Dynamics of sub-hyperbolic and semi-hyperbolic rational semigroups and skew products*

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Abstract

We consider dynamics of sub-hyperbolic and semi-hyperbolic semigroups of rational functions on the Riemann sphere and will show some no wandering domain theorems. The Julia set of a rational semigroup in general may have non-empty interior points. We give a sufficient condition that the Julia set has no interior points. From some information about forward and backward dynamics of the semigroup, we consider when the area of the Julia set is equal to 0 or an upper estimate of the Hausdorff dimension of the Julia set.

For a Riemann surface S, let $\mathrm{End}(S)$ denote the set of all holomorphic endomorphisms of S. It is a semigroup with the semigroup operation being composition of maps. A *rational semigroup* is a subsemigroup of $\mathrm{End}(\overline{\mathbb{C}})$ without any constant elements. We say that a rational semigroup G is a *polynomial semigroup* if each element of G is a polynomial.

Definition 0.1. Let G be a rational semigroup. We set

$$F(G) = \{z \in \overline{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\},\ J(G) = \overline{\mathbb{C}} \setminus F(G).$$

$F(G)$ is called the *Fatou set* for G and $J(G)$ is called the *Julia set* for G.

$J(G)$ is backward invariant under G but not forward invariant in general. If G is generated by a compact subset of $\mathrm{End}(\overline{\mathbb{C}})$, then $J(G)$ has the *backward self-similarity.* That is,

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Lemma 0.2. *Let G be a rational semigroup and assume G is generated by a compact subset Λ of $End(\overline{\mathbb{C}})$. Then*

$$J(G) = \bigcup_{f \in \Lambda} f^{-1}(J(G)).$$

We call this property the backward self-similarity of the Julia set.

Proof. Since $J(G)$ is backward invariant under G, we have

$$J(G) \supset \cup_{f \in \Lambda} f^{-1}(J(G)).$$

Suppose there exists a point $x \in J(G)$ that does not belong to $\cup_{f\in\Lambda} f^{-1}(J(G))$. There exists a neighborhood U of x in $\overline{\mathbb{C}}$ such that $f(U) \subset F(G)$ for each $f \in \Lambda$. Take any $x' \in U$. Let $\epsilon > 0$ be any small number. Since $\cup_{f\in\Lambda} f(x')$ is a compact subset of $F(G)$, there exists a number $\delta_1 > 0$ such that if $d(f(x'), \ y) < \delta_1$ for some $f \in \Lambda$, then $d(gf(x'), \ g(y)) < \epsilon$ for each $g \in G \cup \{id\}$. Take $\delta_2 > 0$ such that if $d(x', y) < \delta_2$ then $d(f(x'), \ f(y')) < \delta_1$ for each $f \in \Lambda$. Then we have that if $d(x', y) < \delta_2$, then $d(gf(x'), \ gf(y')) < \epsilon$ for each $g \in G \cup \{id\}$ and each $f \in \Lambda$. Hence we have $x \in F(G)$ and this is a contradiction. $\square$

The Julia set of any rational semigroup is a perfect set, backward orbit of any point of the Julia set is dense in the Julia set and the set of repelling fixed points of the semigroup is dense in the Julia set. In general, the Julia set of a rational semigroup may have non-empty interior points. For example, $J(\langle z^2, 2z\rangle) = \{|z| \leq 1\}$. In fact, in [HM2] it was shown that if G is a finitely generated rational semigroup, then any super attracting fixed point of any element of G does not belong to $\partial J(G)$. Hence we can easily get many examples that the Julia sets have non-empty interior points. For more details about these properties, see [HM1], [HM2], [ZR], [GR], [S1] and [S2]. In this paper we use the notations in [HM1] , [S1] and [S2].

Since the Julia set of a rational semigroup may have non-empty interior points, it is significant for us to get sufficient conditions such that the Julia set has no interior points, to know when the area of the Julia set is equal to 0 or to get an upper estimate of the Hausdorff dimension of the Julia set. We will try that using various information about forward dynamics of the semigroup in the Fatou set or backward dynamics of the semigroup in the Julia set.

In the section 1 of this paper we will define sub-hyperbolic and semi-hyperbolic rational semigroups and show no wandering domain theorems. In particular, we will see that if G is a finitely generated sub-hyperbolic or semi-hyperbolic rational semigroup, then there exists an attractor in the Fatou set for G(Theorem 1.34). By using these theorems, we can show the continuity of the Julia set with respect to the perturbation of the generators(Corollary 1.39).

In Section 2, we will consider the skew products of rational functions or dynamics on $\overline{\mathbb{C}}$-fibrations. The "Julia set" of any skew product is defined to be the closure of the union of the fibrewise Julia sets. We will define hyperbolicity and semi-hyperbolicity. We will show that if a skew product is semi-hyperbolic, then the Julia set is equal to the union of the fibrewise Julia sets and the skew product has the contraction property with respect to the backward dynamics along fibres(Theorem 2.13). The results in section 2 are generalized to those of version of dynamics on $\overline{\mathbb{C}}$-fibrations.

In section 3, we will consider necessary and sufficient conditions to be semi-hyperbolic(Theorem 3.1, Theorem 3.5). We will show that any sub-hyperbolic semigroup without any superattracting fixed point of any element of the semigroup in the Julia set is semi-hyperbolic(Theorem 3.7).

In section 4, we will show that if a finitely generated rational semigroup G is semi-hyperbolic and satisfies the open set condition with an open set O such that $\sharp(\partial O \cap J(G)) < \infty$, then 2-dimensional Lebesgue measure of the Julia set is equal to 0(Theorem 4.4).

In section 5, we will consider constructing δ-subconformal measures. If a rational semigroup has at most countably many elements and the δ-Poincaré series converges, then we can construct δ-subconformal measures. We will see that if G is a finitely generated semi-hyperbolic rational semigroup, then the Hausdorff dimension of the Julia set is less than the exponent δ(Theorem 5.6, Theorem 5.7). To show those results, the contracting property of backward dynamics will be used.

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1 No Wandering Domain

Definition 1.1. Let G be a rational semigroup. We set

$$P(G) = \overline{\bigcup_{g \in G} \{ \text{ critical values of } g \}}.$$

We call $P(G)$ the post critical set of G. We say that G is *hyperbolic* if $P(G) \subset F(G)$. Also we say that G is *sub-hyperbolic* if $\sharp\{P(G) \cap J(G)\} < \infty$

and $P(G) \cap F(G)$ is a compact set.

We denote by $B(x, \epsilon)$ a ball of center x and radius ϵ in the spherical metric. We denote by $D(x, \epsilon)$ a ball of center $x \in \mathbb{C}$ and radius ϵ in the Euclidian metric. Also for any hyperboplic manifold M we denote by $H(x, \epsilon)$ a ball of center $x \in M$ and radius ϵ in the hyperbolic metric. For any rational map g, we denote by $B_g(x, \epsilon)$ a connected component of $g^{-1}(B(x, \epsilon))$. For each open set U in $\overline{\mathbb{C}}$ and each rational map g, we denote by $c(U, g)$ the set of all connected components of $g^{-1}(U)$. Note that if g is a polynomial and $U = D(x, r)$ then any element of $c(U, g)$ is simply connected by the maximal principle.

For each set A in $\overline{\mathbb{C}}$, we denote by A^i the set of all interior points of A.

Definition 1.2. Let G be a rational semigroup and A a set in $\overline{\mathbb{C}}$. We set $G(A) = \cup_{g \in G} g(A)$ and $G^{-1}(A) = \cup_{g \in G} g^{-1}(A)$.

We can show the following Lemma immediately.

Lemma 1.3. *Let G be a rational semigroup. Assume that $\{f_\lambda\}_{\lambda \in \Lambda}$ is a generator system of G. Then we have*

$$\bigcup_{g \in G} \{\text{critical values of } g\} = \bigcup_{\lambda \in \Lambda} (G \cup \{Id\})(\{\text{critical values of } f_\lambda\}).$$

Definition 1.4. Let G be a rational semigroup and N a positive integer. We set

$$\begin{aligned} & SH_N(G) \\ = \quad & \{x \in \overline{\mathbb{C}} \mid \exists \delta(x) > 0,\ \forall g \in G,\ \forall B_g(x, \delta(x)),\ \deg(g : B_g(x, \delta) \to B(x, \delta)) \leq N\} \end{aligned}$$

and $UH(G) = \overline{\mathbb{C}} \setminus (\cup_{N \in \mathbb{N}} SH_N(G))$.

Remark 1. By definition, $SH_N(G)$ is an open set in $\overline{\mathbb{C}}$ and $g^{-1}(SH_N(G)) \subset SH_N(G)$ for each $g \in G$. Also $UH(G)$ is a compact set and $g(UH(G)) \subset UH(G)$ for each $g \in G$. For each rational map g with $\deg(g) \leq 2$, any parabolic or attracting periodic point of g belongs to $UH(G)$.

Definition 1.5. Let G be a rational semigroup. We say that G is *semi-hyperbolic* (resp. *weakly semi-hyperbolic*) if there exists a positive integer N such that $J(G) \subset SH_N(G)$(resp.$\partial J(G) \subset SH_N(G)$).

Remark 2. 1. If G is semi-hyperbolic and $N = 1$, then G is hyperbolic.

2. If G is hyperbolic, then G is semi-hyperbolic.

3. For a rational map f with the degree at least two, $\langle f \rangle$ is semi-hyperbolic if and only if f has no parabolic orbits and each critical point in the Julia set is non-recurrent([CJY], [Y]). If $\langle f \rangle$ is semi-hyperbolic, then there are neither indifferent cycles, Siegel disks nor Hermann rings.

Definition 1.6. Let V be a domain in $\overline{\mathbb{C}}$ and E a compact subset of V. We set

$$\text{mod } (E, V) = \sup\{ \text{ mod } A\},$$

where the supremum is taken over all annulus A such that E lies in a compact component of $V \setminus A$.

Lemma 1.7 ([CJY]). *For any positive integer N and real number r with $0 < r < 1$, there exists a constant $C = C(N, r)$ such that if $f : D(0,1) \to D(0,1)$ is a proper holomorphic map with* $\deg(f) = N$, *then*

$$H(f(z_0), C) \subset f(H(z_0, r)) \subset H(f(z_0), r)$$

for any $z_0 \in D(0,1)$. Here we can take $C = C(N, r)$ independent of f.

Corollary 1.8. *For any positive integer N and real number r with $0 < r < 1$, there exist constants r_1 and r_2 with $0 < r_1 \leq r_2 < 1$ depending only on r, N such that if $f : D(0,1) \to D(0,1)$ is a proper holomorphic map with* $\deg(f) = N$ *and $f(0) = 0$, then*

$$D(0, r_1) \subset W \subset D(0, r_2)$$

where W is the connected component of $f^{-1}(D(0, r))$ containing 0.

Corollary 1.9 ([Y]). *Let V be a simply connected domain in $\mathbb{C}$, $0 \in V$, $f : V \to D(0,1)$ be a proper holomorphic map of degree N and $f(0) = 0$, W be the component of $f^{-1}(D(0, r))$ containing 0, $0 < r < 1$. Then there exists a constant K depending only on r and N, not depending on V and f, so that*

$$|\frac{x}{y}| \leq K$$

for all $x, y \in \partial W$.

Proof. We will follow Y.Yin's proof([Y]). Let $g : V \to D(0,1)$ be the univalent function such that $g(0) = 0$. ¿From Corollary 1.8,

$$r_1 \leq |g(x)| \leq r_2$$

for all $x \in \partial W$. Applying the Koebe distortion theorem, we have that

$$|(g^{-1})'(0)| \cdot \frac{r_1}{(1 + r_1)^2} \leq |x| \leq |(g^{-1})'(0)| \cdot \frac{r_2}{(1 - r_2)^2}.$$

Then

$$|\frac{x}{y}| \leq \frac{r_2(1 + r_1)^2}{r_1(1 - r_2)^2} =: K.$$

K is a constant depending only on N and r. □

Lemma 1.10. *Let V be a domain in $\overline{\mathbb{C}}$, K a continuum in $\overline{\mathbb{C}}$ with $diam_S K = a$. Assume $V \subset \overline{\mathbb{C}} \setminus K$. Let $f : V \to D(0,1)$ be a proper holomorphic map of degree N. Then there exists a constant $r(N,a)$ depending only on N and a such that for each r with $0 < r \leq r(N,a)$, there exists a constant $C = C(N,r)$ depending only on N and r satisfying that for each connected component U of $f^{-1}(D(0,r))$,*

$$diam_S\, U \leq C,$$

where we denote by $diam_S$ the spherical diameter. Also we have $C(N,r) \to 0$ as $r \to 0$.

Proof. Let r be a number with $0 < r < 1$. Let U be a connected component of $f^{-1}(D(0,r))$ and V' be the connected component of $\overline{\mathbb{C}} \setminus \overline{V}$ containing K. Since V is connected, V' is simply connected. Let U' be the connected component of $\overline{\mathbb{C}} \setminus \overline{U}$ containing V'. Since U' is also simply connected and $\overline{V'} \subset U'$, we have that there exists a connected component of $U' \setminus \overline{V'}$ which is a ring domain.

There exists a sequence $(r_j)_{j=0}^n$ of real numbers with $r_0 = r < r_1 < \cdots < r_n = 1$ such that there exist no critical values of f in $D(0,r_{j+1}) \setminus \overline{D(0,r_j)}$ for $j = 0, \ldots, n-1$. For each $i = 0, \ldots, n$, let U_i'' be the connected component of $f^{-1}(D(0,r_i))$ containing U and let U_i' be the connected component of $\overline{\mathbb{C}} \setminus \overline{U_i''}$ containing V'. Then we have

$$U_0'' = U \subset U_1'' \subset \cdots \subset U_n'' = V \text{ and}$$

$$U_0' = U' \supset U_1' \supset \cdots \supset U_n' = V'.$$

By the construction, $f : U_{i+1}'' \setminus \overline{U_i''} \to D(0,r_{i+1}) \setminus \overline{D(0,r_i)}$ is a proper map for $i = 0, \ldots, n-1$. Since there exist no critical values of f in $D(0,r_{i+1}) \setminus \overline{D(0,r_i)}$, each connected component of $U_{i+1}'' \setminus \overline{U_i''}$ is a ring domain.

Now we claim that for each $i = 0, \ldots, n-1$, there exists a connencted component of $U_{i+1}'' \setminus \overline{U_i''}$ which is included in $U_i' \setminus \overline{U_{i+1}'}$. We will show that. Since $\partial U_i' \subset U_{i+1}''$, there exists a ring domain R_i in $U_{i+1}'' \setminus \overline{U_i''}$ such that $\partial U_i'$ is a connected component of ∂R_i. Let R_i' be the connected component of $U_{i+1}'' \setminus \overline{U_i''}$ containing R_i. Since

$$\partial(U_i' \setminus \overline{U_{i+1}'}) = \partial U_i' \cup \partial U_{i+1}' \subset \partial U_i'' \cup \partial U_{i+1}'',$$

we have $R_i' \cap \partial(U_i' \setminus \overline{U_{i+1}'}) = \emptyset$. Hence $R_i' \subset U_i' \setminus \overline{U_{i+1}'}$ and we have proved the above claim.

From the above claim, we get

$$\text{mod}\ (\overline{U_{i+1}'},\ U_i') \geq \frac{1}{2\pi N} \log \frac{r_{i+1}}{r_i},\ \text{ for } i = 0, \ldots, n-1.$$

It follows that

$$
\begin{aligned}
\text{mod } (\overline{V'},\ U') &\geq \sum_{i=0}^{n-1} \text{mod } (U'_i \setminus \overline{U'_{i+1}}) \\
&\geq \frac{1}{2\pi N} \log \frac{1}{r}.
\end{aligned}
$$

On the other hand, by Lemma 6.1 in p34 in [LV], we have

$$\text{mod } (\overline{V'},\ U') \leq \frac{\pi^2}{2C_1^2},$$

where $C_1 = \min\{a,\ \text{diam}_S\, U\}$. Hence the statement of our lemma holds. □

Lemma 1.11. *Let V and W be simply connected domains in $\overline{\mathbb{C}}$. Suppose that $\overline{W} \subset V$ and mod $(\overline{W},\ V) > c > 0$. Then there exists a constant $0 < \lambda < 1$ depending only on c such that*

$$\frac{\textit{diam } W}{\textit{diam } V} \leq \lambda,$$

here by "diam" we mean the spherical diameter.

Proof. We can assume that $0 \in W$ and diam $V = d(0,1)$ where d is the spherical metric. Let $g : D(0,1) \to V$ be the Riemann map such that $g(0) = 0$. By Theorem 2.4 in [M], there exists a constant c_1 depending only on c such that

$$\text{diam}_H(g^{-1}(W)) \leq c_1,$$

where we denote by diam_H the diameter with respect to the hyperbolic metric in $D(0,1)$. Since diam $V = d(0,1)$, by the Koebe distortion theorem, we have that there exists a constant c_2 not depending on V and W such that $|g'(0)| \leq c_2$. Using the Koebe distortion theorem again, we see that there exists a constant c_3 depending only on c such that for each $z \in g^{-1}(W)$, $|g(z)| \leq c_3$. Hence there exists a constant $0 < c_4 < d(0,1)$ depending only on c such that diam $W \leq c_4$.

