NORM ESTIMATES OF THE PRE-SCHWARZIAN DERIVATIVES FOR CERTAIN CLASSES OF UNIVALENT FUNCTIONS

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Abstract. We introduce a kind of maximal operator associated with the Schwarz-Pick lemma. This will lead to a sharp norm estimate of the pre-Schwarzian derivatives of close-to-convex functions of specified type. We also discuss a relation between the subclasses of close-to-convex functions and the Hardy spaces.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Also let $\mathcal{S}$, $\mathcal{S}^*$ and $\mathcal{K}$ denote the subclasses of $\mathcal{A}$ consisting of functions which are univalent, starlike and convex in $\mathbb{D}$, respectively. For analytic functions g and h in $\mathbb{D}$, g is said to be subordinate to h if there exists an analytic function ω such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $g(z) = h(\omega(z))$ for $z \in \mathbb{D}$. The subordination will be denoted by $g \prec h$, or conventionally, $g(z) \prec h(z)$. In particular, when h is univalent, $g \prec h$ if and only if $g(0) = h(0)$ and if $g(\mathbb{D}) \subset h(\mathbb{D})$.

Now we introduce the terminology needed in the following. Let $\mathcal{M}$ be the class of zero-free analytic functions φ in $\mathbb{D}$ with the normalization condition $\varphi(0) = 1$. Following Ma and Minda [8], we define the subclasses $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ of $\mathcal{A}$ as the sets of functions $f \in \mathcal{A}$ of the forms

$$\frac{zf'(z)}{f(z)} \prec \varphi(z),$$

and

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z),$$

respectively, for each $\varphi \in \mathcal{M}$. By definition, it is obvious that $f \in \mathcal{K}(\varphi)$ if and only if $zf' \in \mathcal{S}^*(\varphi)$. We note that $\mathcal{S}^*(\varphi) \subset \mathcal{S}^*(\psi)$ and $\mathcal{K}(\varphi) \subset \mathcal{K}(\psi)$ for $\varphi \prec \psi$.

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A typical example for φ is given by

$$\varphi_{A,B}(z)=\frac{1+Az}{1+Bz}, \tag{1.1}$$

where A and B are real numbers satisfying $-1\le B<A\le 1$. Note that the Möbius transformation $\varphi_{A,B}$ maps the unit disk onto the disk (or half-plane) with diameter $((1-A)/(1-B),(1+A)/(1+B))$. The corresponding classes $\mathcal{K}(\varphi_{A,B})$ and $\mathcal{S}^*(\varphi_{A,B})$ have been studied by Janowski [4], [5], and Silverman and Silvia [10]. We note that $\mathcal{S}^*=\mathcal{S}^*(\varphi_{1,-1})$ is the class of starlike functions and $\mathcal{K}=\mathcal{K}(\varphi_{1,-1})$ is the class of convex functions.

We now introduce a class of analytic functions defined in a similar way to that of close-to-convex functions. For functions $\varphi,\psi\in\mathcal{M}$, following [6], we denote by $\mathcal{C}(\varphi,\psi)$ the set of all f in $\mathcal{A}$ such that there exists a function $h\in\mathcal{K}(\varphi)$ with

$$\frac{f'}{h'}\prec\psi. \tag{1.2}$$

The pre-Schwarzian derivative T_f of a locally univalent analytic function f is defined by

$$T_f(z)=\frac{f''(z)}{f'(z)}.$$

We also define the norm of T_f by

$$\|T_f\|=\sup_{z\in\mathbb{D}}|T_f(z)|(1-|z|^2).$$

We note that the set $\mathcal{T}_1$ of pre-Schwarzian derivatives T_f of those functions f in $\mathcal{S}$ which extend to quasiconformal automorphisms of the Riemann sphere can be regarded as a model of the universal Teichmüller space (cf. [14]) in analogy with the Schwarzians.

The authors deduced various properties (distortion, growth, growth of the coefficients and so on) of functions $f\in\mathcal{A}$ with $\|T_f\|\le 2\lambda$ for a fixed number $\lambda>0$, and gave norm estimates for a few classes of univalent functions in [7]. The present article is a continuation of the work. The goal is to give (possibly sharp) norm estimates of the pre-Schwarzian derivative for the class $\mathcal{C}(\varphi,\psi)$. To this end, we introduce a maximal operator on the set $C([0,1))$ of continuous functions on the interval $[0,1)$. For $F\in C([0,1))$, we set

$$\hat F(r)=\max_{0\le s\le r}K(r,s)|F(s)|,\quad 0\le r<1, \tag{1.3}$$

where the "kernel" function $K(r,s)$ is defined by

$$K(r,s)=\frac{s}{r}+\frac{r^2-s^2}{r(1-r^2)}=\frac{s(1-r^2)+r^2-s^2}{r(1-r^2)} \tag{1.4}$$

for $0\le s\le r<1$ and we call $\hat F$ the *maximal function* of F. Here, we define $K(0,0)=1$. Note that the above expression of $K(s,r)$ still makes sense when $s>r$ as far as $0\le r<1$. For later convenience (cf. Proposition 4.2), we extend $K(s,r)$ in this way.

