ON THE AREA OF THE COMPLEMENT OF THE INVARIANT COMPONENT OF CERTAIN B-GROUPS AND ON SEQUENCES OF TERMINAL REGULAR B-GROUPS

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Introduction Let G be a finitely generated Fuchsian group of the first kind, and $\partial T(G)$ the Bers boundary of the Teichmüller space of G. Let χ_φ be the canonical isomorphism from G to the b-group corresponding to $\varphi \in \partial T(G)$ with suitable normalizations (cf. Section 1.1), and Δ_φ the invariant component of $\chi_\varphi(G)$. The main result of this paper is the following.

Theorem 1. *Let $\{\varphi_n\}_{n=1}^\infty \subset \partial T(G)$ be a sequence corresponding to terminal regular b-groups such that*

(a) *For any hyperbolic element $g \in G$, there exist $\epsilon(g)$, $N(g) > 0$ such that for $n > N(g)$, if $\chi_{\varphi_n}(g)$ is loxodromic, then $|\mathrm{tr}^2(\chi_{\varphi_n}(g)) - 4| \geq \epsilon(g)$, and*

(b) *The Euclidean area of $\mathbb{C} \setminus \Delta_{\varphi_n}$ tends to 0 as $n \to \infty$.*

Then every accumulation point of the sequence corresponds to a totally degenerate group.

This theorem is proved in Section 3.2. We know that *any* $\varphi_0 \in \partial T(G)$ has a sequence $\{\varphi_m\}_{m=1}^\infty$ corresponding to terminal regular b-groups in $\partial T(G)$ such that φ_m converges to φ_0 and that the area of $\mathbb{C} \setminus \Delta_{\varphi_m}$ tends to zero (cf. Remark (2) in Section 3.3).

This paper is organized as follows: In section 1, we fix our notations and recall some basic definitions and facts. Section 2 deals with the lower estimate of the area of the complement of the invariant component of a b-group which contains triangle groups as component subgroups. This class of b-groups, by definition, involves the set of terminal regular b-groups. In Section 3, we give the proof of Theorem 1 and several remarks about our result.

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1. Preliminaries

1.1. Let G be a finitely generated Fuchsian group of the first kind acting on $\Sigma := \{z \in \hat{\mathbb{C}} : |z| > 1\}$. Let $B(G)$ be the complex Banach space of holomorphic functions $\varphi(z)$, $z \in \Sigma$ with norm $\|\varphi\| = \sup(|z|^2 - 1)^2|\varphi(z)|/4 < \infty$ which satisfy the functional equation of quadratic differentials $\varphi(g(z))g'(z)^2 = \varphi(z)$, $g \in G$. It is well-known that $\dim B(G) < \infty$ and that for every $\varphi \in B(G)$, there exists the local

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univalent function $W_\varphi(z)$, $z \in \Sigma$, such that the Schwarzian derivative $\{W_\varphi, z\}$ of W_φ is equal to $\varphi(z)$ and that W_φ forms

$$W_\varphi(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}$$

near $z = \infty$. For $\varphi \in B(G)$, we denote by χ_φ the homomorphism from G into Möb($\hat{\mathbb{C}}$) defined by the equation $\chi_\varphi(g) \circ W_\varphi = W_\varphi \circ g$, $g \in G$.

Let $T(G)$ be the Teichmüller space of G. $T(G)$ can be identified with a bounded domain in $B(G)$. We know that for $\varphi \in \partial T(G)$, W_φ is univalent and $G_\varphi := \chi_\varphi(G)$ is a b-group with the invariant component $\Delta_\varphi := W_\varphi(\Sigma)$ (see Section 1.2). We call $\partial T(G)$ the *Bers boundary of* $T(G)$ (see [7], [9], and [12]).

1.2. For $E \subset \hat{\mathbb{C}}$, we denote by Möb($E$) the group of Möbius transformations g satisfying that $g(E) = E$. Through this paper, all discrete groups in Möb($\hat{\mathbb{C}}$) are *torsion free*. A finitely generated non-elementary Kleinian group , is called a b-group if it has precisely one simply connected invariant component Δ_Γ of its region of discontinuity $\Omega($, $)$. By Ahlfors's finiteness theorem, a b-group represents a finitely many Riemann surfaces, each with a finite Poincaré area. By Bers's second area inequality (cf.[18]), the total Poincaré area of $\Omega($, $)/$, is at most twice the Poincaré area of $\Delta_\Gamma/$, . If equality holds, , is called *regular*.

Let , be a b-group and f a conformal mapping from Σ to Δ_Γ. Let $G = f^{-1}$, f. If $\{f, -\} \in \partial T(G)$, , is also called a *boundary group*. For a Fuchsian group G with $\infty \in \Omega(G)$, we denote by Δ_G the component of $\Omega(G)$ which contains ∞.

Let , be a b-group. For an accidental parabolic transformation (A.P.T.) $g \in G$, we denote by A_g the axis of g (cf. [14, p.611] and [14, Lemma 1]). Let $\{g_i\}_{i=1}^{s}$ be a basis for A.P.T.s in , . (cf. [14, p.612]). Let π be a projection mapping from Δ_Γ to $\Delta_\Gamma/$, . Then, a system $\mathcal{C}_\Gamma := \{\pi(A_g)\}_{g:\text{A.P.T.}}$ is a partition on $\Delta_\Gamma/$, , that is, $\mathcal{C}_\Gamma$ is the system of mutually disjoint simple closed geodesics (cf. [14, p.613]). The system $\mathcal{C}_\Gamma$ and a components of $R \setminus \mathcal{C}_\Gamma$ are called the *partition with respect to* , and a *block of* , respectively.

Let , be either a b-group or a Fuchsian group with $\infty \in \Omega($, $)$. For $E \subset \Delta_\Gamma/$, , a stabilizer group of a component of $\pi^{-1}(E)$ in , is called a *covering group of* E in , (cf. [16, p.251]). For a b-group, a covering group of a block is called a *structure subgroup*. We say that a set of structure subgroups $\{H_j\}_{j=1}^{s}$ of , is *a basis of structure subgroups* of , if each H_i are not mutually conjugate in , and every structure subgroup is conjugate some H_i in , . A stabilizer subgroup of a component of $\Omega($, $) - \Delta_\Gamma$ in , is called a *component subgroup*. We say that a Kleinian group , is *a b-group with no moduli* if , is a b-group satisfying either $\Omega($, $)$ is connected or each component subgroup of , is a triangle group (cf.[7]), where a Kleinian group is called a triangle group if it conjugate in Möb($\hat{\mathbb{C}}$) to the principal congruence subgroup of level 2:

$$\langle z \mapsto z + 2, z \mapsto z/(-2z + 1)\rangle.$$

A b-group , is called *terminal regular* if , is regular and has no moduli. A b-group , is called *totally degenerate* if $\Omega($, $)$ is connected.