□

Lemma 1.12. *Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated rational semigroup. Let y be a point of $\overline{\mathbb{C}} \setminus UH(G)$. If there exists a neighborhood W of y such that $\overline{\mathbb{C}} \setminus G^{-1}(W)$ contains a continuum, then there exists a neighborhood W_1 of y such that for each simply connected open neighborhood V of y included in W_1 and for each $g \in G$, each element of $c(V, g)$ is simply connected.*

Proof. For each $j = 1, \dots, m$, let C_j be the set of all critical points of f_j. By Lemma 1.10, there exists a $\delta > 0$ such that for each $g \in G$, each element of $c(B(y, \delta), g)$ does not contain any two different points of C_j, $j = 1, \dots m$. Then for any simply connected open neighborhood V of y included in $B(y, \delta)$ and for any $g \in G$, each element of $c(V, g)$ is simply connected. $\square$

Lemma 1.13. *Let G be a rational semigroup and N a positive integer. Then for each $g \in G$, any critical point c of g does not belong to $SH_N(G) \cap \overline{(G \cup \{id\})(g(c))}$.*

Proof. Assume that there exists a critical point c of an element $g \in G$ such that $c \in SH_N(G) \cap \overline{(G \cup \{id\})(g(c))}$. Then there exists a sequence (g_n) in G so that $g_n g(c) \to c$.

There exists a positive number ϵ such that $B(c, \epsilon) \subset SH_N(G)$. Since $g_n g(c) \to c$, we can construct a sequence (n_j) and a sequence (B_j) so that for each j, B_j is a connected component of $((g_{n_1} g)(g_{n_2} g) \cdots (g_{n_j} g))^{-1}(B(c, \epsilon))$ and $c \in B_j$, which contradicts that $c \in SH_N(G)$. $\square$

Lemma 1.14. *Let g be a rational map with $\deg(g) \geq 2$ and N a positive integer. Assume that $x \in J(\langle g \rangle) \cap SH_N(\langle g \rangle)$. Then x belongs to neither boundaries of Siegel disks, boundaries of Hermann rings nor indifferent cycles.*

Proof. By Theorem 1 and Corollary in [Ma] and Lemma 1.13, we can show the statement immediately. $\square$

Definition 1.15. Let G be a rational semigroup and U a component of $F(G)$. For every element g of G, we denote by U_g the connected component of $F(G)$ containing $g(U)$. We say that U is a wandering domain if $\{U_g\}$ is infinite.

Remark 3. In [HM1], A.Hinkkanen and G.J.Martin showed that there exists an infinitely generated polynomial semigroup which has a wandering domain.

Lemma 1.16. *Let G be a rational semigroup which contains an element with the degree at least two. Let x be a point of $F(G)$ and assume that there exists a point $y \in \partial J(G)$ and a sequence (g_n) of elements of G such that $g_n(x) \to y$. Then we have $y \in P(G) \cap \partial J(G)$.*

Proof. We can assume that $\sharp P(G) \geq 3$. Suppose $y \in \overline{\mathbb{C}} \setminus P(G)$. Let δ be a number so that $\overline{B(y, \delta)} \subset \overline{\mathbb{C}} \setminus P(G)$. We can assume that for each n, $g_n(x) \in B(y, \delta)$. For each n, there exists an analytic inverse branch α_n of g_n in U such that $\alpha_n(g_n(x)) = x$. Since $\sharp P(G) \geq 3$, we have $\{\alpha_n\}$is normal in U. Hence if we take an ϵ small enough,

$$\text{diam } \alpha_n(B(y, \ \epsilon\delta)) < d(x, J(G)), \text{ for each } n.$$

But $x \in \alpha_n(B(y, \ \epsilon\delta))$ for large n and $\alpha_n(B(y, \ \epsilon\delta)) \cap J(G) \neq \emptyset$ because $J(G)$ is backward invariant under G. This is a contradiction.

$\square$

Corollary 1.17. *Let G be a rational semigroup which contains an element with the degree at least two. If $P(G) \cap \partial J(G) = \emptyset$, then for each $x \in F(G)$, $\overline{G(x)} \setminus F(G)$ and there is no wandering domain.*

Lemma 1.18. *Let G be a polynomial semigroup, N a positive integer and y a point in $\partial J(G) \cap \mathbb{C}$. Assume that there exists an open neighborhood U of y such that $U \subset SH_N(G)$ and $\sharp(\overline{\mathbb{C}} \setminus G^{-1}(U)) \geq 3$. Then for each $x \in F(G)$, $\overline{G(x)} \subset \overline{\mathbb{C}} \setminus \{y\}$.*

Proof. We can assume that $\infty \in \overline{\mathbb{C}} \setminus G^{-1}(U)$. Suppose that there exists a point $x \in F(G)$ and a sequence (g_n) in G such that $g_n(x) \to y$ as $n \to \infty$. Let δ be a positive number so that for each $g \in G$,

$$\deg(g : V \to D(y, \delta)) \leq N,$$

for each $V \in c(D(y, \ \delta), \ g)$. For any r with $0 < r \leq \delta$ there exists a positive integer $n(r)$ such that for each integer n with $n \geq n(r)$, $g_n(x) \in D(y, r)$. Let $D_{g_n}(y, r)$ be the connected component of $g_n^{-1}(D(y, r))$ containing x. For each n with $n \geq n(r)$, there exists a conformal map φ_n from $D(0, 1)$ onto $D_{g_n}(y, \delta)$ such that $\varphi_n(0) = x$. From Lemma 1.10, there exists a constant $C(r)$ with $C(r) \to 0$ as $r \to 0$ such that for each integer n with $n \geq n(r)$,

$$\text{diam } \varphi_n^{-1}(D_{g_n}(y, r)) \leq C(r).$$

Since $\sharp(\overline{\mathbb{C}} \setminus G^{-1}(U)) \geq 3$, the family $\{\varphi_n\}$ is normal in $D(0, 1)$. Hence if r is sufficiently small, then for each integer n with $n \geq n(r)$,

$$\text{diam}_S D_{g_n}(y, r) < d(J(G), x),$$

where we denote by diam_S the spherical diameter and by d the spherical distance. On the other hand, since $J(G)$ is backward invariant under G and $y \in J(G)$, we have that for each n with $n \geq n(r)$, $D_{g_n}(y, r) \cap J(G) \neq \emptyset$. This is a contradiction. Therefore we have for each $x \in F(G)$, $\overline{G(x)} \subset \overline{\mathbb{C}} \setminus \{y\}$. $\square$

Lemma 1.19. *Let G be a polynomial semigroup. Assume that there exists a point $x \in F(G)$, a point $y \in \partial J(G)$ and a sequence (g_n) in G such that $g_n(x) \to y$ as $n \to \infty$. Then at least one of the following holds.*

1. *$UH(G) = \emptyset$ and each element of G is a Möbius transformation. For each $z \in F(G)$, $y \in \overline{G(z)}$.*
2. *$\sharp(UH(G)) = 1 or 2$, $UH(G) \subset J(G)$ and $UH(G) \cap \partial J(G) \neq \emptyset$. For each $z \in F(G)$, $y \in \overline{G(z)}$.*

3. $y \in UH(G)$.

Proof. Suppose that $\sharp(UH(G)) \geq 3$. From Lemma 1.18, we have $y \in UH(G)$.

Suppose there exists a point $z \in F(G)$ such that $\overline{G(z)} \subset \overline{\mathbb{C}} \setminus \{y\}$. Then there exists a neighborhood V of z such that $G(V) \subset \overline{\mathbb{C}} \setminus \{y\}$. By Lemma 1.18, $y \in UH(G)$.

Now we consider the case $\sharp(UH(G)) = 1$ or 2. Then $\infty \in UH(G)$. If $\infty \in F(G)$, then since $G(\infty) = \{\infty\}$, from Lemma 1.18 the condition 3. holds. Now suppose $\infty \in J(G)$. There exists an element $g \in G$ with the degree at least two. From Corollary 1.14, g has no Siegel disks. Let z be a point in $F(G)$. Since $F(G) \subset F(\langle g \rangle)$, $z \in F(\langle g \rangle)$. From no wandering domain theorem and the fact that g has no Siegel disks, there exists an attracting or parabolic periodic point $\zeta \in \overline{F(G)}$ of g and a sequence (n_j) of positive integers such that $g^{n_j}(z) \to \zeta$. We have $\zeta \in UH(G)$. If $\zeta \in \partial J(G)$, then the condition 2. holds. If $\zeta \in F(G)$, then since G is a polynomial semigroup, we have $G(\{\zeta\}) = \{\zeta\} \subset F(G)$ and it implies $y \in UH(G)$ from Lemma 1.18. Hence the condition 3. holds.

Finally we consider the case $UH(G) = \emptyset$. Assume there exists an element $h \in G$ with the degree at least two. Since $F(G) \neq \emptyset$, we have $F(\langle g \rangle) \neq \emptyset$. By the no wandering domain theorem, g has (super)attracting cycles, parbolic cycles, Siegel disks or Hermann rings. Since $UH(G) = \emptyset$, this is a contradiction. □

Theorem 1.20. *Let G be a rational semigroup containing an element with the degree at least two and U a connected component. Assume that there exists a sequence (g_n) of elements of G such that $U_{g_n} \cap U_{g_m} = \emptyset$ if $n \neq m$ (in pariticular, U is a wandering domain). Then there exists a subsequence (g_{n_j}) of (g_n) and a point $y \in P(G) \cap \partial J(G)$ such that (g_{n_j}) converges to y locally uniformly on U.*

Proof. By the method in the proof of Theorem 2.2.3 in [S1], we can show that there exists a subsequence (g_{n_j}) of (g_n) and a point $y \in \partial J(G)$ such that (g_{n_j}) converges to y locally uniformly on U. Hence the statement of our theorem holds from Lemma 1.16. □

Theorem 1.21. *Let G be a polynomial semigroup and U a connected component of $F(G)$. Assume that there exists a sequence (g_n) of elements of G such that $U_{g_n} \cap U_{g_m} = \emptyset$ if $n \neq m$ (in pariticular, U is a wandering domain). Then at least one of the following holds.*

1. *$UH(G) = \emptyset$ and each element of G is a Möbius transformation. For each $z \in F(G)$, $\overline{G(z)} \cap \partial J(G) \neq \emptyset$.*

2. *$\sharp(UH(G)) = 1$ or 2, $UH(G) \subset J(G)$ and $UH(G) \cap \partial J(G) \neq \emptyset$. For each $z \in F(G)$, $\overline{G(z)} \cap \partial J(G) \neq \emptyset$.*

3. *There exists a subsequence* (g_{n_j}) *of* (g_n) *and a point* $y \in UH(G) \cap \partial J(G)$ *such that* (g_{n_j}) *converges to* y *locally uniformly on* U.

Proof. Using Lemma 1.19, we can show the statement in the same way as the proof of Theorem 1.20. □

By Lemma 1.10 and using the method of the proof in Lemma 1.18, we can show the next lemma immediately.

Lemma 1.22. *Let* G *be a rational semigroup and* y *a point of* $\partial J(G) \setminus UH(G)$. *Assume that there exists an open neighborhood* U *of* y *such that* $\overline{\mathbb{C}} \setminus G^{-1}(U)$ *contains a continuum* K. *Then for each* $x \in F(G)$, $\overline{G(x)} \subset \overline{\mathbb{C}} \setminus \{y\}$.

Lemma 1.23. *Let* G *be a rational semigroup. Assume that there exists a point* $x \in F(G)$, *a point* $y \in \partial J(G)$ *and a suquence* (g_n) *in* G *such that* $g_n(x) \to y$ *as* $n \to \infty$. *Then at least one of the following holds.*

1. $UH(G) = \emptyset$ *and each element of* G *is a Möbius transformation. For each* $z \in F(G)$, $y \in \overline{G(z)}$.

2. $UH(G)$ *is totally disconnected,* $UH(G) \subset J(G)$ *and* $UH(G) \cap \partial J(G) \neq \emptyset$. *For each* $z \in F(G)$, $y \in \overline{G(z)}$.

3. $y \in UH(G)$.

Proof. Suppose $UH(G)$ is empty. Then we can show that each element of G is a Möbius transformation in the same way as the proof of Lemma 1.19.

Suppose there exists a point $z \in F(G)$ such that $\overline{G(z)} \subset \overline{\mathbb{C}} \setminus \{y\}$. Then there exists a neighborhood V of z such that $G(V) \subset \overline{\mathbb{C}} \setminus \{y\}$. By Lemma 1.22, $y \in UH(G)$.

Suppose $UH(G) \cap F(G) \neq \emptyset$. Let $z \in UH(G) \cap F(G)$. If $\overline{G(z)} \subset \overline{\mathbb{C}} \setminus \{y\}$, then by the previous arguments, $y \in UH(G)$. If $y \in \overline{G(z)}$, we have also $y \in UH(G)$.

If $UH(G)$ contains a continuum, then from Lemma 1.22, we have $y \in UH(G)$.

Suppose that $\emptyset \neq UH(G) \subset J(G)$ and $UH(G)$ is totally disconnected. There exists an element $g \in G$ of degree at least two. Since $UH(G)$ is totally disconnected and $F(G) \neq \emptyset$, by no wandering domain theorem we can show that g has an (super) attracting or parabolic periodic point ζ in $\partial J(G)$. We have $\zeta \in UH(G)$. □

By Lemma 1.23, we can show the next result in the same way as the proof of Theorem 1.20.

Theorem 1.24. *Let G be a rational semigroup and U a connected component of $F(G)$. Assume that there exists a sequence (g_n) of elements of G such that $U_{g_n} \cap U_{g_m} = \emptyset$ if $n \neq m$ (in pariticular, U is a wandering domain). Then at least one of the following holds.*

1. *$UH(G) = \emptyset$ and each element of G is a Möbius transformation. For each $z \in F(G)$, $\overline{G(z)} \cap \partial J(G) \neq \emptyset$.*
2. *$UH(G)$ is totally disconnected, $UH(G) \subset J(G)$ and $UH(G) \cap \partial J(G) \neq \emptyset$. For each $z \in F(G)$, $\overline{G(z)} \cap \partial J(G) \neq \emptyset$.*
3. *There exists a subsequence (g_{n_j}) of (g_n) and a point $y \in UH(G) \cap \partial J(G)$ such that (g_{n_j}) converges to y locally uniformly on U.*

By Lemma 1.22, we can show the next result immediately.

Theorem 1.25. *Let G be a rational semigroup. Assume that G is weakly semi-hyperbolic and there is a point $z \in F(G)$ such that the closure of the G-orbit $\overline{G(z)}$ is included in $F(G)$. Then for each $x \in F(G)$, $\overline{G(x)} \subset F(G)$ and there is no wandering domain.*

Next theorem follows from Lemma 1.23.

Theorem 1.26. *Let G be a rational semigroup containing an element $g \in G$ with $\deg(g) \geq 2$. Assume that G is weakly semi-hyperbolic. If $F(G) \neq \emptyset$, then for each $x \in F(G)$, $\overline{G(x)} \subset F(G)$ and there is no wandering domain.*

Definition 1.27. Let G be a rational semigroup. We set

$$A_0(G) = \overline{G(\{z \in \overline{\mathbb{C}} \mid \exists g \in G \text{ with } \deg(g) \geq 2,\ g(x) = x \text{ and } |g'(x)| < 1.\})},$$

$$\tilde{A}_0(G) = \overline{G(\{z \in F(G) \mid \exists g \in G \text{ with } \deg(g) \geq 2,\ g(x) = x \text{ and } |g'(x)| < 1.\})},$$

$$A(G) = \overline{G(\{z \in \overline{\mathbb{C}} \mid \exists g \in G,\ g(x) = x \text{ and } |g'(x)| < 1.\})},$$

$$\tilde{A}(G) = \overline{G(\{z \in F(G) \mid \exists g \in G,\ g(x) = x \text{ and } |g'(x)| < 1.\})},$$

where the closure in the definition of $\tilde{A}_0(G)$ and $\tilde{A}(G)$ is considered in $\overline{\mathbb{C}}$.

Remark 4. By definition, $A_0(G) \subset A(G) \cap P(G)$. For each $g \in G$, $g(A_0(G)) \subset A_0(G)$ and $g(A(G)) \subset A(G)$. We have also similar statements for $\tilde{A}_0(G)$ and $\tilde{A}(G)$.

Lemma 1.28. *Let G be a rational semigroup. If $\tilde{A}_0(G)$ is a non-empty compact subset of $F(G)$, then*

$$\emptyset \neq \tilde{A}_0(G) = \tilde{A}(G) \subset P(G) \cap F(G).$$

Proof. Let g be any Möbius transformation in G and $x \in \overline{\mathbb{C}}$ a fixed point of g with $|g'(x)| < 1$. Since $g(\tilde{A}_0(G)) \subset \tilde{A}_0(G) \cap F(G)$ and $\tilde{A}_0(G) \neq \emptyset$, we have that $x \in \tilde{A}_0(G)$. Therefore the statement follows. $\square$

Lemma 1.29. *Let G be a rational semigroup containing an element with the degree at least two. Assume that G is semi-hyperbolic and $F(G) \neq \emptyset$. Then*

$$\emptyset \neq A_0(G) = \tilde{A}_0(G) = A(G) = \tilde{A}(G) \subset F(G).$$

Proof. Let $g \in G$ be an element with the degree at least two. Since $F(G) \neq \emptyset$, the element g has a (super)attracting periodic point x in $\overline{F(G)}$. By Remark 1, we have that $A_0(G) \subset F(G)$. Hence the statement follows from the proof of Lemma 1.28. $\square$

Lemma 1.30. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. Assume that each element of G with the degree at least two has neither Siegel disks nor Hermann rings and each element of $\mathrm{Aut}\,\overline{\mathbb{C}} \cap G$ (if this is not empty) is loxodromic. Also assume that $\sharp J(G) \geq 3$. Let $U_1, \dots, U_s$ be some connected components of $F(G)$ and K a non-empty compact subset of $V = \cup_{j=1}^{s} U_j$ such that $U_j \cap K \neq \emptyset$ for each $j = 1, \dots, s$ and $g(K) \subset K$ for each $g \in G$. Then for each compact subset L of V there exists a constant c with $c > 0$ and a constant λ with $0 < \lambda < 1$ such that*

1. $\sup\{\|(f_{i_n} \cdots f_{i_1})'(z)\| \mid z \in L,\ (i_n, \dots, i_1) \in \{1, \dots, m\}^n\} \leq c\lambda^n$, *where we denote by $\|\cdot\|$ the norm of the derivative of with respect to the hyperbolic metric on V.*
2. $\sup\{d(f_{i_n} \cdots f_{i_1}(z), K) \mid z \in L,\ (i_n, \dots, i_1) \in \{1, \dots, m\}^n\} \leq c\lambda^n$, *where we denote by d the spherical metric.*

Proof. Let a be a large positive number. For each $j = 1, \dots, s$, let K_j be the compact a-neighborhood of $K \cap U_j$ in U_j with respect to the distance induced by the hyperbolic metric in U_j. We set $K_0 = \cup_{j=1}^{s} K_j$. Then for each $g \in G$, $g(K_0) \subset K_0$. If a is large enough, we have that $L \subset K_0$.