Apart from the obvious subadditivity $(F+G)^\wedge\le\hat F+\hat G$, the following estimates constitute basic properties of the operator $F\mapsto\hat F$.

Lemma 1.1. *Let F be a continuous function on the interval $[0,1)$. Then*

$$(1.5)\qquad (1-r^2)|F(r)| \le (1-r^2)\hat{F}(r) \le \max_{0\le s\le r}(1-s^2)|F(s)|.$$

and, in particular,

$$(1.6)\qquad \sup_{0\le r<1}(1-r^2)\hat{F}(r) = \sup_{0\le r<1}(1-r^2)|F(r)|.$$

We prove this lemma in Section 2. For an analytic function g on $\mathbb{D}$, we denote by $\hat{M}(r,g)$ the maximal function of $M(r,g)=\max\{|g(z)| : |z|=r\}$. The main result of the present article is stated as follows.

Theorem 1.2. *Let $\varphi,\psi\in\mathcal{M}$ and suppose that φ is univalent and that the image $\varphi(\mathbb{D})$ is starlike with respect to 1. Then the inequality*

$$(1.7)\qquad \|T_f\| \le \sup_{0\le r<1}(1-r^2)\left[\hat{M}(r,(\varphi(z)-1)/z)+\hat{M}(r,\psi'/\psi)\right]$$

holds for every $f\in\mathcal{C}(\varphi,\psi)$. Moreover, the inequality is sharp if the inequalies

$$(1.8)\qquad \left|\frac{\varphi(z)-1}{z}\right| \le \frac{\varphi(\varepsilon|z|)-1}{\varepsilon|z|} \quad \text{and} \quad |\psi'(z)/\psi(z)| \le \psi'(\varepsilon|z|)/\psi(\varepsilon|z|)$$

hold simultaneously for all $z\in\mathbb{D}$, where ε is a unimodular constant.

Though it is generally difficult to compute the right-hand side in (1.7), simpler estimates can be deduced from Theorem 1.2. For instance, with the aid of (1.6), taking supremum term by term, we obtain the following corollary.

Corollary 1.3. *Under the same hypothesis as in Theorem 1.2, the inequality*

$$\|T_f\| \le \sup_{|z|<1}(1-|z|^2)\left|\frac{\varphi(z)-1}{z}\right| + \sup_{|z|<1}(1-|z|^2)\left|\frac{\psi'(z)}{\psi(z)}\right|$$

holds for every $f\in\mathcal{C}(\varphi,\psi)$.

We will prove Theorem 1.2 in Section 3. In Section 4, we make some attempts to compute the right-hand side of (1.7) when φ and ψ are of the form $\varphi_{A,B}$. More specific examples of computation will be given in Section 5. We also provide inclusion relations between the class $\mathcal{C}(\varphi,\psi)$ and the Hardy spaces in the final section.

Finally, we mention a couple of related results. S. Yamashita [13] investigated the norm of pre-Schwarzian derivatives of Gelfer-starlike, -convex, and -close-to-convex functions (see also [12] for Gelfer functions). Recently, Y. Okuyama [9] gave a sharp norm estimate for the class of β-spirallike functions.

2. An extremal problem and the associated maximal operator

As a preparation of the proof of our main theorem, we introduce an extremal problem and deduce fundamental properties of the adapted maximal operator defined by (1.3).

We first consider the extremal problem: for a given pair of points z_0, w_0 with $|w_0|\le |z_0|<1$, find the maximum of values $|\omega'(z_0)|$ or, more precisely, the region of values $\omega'(z_0)$, for holomorphic mappings $\omega:\mathbb{D}\to\mathbb{D}$ with $\omega(0)=0$ and $\omega(z_0)=w_0$. A complete solution to this problem was given by J. Dieudonné in 1931. The following is known as Dieudonné's Lemma (see [3, p.198]).