1.3. Assume that $f(z) = z + b_0 + \sum_{k=1}^{\infty} b_k z^{-k}$ is univalent on Σ. Then the following inequality, called *the Golusin's inequality*, holds:

$$\sum_{k=1}^{\infty} k \left| \sum_{l=1}^{N} b_{kl}\lambda_l \right|^2 \leq \sum_{l=1}^{N} \frac{|\lambda_l|^2}{l}, \tag{1}$$

for any $\lambda_l \in \mathbb{C}$, $(l = 1, 2, \ldots, N)$, and

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{k,l=1}^{\infty} b_{kl} z^{-k} \zeta^{-l} \quad (z, \zeta \in \Sigma).$$

(cf. [3, p.91]). The coefficients $\{b_{kl}\}_{k,l=1}^{\infty}$ are called the *Grunsky coefficients of* f. This inequality induces the following:

$$\left| \sum_{k,l=1}^{\infty} b_{kl}\lambda_k \mu_l \right|^2 \leq \sum_{l=1}^{\infty} \frac{|\lambda_l|^2}{l} \sum_{l=1}^{\infty} \frac{|\mu_l|^2}{l}. \tag{2}$$

for $\lambda_k, \mu_k \in \mathbb{C}$ $(k = 1, 2, \ldots)$ such that$\{k^{-1/2}\lambda_k\}_{k=1}^{\infty}$, $\{k^{-1/2}\mu_k\}_{k=1}^{\infty} \in l^2$.

2. The area of the complement of an invariant component

In this section, we will give the lower estimate of the Euclidean area of the complement of the invariant component of a b-group containing triangle groups as component groups. For a measurable set $E \subset \mathbb{C}$, we denote by Area(E) the Lebesgue measure of E.

2.1. Let $F < \text{Möb}(\hat{\mathbb{C}})$ be a triangle group so that $\infty \in \Omega(F)$. Let $\{A, B\}$ be a generator of F such that A, B, and AB are parabolic. Then, we have

Lemma 1. *For* $g(z) = (az + b)/(cz + d), (ad - bc = 1)$*, let* $c_g = |c|$*. Then*

$$\begin{aligned}\text{Area}(\Omega(F) \setminus \Delta_F) &= 4\pi\{2(c_A c_B + c_B c_{AB} + c_{AB} c_A) - (c_A^2 + c_B^2 + c_{AB}^2)\}^{-1} \\ &\geq 4\pi(c_A^2 + c_B^2 + c_{AB}^2)^{-1}\end{aligned}$$

Proof. The direct calculation gives that the interior of the circumscribing circle of the triangle whose edges has lengths x,y, and z has the area

$$\pi \left\{ 2\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) - \left(\frac{y^2}{z^2x^2} + \frac{z^2}{x^2y^2} + \frac{x^2}{y^2z^2}\right)\right\}^{-1}.$$

Let a, b and c be fixed points of A, B and AB respectively. By Proposition 12.1 in [11, p.571],

$$c_A = \frac{2|c-b|}{|c-a||a-b|}, \qquad c_B = \frac{2|a-c|}{|a-b||b-c|}, \text{ and } \quad c_{AB} = \frac{2|b-a|}{|b-c||c-a|}.$$

Therefore, we have the assertion. □

2.2. We have the following lemma (cf. [22, p.372, Section 4]).

Lemma 2. *Let* $A \in \text{Möb}(\Sigma)$ *with* $A(\infty) \neq \infty$ *and* $g \in \text{Möb}(\hat{\mathbb{C}})$ *a parabolic element. Suppose that there exists a univalent function* f *from* Σ *into* $\hat{\mathbb{C}}$ *with normalization* $f(z) = z + O(1)$ *near* $z = \infty$ *such that* $g \circ f = f \circ A$*. Then*

$$c_g^2 \leq 4(1 - |A'(0)|)/ \operatorname{tr}^2(A)|A'(0)|^3 \tag{3}$$

Proof. Let $\{b_{kl}\}_{k,l=1}^{\infty}$ be the Grunsky coefficient of f. By definition,

$$g(\infty) - g^{-1}(\infty) = f(A(\infty)) - f(A^{-1}(\infty))$$
$$= (A(\infty) - A^{-1}(\infty)) \exp\Big\{ - \sum_{k,l=1}^{\infty} b_{kl} A(\infty)^{-k} (A^{-1}(\infty))^{-l} \Big\}.$$

Since $A(\infty) = \{\overline{A(0)}\}^{-1}$, $|A(0)| < 1$, and $|A(0)| = |A^{-1}(0)|$, $\{k^{-1/2}\overline{A(0)}^{k}\}_{k=1}^{\infty}$, $\{k^{-1/2}\overline{A^{-1}(0)}^{k}\}_{k=1}^{\infty}$ are contained in l^2. By (2), we have

$$\frac{|\mathrm{tr}^2(g)|}{c_g^2} = |g(\infty) - g^{-1}(\infty)|^2$$
$$= |A(\infty) - A^{-1}(\infty)|^2 \exp\Big\{ -2\mathrm{Re}\Big(\sum_{k,l=1}^{\infty} b_{kl} \overline{A(0)}^{k} (\overline{A^{-1}(0)})^{l} \Big) \Big\}$$
$$\geq \frac{\mathrm{tr}^2(A)|A'(0)|}{1 - |A'(0)|} \exp\Big\{ -2\Big(\sum_{k=1}^{\infty} \frac{|A(0)|^{2k}}{k} \sum_{k=1}^{\infty} \frac{|A^{-1}(0)|^{2k}}{k} \Big)^{1/2} \Big\}$$
$$= \frac{\mathrm{tr}^2(A)|A'(0)|}{1 - |A'(0)|} \exp\Big\{ 2\log(1 - |A(0)|^2) \Big\} = \frac{\mathrm{tr}^2(A)|A'(0)|^3}{1 - |A'(0)|}.$$

Since g is parabolic, we conclude (3). □

2.3. For a parabolic $A \in \mathrm{M\ddot{o}b}(\Sigma)$ and $\epsilon > 0$, the *ϵ-horocycle* of A is, by definition, the cycle C in $\overline{\Sigma}$ through the fixed point of A such that the hyperbolic distance between z and $A(z)$ is equal to ϵ for $z \in C$. We denote by $d(A, \epsilon)$ the hyperbolic distance between ∞ to the ϵ-horocycle of A in Σ. For a hyperbolic $A \in \mathrm{M\ddot{o}b}(\Sigma)$, we denote by $d(A)$ the hyperbolic distance between ∞ to the axis of A in Σ.