We claim that there exists a constant $c > 0$ and a constant $\lambda < 1$ such that

$$\sup\{\|(f_{i_n} \cdots f_{i_1})'(z)\| \mid z \in L,\ (i_n, \dots, i_1) \in \{1, \dots, m\}^n\} \leq c\lambda^n, \qquad (1)$$

where we denote by $\|\cdot\|$ the norm of the derivative of with respect to the hyperbolic metric on V. To show the claim, let z be a point of K_j and $(i_{s+1}, \dots, i_1)$ an element of $\{1, \dots, m\}^{s+1}$. Then there exists an integer t with $1 \leq t \leq s$ such that $(f_{i_{s+1}} \cdots f_{i_{t+1}})(U_{j_t}) \subset U_{j_t}$, where U_{j_t} is the component of V containing $(f_{i_t} \cdots f_{i_1})(U_j)$. From the assumption, we have that for each $x \in K_{j_t}$, $\|(f_{i_{s+1}} \cdots f_{i_{t+1}})'(x)\| < 1$. Hence

$$\|(f_{i_{s+1}} \cdots f_{i_1})'(z)\| < 1.$$

Therefore the claim holds.

From the above claim, we can show the statement of our lemma immediately. $\square$

Definition 1.31. Let G be a rational semigroup and U a open set in $\overline{\mathbb{C}}$. We say that a non-empty compact subset K of U is an *attractor* in U for G if $g(K) \subset K$ for each $g \in G$ and for any open neighborhood V of K in U and each $z \in U$, $g(z) \in U$ for all but finitely many $g \in G$.

Lemma 1.32. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup and E a finite subset of $\overline{\mathbb{C}}$. Assume that each $x \in E$ is not a non-repelling fixed point of any element of G. Then for any $M > 0$, there exists a positive integer n_0 such that for any integer n with $n \geq n_0$ if $z,\ f_{w_1}(z),\ f_{w_2}f_{w_1}(z),\ \dots, (f_{w_{n-1}} \cdots f_{w_1})(z)$ and $(f_{w_n} \cdots f_{w_1})(z)$ belong to E and $|(f_{w_n} \cdots f_{w_1})'(z)| \neq 0$, then $|(f_{w_n} \cdots f_{w_1})'(z)| > M$.*

Proof. We will show the statement by induction on $\sharp E$. When $\sharp E = 1$, it easy to see that the statement holds. Now assume that for each finite subset E of $\overline{\mathbb{C}}$ with $\sharp E \leq s$ the statement holds. Let E' be a finite subset of $\overline{\mathbb{C}}$ with $\sharp E' = s+1$ and assume that each $x \in E'$ is not a non-repelling fixed point of any element of G. Take a number M_0 so that

$$M_0(\inf\{|(f_j)'(\zeta)| \mid \zeta \in E',\ (f_j)'(\zeta) \neq 0,\ j = 1, \dots, m.\})^2 > 1.$$

From the hypothesis of the induction, there exists a positive integer n_0 such that for any subset E of E' with $E \neq E'$ and for any integer n with $n \geq n_0$, if $x,\ f_{w_1}(x),\ f_{w_2}f_{w_1}(x),\ \dots, (f_{w_{n-1}} \cdots f_{w_1})(x)$ and $(f_{w_n} \cdots f_{w_1})(x)$ belong to E and $|(f_{w_n} \cdots f_{w_1})'(x)| \neq 0$, then $|(f_{w_n} \cdots f_{w_1})'(x)| > M_0$. For each $y \in E$ and postive integer t with $t \leq n_0 + 1$, we set

$$G_{y,t} = \{g \in G \mid g(y) = y,\ g\text{: a product of } t \text{ generators }\}.$$

Then we have that $\sharp G_{y,t} < \infty$ and for each $g \in G_{y,t}$, y is a repelling fixed point of g.

Now assume that $z,\ f_{w_1}(z),\ f_{w_2}f_{w_1}(z),\ \dots, (f_{w_{n-1}} \cdots f_{w_1})(z)$ and $(f_{w_n} \cdots f_{w_1})(z)$ belong to E', $(f_{w_n} \cdots f_{w_1})(z) = z$, $(f_{w_n} \cdots f_{w_1})'(z) \neq 0$ and $(f_{w_j} \cdots f_{w_1})(z) \neq z$ for each $j = 1, \dots, n-1$. If $n \leq n_0 + 1$, we have

$$|(f_{w_n} \cdots f_{w_1})'(z)| > \inf\{|g'(z)| \mid g \in G_{z,t},\ 1 \leq t \leq n_0 + 1\} > 1.$$

If $n \geq n_0 + 2$, then we have

$$|(f_{w_n} \cdots f_{w_1})'(z)| > M_0(\inf\{|(f_j)'(\zeta)| \mid \zeta \in E',\ f_j'(\zeta) \neq 0,\ j = 1, \dots, m.\})^2 > 1.$$

From these results, we can show that for any $M > 0$, there exits a positive integer n_1 such that for any integer u with $u \geq n_1$ if $z,\ f_{w_1}(z),\ f_{w_2}f_{w_1}(z),\ \dots,$

$(f_{w_{u-1}} \cdots f_{w_1})(z)$ and $(f_{w_u} \cdots f_{w_1})(z)$ belong to E' and $|(f_{w_u} \cdots f_{w_1})'(z)| \neq 0$, then

$$(f_{w_u} \cdots f_{w_1})'(z)| > M.$$

Hence we have completed the induction.

$\square$

Lemma 1.33. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup and E a finite subset of $\overline{\mathbb{C}}$. Assume that each $x \in E$ is not any non-repelling fixed point of any element of G. Then there exists an open neighborhood V of E in $\overline{\mathbb{C}}$ such that for each $z \in V$, if there exists a word $w = (w_1, w_2, \dots) \in \{1, \dots, m\}^{\mathbb{N}}$ satifying that:*

1. *for each n, $(f_{w_n} \cdots f_{w_1})(z) \in V$,*
2. *$(f_{w_n} \cdots f_{w_1}(z))$ accumulates only in E and*
3. *for each n, $(f_{w_n} \cdots f_{w_1})(\zeta) \in E$ and $(f_{w_n} \cdots f_{w_1})'(\zeta) \neq 0$ where ζ is the closest point to z in E,*

then z is equal to the point $\zeta \in E$.

Proof. Let ϵ be a small number so that $B(x, \epsilon) \cap B(y, \epsilon) = \emptyset$ if $x, y \in E$ and $x \neq y$. Take an ϵ smaller, if necesarry, so that if $z_0 \in E$ and $f_j'(z_0) \neq 0$ for some j, then $f_j|_{B(z_0, \epsilon)}$ is injective. We set $V = \cup_{z \in E} B(z, \epsilon)$.

Let $z \in V$ be a point. Assume that there exists a word $w = (w_1, w_2 \dots) \in \{1, \dots, m\}^{\mathbb{N}}$ satisfying the conditions 1, 2 and 3. We set $\alpha_n = f_{w_n} f_{w_{n-1}} \cdots f_{w_1}$. From the conditions 2 and 3, there exists a point $a \in E$ and a sequence (n_j) such that $\alpha_{n_j}(z) \to a$ as $j \to \infty$ and $a_{n_j}(\zeta) = a$ for each j. By lemma 1.32, we have $|(\alpha_n)'(\zeta)| \to \infty$ as $n \to \infty$. Hence by the Koebe distortion theorem, there exists a number $\eta > 0$ such that for each positive integer j, there exists an analytic inverse branch β_j of α_{n_j} on $B(a, \eta)$ so that $\beta_j(a) = \zeta$ and $\beta_j(B(a, \eta)) \subset V$ and diam $\beta_t(B(a, \eta)) \to 0$ as $t \to \infty$.

We set $y_j = \beta_j(\alpha_{n_j}(z))$ for each large j. We claim that for each integer l with $0 \leq l \leq n_j - 1$, if $(f_{w_{l+1}} f_{w_l} \cdots f_{w_1})(y_j) = (f_{w_{l+1}} f_{w_l} \cdots f_{w_1})(z)$, then $(f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(y_j) = (f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(z)$. Let us show the claim above. Assume that $(f_{w_{l+1}} f_{w_l} \cdots f_{w_1})(y_j) = (f_{w_{l+1}} f_{w_l} \cdots f_{w_1})(z)$. We have that

$$f_{w_l} f_{w_{l-1}} \cdots f_{w_1} \circ \beta_j : B(a, \eta) \to \overline{\mathbb{C}}$$

is an analytic inverse branch of $f_{w_{n_j}} f_{w_{n_j - 1}} \cdots f_{w_{l+1}}$ satisfying

$$(f_{w_l} f_{w_{l-1}} \cdots f_{w_1} \beta_j)(a) = (f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(\zeta).$$

By Lemma 1.32 and the Koebe distortion theorem, we can assume that

$$(f_{w_l} f_{w_{l-1}} \cdots f_{w_1} \beta_j)(B(a, \eta)) \subset B((f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(\zeta),\ \epsilon).$$

Since

$$(f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(z) \in B((f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(\zeta), \epsilon), \tag{2}$$

$$(f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(y_j) = (f_{w_l} f_{w_{l-1}} \cdots f_{w_1} \beta_j)(\alpha_{n_j}(z)) \in B((f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(\zeta),\ \epsilon) \tag{3}$$

and $f_{w_{l+1}}|_{B(f_{w_l} f_{w_{l-1}} \cdots f_{w_1}(\zeta),\ \epsilon)}$ is injective,

$$(f_{w_{l+1}} f_{w_l} \cdots f_{w_1})(y_j) = (f_{w_{l+1}} f_{w_l} \cdots f_{w_1})(z)$$

implies that $(f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(y_j) = (f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(z)$. Hence the claim above holds.

From this claim, it follows that $y_j = z$ for each large j. Since diam $\beta_j(B(a, \eta)) \to 0$ as $j \to \infty$, we have $z = \zeta$.

□

Theorem 1.34. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup. Assume that $F(G) \neq \emptyset$, there is an element $g \in G$ such that $\deg(g) \geq 2$ and each element of $Aut\, \overline{\mathbb{C}} \cap G$ (if this is not empty) is loxodromic. Also we assume all of the following conditions;*

1. *$\tilde{A}_0(G)$ is a compact subset of $F(G)$,*
2. *any element of G with the degree at least two has neither Siegel disks nor Hermann rings.*
3. *$\sharp(UH(G) \cap \partial J(G)) < \infty$ and each point of $UH(G) \cap \partial J(G)$ is not a non-repelling fixed point of any element of G.*

Then $\tilde{A}_0(G) = \tilde{A}(G) \neq \emptyset$ and for each compact subset L of $F(G)$,

$$\sup\{d(f_{i_n} \cdots f_{i_1}(z),\ \tilde{A}(G)) \mid z \in L,\ (i_n, \ldots, i_1) \in \{1, \ldots, m\}^n\} \to 0,$$

as $n \to \infty$, where we denote by d the spherical metric. Also $\tilde{A}(G)$ is the smallest attractor in $F(G)$ for G. Moreover we have that if (h_n) is a sequence in G consisting of mutually disjoint elements and converges to a map ϕ in a subdomain V of $F(G)$, then ϕ is constant taking its value in $\tilde{A}(G)$.

Proof. First we will show that $\tilde{A}_0(G) = \tilde{A}(G) \neq \emptyset$. By the condition 2, g has neither Siegel disks nor Hermann rings. Since $F(G) \neq \emptyset$ and by the condition 3, applying the no wandering domain theorem for $\langle g \rangle$, we see that the element g has an attracting periodic point x in $F(G)$. Hence $\tilde{A}_0(G) \neq \emptyset$. By Lemma 1.28, we get $\tilde{A}_0(G) = \tilde{A}(G) \neq \emptyset$.

Next we will show that for each $x \in F(G)$, $\overline{G(x)} \subset F(G)$. Assume that there exists a connected component U of $F(G)$, a sequence (g_n) of elements of G and a point $y \in \partial J(G)$ such that (g_n) converges to y locally uniformly on U. We take a subsequence $(g_{1,n})$ of (g_n) satisfying that there exists a generator f_{i_1} so that

$$g_{1,n} = \cdots f_{i_1},$$

for each n. Inductively when we get a sequence $(g_{j,n})_n$ satisfying that there exists a word $(i_1, \ldots, i_j) \in \{1, \ldots, m\}^j$ so that $g_{j,n} = \cdots f_{i_j} \cdots f_{i_1}$ for each n, we take a subsequence $(g_{j+1,n})_n$ of $(g_{j,n})_n$ satisfying that there exists a generator $f_{i_{j+1}}$ so that

$$g_{j+1,n} = \cdots f_{i_{j+1}} \cdots f_{i_1}$$

for each n. By the diagonal method, we get a subsequence $(g_{n,n})_n$ of (g_n) satisfying that there exists a word $(i_1, i_2, \ldots) \in \{1, \ldots, m\}^{\mathbb{N}}$ so that for each n,

$$g_{n,n} = \alpha_n f_{i_n} \cdots f_{i_1},$$

where α_n is an element of G. We consider the sequence (β_n) where $\beta_n = f_{i_n} \cdots f_{i_1}$. We see that $U_{\beta_n} \neq U_{\beta_m}$ if $n \neq m$. For, if there exists n and m with $n > m$ such that $U_{\beta_n} = U_{\beta_m}$, then

$$(f_{i_n} \cdots f_{i_{m+1}})(U_{\beta_m}) \subset U_{\beta_m}$$

and the element $f_{i_n} \cdots f_{i_{m+1}}$ has an (super)attracting fixed point x_0 in $\overline{U_{\beta_m}}$. By the condition 3, we have $x_0 \in \tilde{A}(G)$. From Lemma 1.30, it contradicts to that (g_n) converges to $y \in \partial J(G)$ in U. Hence $U_{\beta_n} \neq U_{\beta_m}$ if $n \neq m$. Now let z be a point of U. Since $U_{\beta_n} \neq U_{\beta_m}$ if $n \neq m$, we have $(\beta_n(z))$ accumulates only in $\partial J(G)$. By Theorem 1.24, we can show that $(\beta_n(z))$ accumulates only in $\partial J(G) \cap UH(G)$. For each large n, let ζ_n be the closest point to $\beta_{i_n}(z)$ in $\partial J(G) \cap UH(G)$. Since $\sharp(\partial J(G) \cap UH(G)) < \infty$ and there is no super attracting fixed point of any element of G in $\partial J(G)$, there exists an integer n_0 such that for each integer n with $n \geq n_0$,

$$(f_{i_n} \cdots f_{i_{n_0}+1})'(\zeta_{n_0}) \neq 0.$$

¿From Lemma 1.33, we get a contradiction. Therefore we have for each $x \in F(G)$, $\overline{G(x)} \subset F(G)$.

Now let x be a point of $F(G)$. We have $\overline{G(x)} \subset F(G)$. Let $\{U_1, \ldots, U_s\}$ be the set of all connected components of $F(G)$ having non-empty intersection with $\overline{G(x)}$. We set $V = \cup_{j=1}^{s} U_j$. Suppose that $x \in U_j$. For each $(i_{s+1}, i_s, \ldots, i_1) \in \{1, \ldots, m\}^{s+1}$, there exists an integer t with $1 \leq t \leq s$ such that $(f_{i_{s+1}} \cdots f_{i_{t+1}})(U_{j_t}) \subset U_{j_t}$, where U_{j_t} is the component of V containing $(f_{i_t} \cdots f_{i_1})(U_j)$. From our assumption, the element $f_{i_{s+1}} \cdots f_{i_{t+1}}$ has an attracting fixed point in $U_{j_t} \cap \tilde{A}(G)$. Hence, from Lemma 1.30, we have

$$\sup\{d(f_{i_n} \cdots f_{i_1}(z),\ \tilde{A}(G)) \mid (i_n, \ldots, i_1) \in \{1, \ldots, m\}^n\} \to 0,$$

as $n \to \infty$. Therefore for each compact subset L of $F(G)$, the similar result holds.

Next we will show that $\tilde{A}(G)$ is the smallest attractor in $F(G)$ for G. From the argument above, $\tilde{A}(G)$ is an attractor in $F(G)$ for G. Let K be any attractor in $F(G)$ for G. It is easy to see that each attracting fixed point of any element of G in $F(G)$ belongs to the set K. It implies that $\tilde{A}(G) \subset K$.

Finally assume (h_n) is a sequence in G consisting of mutually disjoint elements and converges to a map ϕ in a subdomain V of $F(G)$. Then by Lemma 1.30 we see that ϕ is constant taking its value in $\tilde{A}(G)$. $\square$

By Theorem 1.34 and Lemma 1.29, we get the next theorem.