Lemma 2.1 (Dieudonné). *Let $\mathcal{F}$ be the family of analytic functions ω on the unit disk with $|\omega| < 1, \omega(0) = 0$ and $\omega(z_0) = w_0$, where z_0 and w_0 are points in $\mathbb{D}$ with $|w_0| \le |z_0| \ne 0$. Then the set $\{\omega'(z_0) : \omega \in \mathcal{F}\}$ is the closed disk centered at w_0/z_0 with radius $(|z_0|^2 - |w_0|^2)/|z_0|(1-|z_0|^2)$. Furthermore, if $\omega'(z_0)$ lies on the boundary of the disk, then ω has the form*

$$\omega(z) = z\,\frac{\lambda\frac{z-z_0}{1-\bar{z}_0 z} + \frac{w_0}{z_0}}{1 + \lambda\frac{\bar{w}_0}{\bar{z}_0}\frac{z-z_0}{1-\bar{z}_0 z}} \tag{2.1}$$

for a constant λ with $|\lambda| = 1$.

In particular, we obtain the sharp inequality

$$|\omega'(z_0)| \le \left|\frac{w_0}{z_0}\right| + \frac{|z_0|^2 - |w_0|^2}{|z_0|(1-|z_0|^2)} = K(|z_0|, |w_0|) \tag{2.2}$$

for such a function ω, where the function $K(r,s)$ is given by (1.4). In this way, our kernel function $K(r,s)$ connects with the above extremal problem. Note that equality holds in (2.2) if and only if $\lambda = w_0|z_0|^2/|w_0|z_0^2$.

Here is a good point to give a proof of the basic lemma given in Introduction.

Proof of Lemma 1.1. First, by the identity

$$r(1-s^2) - \left[s(1-r^2) + r^2 - s^2\right] = (r-s)(1-r)(1-s),$$

we obtain the following estimate of the kernel $K(r,s)$ given in (1.4):

$$K(r,s) \le \frac{1-s^2}{1-r^2} \tag{2.3}$$

for $0 \le s \le r < 1$. Therefore, the right-hand inequality in (1.5) follows. The left-hand one is obvious because $K(r,r) = 1$. The relation in (1.6) is an immediate consequence of (1.5). □

Remark. In view of the Schwarz-Pick lemma: $|\omega'(z)| \le (1-|\omega(z)|^2)/(1-|z|^2)$ for a holomorphic function $\omega : \mathbb{D} \to \mathbb{D}$, the inequality (2.3) is a natural conclusion.

3. Proof of the main theorem

For $\varphi \in \mathcal{M}$, we define the functions h_φ and k_φ in $\mathcal{A}$ by the relations

$$\frac{zh_\varphi'(z)}{h_\varphi(z)} = \varphi(z) \quad \text{and} \quad 1 + \frac{zk_\varphi''(z)}{k_\varphi'(z)} = \varphi(z), \tag{3.1}$$

namely,

$$h_\varphi(z) = z\exp\int_0^z \frac{\varphi(t)-1}{t}dt \quad \text{and} \quad k_\varphi(z) = \int_0^z \left(\exp\int_0^\zeta \frac{\varphi(t)-1}{t}dt\right)d\zeta. \tag{3.2}$$

For instance, we can compute $h_{\varphi_{A,B}}$ and $k_{\varphi_{A,B}}$ for $-1 \le B < A \le 1$ as follows:

$$h_{\varphi_{A,B}}(z) = zk_{\varphi_{A,B}}'(z) = \begin{cases} z(1+Bz)^{(A-B)/B} & (B \ne 0) \\ ze^{Az} & (B = 0), \end{cases} \tag{3.3}$$

and

$$k_{\varphi_{A,B}}(z) = \begin{cases} \frac{1}{A}((1+Bz)^{A/B}-1) & (A \neq 0, B \neq 0) \\ \frac{1}{B}\log(1+Bz) & (A=0) \\ \frac{1}{A}(e^{Az}-1) & (B=0). \end{cases} \tag{3.4}$$

Under some additional assumptions on φ, Ma and Minda showed in [8] that these functions are extremal in $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$, respectively, in various aspects. Among them, they obtain the following lemma. In order to clarify what assumptions are necessary for φ, we will reproduce the proof of it.