For a Fuchsian group G acting on Σ and $\epsilon > 0$, by the *ϵ-thick part* $\mathrm{thick}_\epsilon(G)$ for G we mean that the set of points $z \in \Sigma$ such that the hyperbolic distance between z and $g(z)$ is more than ϵ for all parabolic $g \in G$. For b-group , , let f be a conformal mapping from Σ to Δ_Γ. We define the ϵ-thick part $\mathrm{thick}_\epsilon($,) for , by $f(\mathrm{thick}_\epsilon(f^{-1}$, $f))$.

Let P be a subset of Σ/G, where G is a Fuchsian group. For $\epsilon > 0$, we denote $P_\epsilon := (\mathrm{thick}_\epsilon(G)/G) \cap P$. We say that a closed curve in Σ/G is the *ϵ-horocycle* if there exists a primitive parabolic $g \in G$ so that C is the image of the ϵ-horocycle of g by the projection. For a rectifiable curve C in $R := \Sigma/G$, we denote by $l_R(C)$ the hyperbolic length of C on R. Let P be the Nielsen kernel of a Riemann surface homeomorphically to a three punctured sphere (cf.[6]). Let j be the number of cusps of P. In this paper, P is called the *pair of pants of type* $(3-j, j)$ (cf. Figure 1.)

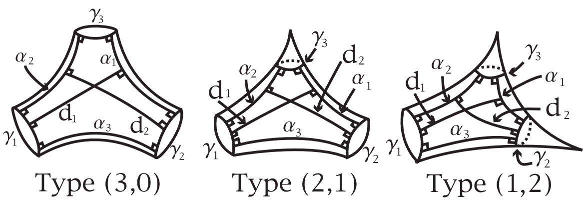

FIGURE 1. Pairs of Pants

Let $\epsilon_0 = 2\mathrm{arcsinh}1$ and $\epsilon < \epsilon_0$. Let P be a pair of pants pf type $(3-j, j)$. Then there exist geodesics $\{\gamma_i\}_{i=1}^{3-j}$, $\{\alpha_i\}_{i=1}^{3}$, $\{d_i\}_{i=1}^{2}$, ϵ-horocycles $\{\gamma_i\}_{j=3-j+1}^{3}$, if $j \neq 0$, and the point $q \in P$ as in Figure 1 by the Collar theorem (cf. [8, Theorem 4.4.6]). From now on, we fix $\epsilon < \epsilon_0$.

The following lemma can be proved in the argument similar to that of Theorem 2.4.3 and 3.1.8 in [8] and Lemma 4.4 in [20]. Hence we omit the proof. The author would like to thank Professor Toshihiro Nakanishi for teaching about the joint work [20] with Professor Marjatta Näätänen.

Lemma 3. *Let P be a pair of pants of type $(3-j, j)$. Let $\{\gamma_i\}_{i=1}^{3}$, $\{\alpha_i\}_{i=1}^{3}$, $\{d_i\}_{i=1}^{2}$, and $q \in P$ as in Figure 1. Let d_3 be the shortest geodesic connecting γ_3 and α_3. Let l_i and $l(d_i)$ be the lengths of γ_i and d_i respectively. Then*

(a) *d_3 passes through q.*

(b) *Let $L_i = \cosh(l_i/2)$ $(1 \leq j \leq 3-j)$, then*

(b-1) *If P is of type $(3,0)$, then*

$$\cosh(l(d_i)) = \frac{(L_1^2 + L_2^2 + L_3^2 + 2L_1L_2L_3 - 1)^{1/2}}{\sinh(l_i/2)}, \text{ for } i = 1, 2, 3.$$

(b-2) *If P is of type $(2,1)$, then*

$$\cosh(l(d_i)) = \frac{L_1 + L_2}{\sinh(l_i/2)}, \text{ for } j = 1, 2, \text{ and } e^{l(d_3)} = \frac{L_1 + L_2}{\sinh(\epsilon/2)}.$$

(b-3) *If P is of type $(1,2)$, then*

$$\cosh(l(d_1)) = \frac{L_1 + 1}{\sinh(l_1/2)}, \text{ and } e^{l(d_i)} = \frac{L_1 + 1}{\sinh(\epsilon/2)}, \text{ for } i = 2, 3. \square$$

2.4. Let , be a b-group which contains triangle groups as structure groups. We denote by $\{P_k\}_{k=1}^{s_0}$ the blocks of , . Let π is the projection from Δ_Γ to R and f the conformal mapping from Σ to Δ_Γ.

We may assume that for $1 \leq k \leq s$, P_k is a pair of pants of type $(3-j_k, j_k)$. Let $\{\gamma_{k,j}\}_{j=1}^{3}$ be boundary curves of $(P_k)_\epsilon$, We assume that for $1 \leq j \leq 3-j_k$, $\gamma_{k,j}$ is a geodesic (see Figure 1).

Lemma 4. *Fix $0 < \epsilon < \epsilon_0$ so that $p_0 := \pi \circ f(\infty) \in R_\epsilon$. Then, for $k = 1, \ldots, s$, there exist a structure group $,_k$ corresponding to P_k and generators $\{C_{k,i}\}_{i=1}^{3}$ of $H_k := f^{-1}, _k f$ such that*

(i) *For $j = 1, 2, 3$, if $C_{k,i}$ is hyperbolic (resp. parabolic), the the axis (resp. ϵ-horocycle) of $\chi_\varphi^{-1}(C_{k,i})$ maps to $\gamma_{k,i}$ by $\pi \circ f$.*

(ii) *$C_{k,3}C_{k,2}C_{k,1} = id$*

(iii) *$d(C_{k,i})$ (resp. $d(C_{k,i}, \epsilon)$) $\leq \mathrm{diam}(R_\epsilon) + \delta(k, i)$,*

where $\delta(k,i)$ is $l(d_i)$ as in Lemma 3 with respect to curves $\gamma_i := \gamma_{k,i}$, $i = 1, 2, 3$ and $P := P_k$, and $\mathrm{diam}(E)$ is the hyperbolically diameter of $E \subset R$.

Proof. Fix $k \in \{1, \ldots, s\}$. We only show the case where P_k is of type $(3,0)$. Another cases are proved in the similar manner.