Theorem 1.35. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup which is semi-hyperbolic. Assume that there is an element $g \in G$ such that* $\deg(g) \geq 2$ *and each element of Aut $\overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic. If $F(G) \neq \emptyset$, then $\emptyset \neq A(G) = A_0(G) \subset F(G)$ and for each compact subset L of $F(G)$,*

$$\sup\{d(f_{i_n} \cdots f_{i_1}(z),\ A(G)) \mid z \in L,\ (i_n, \dots, i_1) \in \{1, \dots, m\}^n\} \to 0,$$

as $n \to \infty$, where we denote by d the spherical metric. Also $A(G)$ is the smallest attractor in $F(G)$ for G. Moreover we have that if (h_n) is a sequence in G consisting of mutually disjoint elements and converges to a map ϕ in a subdomain V of $F(G)$, then ϕ is constant taking its value in $\tilde{A}(G)$.

Theorem 1.36. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup which is sub-hyperbolic. Assume that there is an element $g \in G$ such that* $\deg(g) \geq 2$ *and each element of Aut $\overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic. If $F(G) \neq \emptyset$, then $\emptyset \neq \tilde{A}(G) = \tilde{A}_0(G) \subset F(G)$ and for each compact subset L of $F(G)$,*

$$\sup\{d(f_{i_n} \cdots f_{i_1}(z),\ \tilde{A}(G)) \mid z \in L,\ (i_n, \dots, i_1) \in \{1, \dots, m\}^n\} \to 0,$$

as $n \to \infty$, where we denote by d the spherical metric. Also $\tilde{A}(G)$ is the smallest attractor in $F(G)$ for G. Moreover we have that if (h_n) is a sequence in G consisting of mutually disjoint elements and converges to a map ϕ in a subdomain V of $F(G)$, then ϕ is constant taking its value in $\tilde{A}(G)$.

Proof. Since $\tilde{A}_0(G) \subset P(G)$ and G is sub-hyperbolic, we have that $\tilde{A}_0(G)$ is a compact subset of $F(G)$ and $\sharp(UH(G) \cap J(G)) < \infty$. Now let x be a point of $UH(G) \cap \partial J(G)$. Assume that there exists an element $h \in G$ such that $h(x) = x$. Since G is sub-hyperbolic, x is neither attracting nor indifferent fixed point of h. Since G is finitely generated, by [HM2], we have that there exists no superattracting fixed point of any element of G in $\partial J(G)$. Hence x is a repelling fixed point of h.

From Theorem 1.34, the statement of our theorem holds. $\square$

Proposition 1.37. *Let G be a finitely generated rational semigroup which contains an element with the degree at least two. Assume that $\sharp P(G) < \infty$ and $P(G) \subset J(G)$. Then $J(G) = \overline{\mathbb{C}}$.*

Proof. Suppose $F(G) \neq \emptyset$. Let $g \in G$ be an element with the degree at least two. By the assumption of our Proposition, g has a super attracting periodic point in $\partial J(G)$. On the other hand, since G is finitely generated, by [HM2], there exist no super attracting fixed points of any element of G in $\partial J(G)$. This is a contradiction. $\square$

Definition 1.38. Let M be a complex manifold. Suppose the map

$$(z,a) \in \overline{\mathbb{C}} \times M \mapsto f_{j,a}(z) \in \overline{\mathbb{C}}$$

is holomorphic for each $j = 1, \ldots, n$. We set $G_a = \langle f_{1,a}, \cdots, f_{n,a} \rangle$. Then we say that $\{G_a\}_{a \in M}$ is a holomorphic family of rational semigroups.

By Theorem 1.34 and Theorem 2.3.4 in [S1], we get the following result.

Corollary 1.39. *Let M be a complex manifold. Let $\{G_a\}_{a \in M}$ be a holomorphic family of rational semigroups where $G_a = \langle f_{1,a}, \cdots, f_{n,a} \rangle$. Let b be a point of M. We assume that G_b satisfies the assumption in Theorem 1.34. Then the map*

$$a \mapsto J(G_a)$$

is continuous at the point $a = b$ with respect to the Hausdorff metric.

Corollary 1.40. *Let M be a complex manifold. Let $\{G_a\}_{a \in M}$ be a holomorphic family of rational semigroups where $G_a = \langle f_{1,a}, \cdots, f_{n,a} \rangle$. Let b be a point of M. Assume that G_b contains an element of degree at least two and that each element of $Aut\, \overline{\mathbb{C}} \cap G_b$(if this is not empty) is loxodromic. If G_b is semi-hyperbolic or sub-hyperbolic, then the map*

$$a \mapsto J(G_a)$$

is continuous at the point $a = b$ with respect to the Hausdorff metric.

2 Rational Skew Product

Definition 2.1 (rational skew product). Let X be a topological space. If a continuous map $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ is represented by the following form:

$$\tilde{f}((x,y)) = (p(x),\ q_x(y)),$$

where $p : X \to X$ is a continuous map and $q_x : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational map with the degree at least 1 for each $x \in X$, then we say that $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ is a rational skew product. In this paper we always assume that X is a compact metric space.

O.Sester investigated polynomial skew products(in particular, quadratic case) in [Se]. M.Jonsson investigated dynamics on $\overline{\mathbb{C}}$-fibration whose fiber-wise maps are rational maps of degree $d,\ d \geq 2$ in [J2].

Definition 2.2. Let G be a rational semigroup generated by $\{f_\lambda\}_{\lambda \in \Lambda}$. Let $X = \Lambda^{\mathbb{N}}$. Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be the map defined by:

$$\tilde{f}((x, y)) = (p(x),\ f_{x_1}(y)),$$

where $p : X \to X$ is the shift map and $x \in X$ is represented by: $x = (x_1, x_2, \dots)$. Then we say that $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ is the rational skew product constructed by the generator system $\{f_\lambda\}_{\lambda \in \Lambda}$.

Definition 2.3. Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. For each $n \in \mathbb{N}$ and $x \in X$, we set $q_x^{(n)} := q_{p^{n-1}(x)} \circ \cdots \circ q_x$ and $\tilde{f}_x^n := \tilde{f}^n|_{\pi_X^{-1}(\{x\})}$. We define the following sets. For each $x \in X$,

$$d(x) = \deg(q_x),$$

$$F_x = \{y \in \overline{\mathbb{C}} \mid \{q_x^{(n)}\}_n \text{ is normal in a neigborhood of } y\},$$

$$J_x = \overline{\mathbb{C}} \setminus F_x,\ \tilde{J}_x = \{x\} \times J_x.$$

Further we set

$$\tilde{J}(\tilde{f}) = \overline{\bigcup_{x \in X} \tilde{J}_x},\ \tilde{F}(\tilde{f}) = (X \times \overline{\mathbb{C}}) \setminus \tilde{J}(\tilde{f}).$$

$$C(\tilde{f}) = \{(x, y) \in X \times \overline{\mathbb{C}} \mid q_x'(y) = 0\},\ P(\tilde{f}) = \overline{\bigcup_{n \in \mathbb{N}} \tilde{f}^n(C(\tilde{f}))}.$$

$C(\tilde{f})$ is called the critical set for $\tilde{f}$ and $P(\tilde{f})$ is called the post critical set for $\tilde{f}$. Moreover we set

$$(\tilde{f}^n)'((x, y)) = (q_x^{(n)})'(y).$$

If (x, y) is a period point of $\tilde{f}$ with the period n, then we say that (x, y) is repelling(resp. indifferent, attracting, etc.) if $|(\tilde{f}^n)'((x, y))| > 1$(resp.$=1,\ < 1$, etc.).

Lemma 2.4. *Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product represented by $\tilde{f}((x, y)) = (p(x),\ q_x(y))$. Then the following hold.*

1. *if $x \in X$, then $q_x^{-1}(F_{p(x)}) = F_x,\ q_x^{-1}(J_{p(x)}) = J_x,\ \tilde{f}(\tilde{J}(\tilde{f})) \subset \tilde{J}(\tilde{f})$.*
2. *if $p : X \to X$ is surjective, then $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ is surjective.*
3. *if $p : X \to X$ is a surjective and open map, then $\tilde{f}^{-1}(\tilde{J}(\tilde{f})) = \tilde{f}(\tilde{J}(\tilde{f})) = \tilde{J}(\tilde{f})$.*

Proof. We will show the last statement. Suppose $\tilde{f}((x,y)) \in \tilde{J}(\tilde{f})$. Then there exists a sequence $((x_i,\ y_i))$ converging to $\tilde{f}((x,y))$ such that $y_i \in F_{x_i}$ for each i. Since p is an open map, there exists a sequence $(\tilde{x}_i)$ converging to x such that $p(\tilde{x}_i) = x_i$. Then there exists a sequence $(\tilde{y}_i)$ converging to y such that $q_{\tilde{x}_i}(\tilde{y}_i) = y_i$ for each i. Then $\tilde{y}_i \in J_{\tilde{x}_i}$. Hence $(x,y) \in \tilde{J}(\tilde{f})$. Hence $\tilde{f}^{-1}(\tilde{J}(\tilde{f})) \subset \tilde{J}(\tilde{f})$. Since $\tilde{f}(\tilde{J})(\tilde{f}) \subset \tilde{J}(\tilde{f})$ and $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ is surjective, we have $\tilde{f}^{-1}(\tilde{J}(\tilde{f})) = \tilde{f}(\tilde{J}(\tilde{f})) = \tilde{J}(\tilde{f})$. □

Now we need some notations from [J2], concerning potential theoritic aspects. Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product represented by $\tilde{f}((x,y)) = (p(x),\ q_x(y))$. Let ω be the spherical probability measure on $\overline{\mathbb{C}}$. Let $\omega_x = (i_x)_*\omega$ for each $x \in X$ where we denote by $i_x : \overline{\mathbb{C}} \to \pi_X^{-1}(\{x\})$ the natural isomorphism. For each continuous function φ on $\pi_X^{-1}(\{x\})$ let $(\tilde{f}_x^n)^*\varphi$ be the continuous function on $\pi_X^{-1}(\{p^n(x)\})$ defined by $((\tilde{f}_x^n)^*\varphi)(z) = \sum_{\tilde{f}_x^n(w)=z} \varphi(w)$ for each $n \in \mathbb{N}$. Let $\mu_{x,n}$ be the probability measure on $\pi_X^{-1}(\{x\})$ defined by $\langle \mu_{x,n},\ \varphi \rangle = \frac{1}{\prod_{j=0}^{n-1} d(p^j(x))} \langle \omega_{p^n(x)},\ (\tilde{f}_x^n)^*\varphi \rangle$. For each $x \in X$, we denote by $R_x : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}^2 \setminus \{0\}$ the homogenious polynomial mapping of degree $d(x)$ such that $q_x \circ \pi' = \pi' \circ R_x$ where $\pi' : \mathbb{C}^2 \setminus \{0\} \to \overline{\mathbb{C}}$ is the natural projection and $\sup\{|R_x(z,w)| \mid |(z,w)| = 1\} = 1$. R_x is determined uniquely up to multiplication by a complex number of units. We can assume $x \mapsto R_x$ is continuous. For each $x \in X$ and $n \in \mathbb{N}$ let $G_{x,n} := \frac{1}{\prod_{j=0}^{n-1} d(p^j(x))} \log |R_x^n|$ where $R_x^n := R_{p^{n-1}(x)} \circ \cdots \circ R_x$. Then the following results hold.

Proposition 2.5. *Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product represented by $\tilde{f}((x,y)) = (p(x),\ q_x(y))$ and assume $d(x) \geq 2$ for each $x \in X$. Then we have the following.*

1. *$\mu_{x,n}$ converges to a probability measure μ_x on $\pi_X^{-1}(\{x\})$ weakly as $n \to \infty$ for each $x \in X$.*

2. *$G_{x,n}$ converges to a continuous plurisubharmonic function G_x locally uniformly on $\mathbb{C}^2 \setminus \{0\}$ as $n \to \infty$ for each $x \in X$.*

3. *$\mu_x = (i_x^{-1})_*(dd^c(G_x \circ s))$ where s is a local section of $\pi' : \mathbb{C}^2 \setminus \{0\} \to \overline{\mathbb{C}}$. Further $G_x(z,w) \leq \log |(z,w)| + O(1)$ as $|(z,w)| \to \infty$ and $G_x(\lambda z, \lambda w) = G_x(z,w) + \log \lambda$ for each $\lambda \in \mathbb{C}$, for each $x \in X$.*

4. *$G_{p(x)} \circ R_x = d(x) \cdot G_x$ for each $x \in X$.*

5. *if $x \to x'$ then $G_x \to G_{x'}$ uniformly on $\mathbb{C}^2 \setminus \{0\}$.*

6. *$(\tilde{f}_x)_*\mu_x = \mu_{p(x)},\ (\tilde{f}_x)^*\mu_{p(x)} = d(p(x)) \cdot \mu_x$ for each $x \in X$.*

7. *μ_x puts no mass on polar subsets of $\pi_X^{-1}(\{x\})$ for each $x \in X$.*

8. *$x \mapsto \mu_x$ is continuous with respct to the weak topology of measures in $X \times \overline{\mathbb{C}}$.*

9. *$supp(\mu_x) = \tilde{J}_x$ for each $x \in X$.*

10. *$\tilde{J}_x$ has no isolated points for each $x \in X$.*

11. *$x \mapsto \tilde{J}_x$ is lower semicontinuous with respect to the Hausdorff metric in the space of compact subsets of $X \times \overline{\mathbb{C}}$.*

Proof. Since $d(x) \geq 2$ for each $x \in X$, we can show the statements in the same way as that in section 3 in [J2]. $\square$

Definition 2.6 (hyperbolicity). Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. We say that $\tilde{f}$ is hyperbolic along fibres if $P(\tilde{f}) \subset \tilde{F}(\tilde{f})$.

Definition 2.7. Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. We say that $\tilde{f}$ is expanding along fibres if there exists a positive constant C and a constant λ with $\lambda > 1$ such that for each $n \in \mathbb{N}$,

$$\inf_{z \in \tilde{J}(\tilde{f})} \|(\tilde{f}^n)'(z)\| \geq C\lambda^n,$$

where we denote by $\| \cdot \|$ the norm of the derivative with respect to the spherical metric.

Definition 2.8 (semi-hyperbolicity). Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. Let N be a positive integer. We say that a point $(x_0, y_0) \in X \times \overline{\mathbb{C}}$ belongs to $SH_N(\tilde{f})$ if there exists a neighborhood U of x_0 and a positive number δ satisfying that for any $x \in U$, any $n \in \mathbb{N}$, any element $x_n \in p^{-n}(x)$ and any element V of $c(B(y_0, \delta), q_{x_n}^{(n)})$,

$$\deg(q_{x_n}^{(n)} : V \to B(y_0, \delta)) \leq N.$$

We set

$$UH(\tilde{f}) = (X \times \overline{\mathbb{C}}) \setminus \cup_{N \in \mathbb{N}} SH_N(\tilde{f}).$$

We say that $\tilde{f}$ is semi-hyperbolic along fibres if for any $(x_0, y_0) \in \tilde{J}(\tilde{f})$ there exists a positive integer N such that $(x_0, y_0) \in SH_N(\tilde{f})$.

Lemma 2.9. *Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. If $\tilde{f}$ is hyperbolic along fibres, then it is semi-hyperbolic along fibres.*

Lemma 2.10. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup. Then G is semi-hyperbolic if and only if the rational skew product $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ constructed by the generator system $\{f_1, f_2, \ldots, f_m\}$ is semi-hyperbolic along fibres. G is hyperbolic if and only if $\tilde{f}$ is hyperbolic along fibres.*

Definition 2.11 (Condition(C1)). Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. We say that $\tilde{f}$ satisfies the condition (C1) if there exists a family $\{D_x\}_{x \in X}$ of discs in $\overline{\mathbb{C}}$ such that the following three conditions are satisfied:

1. $\overline{\bigcup_{n \geq 0} \tilde{f}^n(\{x\} \times D_x)} \subset \tilde{F}(\tilde{f})$.
2. for any $x \in X$, we have that $\mathrm{diam}(q_x^{(n)}(D_x)) \to 0$, as $n \to \infty$.
3. $\inf_{x \in X}$ diam $(D_x) > 0$.

Now we will show the following lemma and theorem.

Lemma 2.12. *Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product satisfying the condition (C1). Assume that there exists a point $(x_0, y_0) \in X \times \overline{\mathbb{C}}$ with $y_0 \in F_{x_0}$, a connected open neighborhood U of y_0 in $\overline{\mathbb{C}}$ and a sequence (n_j) of positive integers such that $R_j := q_{x_0}^{(n_j)}$ converges to a non-constant map ϕ uniformly on U as $j \to \infty$. Let $(x_j, y_j) = \tilde{f}^{n_j}(x_0, y_0)$ and $(x_\infty, y_\infty) = \lim_{j \to \infty}(x_j, y_j)$. Let $S_{i,j} = q_{x_i}^{(n_j - n_i)}$ for $1 \leq i < j$. Let*

$$V = \{y \in \overline{\mathbb{C}} \mid \exists \epsilon > 0,\ \lim_{i \to \infty} \sup_{j > i} \sup_{d(\xi, y) \leq \epsilon} d(S_{i,j}(\xi),\ \xi) = 0\}.$$

Then V is a non-empty open set and for any $y \in \partial V$, we have that

$$(x_\infty,\ y) \in \tilde{J}(\tilde{f}) \cap UH(\tilde{f}). \tag{4}$$

Theorem 2.13. *Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. Assume $\tilde{f}$ is semi-hyperbolic along fibres and satisfies the condition (C1). Then the following hold.*

1. *Let $(x_0, y_0) \in X \times \overline{\mathbb{C}}$ be any point with $y_0 \in F_{x_0}$. Then for any open connected neighborhood U of y_0 in $\overline{\mathbb{C}}$, there exists no subsequence of $(q_{x_0}^{(n)})_n$ converging to a non-constant map locally uniformly on U.*
2. $$\tilde{J}(\tilde{f}) = \bigcup_{x \in X} \tilde{J}_x.$$
3. *If there exists a disc D in $\overline{\mathbb{C}}$ such that $D_x = D$ for all $x \in X$ in the condition (C1), then there exist positive constants δ, L and $\lambda (0 < \lambda < 1)$ such that for any $n \in \mathbb{N}$,*

   $$\sup\{\mathit{diam}\ U \mid U \in c(B(y, \delta),\ q_{x_n}^{(n)}),\ (x, y) \in \tilde{J}(\tilde{f}),\ x_n \in p^{-n}(x)\} \leq L\lambda^n.$$
4. *Assume $d(x) \geq 2$ for each $x \in X$. Then we have that $x \mapsto \tilde{J}_x$ is continuous with respect to the Hausdorff metric in the space of compact subsets of $X \times \overline{\mathbb{C}}$.*

5. *Assume* $d(x) \geq 2$ *for each* $x \in X$. *Then for any compact subset* K *of* $\tilde{F}(\tilde{f})$, *we have that* $\overline{\cup_{n\geq 0}\tilde{f}^n(K)} \subset \tilde{F}(\tilde{f})$ *and there exist constants* $C > 0$ *and* $\tau < 1$ *such that for each* n, $\sup_{z\in K} \|(\tilde{f}^n)'(z)\| \leq C\tau^n$.