Lemma 3.1 (Ma and Minda [8, Theorem 1]). *Suppose that a function $\varphi \in \mathcal{M}$ is univalent and $\varphi(\mathbb{D})$ is starlike with respect to 1. Then $f' \prec k_\varphi'$ holds for every $f \in \mathcal{K}(\varphi)$.*

Proof. Let $g = c\log k_\varphi'$, where $c = 1/\varphi'(0)$. Since $c(\varphi - 1) \in \mathcal{A}$ is starlike, we can see that $1 + zg'(z)/g''(z) = z\varphi'(z)/(\varphi(z)-1)$ has positive real part, in other words, g is convex. By assumption, the relation $czf''/f' \prec c(\varphi - 1) = czk_\varphi''/k_\varphi' = zg'$ holds, and hence, one obtains $c\log f' \prec g = c\log k_\varphi'$, equivalently, $f' \prec k_\varphi'$ by Suffridge's Theorem [11, Theorem 3]. (Recall that convexity of g was essential in this theorem.) □

In general, for $f, g \in \mathcal{A}$, the condition $f' \prec g'$ implies the inequality $\|T_f\| \leq \|T_g\|$, see [7]. Hence, we obtain the following as a corollary.

Theorem 3.2. *Let φ be as in Lemma 3.1. If $f \in \mathcal{K}(\varphi)$, then $\|T_f\| \leq \|T_{k_\varphi}\|$ holds, where k_φ is the function given in* (3.2).

We now prove Theorem 1.2. It is convenient below to introduce the class $\mathcal{B}$ of analytic functions ω on the unit disk with $|\omega(z)| \leq |z|$. Let $f \in \mathcal{C}(\varphi, \psi)$. Then, by definition, there is a function $h \in \mathcal{K}(\varphi)$ such that $f'/h' \prec \psi$. By Lemma 3.1, we see that $h' \prec k_\varphi'$. Let ω_1 and ω_2 be analytic functions in $\mathcal{B}$ satisfying $h' = k_\varphi' \circ \omega_1$ and $f'/h' = \psi \circ \omega_2$. Conversely, for any pair of functions $\omega_1, \omega_2 \in \mathcal{B}$, the function f is uniquely determined so that the above relations hold. We occasionally write $f = f[\omega_1, \omega_2]$. By taking the logarithmic derivative, these relations yield

$$\begin{aligned} T_f &= T_h + \frac{\psi' \circ \omega_2 \cdot \omega_2'}{\psi \circ \omega_2} \\ &= \frac{(\varphi \circ \omega_1 - 1)\omega_1'}{\omega_1} + \frac{\psi' \circ \omega_2 \cdot \omega_2'}{\psi \circ \omega_2} \\ &= \omega_1' \cdot \Phi \circ \omega_1 + \omega_2' \cdot \Psi \circ \omega_2, \end{aligned}$$

where we have set $\Phi(z) = (\varphi(z) - 1)/z$ and $\Psi(z) = \psi'(z)/\psi(z)$. Fix a point $z_0 \in \mathbb{D}$ with $r = |z_0| > 0$. For any pair of points w_1, w_2 with $r_j = |w_j| \leq r$, consider functions $\omega_1, \omega_2 \in \mathcal{B}$ with $\omega_j(z_0) = w_j$ for $j = 1, 2$. By (2.2), we observe that

$$\begin{aligned} |T_{f[\omega_1,\omega_2]}(z_0)| &\leq K(r, r_1)|\Phi(w_1)| + K(r, r_2)|\Psi(w_2)| \\ &\leq K(r, r_1)M(r_1, \Phi) + K(r, r_2)M(r_2, \Psi) \\ &\leq \hat{M}(r, \Phi) + \hat{M}(r, \Psi). \end{aligned}$$

From this inequality, the required one (1.7) follows immediately.

We next show the sharpness under the additional assumption (1.8). For a given $0 \le r < 1$, we choose $r_1, r_2 \in [0, r]$ so that $\hat{M}(r, \Phi) = K(r, r_1)M(r, \Phi)$ and $\hat{M}(r, \Psi) = K(r, r_2)M(r, \Psi)$ hold. For each $j = 1, 2$, let ω_j be the function of the form (2.1) with $w_0 = \varepsilon r_j$ and $\lambda = \varepsilon |z_0|^2/z_0^2$. Then equality holds at each step of the above estimation. Hence,

$$\max_{f \in \mathcal{C}(\varphi,\psi)} M(T_f, r) = \hat{M}(r, \Phi) + \hat{M}(r, \Psi)$$

holds for each $r < 1$. We remark that the extremal function attaining the above maximum is uniquely determined for each $r < 1$. Now it is evident that the estimate (1.7) is best possible if (1.8) is satisfied.

4. Applications to the class $\mathcal{C}(\varphi_{A_1,B_1}, \varphi_{A_2,B_2})$

As an application of Theorem 1.2, we consider the case when $\varphi = \varphi_{A_1,B_1}$ and $\psi = \varphi_{A_2,B_2}$ for some real numbers A_1, B_1, A_2, B_2 with $-1 \le A_j < B_j \le 1$ for $j = 1, 2$, where $\varphi_{A,B}$ is the function given in (1.1). First we need the next elementary lemma.

