A condition of quasiconformal extendability

Y. Gotoh and M. Taniguchi

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Recently, Heinonen and Koskela showed, as a corollary of their deep result, the following extension theorem.

Proposition 1 ([3],4.2 Theorem) *Suppose that f is a quasiconformal map of the complement of a closed set E in $\mathbf{R}^n$ into $\mathbf{R}^n$, $n \geq 2$, and suppose that each point $x \in E$ has the following property: there is a sequence of radii r_j, $r_j \to 0$ as $j \to \infty$, such that the annular region $B(x, ar_j) - B(x, r_j/a)$ does not meet E for some $a > 1$ independent of x. Then f has a quasiconformal extension to $\hat{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$. Moreover, the dilatation of the extension agrees with the dilataiton of f.*

There, they remarked that this result may be new even for conformal maps in the plane. So it is noteworthy to give a different proof of a more general extension theorem on 2-dimensional quasiconformal maps of the plane based on some classical results in the function theory.

We begin with the following definition, which weakens the condition in the above theorem to a conformally invariant one.

Definition We say that a closed set E in the complex plane is *annularly coarse* if each point $x \in E$ has the following property: there is a sequence of mutually disjoint nested annuli $\{R_k\}_{k=1}^{\infty}$ such that the modulus $m(R_k)$ of R_k satisfies

$$m(R_k) \geq c$$

with a positive c. Here we say that a sequence of annuli $\{R_k\}_{k=1}^{\infty}$ is *nested* if every R_k $(k > 1)$ separates R_{k-1} from x.

Also note that the positive constant c can depend on x.

Now we will prove the following

Theorem 2 *Suppose that f is a quasiconformal map of the complement of a closed set E in the complex plane $\mathbf{C}$ into $\mathbf{C}$ and suppose that E is annularly coarse. Then f has a quasiconformal extension to $\hat{\mathbf{C}}$. Moreover, the dilatation of the extension agrees with the dilataiton of f.*

1 Known facts and basic lemmas

In 2-dimensional case, we have the following

Proposition 3 *Let E be a compact set in $\mathbf{C}$. Then the following conditions are mutually equivalent.*

1. *Every conformal map of $D = \mathbf{C} - E$ is the restriction of a Möbius transformaiton.*
2. *Every quasiconformal map of $D = \hat{\mathbf{C}} - E$ has a quasiconformal extension to the whole $\hat{\mathbf{C}}$.*
3. *For every relatively compact neighborhood U of E, every quasiconformal map of $U - E$ has a quasiconformal extension to U.*

Proof. First assume the condition 1) and take any quasiconformal map f of $D = \mathbf{C} - E$. Here we may assume that $f(\infty) = \infty$. Let μ be the Beltrami coefficient of f^{-1} on $f(\mathbf{C} - E)$. Set $\mu = 0$ on $\mathbf{C} - f(\mathbf{C} - E)$, and we have a quasiconformal map g of $\hat{\mathbf{C}}$ with the complex dilatation μ (cf. [1] and [4]). Then, $g \circ f$ has vanishing complex dilatation on $\mathbf{C} - E$, and hence the assumption implies that it is a Möbius transformation T. Thus f can be extended a quasiconformal map $g^{-1} \circ T$ of the whole $\hat{\mathbf{C}}$.

Next assume the condition 2) and take a relatively compact neighborhood U of E and a quasiconformal map f of $U - E$ arbitrarily. Since E is compact, the famous extention theorem ([6] II Theorem 8.1) gives a neighborhood V of E in U and a quasiconformal map g of $\hat{\mathbf{C}} - E$ which coincides with f on $V - E$. Then the assumption implies that g can be extended to a quasiconformal map of $\hat{\mathbf{C}}$, which clearly gives a quasiconformal extension of f to U.

Finally, assume the condition 3) and take any conformal map f of $D = \mathbf{C} - E$. Then f can be extended to a quasiconformal map g of $\mathbf{C}$. Hence if E has vanishing area, then this g is actually conformal, and hence is a Möbius transformation. If not, consider the extremal (horizontal) slit map h of $\mathbf{C} - E$. Then h should be extended a quasiconformal map of $\mathbf{C}$. But this is impossible, for $\mathbf{C} - f(\mathbf{C} - E)$ has vanishing area by Koebe's uniformization theorem. ■

Remark Koebe's uniformization theorem asserts that every planar domain Ω can map conformally onto the complement of some union of horizontal slits and points whose total area vanishes. And an example of such univalent holomorphic maps are the extremal slit maps. (See for instance, [5].)