To show Lemma 2.12 and Theorem 2.13, we need the following lemma.

Lemma 2.14. *Let* $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ *be a rational skew product satisfying the condition (C1). Assume* $(x_0, y_0) \in SH_N(\tilde{f})$ *for some* $N \in \mathbb{N}$. *Then there exists a positive number* δ_0 *such that for each* δ *with* $0 < \delta < \delta_0$ *there exists a neighborhood* U *of* x_0 *in* $\overline{\mathbb{C}}$ *satisfying that for each* $n \in \mathbb{N}$, *each* $x \in U$ *and each* $x_n \in p^{-n}(x)$, *we have that each element of* $c(B(y_0, \delta),\ q_{x_n}^{(n)})$ *is simply connected.*

Proof. Take a positive number δ_1 such that for each $x \in X$ and each $x_1 \in p^{-1}(x)$, we have that each connected component of $q_{x_1}^{-1}(D_x)$ contains a ball with the radius at least δ_1.

By the semi-hyperbolicity and Lemma 1.10, we can take a positive number δ_0 and a open neighborhood U of x_0 in $\overline{\mathbb{C}}$ such that for each δ with $0 < \delta < \delta_0$, each $x \in U$, each $n \in \mathbb{N}$, and each $x_n \in p^{-n}(x)$, we have that the diameter of each element of $c(B(y_0, \delta),\ q_{x_n}^{(n)})$ is less than δ_1.

Now we will show each element of $c(B(y_0, \delta),\ q_{x_n}^{(n)})$ is simply connected by induction on n. Assume an element W of $c(B(y_0, \delta),\ q_{x_n}^{(n)})$ is simply connected. Let W_1 be a connected component of $q_{x_{n+1}}^{-1}(W)$ where x_{n+1} is an element of $p^{-1}(x_n)$. Suppose W_1 is not simply connected. Each connected component of ∂W_1 is mapped onto ∂W by $q_{x_{n+1}}$. Hence the image of each connected component of $\overline{\mathbb{C}} \setminus W_1$ by $q_{x_{n+1}}$ contains D_{x_n}. Hence we have that diam $W_1 \geq \delta_1$, which contradicts to the choice of δ_0 and U. $\square$

Now we will show the Lemma 2.12.

Proof. We will show the statement developing a method in M.Jonsson's Thesis([J1]). By the definition, V is an open set. Since ϕ is non-constant, there exists a positive number a such that

$$R_j(U) \supset B(y_\infty,\ a)$$

for each $j \in \mathbb{N}$. We have that $B(y_\infty,\ a) \subset V$. For, if $y \in B(y_\infty,\ a)$ then $y = R_i(\xi_i)$ for some $\xi_i \in U$ and so $d(S_{i,j}(y),\ y) = d(R_j(\xi_i),\ R_i(\xi_i))$ which is small if i is large. Hence V is a non-empty open set.

Take any $y \in \partial V$. We will show

$$(x_\infty,\ y) \in \tilde{J}(\tilde{f}). \tag{5}$$

Assume this is false. If there exists a positive integer i_0 such that $\{S_{i,j}\}_{j\geq i\geq i_0}$ is normal in a neighborhood of y, then since $S_{i,j}\to Id$ on $V\cap W$, we have that $W\subset V$ and it is a contradiction. Hence there exist sequences (i_k), (j_k) and (ξ_k) such that $i_k\leq j_k$, $j_k-i_k\to\infty$, $\xi_k\to y$ and

$$S_{i_k,\ j_k}(\xi_k)\in D_{x_{i_k}}, \tag{6}$$

where we denote by $(D_x)_{x\in X}$ a family of discs in $\overline{\mathbb{C}}$ in the definition of condition (C1). Since we are assuming $(x_\infty,\ y)\in\tilde{F}(\tilde{f})$, we have that there exists an disc B around y such that $B\subset F_{x_{i_k}}$ and $\xi_k\in B$ for large k. By (6), the condition (C1) and the definition of V, we get a contradiction. Hence (5) holds.

Now we will show $(x_\infty,y)\in UH(\tilde{f})$. Suppose this is false. Then there exists a positive integer N such that $(x_\infty,y)\in SH_N(\tilde{f})$. Let δ_0 be a number for (x_∞,y) in Lemma 2.14 and let $\delta=\delta_0/2$. We can assume that there exists a neighborhood U' of x_∞ satisfying that for any $x\in U'$, any $n\in\mathbb{N}$, any element $x_n\in p^{-n}(x)$ and any element V of $c(B(y,\ \delta_0),\ q_{x_n}^{(n)})$,

$$\deg(q_{x_n}^{(n)}:V\to B(y,\ \delta_0))\leq N.$$

Take two domains V_1 and V_2 such that

$$y_\infty\in V_2\subset\subset V_1\subset\subset V,\ B(y,\ \delta)\cap V_2\neq\emptyset. \tag{7}$$

If $j>i$ and i is large enough, then $S_{i,j}$ is close to id_{V_1} on V_1. Hence $S_{i,j}$ is biholomorphic on V_1 and $S_{i,j}(V_1)\supset V_2$. Let $h_{i,j}:V_2\to V_1$ be a map such that $S_{i,j}\circ h_{i,j}=id$ on V_2 and $h_{i,j}\circ S_{i,j}=id$ on $S_{i,j}^{-1}(V_2)\cap V_1$. Then we have that

$$\lim_{i\to\infty}\sup_{j>i}\sup_{\xi\in V_2}d(h_{i,j}(\xi),\ \xi)=0. \tag{8}$$

For each (i,j) such that $j>i$ and i is large enough, let $B_{i,j}\in c(B(y,\delta),\ S_{i,j})$ be an element such that $h_{i,j}(V_2\cap B(y,\delta))\subset B_{i,j}$. By the choice of δ_0, we have that $B_{i,j}$ is simply connected. By semi-hyperbolicity, there exists a positive integer M such that for each (i,j) with $j>i$ where i is large enough,

$$\sharp(\ \mathrm{cv}(S_{i,j}|_{B_{i,j}})\cap B(y,\delta))\leq M, \tag{9}$$

where we denote by "cv" the set of critical values. Hence there exists a positive number θ with $0<\theta<2\pi$ such that for each (i,j) with $i<j$ there exists a sector $U_{i,j}$ in $B(y,\delta)$ of angle θ with the center y such that $U_{i,j}\cap\ \mathrm{cv}(S_{i,j}|_{B_{i,j}})=\emptyset$ and $U_{i,j}\cap V_2\neq\emptyset$. Let $g_{i,j}:U_{i,j}\to B_{i,j}$ be the analytic continuation of $h_{i,j}$ on $V_2\cap U_{i,j}$. Let $y_{i,j}\in B_{i,j}\cap\overline{g_{i,j}(U_{i,j})}$ such that $S_{i,j}(y_{i,j}))=y$. By (8), Corollary 1.9, Condition (C1) and the fact $V_2\cap U_{i,j}\neq\emptyset$, we have that there exists a positive number δ_1 such that

$$B(y_{i,j},\ \delta_1)\subset B_{i,j}, \tag{10}$$

for each (i,j) such that $j > i$ and i is large enough. Now we will show the following claim:

$$\text{Claim: } \lim_{i\to\infty} \sup_{j>i} d(y_{i,j},\ y) = 0. \tag{11}$$

Suppose this is false. Then there exists a sequence $((i_k, j_k))$ with $j_k > i_k$ and a positive number δ_2 such that $d(y_{i_k,j_k},\ y) > \delta_2$, for each k. We can assume that (y_{i_k,j_k}) converges to a point $\tilde{y}$ as $k \to \infty$ and that there exists a sector U_0 with the center y such that $U_{i_k,j_k} = U_0$ for each k. Then we have that

$$B(\tilde{y},\ \delta_1/2) \cap V = \emptyset. \tag{12}$$

For, assume the left hand side is not empty. Then since $S_{i_k,j_k} \to id$ in$V$ and the family $\{S_{i_k,j_k}|_{B(\tilde{y},\delta_1/2)}\}_k$ is normal, we have that $S_{i_k,j_k} \to id$ locally uniformly on $B(\tilde{y}, \delta_1/2)$. But this is a contradiction because $\tilde{y} \neq y$. Hence we have (12).

By Lemma 1.10, there exists a positive number δ_3 with $\delta_3 < \delta_1/4$ such that for each k, the diameter of each element of $c(B(y,\delta_3),\ S_{i_k,j_k})$ is less than δ_1. Hence if we take a fixed point $z \in B(y,\delta_3) \cap U_0$, then we have that for each large k,

$$d(\tilde{y},\ g_{i_k,j_k}(z)) < \delta_1/4. \tag{13}$$

On the other hand, since $g_{i_k,j_k} \to id$ locally uniformly on V_2 and $(g_{i_k,j_k})_k$ is normal in U_0, we have that $g_{i_k,j_k} \to id$ locally uniformly on U_0. Hence we have that $d(g_{i_k,j_k}(z),\ y) < \delta_1/4$ for each large k. Together with (13) and $B(\tilde{y},\ \delta_1/2) \cap V = \emptyset$, we get a contradiction. Hence we have shown the claim (11).

Since $B(y_{i,j},\ \delta_1) \subset B_{i,j}$ for each (i,j) such that $j > i$ and i is large enough, by the above claim we have that there exists a positive integer i_0 such that for each (i,j) with $j > i \geq i_0$,

$$S_{i,j}(B(y,\ \delta_1/2)) \subset B(y,\ \delta).$$

Hence $(S_{i,j})_{j>i\leq i_0}$ is normal in $B(y,\ \delta_1/2)$. Since $S_{i,j} \to id$ on $B(y,\ \delta_1/2) \cap V$, we have that $y \in V$ and this is a contradiction. Hence we have shown the first statement of our lemma. $\square$

Now we will show Theorem 2.13.

Proof. The statement 1 follows from Lemma 2.12.

Now we will show the statement 2 of our theorem. Suppose the statement is false. Then there exists a point $(x_0, y_0) \in \tilde{J}(\tilde{f})$ with $y_0 \in F_{x_0}$, a connected component U of y_0 in $\overline{\mathbb{C}}$ and a sequence (n_j) of positive integers such that $R_j := q_{x_0}^{(n_j)}$ converges to a map ϕ uniformly on U as $j \to \infty$. Let $(x_j, y_j) = \tilde{f}^{n_j}(x_0, y_0)$ and $(x_\infty, y_\infty) = \lim_{j\to\infty}(x_j, y_j)$. By Lemma 1.10, there exists a positive number a such that $y_j \in B(y_\infty,\ a)$ and the element $B_j \in c(B(y_\infty,\ a),\ R_j)$ containing y_j satisfies that $B_j \subset U$ for each large j. Hence $R_j(U) \supset B(y_\infty,\ a)$ for each large j and it implies that ϕ is non-constant. By Lemma 2.12, it is a contradiction. Hence we have shown the statement 2 of our theorem.

Next we will show the statement 3 of our theorem. By semi-hyperbolicity and Lemma 2.14, there exists a positive integer N and a positive real number δ_0 such that for any $(x', y') \in \tilde{J}(\tilde{f})$ there exists a neighborhood U' of x' satisfying that for any real number τ with $0 < \tau \leq \delta_0$, any $x \in U'$, any $n \in \mathbb{N}$, any element $x_n \in p^{-n}(x)$ and any element V of $c(B(y', \tau),\ q_{x_n}^{(n)})$, we have that V is simply connected and

$$\deg(q_{x_n}^{(n)} : V \to B(y',\ \tau)) \leq N.$$

Let $\delta = \delta_0/2$. We set

$$A_n = \sup\{\operatorname{diam} U \mid U \in c(B(y, \delta),\ q_{x_n}^{(n)}),\ (x, y) \in \tilde{J}(\tilde{f}),\ x_n \in p^{-n}(x)\}.$$

First we will show

$$A_n \to 0, \quad \text{as } n \to \infty. \tag{14}$$

Suppose this is false. Then there exists a positive constant C, a sequence $((x^k,\ y^k))$ of points in $\tilde{J}(\tilde{f})$, a sequence $((\tilde{x}^k,\ \tilde{y}^k))$ with $\tilde{f}^{n_k}((\tilde{x}^k,\ \tilde{y}^k)) = (x^k,\ y^k)$ for some $n_k \in \mathbb{N}, \to \infty$ and a sequence $(U_k)_k$ with $U_k \in c(B(y_k,\ \delta),\ q_{\tilde{x}^k}^{(n_k)})$ and $\tilde{y}^k \in U_k$ for each k such that

$$\operatorname{diam} U_k \geq C, \quad \text{for each } k.$$

We can assume that $((x^k,\ y^k))$ tends to a point $(x^0,\ y^0) \in \tilde{J}(\tilde{f})$ and that $((\tilde{x}^k,\ \tilde{y}^k))$ tends to a point $(\tilde{x}^0,\ \tilde{y}^0) \in \tilde{J}(\tilde{f})$. By Corollary 1.9, there exists a positive number r such that $B(\tilde{y}^0,\ r) \subset U_k$ for each large k. Hence

$$q_{\tilde{x}^k}^{(n_k)}(B(\tilde{y}^0,\ r)) \subset B(y^0,\ r), \tag{15}$$

for each large k. By the second statement of our theorem, we have that $\tilde{y}^0 \in J_{\tilde{x}^0}$. Hence there exists a positive integer j and a point $z \in B(\tilde{y}^0,\ r)$ such that

$$q_{\tilde{x}^0}^{(j)}(z) \in D.$$

Hence $q_{\tilde{x}^k}^{(j)}(z) \in D$ for each large k. On the other hand by the condition (C1) if we take δ_0 so small then we can assume

$$\overline{\bigcup_{n\geq 0} \tilde{f}^n(\{x\} \times D)} \cap (\{y^0\} \times B(y^0,\ \delta_0)) = \emptyset.$$

Since $n_k \to \infty$, by (15) we get a contradiction. Hence we have (14).

Take a positive integer n_0 such that for each $n \in \mathbb{N}$ with $n \geq n_0$,

$$A_n \leq \delta/2. \tag{16}$$

Fix any positive integer k. Let $((x_n, y_n))$ be a sequence such that $\tilde{f}((x_{n+1}, y_{n+1})) = (x_n, y_n)$ for each n and $(x_0, y_0) \in \tilde{J}(\tilde{f})$. For each $j = 0, \ldots, k$, let W_j be the element of $c(B(y_{(k-j)n_0}, \ \delta), \ q^{(jn_0)}_{x_{kn_0}})$ containing y_{kn_0}. By (16) we have that

$$W_0 \supset \cdots \supset W_k. \tag{17}$$

For each $j = 1, \ldots, k$,

$$q^{(jn_0)}_{x_{kn_0}} : W_j \to B(y_{(k-j)n_0}, \ \delta)$$

is a proper holomorphic map with the degree at most N. Since $q^{(jn_0)}_{x_{kn_0}}(W_{j+1})$ is a connected component of $(q^{(n_0)}_{x_{(k-j)n_0}})^{-1}(B(y_{(k-j-1)n_0}, \ \delta))$, which is included in $B(y_{(k-j)n_0}, \ \delta/2)$ by (16), we have that for each $j = 0, \ldots, k-1$,

$$\text{mod}\ (\overline{W_{j+1}}, \ W_j) \geq c > 0, \tag{18}$$

where c is a constant number depending only on N. By Lemma 1.11, there exists a λ with $0 < \lambda < 1$ depending only on N such that

$$\text{diam}\ W_{j+1}/\text{diam}\ W_j \leq \lambda, \ \text{for each}\ j = 0, \ldots, k-1.$$

Hence we get that diam $W_k \leq \lambda^k$ diam $B(y_0, \ \delta)$. Therefore the statement 3 of our theorem holds.

The statement 4 of our theorem follows from the statement 2 of Theorem 2.13 and 11 in Proposition 2.5.