On P_k, let $\{\gamma_{k,i}\}_{i=1}^{3}$, $\{\alpha_{k,i}\}_{i=1}^{3}$, and $\{d_{k,i}\}_{i=1}^{3}$ be geodesics as in Lemma 3. Let p_k be a intersection point of $d_{k,1}$ and $d_{k,2}$. By Lemma 3, $l_R(d_{k,i}) = \delta(k,i)$. Take a geodesic β_k in R connecting p_0 and p_k such that $l_R(\beta_k) \leq \mathrm{diam}(R_\epsilon)$. We define the curve $\beta_{k,i} \subset \beta_k \cup d_{k,i}$ connecting p_0 and $\gamma_{k,i}$ so that

$$l_R(\beta_{k,i}) \leq \mathrm{diam}(R_\epsilon) + \delta(k,i).$$

We construct a loop $c'_{k,i} := \beta_{k,i}\gamma_{k,i}\beta_{k,i}^{-1}$ with an initial point p_0. We give an orientation for $c'_{k,i}$ such that $[c'_{k,1}][c'_{k,2}][c'_{k,3}] = 1$, where $[-]$ is an equivalence class in $\pi_1(R, p_0)$, the fundamental group of R with the base point p_0.

We take $C_{k,i} \in f, f^{-1}$ corresponding to $[c'_{k,i}]$ by the canonical isomorphism between $\pi_1(R, p_0)$ and f, f^{-1}. Let $z_k \in \Sigma$ be the end point of the lift of β_k whose initial point is the point at ∞. Let $,_k$ be a structure group which stabilizes the component of $\Delta_\Gamma \setminus \cup_{g:\text{A.P.T}} A_g$ containing $f(z_k)$. Then, by definition, the system $\{C_{k,i}\}_{i=1}^3$ generates H_k and satisfies the assertion of lemma. □

2.5. We now prove the following theorem.

Theorem 2. *Let , be a b-group such that $\infty \in \Delta_\Gamma$ and that the logarithmic capacity of the limit set of , is equal to* 1. *Take $0 < \epsilon < \epsilon_0$ such that $\infty \in \text{thick}_\epsilon(,)$. Let $\{P_k\}_{k=1}^{s_0}$ be blocks of , each of which is a pair of pants. Then*

$$\text{Area}(\mathbb{C} \setminus \Delta_\Gamma) \geq 64\pi \sum_{k=1}^{s_0} A(R, P_k, \epsilon),$$

where $R = \Delta_\Gamma/,$,

$$A(R, P_k, \epsilon) := \left\{ \textstyle\sum_{i=1}^3 S_{k,i}(\sinh^2(l_{k,i}/2), \text{diam}(R_\epsilon) + \delta(k,i)) \right\}^{-1}, \textit{ and}$$

$$S_{k,i}(x,d) := \begin{cases} (x-4)\cosh^2 d\,((x-4)\cosh^2 d + 4)^2/x, & \textit{if } 1 \leq i \leq 3 - j_k \\ 16\sinh^2(\epsilon/2)e^{2d}(1 + \sinh^2(\epsilon/2)e^{2d})^2, & \textit{otherwise.} \end{cases}$$

Here $(3 - j_k, j_k)$ is the type of P_k, $\{l_{k,i}\}_{i=1}^{3-j_k}$ and $\{\delta(k,i)\}_{i=1}^3$ are defined as in Lemma 4, and set $l_{k,i} = 0$ if $3 - j_k + 1 \leq i \leq 3$. Especially, for $M > 0$ and some k, if lengths of all closed geodesics in the boundary of P_k are less than M, then there exists $A > 0$, depend only on R, M, and ϵ, so that $\text{Area}(\mathbb{C} \setminus \Delta_\Gamma) \geq A$.

Proof. The direct calculations shows that if $A \in \text{Möb}(\Sigma)$ is hyperbolic,

$$|A'(0)| = 4/(\text{tr}^2(A) - 4\tanh^2(d(A)))\cosh^2(d(A)), \tag{4}$$

and if A is parabolic and $\infty \in \text{thick}_\epsilon(\langle A \rangle)$,

$$|A'(0)| = 1/(1 + \sinh^2(\epsilon/2)e^{2d(A,\epsilon)}). \tag{5}$$

Let f be the conformal mapping from Σ to Δ_Γ such that $f(z) = z + O(1)$ near $z = \infty$ (cf. [21, p.207, Corollary 9.9]). For $k \in \{1, \ldots, s\}$. let H_k and $\{C_{k,i}\}_{i=1}^3$ be as in Lemma 4. Since $\infty \in \text{thick}_\epsilon(\langle g \rangle)$ for every parabolic $g \in G$, by Lemma 2, and (4) and (5), we have

$$(c_{fC_{k,i}f^{-1}})^2 \leq S_{k,i}(\text{tr}^2(C_{k,i}), d(C_{k,i}))/16 \quad \text{for } i = 1, 2, 3.$$

Hence, by Lemma 1 and 4, we conclude the assertion. □

Corollary 1. *For a b-group , , let f be a conformal mapping from Σ to Δ_Γ. Then it holds that*

$$\|\{f, -\}\| \leq \frac{3}{2}\left\{1 - 64\textstyle\sum_k A(\Delta_\Gamma/,, P_k, \epsilon_0)\right\}^{1/2},$$

where each P_k is a block of , which is a pair of pants.

Proof. Let $G := f^{-1}, f$. Since G is torsion free, for $\varphi \in B(G)$, it holds $\|\varphi\| = \sup\{(|z|^2-1)^2|\varphi(z)|/4 \mid z \in \mathrm{thick}_{\epsilon_0}(G)\}$ (cf. [23, Lemma 1] and [4, p.198,Exercise 8.2]). Hence by an argument similar to that of Lemma 6.7 in [9, p.151] (the Nehari-Kraus theorem), we conclude the assertion. □

3. Sequences of terminal regular b-groups

In this section, by using Theorem 2, we study a behavior of a sequence corresponding to terminal regular b-groups contained in a Bers boundary.

3.1. Let G be a finitely generated Fuchsian group of the first kind acting on Σ. For $\varphi \in \partial T(G)$, we denote by $\mathcal{C}_\varphi$ the partition with respect to G_φ. We show the following lemma.