As a condition which assures these extension properties, we know the following; we say that a compact set E has *absolutely vanishing area* if $\mathbf{C} - g(\mathbf{C} - E)$ has vanishing area for every univalent holomorphic map g of $\mathbf{C} - E$.

Actually, the following fact is classically well-known.

Lemma 4 *Let E be a compact set in $\mathbf{C}$ with absolutely vanishing area. Then every conformal map of $D = \mathbf{C} - E$ is the restriciton of a Möbius transformaiton.*

Proof. Ahlfors and Beurling ([2] showed that D belongs to O_{AD} if and only if E has absolutely vanishing area, which is also equivalent the condition 1) in Proposition 3. (Also see [8] VI and [7] I §2.) ■

Now, it is clear from the definition that an annularly coarse compact set is totally disconnected (or even absolutely disconnected). Furthermore, we see the following

Lemma 5 *Every annularly coarse compact set E has absolutely vanishing area.*

Proof. It suffices to show that E has vanishing area. For this purpose, fix a point $a \in E$ arbitrarily. Then there is a sequence of mutually disjoint nested annuli R_k such that $m(R_k) \geq c$ for every k with a positive constant c.

Let d_k be the diameter of the bounded component F_k of $\mathbf{C} - R_k$. Then we can find a positive constant η (depending only on c) such that $R_k \cap B(a, 2d_k)$ contains a ball B_k with radius ηd_k, where and in the sequel, we set $B(a, r) = \{|z - a| \leq r\}$.

For the sake of convenience, we include a direct proof of this assertion. Let A_k be the distance between F_k and $F'_k = \mathbf{C} - (F_k \cup R_k)$, and $z_k \in F_k$ and $z'_k \in F'_k$ satisfy $|z_k - z'_k| = A_k$. Also take two points $w_k, w'_k \in F_k$ satisfing $|w_k - w'_k| = d_k$. Further we may assume that $|w_k - z_k| \geq d_k/2$. Then R_k separates w_k and z_k from z'_k and ∞, and hence

$$T_k(z) = -\frac{z - z_k}{w_k - z_k}$$

maps R_k onto a region admissible to the extremal problem of Teichmüller (see [1]). Under the notation of [1], we have

$$\begin{aligned} c &< M(R_k) \leq \frac{1}{2\pi} \log \Psi(|(z_k - z'_k)/(z_k - w_k)|) \\ &\leq \frac{1}{2\pi} \log \Psi(2A_k/d_k), \end{aligned}$$

and since $\log \Psi(x) \to 0$ as $x \to 0$, we can find a positive $\eta = \eta(c)$ such that

$$A_k \geq \eta 2 d_k.$$

for every k, which gives the assertion.

Now set $r_k = 2d_k$ for every k, and we have

$$\frac{Area(E \cap B(a, r_k))}{Area(B(a, r_k))} < 1 - \frac{\pi(\eta d_k)^2}{\pi r_k^2} = 1 - \frac{\eta^2}{4}.$$

This implies that a is not a density point of E. Since a is arbitrary, we conclude that the area of E vanishes. ■

2 Proof of Theorem 2

First fix an annularly coarse closed set E arbitrarily. For every n, set $E_n = E \cap B(0, n)$. Then every E_n is compact and the assumption implies that there is a neighborhood U_n of E_n such that the boundary of U_n is a comapct set in $D = \mathbf{C} - E$.

Let f be a quasiconformal map of $\mathbf{C} - E$. Then Proposition 3 implies that f can be extended uniquely to a quasiconformal map, say f_n, of $\mathbf{C} - E \cap U_n$ and the maximal dilataiton of f_n is the same as that of f by Lemma 5.

Since K-quasiconformal maps are sequencially compact, we conclude that f has a desired extension to the whole $\mathbf{C}$.

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