Next we will show the statement 5 of our theorem. Let K be a compact subset of $\tilde{F}(\tilde{f})$. Suppose there exists a sequence (x_k, y_k) of points in K and a sequence (n_k) of positive integers such that $(\tilde{f}^{n_k}((x_k, y_k)))$ converges to a point $(x', y') \in \tilde{J}(\tilde{f})$. By 11 in Proposition 2.5, we have that $x \mapsto \tilde{J}_x$ is lower semi-continuous. Hence we have that for each k there exists a point $z_k \in J_{p^{n_k}(x_k)}$ and (z_k) converges to y'. Since $K \cap \tilde{J}(\tilde{f}) = \emptyset$, by the condition (C1) and Lemma 1.10 we get a contradiction. Hence we get that $\overline{\cup_{n \geq 0} \tilde{f}^n(K)} \subset \tilde{F}(\tilde{f})$. Let $K' = \overline{\cup_{n \geq 0} \tilde{f}^n(K)}$. By the statement 1 of Theorem 2.13, we have that for each $z = (x, y) \in K'$, there exists a neighoborhoood $U(z)$ of x in X, a neighborhood $V(z)$ of y in $\overline{\mathbb{C}}$ and a positive integer $n(z)$ such that $\|(\tilde{f}^{(n(z))})'(z')\| < 1/2$ for each $z' \in U(z) \times V(z)$. Since K' is a compact set, we get the statement . $\square$

Corollary 2.15. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup which is semi-hyperbolic. Assume G contains an element with* *the degree at least two and each element of* $Aut\,\overline{\mathbb{C}} \cap G$*(if this is not empty) is loxodromic. Also assume* $F(G) \neq \emptyset$*. Then there exists a* $\delta > 0$*, a constant* L *with* $L > 0$ *and a constant* λ *with* $0 < \lambda < 1$ *such that*

$$\sup\{diam\ U \mid U \in c(B(x,\delta),\ f_{i_n}\cdots f_{i_1}),\ x \in J(G),\ (i_1,\dots,i_n) \in \{1,\dots,m\}^n\}$$

$$\leq L\lambda^n, \quad \textit{for each } n.$$

Proof. Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be the rational skew product constructed by the generator system $\{f_1, f_2, \dots, f_m\}$. Then this is semi-hyperbolic along fibres. By the existence of an attractor in $F(G)$ for G (Theorem 1.35) we have that if we set $D_x = D$ for each $x \in X$ where D is a small disc around a point of the attractor, then $\tilde{f}$ satisfies the condition (C1) with that family of discs. By Theorem 2.13, the statement of our Corollary holds.

□

Theorem 2.16. *Let* $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ *be a rational skew product. Assume* $\tilde{f}$ *is hyperbolic along fibres and satisfies the condition (C1) with a family of discs* $(D_x)_{x\in X}$ *such that there exists a disc* D *satisfying* $D_x = D$ *for all* $x \in X$*. Then* $\tilde{f}$ *is expanding along fibres.*

Proof. We have only to show that there exists a positive integer n_0 such that for each $n \in \mathbb{N}$ with $n \geq n_0$ and $z \in \tilde{J}(\tilde{f})$,

$$\|(\tilde{f}^n)'(z)\| \geq 2.$$

Suppose this is false. Then there exists a sequence (n_j) of positive integers and a sequence $(z_j) = ((x_j, y_j))$ in $\tilde{J}(\tilde{f})$ such that

$$\|(\tilde{f}^{n_j})'(z_j)\| \leq 2. \tag{19}$$

We can assume that $\tilde{f}^{n_j}(z_j)$ converges to a point $(x, y) \in \tilde{J}(\tilde{f})$. Let δ be a small positive number. For each j let $B_j \in c(B(y,\delta),\ q_{x_j}^{(n_j)})$ be the element containing y_j. By (19) and Koebe distortion theorem, there exists a positive constant c such that for each j, diam $B_j \geq c$. But this contradicts to the statement 3 of Theorem 2.13. □

Remark 5. We can show that the results in this section are generalized to the version of dynamics on $\overline{\mathbb{C}}$-fibration. For the definition of $\overline{\mathbb{C}}$-fibration, see [J2].

3 Conditions to be semi-hyperbolic

Theorem 3.1. *Let* $G = \langle f_1, f_2, \dots, f_m \rangle$ *be a finitely generated rational semigroup. Let* $z_0 \in J(G)$ *be a point. Assume all of the following conditions:*

1. *there exists a neighborhood U_1 of z_0 in $\overline{\mathbb{C}}$ such that for any sequence $(g_n) \subset G$, any domain V in $\overline{\mathbb{C}}$ and any point $\zeta \in U_1$, we have that the sequence (g_n) does NOT converge to ζ locally uniformly on V.*

2. *there exists a neighborhood U_2 of z_0 in $\overline{\mathbb{C}}$ and a positive real number $\tilde{\epsilon}$ such that if we set*

$$T = \{c \in \overline{\mathbb{C}} \mid \exists j,\ f_j'(c) = 0,\ (G \cup \{id\})(f_j(c)) \cap U_2 \neq \emptyset\}$$

then for each $c \in T \cap C(f_j)$, we have $d(c,\ (G \cup \{id\})(f_j(c))) > \tilde{\epsilon}$.

3. $F(G) \neq \emptyset$.

Then $z_0 \in SH_N(G)$ for some $N \in \mathbb{N}$.

Notation: For any family $\{g_\lambda\}_{\lambda \in \Lambda}$ of rational functions, we denote by $F(\{g_\lambda\})$ the set of all points $z \in \overline{\mathbb{C}}$ such that z has a neighborhood where the family $\{g_\lambda\}$ is normal. We set $J(\{g_\lambda\}) = \overline{\mathbb{C}} \setminus F(\{g_\lambda\})$. $F(\{g_\lambda\})$ is called the Fatou set and $J(\{g_\lambda\})$ is called the Julia set for the family.

Corollary 3.2. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. Let $z_0 \in J(G)$ be a point. Assume all of the following conditions:*

1. *there exists a neighborhood U_1 of z_0 in $\overline{\mathbb{C}}$ such that for any sequence $(g_n) \subset G$ consisting of mutually distinct elements and any domain V in $F((g_n))$, there exists a point $x \in V$ such that the sequence $\overline{\cup_n \{g_n(x)\}} \cap \overline{\mathbb{C}} \setminus U_1 \neq \emptyset$.*

2. *there exists a neighborhood U_2 of z_0 in $\overline{\mathbb{C}}$ and a positive real number $\tilde{\epsilon}$ such that if we set*

$$T = \{c \in \overline{\mathbb{C}} \mid \exists j,\ f_j'(c) = 0,\ (G \cup \{id\})(f_j(c)) \cap U_2 \neq \emptyset\}$$

then for each $c \in T \cap C(f_j)$, we have $d(c,\ (G \cup \{id\})(f_j(c))) > \tilde{\epsilon}$.

3. $F(G) \neq \emptyset$.

Then $z_0 \in SH_N(G)$ for some $N \in \mathbb{N}$ and there exists a neighborhood W of z_0 in $\overline{\mathbb{C}}$ such that for any sequence $(g_n) \subset G$ consisting of mutually distinct elements, we have

$$\sup\{\ diam\ S \mid S \in c(W, g_n)\} \to 0,\ as\ n \to \infty.$$

We will consider the proof of Theorem 3.1. We may assume $U_1 = U_2 = U$ for some small disc U . By condition 1 and 3, we may assume $\infty \in F(G)$ and $g^{-1}(U) \subset \mathbb{C}$ for each $g \in G$. Now we will show the above theorem by developing a lemma in [Ma] and using the methods in [KS]. The stories are almost same as those in [KS], except some modifications.

First we need some new notations. An “square” is a set S of the form

$$S = \{z \in \mathbb{C} \mid |\Re(z-p)| < \delta,\ |\Im(z-p)| < \delta\}.$$

The point p is called the center of S and δ its radius . For each $k > 0$, given a square S with center p and radius δ, we denote by S^k the square with the center p and radius $k\delta$. Take a $\sigma > 0$ such that U contains a closed square Q' with the center a point in U and its radius 2σ. Let $Q'' = (Q')^{1/2}$. Q'' is called “admissable square at level 1.” We will define asmissable squares at level n for each $n \in \mathbb{N}$. Let Q be an admissable square at level n with the radius a. Then Q is covered by 16 squares with the radius $a/8$. We have 20 squares with the radius $a/8$ adjacent to Q. We call all these 36 squares admissable at level $n+1$. These squares are denoted by $\{Q_{\mu,n+1}\}$. The union of these 36 squares is denoted by $\tilde{Q}$, which is called the “square attached to Q.” Each admissable and each attached square is a relative compact subset of U.

Notation: For any open set V_1 and for any rational map g, if $V_2 \in c(V_1,\ g)$ then we set $\triangle(V_1,\ g) = \sharp\{x \in V_1 \mid g'(x) = 0\}$, counting the multiplicity.

We need some lemmas to show Theorem 3.1.

Lemma 3.3. *For given $\epsilon > 0$ and $N \in \mathbb{N}$, there exists some $n_0 \in \mathbb{N}$ such that the following holds: If Q is an admissable square at some level $n \geq n_0$, $\tilde{Q}$ the corresponding attached square, V an element of $c(\tilde{Q},\ f)$ for some $f \in G$, and $\triangle(V, f) \leq N$, then diam $(K) \leq \epsilon$ for each element $K \in c(Q, f)$ contained in V.*

Proof. Fix $\epsilon > 0$ and $N \in \mathbb{N}$. If the lemma is false then there exists a sequence $(n_k)_{k\in\mathbb{N}}$ converging to ∞, admissable squares Q_{μ_k,n_k} and functions $g_k \in G$ such that diam $(K_k) \geq \epsilon > 0$ and $\triangle(V_k,\ g_k) \leq N$ for some element $V_k \in c(\tilde{Q}_{\mu_k,n_k},\ g_k)$ and some element $K_k \in c(Q_{\mu_k,n_k},\ g_k)$ contained in V_k. Take $\gamma > 1$ such that $\gamma^{N+1} = \frac{3}{2}$. Then there exists $0 < j \leq N+1$ such that, denoting by $\hat{R}_k = Q^{\gamma^j}_{\mu_k,n_k}$, the set $\hat{R}_k^{\gamma} - \hat{R}_k$ does not contain any critical values of $g_k|_{V_k}$. We have $\hat{R}_k \supset Q_{\mu_k,n_k}$. Take $\hat{K}_k \in c(\hat{R}_k,\ g_k)$ such that $K_k \subset \hat{K}_k \subset V_k$. Then diam $\hat{K}_k > \epsilon$. The element g_k is represented by the following form: $g_k = f_{s_l} \circ \cdots \circ f_{s_1}$. Then there exists a positive integer i such that

$$\text{diam } f_{s_i} \circ \cdots \circ f_{s_1}(\hat{K}_k) > \epsilon,\ \text{diam } f_{s_{i+1}} \circ \cdots \circ f_{s_1}(\hat{K}_k) \leq \epsilon.$$

We may assume i is the largest one satisfying the above. Then taking $\epsilon > 0$ small enough, since the cardinality of generator system of G is finite we see that $f_{s_i} \circ \cdots \circ f_{s_1}(\hat{K}_k)$ is simply connected. Set $\tilde{K}_k = f_{s_i} \circ \cdots \circ f_{s_1}(\hat{K}_k)$ and $\tilde{g}_k = f_{s_l} \circ \cdots \circ f_{s_{i+1}}$. Since $\hat{R}_k^{\gamma} - \hat{R}_k$ does not contain any critical values of $g_k|_{V_k}$, the element $\tilde{K}_k^{+} \in c(\hat{R}_k^{\gamma},\ \tilde{g}_k)$ containing $\tilde{K}_k$ is also simply connected.

Since $\deg(\tilde{g}_k|_{\tilde{K}_k^+}) \leq 1+N$ and diam $\tilde{K}_k > \epsilon$, by Corollary 1.9 we see that there exists a positive real number r such that for each k,

$$B(z_k,\ r) \subset \tilde{K}_k,$$

for some $z_k \in \overline{\mathbb{C}}$. We can assume that (z_k) converges to a point $z \in \overline{\mathbb{C}}$ and $(\hat{R}_k)$ converges to a point $y \in U$. Then $(\tilde{g}_k)$ is normal in $B(z,\ r/2)$ and we can assume that $(\tilde{g}_k)$ converges to y locally uniformly on $B(z,\ r/2)$. But this contradicts to the assumption 1. Hence the lemma holds. $\square$

Now, let $t = \sharp T,\ N = (\max_{j=1,\dots,m} \deg(f_j))^t$ and $\epsilon < \frac{\tilde{\epsilon}}{36N}$. We can assume that $\tilde{\epsilon}$ is sufficiently small and diam $U \leq \tilde{\epsilon}$. Let $n_0 \in \mathbb{N}$ be an integer in Lemma 3.3 for these ϵ and N.

Lemma 3.4. *Let G be an element of the form $f = f_{w_1} \circ \cdots \circ f_{w_k}$. Let B be a simply connected subdomain of U, $B' \in c(B, f)$ an element such that $\triangle(B', f) > N$. Then there exists some $\nu \in \{0, \dots, k-1\}$ such that if we set $B_\nu = f_{w_{k-\nu}} \circ \cdots \circ f_{w_k}(B')$, then B_ν is simply connected, diam $(B_\nu) \geq \tilde{\epsilon}$, and*

$$\deg(f_{w_1} \circ \cdots f_{w_{k-\nu-1}}|B_\nu : B_\nu \to B) \leq N.$$

Proof. Suppose diam $B_\nu < \tilde{\epsilon}$ for each $\nu = 1,\ \dots k-1$, then B' is simply connected(Note that we can assume $\tilde{\epsilon}$ is sufficiently small) and $\deg(f|_{B'} : B' \to B) \leq N$. Hence $\triangle(B', f) \leq N$ and it is a contradiction. Hence there exists a $\nu \in \{1, \dots, k-1\}$ such that

$$\text{diam } B_\nu \geq \tilde{\epsilon}.$$

Take the maximal $\nu (1 \leq \nu \leq k-1)$ satisfying the above. Then B_ν is simply connected and

$$\deg(f_{w_1} \circ \cdots \circ f_{w_{k-\nu-1}}|_{B_\nu} : B_\nu \to B) \leq N.$$

$\square$

Now we will show the Theorem 3.1.

Proof. Take $\tilde{\epsilon}$, ϵ and N as before. Take n_0 in Lemma 3.3 for ϵ and N. Let k be the smallest integer such that there exists some admissable square $Q = Q_{\mu,n}$ at level $n \geq n_0$ with diam $(K) > \epsilon$ for some element K of $c(Q,\ f_{w_1} \circ \cdots \circ f_{w_k})$ where $(w_1, \dots, w_k)$ is some word of length k. We have $k \geq 1$. Let $\tilde{Q}$ be the square attached to Q. By lemma 3.3, there exists an element $S \in c(\tilde{Q},\ f_{w_1} \circ \cdots \circ f_{w_k})$ such that $\triangle(S,\ f_{w_1} \circ \cdots \circ f_{w_k}) > N$. Take an integer ν with $1 \leq \nu < k$ in Lemma 3.4. Then we have

$$\text{diam } (f_{w_{k-\nu}} \circ \cdots \circ f_{w_k}(S)) > \tilde{\epsilon}.$$

If we set $\tilde{S} = f_{w_{k-\nu}} \circ \cdots \circ f_{w_k}(S)$ then

$$\deg(f_{w_1} \circ \cdots \circ f_{w_{k-\nu-1}}|_{\tilde{S}}) \le N$$

and

$$\tilde{S} \subset \bigcup_{\mu} (f_{w_1} \circ \cdots \circ f_{w_{k-\nu-1}})^{-1}(Q_{\mu,\ n+1}).$$

By the minimality of k, we have that the diameter of each element of $c(Q_{\mu,\ n+1},\ f_{w_1} \circ \cdots \circ f_{w_{k-\nu-1}})$ is less than ϵ. Since $\deg(f_{w_1} \circ \cdots \circ f_{w_{k-\nu-1}}|_{\tilde{S}}) \le N$, we have that

$$\tilde{\epsilon} < \text{ diam } \tilde{S} \le 36N\epsilon.$$

This contradicts to $\epsilon < \frac{\tilde{\epsilon}}{36N}$. Hence we have proved that for each admissable square $Q_{\mu,n}$ with $n \ge n_0$ and each $g \in G$, each element $K \in c(Q_{\mu,n},\ g)$ satisfies that diam $(K) < \epsilon$. Since ϵ is sufficiently small, K is simply connected. By Lemma 3.4, we have that

$$\deg(f|_K : K \to Q_{\mu,n}) \le N + 1.$$

Hence $z_0 \in SH_{N+1}$. □

Now we will show the Corollary 3.2.

Proof. If we assume the conditions in the assumption of Corollary 3.2, then clearly the conditions in the assumption of Theorem 3.1 are satisfied. Hence we have $z_0 \in SH_N(G)$ for some $N \in \mathbb{N}$. Now take a small disc W around z_0 contained in $SH_N(G) \cap U_2$. If there exists a constant $C > 0$, a sequence $(g_n) \subset G$ consisting of mutually distinct elements and a sequence (W_n) with $W_n \in c(W, g_n)$ such that diam $W_n > C$ for each n, then by Corollary 1.9, there exists a positive real number r such that for each n, we have $B(z_n,\ r) \subset W_n$ for some $z_n \in \overline{\mathbb{C}}$. We can assume (z_n) converges to a point $y \in \overline{\mathbb{C}}$. Then (g_n) is normal in $B(y,\ r/2)$. Since $g_n(B(y,\ r/2)) \subset W \subset U_1$ for each large $n \in \mathbb{N}$, we get a contradiction. Hence the statement of the Corollary holds. □

Theorem 3.5. *Let $G = \langle f_1, f_2, \dots f_n \rangle$ be a finitely generated rational semigroup. Assume that there exists an element of G with the degree at least two, that each element of $Aut\,\overline{\mathbb{C}} \cap G$ (if this is not empty) is loxodromic and that $F(G) \neq \emptyset$. Then G is semi-hyperbolic if and only if all of the following conditions are satisfied.*

1. *for each $z \in J(G)$ there exists a neighborhood U of z in $\overline{\mathbb{C}}$ such that for any sequence $(g_n) \subset G$, any domain V in $\overline{\mathbb{C}}$ and any point $\zeta \in U$, we have that the sequence (g_n) does NOT converge to ζ locally uniformly on V*

2. for each $j = 1, \dots, m$ each $c \in C(f_j) \cap J(G)$ satisfies

$$d(c,\ (G \cup \{id\})(f_j(c))) > 0$$

Proof. First assume the conditions 1 and 2. Then by Theorem 3.1, we have that G is semi-hyperbolic.