Lemma 4.1. *For real numbers A, B with $-1 \le B < A \le 1$, the inequality*

$$|1 + Az||1 + Bz| \ge (1 + \varepsilon A|z|)(1 + \varepsilon B|z|)$$

holds for every $z \in \mathbb{D}$. Here, $\varepsilon = 1$ when $A + B \le 0$ and $\varepsilon = -1$ when $A + B \ge 0$.

Proof. First assume that $A + B \le 0$. If $AB \ge 0$, then $A \le 0$ and $B \le 0$, and thus, the claim is obvious. If $AB < 0$, the assumptions imply $B < 0 < A$ and

$$\begin{aligned} &\min_{|z|=r} |1 + Az|^2 |1 + Bz|^2 \\ = \; &\min_{-r \le x \le r} (1 + A^2r^2 + 2Ax)(1 + B^2r^2 + 2Bx) \\ = \; &(1 - Ar)^2(1 - Br)^2. \end{aligned}$$

Hence, the required inequality follows. The other case when $A + B \ge 0$ can be treated similarly. □

Noting the expressions

$$\frac{\varphi_{A,B}(z) - 1}{z} = \frac{A - B}{1 + Bz} \quad \text{and} \quad \frac{\varphi'_{A,B}(z)}{\varphi_{A,B}(z)} = \frac{A - B}{(1 + Az)(1 + Bz)}$$

and using Lemma 4.1, we see that the condition (1.8) is fulfilled for $\varphi = \varphi_{A_1,B_1}$ and $\psi = \varphi_{A_2,B_2}$ if either

$$B_1 \le 0 \quad \text{and} \quad A_2 + B_2 \le 0 \quad (\text{with } \varepsilon = 1) \tag{4.1}$$

or

$$B_1 \ge 0 \quad \text{and} \quad A_2 + B_2 \ge 0 \quad (\text{with } \varepsilon = -1). \tag{4.2}$$

Put $p_{A,B}(z) = 1/(1+Az)(1+Bz)$ for $-1 \le B < A \le 1$. Then we may write $\hat{M}(r, (\varphi_{A,B} - 1)/z) = (A - B)\hat{M}(r, p_{0,B})$ and $\hat{M}(r, \varphi'_{A,B}/\varphi_{A,B}) = (A - B)\hat{M}(r, p_{A,B})$. We further set

$$\mu(r; A, B) = \hat{M}(r, p_{A,B}) = \hat{M}\left(r, \frac{1}{(1 + Az)(1 + Bz)}\right). \tag{4.3}$$

We note that Theorem 1.2 implies that the best bound of the norm $\|T_f\|$ for functions $f \in \mathcal{C}(\varphi_{A_1,B_1}, \varphi_{A_2,B_2})$ is given by

$$W(A_1,B_1,A_2,B_2) = \sup_{0\le r<1} \big[(A_1-B_1)\mu(r;0,B_1) + (A_2-B_2)\mu(r;A_2,B_2)\big]. \tag{4.4}$$

In view of the obvious symmetry $\mu(r;A,B) = \mu(r;-B,-A)$, we may always assume that $A+B \le 0$. The following result provides sufficient information for computation of the above maximal functions.

Proposition 4.2. *Suppose that $-1 \le B < A \le 1$ and that $A+B \le 0$. Then, the quantity $\mu(r;A,B)$ is given by*

$$\begin{cases} p_{A,B}(r) = \dfrac{1}{(1+Ar)(1+Br)} & \text{if } 0 \le r < r_0 \\[2ex] \dfrac{2(1-ABr^2) + (A+B)(1-r^2) - 2\sqrt{(1+A)(1+B)(1-Ar^2)(1-Br^2)}}{(A-B)^2 r(1-r^2)} & \text{if } r_0 \le r < 1 \end{cases}$$

for $r_0 \le r < 1$, where r_0 is the unique root of the equation $\sigma(r) = r$ in $0 < r \le 1$ and $\sigma(r)$ is defined by

$$\sigma(r) = \frac{1 + ABr^2 - \sqrt{(1+A)(1+B)(1-Ar^2)(1-Br^2)}}{-(A+B) - AB(1-r^2)}.$$

Furthermore, for $r \in [0,1)$, the condition $r < r_0$ is equivalent to $r < \sigma(r)$.