Lemma 5. *Let $\{\varphi_m\}_{m=1}^\infty \subset \partial T(G)$ be a sequence corresponding to terminal regular b-groups. Then there exist a subsequence $\{\varphi_{m_j}\}_{j=1}^\infty$, a maximal partition $\{C_k\}_{k=1}^{3p-3+n}$ on R, a number $k_0 \in \{0, 1, \ldots, 3p-3+n\}$, and homeomorphisms $\{f_j\}_{j=1}^\infty$ of R onto itself such that*

(1) *For $j \geq 1$, $\mathcal{C}_{\varphi_{m_j}} = \{f_j(C_k)\}_{k=1}^{3p-3+n}$,*
(2) *If $k_0 > 0$, then $l_R(f_j(C_k)) \to \infty$ as $j \to \infty$ for $1 \leq k \leq k_0$, and*
(3) *If $k_0 < 3p-3+n$, then $f_j(C_k) = C_k$ for $k > k_0$.*

If, in addition, $\mathrm{Area}(\mathbb{C} \setminus \Delta_{\varphi_{m_j}}) \to 0$ as $j \to \infty$, then it also holds that

(4) *No component of $R \setminus \cup_{k>k_0} C_k$ is a pair of pants, and hence $k_0 > 0$.*

Proof. Since the number of graphs induced by the maximal partition on R is finite (cf.[2],[11]), we may assume that all graphs induced from $\{\mathcal{C}_{\varphi_m}\}_{m=1}^\infty$ are the same. Let us denote $\mathcal{C}_{\varphi_1} = \{C'_k\}_{k=1}^{3p-3+n}$. Then, there exist homeomorphisms $\{h_m\}_{m=1}^\infty$ of R onto itself such that $\mathcal{C}_{\varphi_m} = \{h_m(C'_k)\}_{k=1}^{3p-3+n}$ (cf.[8, Appendix]). By taking the subsequence of $\{h_m\}_{m=1}^\infty$ and renumbering the curves $\{C'_k\}_{k=1}^{3p-3+n}$ if necessary, we may suppose that there exist $k_0 \in \{0, 1, \ldots, 3p-3+n\}$ and $M > 0$ such that if $k_0 > 0$, then $l_R(h_m(C'_k)) \to \infty$ as $m \to \infty$ for $1 \leq k \leq k_0$, and that if $k_0 < 3p-3+n$, then $l_R(h_m(C'_k)) < M$ for $k_0 < k \leq 3p-3+n$ and $m \geq 1$.

Since the number of closed geodesics in R whose hyperbolic length are less than M is finite (cf.[2]), there exists a subsequence $\{\varphi_{m_j}\}_{j=1}^\infty$ such that $h_{m_j}(C'_k) = h_{m_l}(C'_k)$ for $j, l \geq 1$ and $k_0 < k \leq 3p-3+n$.

Let $f_j = h_{n_j} \circ (h_{n_1})^{-1}$ and $C_k = h_{n_1}(C'_k)$ for $j \geq 1$ and $1 \leq k \leq 3p-3+n$. Then, by definition, the subsequence $\{\varphi_{m_j}\}_{j=1}^\infty$, the partition $\{C_k\}_{k=1}^{3p-3+n}$ on R, the number k_0, and homeomorphisms $\{f_j\}_{j=1}^\infty$ satisfy (1)-(3) in this lemma.

From now on, we assume that $\mathrm{Area}(\mathbb{C} \setminus \Delta_{\varphi_{m_j}}) \to 0$ as $j \to \infty$. Suppose that there exists a component P_i of $R \setminus \cup_{k>k_0} C_k$ such that P_i is a pair of pants. Since $\{C_k\}_{k>k_0} \subset \mathcal{C}_{\varphi_{m_j}}$ for each $j \geq 1$ and P_i does not contain the simple closed geodesic which is not homotopic to boundary components of P_i, P_i is a block of $G_{\varphi_{m_j}}$ for every $j \geq 1$. Hence each $G_{\varphi_{m_j}}$ contains a triangle group as a structure group corresponding to P_i.

Take $\epsilon > 0$ so that $\infty \in \mathrm{thick}_\epsilon(G)$. Since the lengths of all closed geodesics in the boundary of P_i are less than M, by Theorem 2, there exists $A > 0$, depend only on R, M, and ϵ such that

$$\mathrm{Area}(\mathbb{C} \setminus \Delta_{\varphi_{m_j}}) \geq A$$

for $j \geq 1$. This contradicts the assumption. □

3.2. To prove Theorem 1, it suffices to show the following proposition.

Proposition 6. *Let $\{\varphi_m\}_{m=1}^{\infty}$ be a sequence corresponding to terminal regular b-groups in $\partial T(G)$ satisfying* (1)-(3) *in Lemma 5 with respect to a partition $\{C_k\}_{k=1}^{3p-3+n}$, a number k_0, and homeomorphisms $\{f_m\}_{m=1}^{\infty}$ of R onto itself. Suppose that the sequence satisfies* (a) *in Theorem 1 and converges to $\varphi_0 \in \partial T(G)$. Then G_{φ_0} is a b-group with no moduli such that $\mathcal{C}_{\varphi_0} = \{C_k\}_{k>k_0}$. Especially, if no component of $R \setminus \cup_{k>k_0} C_k$ is a pair of pants, then G_{φ_0} is totally degenerate.*

Proof. We prove the case $k_0 < 3p-3+n$. The case where $k_0 = 3p-3+n$ is proved by the similar manner.

Let $g_k \in G$ be primitive hyperbolic elements corresponding to C_k for $1 \leq k \leq 3p-3+n$. We denote by $\{P_i\}_{i=1}^{s_0}$ the components of $R \setminus \cup_{k>k_0} C_k$ each of which is not a pair of pants. Let $\{P_i\}_{i=s+1}^{s_1}$ be components of $R \setminus (\cup_{k>k_0} C_k \cup \cup_{i=1}^{s_0} P_i)$. Fix a stabilizer group H_i corresponding to P_i in G. Let $G_{i,m} = \chi_{\varphi_m}(H_i)$ for $m \geq 0$. For $m \geq 1$, since G_{φ_m} is a terminal regular b-group and, P_i is not a pair of pants for $1 \leq i \leq s_0$, $G_{i,m}$ is also a terminal regular b-group such that $\Delta_{G_{i,m}}/G_{i,m}$ is homeomorphic to P_i if $1 \leq i \leq s_0$ (cf.[11]). By definition, for $i > s_0$, $G_{i,m}$ is a triangle group.