Conversely, suppose G is semi-hyperbolic. By Lemma 1.13, the condition 2. holds. Now we will show the condition 1.holds. By Theorem 1.35, there exists an attractor K in $F(G)$ for G. Let z_0 be any point and U a neighborhood of z_0 such that $\overline{U} \cap K = \emptyset$. Suppose that there exists a sequence $(g_n) \subset G$, a domain V in $\overline{\mathbb{C}}$ and a point $\zeta \in U$ such that $g_n \to \zeta$ as $n \to \infty$ locally uniformly on V. We will deduce a contradiction. We can assume that there exists a word $w \in \{1, \dots, m\}^{\mathbb{N}}$ such that for each n,

$$g_n = \alpha_n \circ f_{w_n} \circ \cdots \circ f_{w_1},$$

where $\alpha_n \in G$ is an element. Then from Theorem 1.35 and that $\overline{U} \cap K = \emptyset$, we have that

$$f_{w_n} \circ \cdots \circ f_{w_1}(V) \subset J(G), \tag{20}$$

for each n. Hence $(f_{w_n} \circ \cdots \circ f_{w_1})_n$ is normal in V. Now let us consider the rational skew product $\tilde{f}$ constructed by the generator system $\{f_1, \dots, f_m\}$. By the second statement of Theorem 2.13, we have that $\{w\} \times V \subset \tilde{F}(\tilde{f})$. Hence there exists a positive integer n such that $f_{w_n} \circ \cdots \circ f_{w_1}(V) \subset F(G)$, if we take V sufficiently small. But this contradicts to (20). Hence we have shown that the condition 1. holds. □

Theorem 3.6. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated sub-hyperbolic rational semigroup. Assume that there exists an element of G with the degree at least two, that each element of $Aut\ \overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic and that there is no super attracting fixed point of any element of G in $J(G)$. Then there exists a Riemannian metric ρ on a neighborhood V of $J(G) \setminus P(G)$ such that for each $z_0 \in J(G) \setminus G^{-1}(P(G) \cap J(G))$, if there exists a word $w = (w_1, w_2, \dots,) \in \{1, \dots, m\}^{\mathbb{N}}$ satisfying $(f_{w_n} \cdots f_{w_1})(z_0) \in J(G)$ for each n, then*

$$\|(f_{w_n} \cdots f_{w_1})'(z_0)\| \to \infty, \text{ as } n \to \infty,$$

where $\|\cdot\|$ is the norm of the derivative measured from ρ on V to it.

Proof. By Theorem 1.36, there exists an attractor K in $F(G)$ for G such that $K^i \supset P(G) \cap F(G)$. Let $\{V_1, \dots, V_t\}$ be the set of all connected components of $\overline{\mathbb{C}} \setminus K$ having non-empty intersection with $J(G)$. We take the hyperbolic metric in $V_i \setminus P(G)$ for each $i = 1, \dots, t$. We denote by ρ the Riemannian metric in $V = \cup_{i=1}^{t} V_i \setminus P(G)$. First we will show the following.

- Claim 1. there exists a $k \in \mathbb{N}$ such that for each n,

$$\|(f_{w_{n+k}} \cdots f_{w_n})'(f_{w_n} \cdots f_{w_1}(z_0))\| > 1,$$

where $\|\cdot\|$ is the norm of the derivative measured from ρ to it. For each $i = 1, \dots, t$, let x_i be a point of $V_i \cap F(G)$. Since K is an attractor in $F(G)$ for G, there exists a $k \in \mathbb{N}$ such that for each $(i_1, \dots, i_k) \in \{1, \dots, m\}^k$,

$$(f_{i_k} \cdots f_{i_1})(x_i) \in K, \quad \text{for } i = 1, \dots, t. \tag{21}$$

Let x be a point of $J(G) \cap V_i \setminus P(G)$. Suppose $(f_{i_k} \cdots f_{i_1})(x) \in V_j \setminus P(G)$ for some $(i_1, \dots, i_k) \in \{1, \dots, m\}^k$ and j. Let U be the connected component of $(f_{i_k} \cdots f_{i_1})^{-1}(V_j \setminus P(G)) \cap (V_i \setminus P(G))$ containing x. Then

$$(f_{i_k} \cdots f_{i_1}) : U \to V_j \setminus P(G)$$

is a covering map. Hence we have

$$\|(f_{i_k} \cdots f_{i_1})'(z)\|_{U,\ V_j \setminus P(G)} = 1, \quad \text{for each } z \in U, \tag{22}$$

where we denote by $\|\cdot\|_{U,\ V_j \setminus P(G)}$ the norm of the derivative measured from the hyperbolic metric on U to that on $V_j \setminus P(G)$. On the other hand, by (21), $U \neq V_i \setminus P(G)$. Therefore the inclusion map $i : U \to V_i \setminus P(G)$ satisfies that

$$\|i'(z)\|_{U,\ V_i \setminus P(G)} < 1, \text{ for each } z \in U, \tag{23}$$

where we denote by $\|\cdot\|_{U,\ V_i \setminus P(G)}$ the norm of the derivative measured from the hyperbolic metric on U to that on $V_i \setminus P(G)$. By (22) and (23), we get

$$\|(f_{i_k} \cdots f_{i_1})'(z)\|_{V_i \setminus P(G),\ V_j \setminus P(G)} > 1, \text{ for each } z \in U, \tag{24}$$

where we denote by $\|\cdot\|_{V_i \setminus P(G),\ V_j \setminus P(G)}$ the norm of the derivative measured from the hyperbolic metric on $V_i \setminus P(G)$ to that on $V_j \setminus P(G)$. Hence the Claim 1. holds.

By Claim 1., we get that if the sequence $(f_{w_n} \cdots f_{w_1})(z_0))_{n=1}^{\infty}$ does not accumulate to any point of $P(G) \cap J(G)$, then $\|(f_{w_n} \cdots f_{w_1})'(z_0)\| \to \infty$ as $n \to \infty$. Hence we can assume that the sequence accumulates to a point of $P(G) \cap J(G)$. We set

$$g_n = f_{w_{nk}} \cdots f_{w_1}, \text{ for each } n.$$

We will show the following.

- Claim 2. $\|(g_n)'(z_0)\| \to \infty$ as $n \to \infty$.

Since $z_0 \in J(G) \setminus G^{-1}(P(G) \cap J(G))$, by the same arguments as that in the proof of Theorem 1.34, we can show that there exists an $\epsilon_1 > 0$ and a sequence (n_j) of integers such that

$$g_{n_j}(z_0) \in J(G) \setminus B(P(G), \epsilon_1),\ g_{n_j+1}(z_0) \in J(G) \cap B(P(G), \epsilon_1).$$

Suppose the case there exists a constant ϵ_2 such that for each j,

$$d(g_{n_j+1}(z_0),\ P(G)) \geq \epsilon_2.$$

Then from Claim 1, there exists a constant $c > 1$ such that for each j,

$$\|(f_{w_{(n_j+1)k}} \cdots f_{w_{n_jk+1}})'((f_{w_{n_jk}} \cdots f_{w_1})(z_0))\| > c.$$

Using the Claim 1 again, we can show that $\|(g_n)'(z_0)\| \to \infty$ as $n \to \infty$.

Next suppose the case there exists a subsequence $(h_l)_{l=1}^{\infty}$ of $(g_{n_j+1})_{j=1}^{\infty}$ such that $d(h_l(z_0),\ P(G)) \to 0$ as $l \to \infty$. There exists a subsequence $(\beta_l)_{l=1}^{\infty}$ of $(g_{n_j})_{j=1}^{\infty}$ such that for each l $h_l = \alpha_l \circ \beta_l$ where α_l is an element of G. Then there exists a constant $c_1 \in \mathbb{N}$ such that for each l, $wl_S(\alpha_l) \leq c_1$ where $S = \{f_1, \ldots, f_m\}$. Hence there exists a sequence (x_l) such that $d(x_l,\ \beta_l(z_0)) \to 0$ as $l \to \infty$ and $\alpha_l(x_l) \in P(G)$ for each $l \in \mathbb{N}$. We can assume that $x_l \in B(\beta_l(z_0),\ \epsilon_1)$ for each $l \in \mathbb{N}$. Let γ_l be the analytic inverse branch of β_l in $B(\beta_l(z_0),\ \epsilon_1)$ such that

$$\gamma_l(\beta_l(z_0)) = z_0, \text{ for each } l \in \mathbb{N}.$$

Since $\cup_{l=1}^{\infty} \gamma_l(B(\beta_l(z_0),\ \epsilon_1)) \subset \overline{\mathbb{C}} \setminus K$ and $d(x_l, \beta_l(z_0)) \to 0$, We get $\gamma_l(x_l) \to z_0$ as $l \to \infty$. Hence we have

$$d(z_0,\ h_l^{-1}(P(G))) \to 0, \text{ as } l \to \infty. \tag{25}$$

There exists an i such that $z_0 \in V_i \setminus P(G)$. For each l let V_{j_l} be the element of $\{V_1, \ldots, V_t\}$ such that $h_l(z_0) \in V_{j_l} \setminus P(G)$. Let W_l be the connected component of $h_l^{-1}(V_{j_l} \setminus P(G)) \cap V_i \setminus P(G)$ containing z_0. Then $h_l : W_l \to V_{j_l}$ is a covering map. Hence we have

$$\|(h_l)'(z)\|_{W_l,\ V_{j_l} \setminus P(G)} = 1, \text{ for } z \in W_l,$$

where $\| \cdot \|_{W_l,\ V_{j_l} \setminus P(G)}$ is the norm of the derivative measured from the hyperbolic metric on W_l to that on V_{j_l}. By Theorem 2.25 in [M], (25) implies that

$$\|(i_l)'(z)\|_{W_l,\ V_i \setminus P(G)} \to 0 \text{ as } l \to \infty,$$

where we denote by i_l the inclusion map from W_l into $V_i \setminus P(G)$ for each $l \in \mathbb{N}$. It follows that

$$\|h_l'(z)\|_{V_i \setminus P(G),\ V_{j_l} \setminus P(G)} \to \infty \text{ as } l \to \infty, \tag{26}$$

where $\|\cdot\|_{V_i\setminus P(G),\ V_{j_l}\setminus P(G)}$ is the norm of the derivative measured from the hyperbolic metric on $V_i \setminus P(G)$ to that on $V_{j_l} \setminus P(G)$. By (26) and Claim 1, we get $\|(g_n)'(z_0)\| \to \infty$ as $n \to \infty$. Hence the Claim 2 holds.

In the same way we can show that for each $i = 1, \ \ldots, k-1$,

$$\|(f_{w_{nk+i}} \cdots f_{w_1})(z_0)\| \to \infty \text{ as } n \to \infty.$$

We have thus proved the Theorem. □

Theorem 3.7. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated sub-hyperbolic rational semigroup. Assume that there exists an element of G with the degree at least two, that each element of Aut $\overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic and that there is no super attracting fixed point of any element of G in $J(G)$. Then G is semi-hyperbolic.*

Proof. We will appeal to Theorem 3.5. Since there is no super attracting fixed point of any element of G in $J(G)$, the condition 2. in Theorem 3.5 is satisfied. By Theorem 1.36, there exists an attractor K in $F(G)$ for G. Let z_0 be any point and U a neighborhood of z_0 such that $\overline{U} \cap K = \emptyset$. Suppose that there exists a sequence $(g_n) \subset G$, a domain V in $\overline{\mathbb{C}}$ and a point $\zeta \in U$ such that $g_n \to \zeta$ as $n \to \infty$ locally uniformly on V. We will deduce a contradiction. We can assume that there exists a word $w \in \{1, \ldots, m\}^{\mathbb{N}}$ such that for each n,

$$g_n = \alpha_n f_{w_n} \circ \cdots \circ f_{w_1},$$

where $\alpha_n \in G$ is an element. Then from Theorem 1.35 and that $\overline{U} \cap K = \emptyset$, we have that

$$f_{w_n} \circ \cdots \circ f_{w_1}(V) \subset J(G), \tag{27}$$

for each n. Hence $(f_{w_n} \circ \cdots \circ f_{w_1})_n$ is normal in V. Let $z_1 \in V \cap G^{-1}(P(G) \cap J(G))$ be a point. By the backward self-similarity of $J(G)$ and Lemma 1.33, there exists a sequence (n_j) of positive integers and a neighborhood W of $P(G) \cap J(G)$ in $\overline{\mathbb{C}}$ such that for each j,

$$f_{w_{n_j}} \circ \cdots \circ f_{w_1}(z_1) \in \overline{\mathbb{C}} \setminus W.$$

By Theorem 3.6, we have that

$$\|(f_{w_{n_j}} \circ \cdots \circ f_{w_1})'(z_1)\| \to \infty, \quad \text{as } j \to \infty, \tag{28}$$

where $\|\cdot\|$ denotes the norm of the derivative with respect to the spherical metric. Since $(f_{w_n} \circ \cdots \circ f_{w_1})_n$ is normal in V, this is a contradiction. Hence the condition 1 in Theorem 3.5 is satisfied. By Theorem 3.5, we get that G is semi-hyperbolic. □

4 Open Set Condition and Area 0

Definition 4.1. Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. We say that G satisfies the *open set condition* with respect to the generators $f_1, f_2, \dots, f_m$ if there exists an open set O such that for each $j = 1, \dots, m,\ f_j^{-1}(O) \subset O$ and $\{f_j^{-1}(O)\}_{j=1,\dots,m}$ are mutually disjoint.

Definition 4.2. Let G be a rational semigroup and $S = \{f_\lambda \mid \lambda \in \Lambda\}$ a generator system of G. For each $g \in G$, We set

$$wl_S(g) = \min\{n \in \mathbb{N} \mid g = f_{\lambda_1} \cdots f_{\lambda_n}\}.$$

We call $wl_S(g)$ the word length of g with respect to S.

Proposition 4.3. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. Assume that G satisfies the open set condition with respect to the generators $f_1, f_2, \dots, f_m$ and $O \setminus J(G) \neq \emptyset$ where O is an open set in the definition of the open set condition. Then $J(G)^i = \emptyset$ where we denote by $J(G)^i$ the interior of $J(G)$.*

Proof. Let $S = \{f_1, \dots, f_m\}$. Assume that $J(G)^i \neq \emptyset$.

Then we claim that for each element $g \in G$ and each point $x \in J(G)^i$,

$$g(x) \in \overline{\mathbb{C}} \setminus (O \setminus J(G)).$$

Suppose that there exists a point $y \in J(G)^i$ and an element $g_1 \in G$ such that $g_1(y) \in O \setminus J(G)$. Since $J(G) = \cup_{i=1}^n f_i^{-1}(J(G))$, there exists an element $h \in G$ with $wl_S(h) = wl_S(g_1)$ such that $h(y) \in J(G)$. Since $f_j^{-1}(O) \subset O$ for each $j = 1, \dots, m$, we have $J(G) \subset \overline{O}$ and $J(G)^i \subset O$. Hence $g_1^{-1}(O) \cap h^{-1}(O) \neq \emptyset$. But $g_1 \neq h$ and that is a contradiction because of the open set condition. Therefore the above claim holds.

Now the claim implies that G is normal in $J(G)^i$ but this is a contradiction and so we have $J(G)^i = \emptyset$. □

Theorem 4.4. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup which is semi-hyperbolic, contains an element with the degree at least two and satisfies the open set condition with respect to the generators $f_1, f_2, \dots,\ f_m$. Let O be an open set in Definition 4.1. Assume that $\sharp(\partial O \cap J(G)) < \infty$. Then the 2-dimensional Lebesgue measure of $J(G)$ is equal to* 0.

Proof. We will show the statement using the method of Theorem 1.3 in [Y]. We fix a gemerator system $S = \{f_1, \dots, f_m\}$. By the assumption of our Theorem, we have each element of Aut $\overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic. By Theorem 1.35, $A(G)$ is an attractor in $F(G)$ for G. We can assume $\infty \in A(G)$. Suppose that the 2-dimensional Lebesgue measure of $J(G)$ is positive.

Since $\sharp(\partial O \cap J(G)) < \infty,$ $G^{-1}(G(\partial O \cap J(G)))$ is a countable set. Hence there exists a Lebesgue density point x of $J(G)$ such that $x \in J(G) \setminus (G^{-1}(G(\partial O \cap J(G)))$. Since we have $J(G) = \cup_{j=1}^{m} f_j^{-1}(J(G)),$ there exists a word $w = (w_1, w_2, \dots) \in \{1, \dots, m\}^{\mathbb{N}}$ such that for each positive integer $u,$ $f_{w_u} \cdots f_{w_1}(x) \in J(G)$.

We will show that the sequence $(f_{w_u} \cdots f_{w_1}(x))_u$ has an accumulation point in $J(G) \setminus \partial O$. Assume that is false. For each large $u,$ let ζ_u be the closest point to $f_{w_u} \cdots f_{w_1}(x)$ in $\partial O \cap J(G)$. Since there exists no super attracting fixed point of any point of any element of G in $J(G),$ there exists a positive integer s such that for each integer t with $t \geq s,$ $(f_{w_t} \cdots f_{w_s})'(\zeta_{s-1}) \neq 0.$ Since G is semi-hyperbolic, we have that for each $x \in \partial O \cap J(G),$ if there exists an element $g \in G$ such that $g(x) = x,$ then x is a repelling fixed point of g. Applying Lemma 1.33, we get a contradiction. Hence the sequence $(f_{w_u} \cdots f_{w_1}(x))_u$ has an accumulation point in $J(G) \setminus \partial O$.