Proof. When $r=0$, we have nothing to prove. For a fixed $r \in (0,1)$, we consider the function $F(x) = K(r,x)p_{A,B}(x)$ so that $\mu(r;A,B) = \max_{0\le x\le r} F(x)$. Then the derivative of F is computed by

$$F'(x) = \frac{-[A+B+AB(1-r^2)]x^2 - 2(1+ABr^2)x + 1 - (1+A+B)r^2}{r(1-r^2)(1+Ax)^2(1+Bx)^2}.$$

Thus F' has exactly the two real zeros $\sigma(r)$ and

$$\tau(r) = \frac{1 + ABr^2 + \sqrt{(1+A)(1+B)(1-Ar^2)(1-Br^2)}}{-(A+B) - AB(1-r^2)}.$$

Here, we note that the inequality

$$A + B + AB(1-r^2) < 0 \tag{4.5}$$

holds. Indeed, if $AB < 0$, the inequality holds trivially. If $AB > 0$, then $-1 \le B < A < 0$. Hence, $A+B+AB(1-r^2) < A+B+AB = A(1+B)+B \le B < 0$. Finally, if $AB = 0$, then clearly $A+B+AB(1-r^2) = A+B \le 0$. Equality holds here only when $A+B=0$, which together with $AB=0$ violates the condition $B < A$.

Since

$$\tau(r) - 1 = \frac{(1+A)(1+B) + \sqrt{(1+A)(1+B)(1-Ar^2)(1-Br^2)}}{-(A+B) - AB(1-r^2)} > 0,$$

there is at most one zero of F' within the interval $[0, 1)$. On the other hand,

$$\sigma'(r) = \frac{(1+A)(1+B)r[-A^2(1+B)(1-Br^2)-B^2(1+A)(1-Ar^2)-2AB\sqrt{(1+A)(1+B)(1-Ar^2)(1-Br^2)}]}{(A+B+AB(1-r^2))^2\sqrt{(1+A)(1+B)(1-Ar^2)(1-Br^2)}}$$

is negative by Lemma 4.3 which will be shown below. Therefore, $\sigma(r)$ is decreasing in $0 \le r \le 1$. Since the function $r - \sigma(r)$ increases from $-\sigma(0)(< 0)$ to $1 - \sigma(1)(\ge 0)$, there is only one number r_0 in $(0, 1]$ so that $\sigma(r_0) = r_0$. Observe that $r < \sigma(r)$ for $0 < r < r_0$ and $\sigma(r) < r$ for $r_0 < r < 1$. Since $F'(0) > 0$, for each $0 < r < r_0$ the derivative $F'(x)$ is positive in $0 < x < r$, and therefore, $\mu(r; A, B) = F(r) = p_{A,B}(r)$. When $r_0 \le r < 1$, noting the inequality $\sigma(r) \le r$, we see that $F'(x) > 0$ in $0 < x < \sigma(r)$ and that $F'(x) < 0$ in $\sigma(r) < x < r$. Therefore, we conclude that $\mu(r; A, B) = F(\sigma(r)) = K(r, \sigma(r))p_{A,B}(\sigma(r))$, which has the desired expression in the theorem. □

Lemma 4.3. *Under the same assumptions as in Lemma 4.2, the quantity*

$$-A^2(1 + B)(1 - Br^2) - B^2(1 + A)(1 - Ar^2) - 2AB\sqrt{(1 + A)(1 + B)(1 - Ar^2)(1 - Br^2)}$$

is negative in $0 < r < 1$.

Proof. The assertion is obvious when $AB \ge 0$. We now assume that $AB < 0$, and hence $B < 0 < A$. Denote by $P(r; A, B)$ or P the above quantity. Since

$$\frac{\partial P}{\partial B} = \frac{(A(1 - r^2) - 2ABr^2 + 2Q)(-B(1 + A)(1 - Ar^2) - AQ)}{Q},$$

where we have set $Q = \sqrt{(1 + A)(1 + B)(1 - Ar^2)(1 - Br^2)}$. Clearly, the first factor of the numerator is positive. We next show that the second one is positive, too. To this end, it is enough to see

$$\begin{aligned} & -B(1 + A)(1 - Ar^2) = |B|(1 + A)(1 - Ar^2) > AQ \\ \Leftrightarrow\ & B^2(1 + A)^2(1 - Ar^2)^2 > A^2(1 + A)(1 + B)(1 - Ar^2)(1 - Br^2) \\ \Leftrightarrow\ & B^2(1 + A)(1 - Ar^2) > A^2(1 + B)(1 - Br^2) \\ \Leftrightarrow\ & -(A - B)(A + B + AB(1 - r^2)) > 0. \end{aligned}$$

The last inequality is certainly valid by virture of (4.5), and thus we have shown that $\partial P/\partial B > 0$.