We first show that $\{G_{i,0}\}_{i=1}^{s_1}$ is a basis of the structure groups of G_{φ_0}. It is clear that for $k > k_0$, $\chi_{\varphi_0}(g_k)$ is an A.P.T. in G_{φ_0}. Since $G_{i,0} = \chi_{\varphi_0}(H_i)$, it suffices to show that $\chi_{\varphi_0}(g)$ is loxodromic for any hyperbolic element $g \in G$ which is not conjugate to a power of g_k for any k.

If the geodesic corresponding to g meets C_k for some $k > k_0$, then $\chi_{\varphi_0}(g)$ is loxodromic. Hence we can take $\epsilon(g)$, $N(g)$ satisfying (a) for g in this theorem. Thus we assume that the geodesic corresponding to g is contained in some P_k. By (2) in Lemma 5, there exists $N(g) > 0$ so that $\chi_{\varphi_m}(g)$ is loxodromic for $m \geq N(g)$. By assumption (a), there exist $\epsilon(g) > 0$ such that for $m \geq N(g)$, inequalities

$$|\mathrm{tr}^2(\chi_{\varphi_m}(g)) - 4| \geq \epsilon(g)$$

hold. Since $\chi_{\varphi_m}(g) \to \chi_{\varphi_0}(g)$ as $m \to \infty$, we have that $\mathrm{tr}^2(\chi_{\varphi_0}(g)) \neq 4$. Since χ_{φ_0} is an isomorphism and G is torsion free, $\chi_{\varphi_0}(g)$ is loxodromic.

Thus, if $1 \leq i \leq s_0$, $G_{i,0}$ is either a quasi-Fuchsian group or a totally degenerate group without A.P.T.s (cf.[15], [17, p.225, Theorem D.21], and [17, p.268, Theorem C.25]). We assume that $G_{i,0}$ is a quasi-Fuchsian group for some i. By the arguments above, for $m \geq 1$, the isomorphism $\chi_{\varphi_m} \circ \chi_{\varphi_0}^{-1}$ from $G_{i,0}$ onto $G_{i,m}$ is allowable in the sence of Bers (cf. [5, p.574]). Since $\chi_{\varphi_m} \circ \chi_{\varphi_0}^{-1}$ converges to the identity on $G_{i,0}$, by the quasiconformally stability for quasi-Fuchsian groups (cf. [5, Proposition 6]), $G_{i,m}$ is quasi-Fuchsian for sufficiently large m. This is contradiction. Thus, $G_{i,0}$ is a totally degenerate group without A.P.T.s for $i = 1, \ldots, s_0$. Thus, G_{φ_0} is a b-group with no moduli such that $\mathcal{C}_{\varphi_0} = \{C_k\}_{k>k_0}$. □

3.3. *Remark.* (1) *Any* sequence correponding to terminal regular b-groups which converges to $\varphi_0 \in \partial T(G)$ corresponding to totally degenerate group without A.P.T.s satisfies (a) and (b) in Theorem 1.

(2) For any $\varphi_0 \in \partial T(G)$, there exists $\{\varphi_m\}_{m=1}^{\infty}$ in $\partial T(G)$ corresponding to terminal regular b-groups such that (2-i) Area$(\mathbb{C} \setminus \Delta_{\varphi_m})$ tends to zero, and (2-ii) $\{\varphi_m\}_{m=1}^{\infty}$ converges to φ_0.

(3) Any totally degenerate group G_{φ_0} *with* A.P.T.s has $\{\varphi_m\}_{m=1}^{\infty}$ in $\partial T(G)$ conresponding to terminal regular b-groups which converges to φ_0 such that (3-i) $\mathrm{Area}(\mathbb{C} \setminus \Delta_{\varphi_m})$ tends to zero, and (3-ii) $\{\varphi_m\}_{m=1}^{\infty}$ does not satisfy (a) in Theorem 1.

(4) If $\dim T(G) > 1$, there exists $\{\varphi_m\}_{m=1}^{\infty}$ corresponding to terminal regular b-groups in $\partial T(G)$ satisfying (a) in Theorem 1 such that $\{G_{\varphi_m}\}_{m=1}^{\infty}$ converges to a b-group but not a totally degenerate group.

Proof. Before proving (1)-(4) above, we note that $\{\varphi_m\}_{m=1}^{\infty}$ corresponding to terminal regular b-groups in $\partial T(G)$ which converges to $\varphi_0 \in \partial T(G)$ corresponding to a totally degenerate group without A.P.T.s satisfies that $\mathrm{Area}(\mathbb{C} \setminus \Delta_{\varphi_m}) \to 0$ as $m \to \infty$. Indeed, it follow from the following two facts; (1) The measure of $\mathbb{C} \setminus \Delta_{\varphi_0} = \Lambda(G_{\varphi_0})$ is zero by Thurston's theorem (cf.[18]), and (2) $\{\Delta_{\varphi_m}\}_{m=1}^{\infty}$ converges to Δ_{φ_0} in the sence of kernel convergence with respect to $w_0 = \infty$ (cf. [21, Theorem 1.8]).

Let us prove Remark (1)-(4).

(1) By the argument above, the sequence satisfies (a) in Theorem 1. Since G_{φ_0} has no A.P.T.s, that also satisfies (a) in Theorem 1. □

(2) Since the set of differentials corresponding to terminal regular b-groups and the set of those corresponding to totally degenerate groups without A.P.T.s are dense in $\partial T(G)$ (cf.[19] and [5, Theorem 14]), by the standerd arguments and Remark (1), we find a sequence satisfying (2-i) and (2-ii). This remark was pointed out to the author by Professor Hiroshige Shiga. □

(3) Let $\{g_k\}_{k=1}^{s}$ be hyperbolic elements in G so that $\{\chi_{\varphi_0}(g_k)\}_{k=1}^{s}$ is a basis of A.P.T.s of G_{φ_0}. Take $L_0 > 0$ so that $2\cosh(L_0/2) := \max_{1 \le k \le s} |\mathrm{tr}(g_k)|$. By applying the argument in Lemma 5 for $\{\psi_m\}_{m=1}^{\infty}$ corresponding to totally degenerate groups without A.P.T.s in $\partial T(G)$ which converges to φ_0, there exists a terminal regular b-group G_{φ_m} such that $l_{\Sigma/G}(C) > mL_0$ for each $C \in \mathrm{C}_{\varphi_m}$, $\|\varphi_m - \psi_m\| < 1/m$, and that $\mathrm{Area}(\mathbb{C} \setminus \Delta_{\varphi_m}) < 1/m$ for $m \ge 1$. By the definition of L_0, $\chi_{\varphi_m}(g_k)$ is loxodromic for each $m \ge 1$ and $k = 1, \dots, s$.