By the argument above, we have that there exists an $\epsilon > 0$ and a sequence (g_n) of elements of G such that for each $n,$ $g_{n+1} = h_n g_n$ for some $h_n \in G$ and $g_n(x) \in J(G) \setminus D(\partial O,\ \epsilon)$. Let δ be a small number so that $\delta < \epsilon$ and for each $g \in G$ and each $x \in J(G),$

$$\deg(g : U \to D(x, \delta)) \leq N$$

for each $U \in c(D(x, \delta),\ g),$ where N is a positive integer independent of $x,$ g and U. By Lemma 1.12, we can assume that for each $g \in G$ and each $x \in J(G),$ if V is a simply connected open neighborhood of x contained in $D(x, \delta),$ then each element of $c(D(x, \delta),\ g)$ is simply connected.

For each $n,$ we set $x_n = g_n(x)$. Let U_n be the conncted component of $g^{-1}(D(x_n, \frac{1}{2}\delta))$ containing x. Now we will claim that

$$\lim_{n \to \infty} \frac{m_2(U_n \cap J(G))}{m_2(U_n)} = 1, \tag{29}$$

where we denote by m_2 the 2-dimensional Lebesgue measure. By Corollary 1.9, Proposition 4.3 and Corollary 2.15, there exist a constant $K > 0,$ two sequences (r_n) and (R_n) such that $\frac{1}{K} \leq \frac{r_n}{R_n} < 1,$ $R_n \to 0$ and

$$D(x, r_n) \subset U_n \subset D(x, R_n).$$

Since x is a Lebesgue density point of $J(G),$ the claim holds. Now we get

$$\lim_{n \to \infty} \frac{m_2(U_n \cap F(G))}{m_2(U_n)} = 0. \tag{30}$$

For each $n,$ Let $\phi_n : D(0, 1) \to D_{g_n}(x_n, \delta)$ be the Riemann map such that $\phi_n(0) = x,$ where $D_{g_n}(x_n, \delta)$ is the element of $c(D(x_n, \delta),\ g_n)$ containing U_n. By (30) and the Koebe distortion theorem, we get

$$\lim_{n \to \infty} \frac{m_2(\phi_n^{-1}(U_n \cap F(G)))}{m_2(\phi_n^{-1}(U_n))} = 0. \tag{31}$$

By Corollary 1.8, there exists a constant $0 < c_1 < 1$ such that for each n, the Euclidian diameter of $\phi_n^{-1}(U_n)$ is less than c_1. Since we can assume that $D_{g_n}(x_n, \delta) \subset \mathbb{C}$ for each n and uniformly bounded in $\mathbb{C}$, by Cauchy's formula, we get that there exists a constant c_2 such that

$$|(g_n\phi_n)'(z)| \leq c_2 \text{ on } \phi_n^{-1}(U_n),\ n = 1, 2, \ldots \tag{32}$$

Now we will show

$$D(x_n, \frac{1}{2}\delta) \cap F(G) = g_n(U_n \cap F(G)), \text{ for each } n. \tag{33}$$

It is easy to see that $D(x_n, \frac{1}{2}\delta) \cap F(G) \supset g_n(U_n \cap F(G))$. Now let z be a point of $D(x_n, \frac{1}{2}\delta) \cap F(G)$ and assume that there exists a point $w \in U_n \cap J(G)$ such that $g_n(w) = z$. Since $J(G) = \cup_{j=1}^m f_j^{-1}(J(G))$ and $g_n(w) \in F(G)$, there exists an element $g \in G$ with $wl_S(g) = wl_S(g_n)$ such that $g(w) \in J(G) \subset \overline{O}$. Hence we have $g \neq g_n$ and $g^{-1}(O) \cap g_n^{-1}(O) \neq \emptyset$. But this contradicts to the open set condition. Therefore (33) holds.

By (31) , (32) and (33), we have

$$\begin{aligned}
\frac{m_2(D(x_n, \frac{1}{2}\delta) \cap F(G))}{m_2(D(x_n, \frac{1}{2}\delta))} &= \frac{m_2((g_n \circ \phi_n)(\phi_n^{-1}(U_n \cap F(G)))}{m_2(D(x_n, \frac{1}{2}\delta))} \\
&\leq \frac{\int_{\phi_n^{-1}(U_n \cap F(G))} |(g_n \circ \phi_n)'(z)|^2 dm_2(z)}{m_2(\phi_n^{-1}(U_n))} \frac{m_2(\phi_n^{-1}(U_n))}{m_2(D(x_n, \frac{1}{2}))} \\
&\to 0,
\end{aligned}$$

as $n \to \infty$. Hence we have

$$\lim_{n\to\infty} \frac{m_2(D(x_n, \frac{1}{2}\delta) \cap J(G))}{m_2(D(x_n, \frac{1}{2}\delta))} = 1.$$

We can assume that there exists a point $x_\infty \in J(G)$ such that $x_n \to x_\infty$. Then

$$\frac{m_2(D(x_\infty, \frac{1}{2}\delta) \cap J(G))}{m_2(D(x_\infty, \frac{1}{2}\delta))} = 1.$$

This implies that $D(x_\infty, \frac{1}{2}\delta) \subset J(G)$ but this is a contradiction because we have $J(G)^i = \emptyset$ by Proposition 4.3. $\square$

Corollary 4.5. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup which is sub-hyperbolic, contains an element with the degree at least two and satisfies the open set condition with respect to the generators $f_1, f_2, \ldots, f_m$. Let O be an open set in Definition 4.1. Assume that $\sharp(\partial O \cap J(G)) < \infty$. Then the 2-dimensional Lebesgue measure of $J(G)$ is equal to* 0.

Proof. By Proposition 4.3, $J(G)^i = \emptyset$. Since G is finitely generated, by [HM2], there is no super attracting fixed point of any element of G in $\partial J(G) = J(G)$. Therefore by Theorem 3.7, we have that G is semi-hyperbolic. By Theorem 4.4, the statement holds.

□

5 δ-subconformal measure and Hausdorff dimension of Julia sets

Definition 5.1. Let G be a rational semigroup and δ a non-negative number. We say that a Borel probability measure μ on $\overline{\mathbb{C}}$ is δ-subconformal if for each $g \in G$ and for each Borel measurable set A

$$\mu(g(A)) \leq \int_A \|g'(z)\|^\delta d\mu,$$

where we denote by $\|\cdot\|$ the norm of the derivative with respect to the spherical metric. For each $x \in \overline{\mathbb{C}}$ and each real number s we set

$$S(s,\ x) = \sum_{g \in G} \sum_{g(y)=x} \|g'(y)\|^{-s}$$

counting multiplicities and

$$S(x) = \inf\{s \mid S(s,\ x) < \infty\}.$$

If there is not s such that $S(s,\ x) < \infty$, then we set $S(x) = \infty$.Also we set

$$s_0(G) = \inf\{S(x)\},\ \ s(G) = \inf\{\delta \mid \exists \mu : \delta\text{-subconformal measure}\}$$

It is not difficult for us to prove the next result using the same method as that in [Sul].

Theorem 5.2 ([S2]). *Let G be a rational semigroup which has at most countably many elements. If there exists a point $x \in \overline{\mathbb{C}}$ such that $S(x) < \infty$ then there is a $S(x)$-subconformal measure. In particular, we have $s(G) \leq s_0(G)$.*

Proposition 5.3 ([S2]). *Let G be a rational semigroup and τ a δ-subconformal measure for G where δ is a real number. Assume that $\sharp J(G) \geq 3$ and for each $x \in E(G)$ there exists an element $g \in G$ such that $g(x) = x$ and $|g'(x)| < 1$. Then the support of τ contains $J(G)$.*

Proposition 5.4. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup. Assume that G satisfies the open set condition with respect to the generators $f_1, f_2, \ldots, f_m$ and $O \setminus J(G) \neq \emptyset$ where O is an open set in the definition of the open set condition. If there exists an attractor in $F(G)$ for G, then*

$$s_0(G) \leq 2.$$

Proof. We can assume $m \geq 2$. Let K be an attractor in $F(G)$ for G. There exists a simply connected domain U in $(O \cap F(G)) \setminus (K \cup P(G))$ such that $g(U) \cap U = \emptyset$ for each $g \in G$. By the open set condition, it is easy to see that if $g \neq h$, then $g^{-1}(U) \cap h^{-1}(U) = \emptyset$. Hence we have

$$\sum_S \int_U \|S'(z)\|^2 dm_2(z) < \infty,$$

where S is taken over all holomorphic inverse branches of all elements of G defined on U, $\|\cdot\|$ denotes the norm of the derivative with respect to the spherical metric and m_2 is the 2-dimensional Lebesgue measure on $\overline{\mathbb{C}}$. It follows that for almost every where $x \in U$, $S(2,x) < \infty$. $\square$

Lemma 5.5. *Let G be a rational semigroup. Assume that $\infty \in F(G)$, $\sharp J(G) \geq 3$ and for each $x \in E(G)$ there exists an element $g \in G$ such that $g(x) = x$ and $|g'(x)| < 1$. We also assume that there exist a countable set E in $\overline{\mathbb{C}}$, positive numbers a_1 and a_2 and a constant c with $0 < c < 1$ such that for each $x \in J(G) \setminus E$, there exist two sequences (r_n) and (R_n) of positive real numbers and a sequence (g_n) of elements of G satisfying all of the following conditions:*

1. *$r_n \to 0$ and for each n, $0 < \frac{r_n}{R_n} < c$ and $g_n(x) \in J(G)$.*
2. *for each n, $g_n(D(x, R_n)) \subset D(g_n(x), a_1)$.*
3. *for each n $g_n(D(x, r_n)) \supset D(g_n(x), a_2)$.*

Let δ be a real number with $\delta \geq s(G)$ and μ a δ- subconformal measure. Then δ-Hausdorff measure on $J(G)$ is absolutely continuous with respect to μ such that the Radon-Nikodim derivative is bounded from above. In particular, we have

$$dim_H(J(G)) \leq s(G).$$

Proof. By Proposition 5.3, the support of μ contains $J(G)$. Hence there exists a constant $c_1 > 0$ such that for each $x \in J(G)$, $\mu(D(x, a_2)) > c_1$.

Fix any $x \in J(G) \setminus E$. For each n we set $\tilde{R_n}(z) = R_n z + x$. By the condition 1 and 2, the family $\{g_n \circ \tilde{R_n}\}$ is normal in $D(0,1)$. By Marty's theorem, there exists a constant c_2 such that for each n and each $w \in D(0,c)$,

$$\|(g_n \circ \tilde{R_n})'(w)\| \leq c_2.$$

Note that we can take the constant c_2 independent of $x \in J(G) \setminus E$. Hence we have for each n,

$$\begin{aligned}
c_1 &\le \mu(D(g_n(x), a_2)) \\
&\le \mu(g_n(D(x, r_n))) \\
&\le \int_{D(x,r_n)} \|g_n'(z)\|^\delta d\mu(z) \\
&= \int_{D(x,r_n)} \|(g_n \circ \tilde{R_n} \circ \tilde{R_n}^{-1})'(z)\|^\delta d\mu(z) \\
&\le c_3 \frac{1}{R_n^\delta} \mu(D(x, r_n)) \\
&\le c_3 \frac{1}{r_n^\delta} \mu(D(x, r_n)),
\end{aligned}$$

where c_3 is a constant not depending on n and $x \in J(G) \setminus E$. Therefore we get that there exists a constant c_4 not depending on n and $x \in J(G) \setminus E$ such that

$$\frac{\mu(D(x, r_n))}{r_n^\delta} \ge c_4. \tag{34}$$

Now we can show the statement of our lemma in the same way as the proof of Theorem 14 in [DU]. We will follow it. Let A be any Borel set in $J(G)$. We set $A_1 = A \setminus E$. We denote by H_δ the δ-Hausdorff measure. Since E is a countable set, we have $H_\delta(A) = H_\delta(A_1)$. Fix γ, ϵ. For every $x \in A_1$, denote by $\{r_n(x)\}_{j=1}^\infty$ the sequence constructed in the above paragraph. Since μ is regular, for every $x \in A_1$ there exists a radius $r(x)$ being of the form $r_n(x)$ such that

$$\mu(\bigcup_{x \in A_1} D(x, r(x)) \setminus A_1) < \epsilon.$$

By the Besicovič theorem we can choose a countable subcover $\{D(x_i, r_{x_i})\}_{i=1}^\infty$ from the cover $\{D(x, r(x)\}_{x \in A_1}$ of A_1, of multiplicity bounded by some constant $C \ge 1$, independent of the cover. By (34), we obtain

$$\begin{aligned}
\sum_{i=1}^\infty r(x_i)^\delta &\le c_4^{-1} \sum_{i=1}^\infty \mu(D(x_i, r(x_i))) \\
&\le c_4^{-1} C \mu(\bigcup_{i=1}^\infty D(x_i, r(x_i))) \\
&\le c_4^{-1} C(\epsilon + \mu(A_1)).
\end{aligned}$$

Letting $\epsilon \to 0$ and then $\gamma \to 0$ we get

$$H_\delta(A) = H_\delta(A_1) \le c_4^{-1} C \mu(A_1) \le c_4^{-1} C \mu(A).$$

$\square$

Theorem 5.6. *Let G be a rational semigroup generated by a generator system $\{f_\lambda\}_{\lambda\in\Lambda}$ such that $\cup_{\lambda\in\Lambda}\{f_\lambda\}$ is a compact subset of $End(\overline{\mathbb{C}})$. Let $\tilde{f}$ be a rational skew product constructed by the generator system. Assume $\tilde{f}$ is semi-hyperbolic along fibres and satisfies the condition C1 with a family of discs $\{D_x\}_{x\in X}$ such that $D_x = D,\ \forall x \in X$ with some D. Then we have*

$$\dim_H(J(G)) \leq s(G).$$

Proof. We can assume $\infty \in F(G)$. Let x be any point of $J(G)$. Since we have $J(G) = \cup_{\lambda\Lambda} f_\lambda^{-1}(J(G)),$ for each $n \in \mathbb{N}$ there exists an element $g_n \in G$ which is a product of n generators such that $g_n(x) \in J(G)$. Let δ be a small positive number. For each n, we denote by $D_{g_n}(g_n(x),\delta)$ the element of $c(D(g_n(x),\delta),\ \delta)$ containing x. By Theorem 2.13, if we take a δ smaller, then

$$\text{diam } (D_{g_n}(g_n(x),\delta)) \to 0, \text{ as } n \to \infty. \tag{35}$$

By Lemma 2.14, we can assume that $D_{g_n}(g_n(x),\delta)$ is simply connected for each n. Let $\phi_n : D(0,1) \to D_{g_n}(g_n(x),\delta)$ be the Riemann map such that $\phi_n(0) = x$. By the Koebe distortion theorem, we have that for each n,

$$D_{g_n}(g_n(x),\delta) \supset D(x,\ \frac{1}{4}|\phi_n'(0)|).$$

Since G is semi-hyperbolic, we can assume that $D(J(G),\ \delta) \subset SH_N(G)$ where N is a positive integer. By Corollary 1.10, we get

$$\sup_{n\in\mathbb{N}}\{\text{diam } (\phi_n^{-1}(D_{g_n}(g_n(x),\epsilon\delta)))\} \to 0, \text{ as } \epsilon \to 0.$$

Therefore by the Koebe distortion theorem, there exists an ϵ such that

$$\begin{aligned} D_{g_n}(g_n(x),\ \epsilon\delta) &= \phi_n(\phi_n^{-1}(D_{g_n}(g_n(x),\ \epsilon\delta))) \\ &\subset D(x,\ \frac{1}{8}|\phi_n'(0)|), \text{ for each } n. \end{aligned}$$

By (35), we have $|\phi_n'(0)| \to 0$ as $n \to \infty$. Applying Lemma 5.5, we get

$$\dim_H(J(G)) \leq s(G).$$

□

Theorem 5.7. *Let $G = \langle f_1, f_2, \ldots, f_m\rangle$ be a finitely generated rational semigroup which is semi-hyperbolic. Assume that G contains an element with the degree at least two, each element of $Aut\,\overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic and $F(G) \neq \emptyset$. Then we have*

$$\dim_H(J(G)) \leq s(G) \leq s_0(G).$$

Proof. By Theorem 5.6 and Theorem 5.2. □

Remark 6. Let $G = \langle f_1, f_2, \dots, f_m\rangle$ be a finitely generated hyperbolic rational semigroup which satisfies the strong open set condition (i.e. G satisfies the open set condition with an open set O satisfying $O \supset J(G)$.). We assume that when $n = 1$ the degree of f_1 is at least two. By the results in [S4](Theorem 3.2 and the proof, Theorem 3.4 and Corollary 3.5), we have

$$0 < \dim_H J(G) = s(G) = s_0(G) < 2.$$

Example 5.8. Let $G = \langle f_1, f_2\rangle$ where $f_1(z) = z^2 + 2,\ f_2(z) = z^2 - 2$. Since $P(G) \cap J(G) = \{2, -2\}$ and $P(G) \cap F(G)$ is compact, we have G is sub-hyperbolic. By Theorem 3.7, G is also semi-hyperbolic. Since $f_j^{-1}(D(0,2)) \subset D(0,2)$ for $j = 1,2$ and$f_1^{-1}(D(0,2)) \cap f_2^{-1}(D(0,2)) = \emptyset$, G satisfies the open set condition. Also $J(G)$ is included in $B = \cup_{j=1}^2 f_j^{-1}(\overline{D(0,2)})$. Since $B \cap \partial D(0,2) = \{2, -2, 2i, -2i\}$, we get $\sharp(J(G) \cap \partial D(0,2)) < \infty$. By Corollary 4.4, we have $m_2(J(G)) = 0$, where we denote by m_2 the 2-dimensional Lebesgue measure. By Theorem 5.7 and Proposition 5.4, we have also

$$dim_H(J(G)) \le s(G) \le s_0(G) \le 2.$$

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