Fix $A > 0$ and $0 < r < 1$. Then the range of B is $-1 \le B \le -A$ by the present assumptions. Since P is increasing in B, we obtain

$$P(r; A, B) \le P(r; A, -A) = -2A^2\left[1 - A^2r^2 - \sqrt{(1 - A^2)(1 - A^2r^4)}\right].$$

Since

$$\begin{aligned} & P(r; A, -A) < 0 \\ \Leftrightarrow\ & (1 - A^2)(1 - A^2r^4) < (1 - A^2r^2)^2 \\ \Leftrightarrow\ & 0 < A^2(1 - r^2)^2, \end{aligned}$$

the assertion now follows. □

In the sequel, it is convenient to have an exact value of

$$E(A,B)=\sup_{|z|<1}\frac{1-|z|^2}{|1+Az||1+Bz|} \tag{4.6}$$

for $-1\le B<A\le 1$. Keeping the simple relation

$$E(A,B)=E(-B,-A) \tag{4.7}$$

in mind, we can give a concrete expression of $E(A,B)$ as follows.

Lemma 4.4. *If $-1\le B<A\le 1$, then*

$$E(A,B)=\frac{2}{1-AB+\sqrt{(1-A^2)(1-B^2)}}. \tag{4.8}$$

Proof. First we assume that $A+B\ge 0$. Then, by Lemma 4.1, we obtain the expression

$$E(A,B)=\sup_{0\le r<1} g(r),$$

where we set

$$g(x)=\frac{1-x^2}{(1-Ax)(1-Bx)}.$$

A simple calculation gives $E(A,B)=g(x_0)$, where x_0 is the unique zero of $g'(x)$ in $0\le x<1$, that is,

$$x_0=\frac{A+B}{1+AB+\sqrt{(1-A^2)(1-B^2)}}.$$

Noting the relation $(A+B)x_0^2-2(1+AB)x_0-(A+B)=0$, we get (4.8). The case when $A+B<0$ can be reduced to the previous one by using (4.7). The proof is now complete. □

As an immediate consequence of this together with Theorem 3.2, we obtain

Theorem 4.5. *Let $-1\le B<A\le 1$. If $f\in\mathcal{K}(\varphi_{A,B})$, then*

$$\|T_f\|\le\frac{2(A-B)}{1+\sqrt{1-B^2}}, \tag{4.9}$$

and equality holds when $f=k_{\varphi_{A,B}}$.

Proof. If $f\in\mathcal{K}(\varphi_{A,B})$, by Theorem 3.2, we have

$$\|T_f\|\le\|T_k\|,$$

where k denotes the function $k_{\varphi_{A,B}}$ given in (3.4). Since

$$\frac{k''(z)}{k'(z)}=\frac{\varphi_{A,B}(z)-1}{z}=\frac{A-B}{1+Bz},$$

we obtain $\|T_k\|=(A-B)E(0,B)=2(A-B)/(1+\sqrt{1-B^2})$ by Lemma 4.4. □

5. Examples

First we compute the quantity $\mu(r;A,B)$ given in (4.3) for special choices of A and B. When $B=-1$, then $\sigma(r)=1$ and thus $r_0=1$ in Proposition 4.2. Therefore, we compute

$$\mu(r;A,-1)=\frac{1}{(1+Ar)(1-r)},\quad 0\le r<1.$$

Next consider the case $A=0$. A simple computation yields $r_0=1/(1+\sqrt{2(1+B)})$ and thus

$$\mu(r;0,B)=\begin{cases}\dfrac{1}{1+Br} & \text{if } 0\le r<r_0\\[2ex] \dfrac{2+B(1-r^2)-2\sqrt{(1+B)(1-Br^2)}}{B^2r(1-r^2)} & \text{if } r_0\le r<1.\end{cases}$$

6. Relationship with the Hardy space

The Hardy space $\mathcal{H}^p$ $(0<p\le\infty)$ is the class of all functions f analytic in $\mathbb{D}$ such that

$$\|f\|_p:=\lim_{r\to1^-}M_p(r,f)<\infty,$$

where

$$M_p(r,f)=\begin{cases}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{1/p} & (0<p<\infty)\\ M(r,f)=\max_{|z|\le r}|f(z)| & (p=\infty).\end{cases}$$