Since $\chi_{\varphi_0}(g_k)$ is parabolic, $\{\varphi_m\}_{m=1}^{\infty}$ satisfies (3-i) and (3-ii). □

(4) Let $R = \Sigma/G$ and $\mathcal{C} = \{C_k\}_{k=1}^{d}$ a maximal partition on R. Let $\{P_s\}_{s=1}^{s_1}$ be the components of $R \setminus \cup_{k \ne 1} C_k$ such that $C_1 \subset P_1$. Since $d = \dim T(G) > 1$, we may suppose that $s_1 > 1$ and that P_2 is a pair of pants. Let R_1 be the infinite Nielsen extension of P_1 (cf.[6]), , $_1$ the Fuchsian group of R_1, and $(, _1)_{\psi_0}$ a totally degenerate group without A.P.T.s. We define $\{\varphi_m\}_{m=1}^{\infty}$ of a sequence corresponding to terminal regular b-groups and $\varphi_0 \in \partial T(G)$ satisfying the conditions (a), (b), (c), and (d) in Theorem 3 in Section 3.4 for the partition $\mathcal{C}$, $s_0 = 1$, and the boundary group $F_1 := (, _1)_{\psi_0}$. Then $\{\varphi_m\}_{m=1}^{\infty}$ satisfies the assertion. □

If $\dim T(G) = 1$, then G_{φ} has an A.P.T. if and only if G_{φ} is a terminal regular b-group. By Remark (1) and the proof of Proposition 6, we have

Corollary 2. *Suppose that* $\dim T(G) = 1$. *For a sequence* Φ *corresponding to terminal regular b-groups in* $\partial T(G)$ *which converges to* $\varphi_0 \in \partial T(G)$, *the following three conditions are equivalent:*

(1) G_{φ_0} *is a totally degenerate group.*
(2) Φ *contains a subsequence with* (a) *and* (b) *in Theorem* 1.

(3) Φ *contains a subsequence which consists of mutually distinct elements and satisfies* (a) *in Theorem* 1.

3.4. To complete the proof of Remark (4) in the previous subsection, we will show the following theorem.

Theorem 3. *Let G be a finitely generated Fuchsian group of the first kind acting on Σ and $R = \Sigma/G$. Let $\mathcal{C} = \{C_k\}_{k=1}^{k_0}$ be a partition on R and $\{P_s\}_{s=1}^{s_0}$ the components of $R \setminus \cup_{k=1}^{k_0} C_k$ each of which is not a pair of pants. For $i = 1, \dots, s_0$, let F_i be a boundary group such that Δ_{F_i}/F_i is homeomorphic to P_i. Then there exist $\varphi_0 \in \partial T(G)$ and $\{\varphi_m\}_{m=1}^{\infty}$ corresponding to terminal regular b-groups such that*

(a) $\varphi_m \to \varphi_0$ *as* $m \to \infty$,
(b) $\mathcal{C} \subset \mathcal{C}_{\varphi_m}$ *for* $m \geq 0$, *and*
(c) *A covering group of P_i in G_{φ_0} is quasiconformally conjugate to F_i,*

If, in addition, each F_i is a totally degenerate group without A.P.T.s, then

(d) Φ *satisfies* (a) *in Theorem* 1.

This theorem is proved in Section 3.6.

3.5. The following lemma is well-known. However, the author has never seen what is stated in this form.

Lemma 7. *Let R and S be a hyperbolic Riemann surface of type (p, n). Let P be a domain in R such that P is homeomorphic to R and that the inclusion mapping i from P to R is homotopic to a homeomorphsm of P onto R. Then, for $K \geq 1$, there exists $K_0 = K_0(K, P, p, n) > 1$ such that if a K-quasiconformal(q.c.) mapping h from P into S which is homotopic to a homeomorphism from P onto S exists, there exists a K_0-q.c.mapping g from R to S so that $g \circ i$ is homotopic to h.*

Proof. Let $T(R)$ be a Teichmüller space of R (cf. [9, p.120]). Let M be a Riemann surface of type (p, n). If there exists a K-q.c.mapping h_M from P to M homotopic to a homeomorphism of P onto M, then there exists a q.c.mapping $f_{(P,K,M)}$ from R onto S such that $f_{(P,K,M)} \circ i$ is homotopic to h_M. We denote by $X(P, K)$ the closure of the set of such $[M, f_{(P,K,M)}]$ in $T(R)$. Let $\hat{i}$ be a homeomorphism form P to R homotopic to i. Let $\{\gamma_i\}_{i=1}^{N}$ be a system of simple closed geodesics fill up R (cf. [10, p.249]). By the decreasing property for the hyperbolic metric and Wolpert's Theorem (cf. [8, p.153]), $\sum_{i=1}^{N} l_S(f_{(P,K,M)}(\gamma_i)) \leq \sum_{i=1}^{N} K l_P(\hat{i}^{-1}(\gamma_i))$. Hence $X(P, K)$ is compact (cf. [10, Lemma 3.1]). Let d_0 be the diameter of $X(P, K)$ with respect to the Teichmüller distance of $T(R)$ (cf. [9, p.125]). Then, $K_0 := e^{d_0}$ satisfies the assertion. □

3.6. Let us prove Theorem 3. We only show the case where $s_0 = 1$. Another cases are proved by the similar manner.

Let $\{P_i\}_{i=1}^{s}$ be the components of $R \setminus \cup_{k=1}^{k_0} C_k$ such that P_1 is not a pair of pants. Let $\hat{R}_1$ be the infinite Nielsen extension of P_1. Since $R_1 := \Delta_{F_1}/F_1$ is homeomorphic to P_1, there exists a K_1-q.c.mapping h_0 from $\hat{R}_1$ onto R_1. Let $Q := h_0(P_1)$, i an inclusion mapping from Q to R_1. Then, by definition, i is homotopic to a homeomorphism from Q onto R_1. Let f be a conformal mapping from Σ to Δ_{F_1} and $, _1 = f^{-1} F_1 f$. We take $\{\psi_m\}_{m=1}^{\infty}$ in $\partial T(, _1)$ corresponding to terminal regular b-groups which converges to $\psi_0 := \{f, -\} \in \partial T(\ , \ _1)$. Let $\mathcal{C}_{\psi_m} = \{C'_{k,m}\}_{k=1}^{k_1}$ and $C_{k,m}$ the geodesic in P_1 (and hence in R) such that $i \circ h_0(C_{k,m})$ is homotopic to $C'_{k,m}$ for $k = 1, \dots, k_1$. Then, $\mathcal{C}_m := \{C_{i,m}, C_j\}_{i=1,\dots,k_1, j=1,\dots,k_0}$ is a maximal partition on R for $m \geq 1$. Take the terminal regular b-group G_{φ_m} so that $\mathcal{C}_{\varphi_m} = \mathcal{C}_m$ (cf.[1, Theorem 6]). We may suppose that $\Phi := \{\varphi_m\}_{m=1}^{\infty}$ converges to some $\varphi_0 \in \partial T(G)$. By definition, Φ and φ_0 satisfy (a) and (b) in this Theorem.

