_n]\}_{n\in\mathbf{Z}}$ has a convergent subsequence. On the other hand, since C_X is compact, $\{[\rho]^{\sigma^n}\}_{n\in\mathbf{Z}}$ also has a convergent subsequence. Take representatives $\bar\rho_n$ of $[\bar\rho_n]$ converging to a representation $\bar\rho_\infty$. Then $\rho_n = \bar\rho_n \circ \sigma^{-n}$ are representatives of $[\rho]^{\sigma^n}$. Therefore, there are elements $\psi_n \in \mathrm{PSL}_2(\mathbf{C})$ such that $\psi_n \cdot \rho_n \cdot {\psi_n}^{-1}$ converges to a representation ρ_∞.

Take a component α of λ and let T be a component of $S - \lambda$ containing α in its boundary. Let $\alpha' (\neq \alpha)$ be a component of λ contained in the boundary of T. Choose a base point x in T and regard $\pi_1(S) = \pi_1(S, x)$. By abuse of notation, α and α' also denotes the elements of $\pi_1(S, x)$ contained in T. Note that $\langle \alpha, \alpha' \rangle$ is a rank 2 free subgroup of $\pi_1(S, x)$ which is mapped into $\pi(H_g, x)$ injectively, and that $\bar\rho_n | \langle \alpha, \alpha' \rangle$ are discrete faithful representations. Moreover since $\rho_n | \langle \alpha, \alpha' \rangle = \bar\rho_n | \langle \alpha, \alpha' \rangle$, by Lemma 5.4, the elements $\psi_n \in \mathrm{PSL}_2(\mathbf{C})$ may be taken to be the identity.

One can find non-trivial elements $\gamma_1, \gamma_2 \in \pi_1(S, x)$ each of which intersects α twice in the opposite direction and does not intersect any other components of λ, and that $\langle \gamma_1, \gamma_2 \rangle$ is a rank 2 free subgroup of $\pi_1(S, x)$ which is mapped into $\pi_1(H_g, x)$ injectively. Then

$$\rho_n(\gamma_1) = \bar{\rho}_n(\alpha^{-n}) \cdot \bar{\rho}_n(\gamma_1) \cdot \bar{\rho}_n(\alpha^n)$$

and

$$\rho_n(\gamma_2) = \bar{\rho}_n(\alpha^{-n}) \cdot \bar{\rho}_n(\gamma_2) \cdot \bar{\rho}_n(\alpha^n)$$

holds. Since both ρ_n and $\bar{\rho}_n$ converge on γ_1 and γ_2, Lemma 5.4 again implies that $\bar{\rho}_n(\alpha^n)$ converges to an element $\hat{\alpha}$ in $\mathrm{PSL}_2(\mathbf{C})$. The same argument in the proof of Proposition 5.2 reveals that $\langle \bar{\rho}_\infty(\alpha), \hat{\alpha} \rangle$ is a rank 2 parabolic subgroup in $\mathrm{PSL}_2(\mathbf{C})$. Hence, $\bar{\rho}_\infty(\alpha)$ is parabolic. Recall that $\rho_n(\alpha) = \bar{\rho}_n(\alpha)$ for all n. Therefore, $\rho_\infty(\alpha)$ also would be a parabolic element. The same argument works well for all components of λ. Hence, by Lemma 5.1, we can conclude that $[\rho_\infty]$ is a maximal cusp whose accidental parabolic locus is λ. □

References

[1] W. Abikoff, *On boundaries of Teichmüller spaces and on Kleinian groups:III*, Acta Math. **134** (1975), 211-234.

[2] L. Bers, *On boundaries of Teichmüller spaces and on Kleinian groups:I*, Ann. of Math. **91** (1970), 570–600.

[3] ———, *The action of the modular group on the complex boundary*, Ann. of Math. Stud. **97** (1981), 33–52.

[4] J. F. Brock, *Iteration of mapping classes on a Bers slice: examples of algebraic and geometric limits of hyperbolic 3-manifolds*, Contemporary Math. **211** (1997), 81–106.

[5] R. D. Canary, *The Poincaré metric and a conformal version of a theorem of Thurston*, Duke Math. J. **64** (1991), 349–359.

[6] A. Fathi, F. Raudenbach, and V. Poenaru, *Travaux de Thurston sur le surfaces*, Astérisque **66-67** (1979).

[7] D. M. Gallo, *Some special limits of Schottky groups*, Proc. Amer. Math. Soc. **118** (1993), 877–883.

[8] ———, *Schottky groups and the boundary of Teichmüller space: genus 2*, Contemporary Math. **169** (1994), 283–305.

[9] D. A. Hejhal, *On Schottky and Koebe-like uniformizations*, Duke Math. J. **55** (1987), 267–286.

[10] S. P. Kerckhoff and W. P. Thurston, *Non-continuity of the action of the modular group at Bers' boundary of Teichmuller space*, Invent. Math. **100** (1990), 25–47.

[11] I. Kra, *A generalization of a theorem of Poincaré*, Proc. Amer. Math. Soc. **27** (1971), 299–302.

[12] ———, *Deformations of Fuchsian groups, II*, Duke Math. J. **38** (1971), 499–508.

[13] ———, *On spaces of Kleinian groups*, Comm. Math. Helv. **47** (1972), 53–69.

[14] I. Kra and B. Maskit, *Remarks on projective structures*, Ann. of Math. Stud. **97** (1981), 343–359.

[15] A. Marden, *The geometry of finitely generated Kleinian groups*, Ann. of Math. **99** (1974), 384–462.

[16] B. Maskit, *On boundaries of Teichmüller spaces and on kleinian groups:II*, Ann. of Math. **91** (1970), 607–639.

[17] ———, *Self-maps of Kleinian groups*, Amer. J. Math. **93** (1971), 840–856.

[18] ———, *Kleinian groups*, Springer-Verlag, 1988.

[19] K. Matsuzaki, *Projective structures inducing covering maps*, Duke Math. J. **78** (1995), 413–425.

[20] C. T. McMullen, *Cusps are dense*, Ann. of Math. **133** (1991), 217–247.

[21] K. Ohshika, *Geometrically finite Kleinian groups and parabolic elements*, Proc. Edinburgh Math. Soc. **41** (1998), 141–159.

[22] D. P. Sullivan, *On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions*, Ann. of Math. Stud. **97** (1981), 465–496.
[23] W. P. Thurston, *Hyperbolic structures on 3-manifolds, III: Deformations of 3-manifolds with incompressible boundary*, preprint.

Department of Mathematics, Tokyo Institute of Technology, Tokyo 152-8551 Japan
E-mail address: `itoken@math.titech.ac.jp`