Let BMOA be the family of functions f analytic in $\mathbb{D}$ with finite BMOA norm:

$$\|f\|_*:=\sup_{\alpha\in\mathbb{D}}\|f_\alpha\|_2+|f(0)|<\infty,$$

where $f_\alpha(z)=f((z+\alpha)/(1+\bar\alpha z))-f(\alpha)$. See [1] and [2] for further information. A simple relationship between the class $\mathcal{C}(\varphi,\psi)$ and the Hardy space $\mathcal{H}^p$ is given by

Theorem 6.1. *Let $1\le p<\infty$. Suppose that $\varphi\in\mathcal{M}$ has positive real part and satisfies $k'_\varphi\in\mathcal{H}^1$, where k_φ is given by (3.2). Then $\mathcal{C}(\varphi,\psi)\subset\mathcal{H}^p$ for every $\psi\in\mathcal{H}^p\cap\mathcal{M}$.*

Proof. If $f\in\mathcal{C}(\varphi,\psi)$, from (1.2) we have

$$f(z)=\int_0^z h'(t)\psi(\omega(t))dt,$$

where $h\in\mathcal{K}(\varphi)$ and $|\omega(z)|\le|z|$. By Littlewood's Subordination Theorem [2, Theorem 1.7], it follows that $\psi\circ\omega\in\mathcal{H}^p$. By assumption, $h'\prec k'_\varphi\in\mathcal{H}^1$, and hence $h'\in\mathcal{H}^1$. This implies that $h\in\mathcal{H}^\infty\subset$ BMOA. Now the following theorem yields the desired result. □

Theorem (Aleman and Siskakis [1]). *Let h be an analytic function in the unit disk and $1\le p<\infty$. The operator*

$$f\mapsto\frac{1}{z}\int_0^z f(t)h'(t)dt$$

maps $\mathcal{H}^p$ continuously into itself if and only if $h\in$ BMOA.

Corollary 6.2. *Let $-1 \le B < A \le 1$. If $-1 < B$ or $A \le 0$, then for any number $1 \le p < \infty$ we have $\mathcal{C}(\varphi_{A,B}, \psi) \subset \mathcal{H}^p$ for all $\psi \in \mathcal{M} \cap \mathcal{H}^p$. The range of A and B is sharp.*

Proof. In view of (3.3), we can see that $k'_{\varphi_{A,B}} \in \mathcal{H}^1$ if and only if $-1 < B$ or $A < 0$. Thus, by Theorem 6.1, the statement holds in this case. When $B = -1$ and $A = 0$, $\varphi(z) = \varphi_{0,-1}(z) = 1/(1-z)$, therefore $k'_\varphi(z) = 1/(1-z)$. If $h' \prec k'_\varphi$, then $h' = 1/(1-\omega)$, where $\omega : \mathbb{D} \to \mathbb{D}$ is analytic with $\omega(0) = 0$. Hence $(1-|z|^2)|h'(z)| \le (1-|z|^2)/(1-|\omega(z)|) \le 1+|z| < 2$, which implies $h \in \mathrm{BMOA}$ because a univalent Bloch function is known to be of BMOA. Now the theorem of Aleman and Siskakis implies the desired claim even in this case.

Now suppose $B = -1$ and $A > 0$. We take $\psi(z) = 1 - \log(1-z) \in \mathcal{M}_{\mathrm{su}}$ and define $f \in \mathcal{A}$ by the relation $f'/k'_{\varphi_{A,-1}} = \psi$. Note that $\psi \in \mathcal{H}^p$ for any $p < \infty$. Using (3.4), we can calculate as

$$f(z) = \frac{1}{A}\left(1 - \log(1-z) - \frac{1}{A}\right)\left((1-z)^{-A} - 1\right) - \frac{1}{A}\log(1-z).$$

Therefore we see that $f \in \mathcal{H}^p$ for any $p < 1/A$ but $f \notin \mathcal{H}^{1/A}$. This implies that the above statement fails when $p \ge 1/A$. □

Remark. In general, if $\psi \in \mathcal{M}$ has positive real part, by [2, Theorem 3.2], we have

$$\psi \in \bigcap_{0<p<1} \mathcal{H}^p.$$

We note also that

$$\mathcal{C}(\varphi, \psi) \subset \mathcal{C} \subset \mathcal{S} \subset \bigcap_{0<p<1/2} \mathcal{H}^p$$

for $\varphi \in \mathcal{M}$ with $\operatorname{Re}\varphi > 0$ and $\psi \in \mathcal{M}$ with $\operatorname{Re} e^{i\gamma}\psi > 0$ for some $\gamma \in \mathbb{R}$ (see [2, Theorem 3.16]). The above ranges for p are sharp.

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