We prove that Φ satisfies (c). Let π be the projection from Σ to R and $\tilde{P}_1$ a component of $\pi^{-1}(P_1)$. We may assume that $\infty \in \tilde{P}_1$. Let H_1 be the stabilizer subgroup of $\tilde{P}_1$ in G and $G_m := \chi_{\varphi_m}(H_1)$ for $m \geq 0$. Then for $m \geq 1$, G_m is a covering group of P_1 in G_{φ_m} and is a terminal regular b-group (cf.[11]).

Let $S_m = \Delta_{G_m}/G_m$ and π_m the projection from Δ_{G_m} to S_m. Then there exists the injective holomorphic mapping h_n from P_1 to S_m such that $h_m \circ \pi \mid_{\tilde{P}_1} = \pi_m \circ W_{\varphi_m} \mid_{\tilde{P}_1}$. By definition, h_n is homotopic to a homeomorphism from P_1 to S_m (cf.[14]). Hence, by Lemma 7, there exist $K_0 = K_0(K_1, Q, p, n) > 0$, and the K_0-q.c.mapping g_m from R_1 to S_m so that $g_m \circ i$ is homotopic to $h_m \circ (h_0 \mid_{P_1})^{-1}$.

Fix the lift $\tilde{h}_0$ of $i \circ h_0 \mid_{P_1}$ from $\tilde{P}_1$ into Σ. $\tilde{h}_0$ defines the isomorphism ξ from H_1 to $\ _1$ by $\xi(h) \circ \tilde{h}_0 = \tilde{h}_0 \circ h$ for $h \in H_1$. By definition, h_m induces the isomorphism $\chi_{\varphi_m} \mid_{H_1}$. Since for $m \geq 1$, $g_m \circ i \circ h_0 \mid_{P_1}$ is homotopic to h_m, there exists the lift $\tilde{g}_m$ of g_m from Σ onto Δ_{G_m} so that the isomorphism $\tilde{\eta}_m$ from $\ _1$ to G_m defined by $\tilde{\eta}_m(\gamma) = \tilde{g}_m \gamma \tilde{g}_m^{-1}$ satisfies that $\tilde{\eta}_m \circ \xi = \chi_{\varphi_m} \mid_{H_1}$.

Let $w_m = \tilde{g}_m \circ W_{\psi_m}^{-1}$. Then w_m is a K_0-q.c.mapping from Δ_{ψ_m} onto Δ_{G_m} and defines the isomorphism η_m from G_{ψ_m} to G_m by $\eta_m(g) = w_m g w_m^{-1}$. Then, η_m satisfies that $\eta_m = \chi_{\varphi_m} \mid_{H_1} \circ (\chi_{\psi_m} \circ \xi)^{-1}$. Since $g_m \circ i$ is homotopic to h_m, η_m is type preserving. Since G_{ψ_m} and G_m are terminal regular, by the rigidity of triangle groups, w_m can be extended to the K_0-q.c.mapping on $\hat{\mathbb{C}}$ conpatible with G_{ψ_m}. This extension is denoted by the same symbol w_m for short.

To prove (c) in Theorem 3, it suffices to show that the family $\{w_m\}_{m=1}^{\infty}$ contains a subsequece which converges to a K_0-q.c.mapping w_0 on $\hat{\mathbb{C}}$. Indeed, since $w_m G_{\psi_m} w_m^{-1} = \eta_m(G_{\psi_m}) = G_m$ for $m \geq 1$, $w_0 G_{\psi_0} w_0^{-1} = G_0$.

Take primitive hyperbolic $g_1, g_2 \in H_1$ so that g_1 is not conjugate to g_2 in H_1 and that $\alpha_{i,0} := \chi_{\varphi_0}(g_i)$ and $\beta_{i,0} := \chi_{\psi_0} \circ \xi(g_i)$ are loxodromic. Let $\alpha_{i,m} = \chi_{\varphi_m}(g_i)$ and $\beta_{i,m} := \chi_{\psi_m} \circ \xi(g_i)$. Then there exists $N_1 > 0$ such that for $m \geq N_1$, $\alpha_{i,m}$ and $\beta_{i,m}$ are loxodromic. For $m \geq N_1$, let $\{a_{2i-1,m}, a_{2i,m}\}$ and $\{b_{2i-1,m}, b_{2i,m}\}$ be the set of the fixed points of $\alpha_{i,m}$ and $\beta_{i,m}$ respectively. By discreteness, the cardinality of $\{a_{j,m}\}_{i=1}^{4}$ and $\{b_{i,m}\}_{i=1}^{4}$ are equal to 4 for $m \geq N_1$ or $m = 0$. Since $\alpha_{i,m} \to \alpha_{i,0}$ and $\beta_{i,m} \to \beta_{i,0}$, we may suppose that there exist $N_0 \geq N_1$ and $d > 0$ such that $k(a_{j,m}, a_{j,0})$, $k(b_{j,m}, b_{j,0}) < d$ for $j = 1, \dots, 4$ and $m \geq N_0$ and that $k(a_{i,0}, a_{j,0})$, $k(b_{i,0}, b_{j,0}) > 4d$ for $i \neq j$, where $k(-,-)$ is the spherical distance on $\hat{\mathbb{C}}$. Let $B_i = \{z \in \hat{\mathbb{C}} \mid k(z, a_{i,0}) \leq d\}$. Since $w_m(\{a_{i,m}\}_{i=1}^{4}) = \{b_{i,m}\}_{i=1}^{4}$, by applying an argument similar to that of Theorem 4.2 in [13, p.70] for domains $\{\hat{\mathbb{C}} \setminus B_i \cup B_j\}_{i \neq j}$, there exists a subsequence $\{w_{m_j}\}_{j=1}^{\infty}$ and a K_0-q.c.mapping w_0 so that w_{m_j} converges uniformly to w_0.

It is easy to observe that if each F_i is a totally degenerate group without A.P.T.s, then $\{\varphi_m\}_{m=1}^{\infty}$ satisfies (d). □

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