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The universal Teichmüller space and related topics

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Abstract. In this survey, we give an expository account of the universal Teichmüller space with emphasis on the connection with univalent functions. In the theory, the Schwarzian derivative plays an important role. We also introduce many interesting results involving Schwarzian derivatives and pre-Schwarzian derivatives, as well.

Keywords. universal Teichmüller space, univalent function, Schwarzian derivative, pre-Schwarzian derivative.

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Contents

1. Preliminary

In this section, we prepare basic tools to understand the universal Teichmüller space. The material is more or less standard, but for convenience, an expository account will be given without proofs. The most convenient reference for overall topics is perhaps the recently published handbook [58].

1.1. Quasiconformal mappings. A homeomorhism f of a plane domain D onto another domain D' is called a *quasiconformal map* if f has locally square integrable partial derivatives (in the sense of distribution) and satisfies the inequality

$$|f_{\bar{z}}| \le k|f_z|$$

almost everywhere in D, where k is a constant with $0 \le k < 1$,

$$f_z = \tfrac{1}{2}(f_x - if_y), \quad f_{\bar{z}} = \tfrac{1}{2}(f_x + if_y)$$

and

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}.$$

It turns out that f preserves sets of (2-dimensional) Lebesgue measure zero and, in particular, $f_z \neq 0$ a.e. Thus the quotient $\mu = f_{\bar{z}}/f_z$ is well defined as a Borel measurable function on D and satisfies $\|\mu\|_\infty \le k < 1$. This function is sometimes called the *complex dilatation* of f and denoted by μ_f. More specifically, f is also called a K-quasiconformal map, where $K = (1+k)/(1-k)$. The minimal $K = (1+\|\mu\|_\infty)/(1-\|\mu\|_\infty)$ is called the *maximal dilatation* of f and denoted by $K(f)$. It is known that a 1-quasiconformal map is conformal (i.e., biholomorphic) and vice versa. The composition of a K_1-quasiconformal map and a K_2-quasiconformal map is K_1K_2-quasiconformal map and the inverse map of a K-quasiconformal map is also K-quasiconformal. In particular, K-quasiconformality is preserved under composition with conformal maps. Therefore, K-quasiconformality and, hence, quasiconformality can be defined for homeomorphisms between Riemann surfaces. In particular, we can argue quasiconformality of a homeomorphism of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

More precise information about compositions of quasiconformal maps will be needed later. Let $f : \Omega \to \Omega'$ and $g : \Omega \to \Omega''$ be quasiconformal maps. Then the complex dilatation of $g \circ f^{-1}$ is given by

$$(\mu_{g\circ f^{-1}} \circ f)\frac{\overline{f_z}}{f_z} = \frac{\mu_g - \mu_f}{1 - \overline{\mu_f} \cdot \mu_g}. \tag{1.1.1}$$

In particular, we obtain the following lemma.

Lemma 1.1.2. *Let $f : \Omega \to \Omega'$ and $g : \Omega \to \Omega''$ be quasiconformal maps. Then $g \circ f^{-1}$ is conformal on Ω' if and only if $\mu_f = \mu_g$ a.e. in Ω.*

Fundamental in the theory of quasiconformal maps is the following exsistence and uniqueness theorem.

Theorem 1.1.3 (The measurable Riemann mapping theorem)**.** *For any measurable function μ on $\mathbb{C}$ with $\|\mu\|_\infty < 1$, there exists a unique quasiconformal map $f : \mathbb{C} \to \mathbb{C}$ such that $f(0) = 0, f(1) = 1$ and $f_{\bar z} = \mu f_z$ a.e. in $\mathbb{C}$.*

For the proof of the theorem and for more information about quasiconformal maps, the reader should consult the books [3] and [66] as well as the paper [4] by Ahlfors and Bers.

We denote by Belt(D) the open unit ball of the space $L^\infty(D)$ for a domain (or, more generally, a measurable set) D. An element μ of Belt(D) is called a *Beltrami coefficient* on D. For a Beltrami coefficient μ on $\mathbb{C}$, the function f given in the measurable Riemann mapping theorem will be denoted by f^μ throughout the present survey.

Let μ be a Beltrami coefficient on the outside $\mathbb{D}^*$ of the unit disk. We extend μ to $\mu^* \in$ Belt$(\mathbb{C})$ by setting $\mu^*(z) = \overline{\mu(1/\bar z)}$ for $z \in \mathbb{D}^*$. Let f be a quasiconformal automorphism of $\widehat{\mathbb{C}}$ fixing $1, -1, -i$ with $\mu_f = \mu^*$. Since $f(z)$ and $1/\overline{f(1/\bar z)}$ both have the same complex dilatation μ^* and satisfy the same normalization condition, they must be equal by uniqueness part of the measurable Riemann mapping theorem. In particular, $|f(z)|^2 = 1$ for $z \in \partial\mathbb{D}$, and consequently, f maps $\mathbb{D}^*$ onto itself. We define $w_\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by $w_\mu = f$. Recall that w_μ fixes $1, -1$ and $-i$.

The following fact was observed by Ahlfors and Bers [4].

Theorem 1.1.4. *Let μ_t be a family of Beltrami coefficients on $\mathbb{C}$ holomorphically parametrized over a complex manifold X. Then the map $t \mapsto f^{\mu_t}(z)$ is holomorphic on X for a fixed $z \in \mathbb{C}$.*

We do not explain the meaning of "holomorphically parametrized" here. It is, however, sufficient practically to know that $(t\mu + \nu)/(1 + t\bar\nu\mu)$ is a family of Beltrami coefficients holomorphically parametrized over the unit disk $|t| < 1$, where $\mu, \nu \in$ Belt$(\mathbb{C})$.

1.2. Hyperbolic Riemann surfaces. A connected complex manifold of complex dimension one is called a *Riemann surface*. The Poincaré-Koebe uniformization theorem tells us that every Riemann surface R admits an analytic universal covering projection p of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto R except for the case when R is conformally equivalent to the Riemann sphere $\widehat{\mathbb{C}}$, the complex plane $\mathbb{C}$, the punctured complex plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ or a complex torus (a smooth elliptic curve). Those non-exceptional Riemann surfaces are called *hyperbolic*.

The group of analytic automorphisms of R is denoted by $\mathrm{Aut}(R)$. The group of disk automorphisms $\mathrm{Aut}(\mathbb{D})$ is identified with $\mathrm{PSU}(1,1)$ and isomorphic to $\mathrm{PSL}(2,\mathbb{R})$ through the Möbius transformation $M : \mathbb{H} = \{z : \mathrm{Im}\, z > 0\} \to \mathbb{D}$ defined by $M(z) = (z-i)/(z+i)$. Thus $\mathrm{Aut}(\mathbb{D})$ inherits a structure of real Lie group. A subgroup Γ of $\mathrm{Aut}(\mathbb{D})$ is called *Fuchsian* if Γ is discrete. It is known that Γ is discrete if and only if Γ acts on $\mathbb{D}$ properly discontinuously. Note also that Γ is torsion-free if and only if Γ acts on $\mathbb{D}$ without fixed points. The covering transformation group $\Gamma = \{\gamma \in \mathrm{Aut}(D) : p \circ \gamma = p\}$ is a torsion-free Fuchsian group and will be called the *Fuchsian model* of R. Conversely, for a given torsion-free Fuchsian group Γ the quotient space $\mathbb{D}/\Gamma$ has natural complex structure so that the projection $\mathbb{D} \to \mathbb{D}/\Gamma$ becomes an analytic universal covering. In this way, theory of hyperbolic Riemann surfaces can be translated into that of torsion-free Fuchsian groups.

Since the Poincaré metric $\rho_{\mathbb{D}}(z)|dz| = |dz|/(1-|z|^2)$ is invariant under the pull-back by analytic automorphisms of $\mathbb{D}$, it projects to a smooth metric, denoted by $\rho_R = \rho_R(w)|dw|$, on the hyperbolic Riemann surface R via p. The metric ρ_R is called the *hyperbolic metric* of R. Thus ρ_R is characterized by the relation $\rho_{\mathbb{D}} = p^*(\rho_R) = \rho_R(p(z))|p'(z)||dz|$.

Note that ρ_R has constant Gaussian curvature -4, in other words, $\Delta \log \rho_R = 4\rho_R^2$.

The Schwarz-Pick lemma implies the contraction property $f^*\rho_S \le \rho_R$ for a holomorphic map $f : R \to S$ between hyperbolic Riemann surafaces R and S, where equality holds at some (hence every) point in R iff f is a covering projection of R onto S.

1.3. Quadratic differentials. Let $H(\mathbb{D})$ be the set of analytic functions on the unit disk $\mathbb{D}$ and let n be a non-negative integer. For a Fuchsian group Γ, a function $\varphi \in H(\mathbb{D})$ is said to be *automorphic* for Γ (with weight $-2n$) if φ satisfies the functional equation $(\varphi \circ \gamma)(\gamma')^n = \varphi$ for every $\gamma \in \Gamma$, that is to say, $\varphi(z)dz^n$ is an invariant n-form for Γ. The set of automorphic functions for Γ with weight $-2n$ will be denoted by $H_n(\mathbb{D}, \Gamma)$.

An element φ of $H_n(\mathbb{D}, \Gamma)$ for a torsion-free Fuchsian group Γ projects to a holomorphic n-form $q = q(w)dw^n$ on $R = \mathbb{D}/\Gamma$ so that $p_n^* q = \varphi(z)dz^n$, where $p_n^* q$ means the pull-back $q(p(w))(p'(w))^n$ of the n-form q by the canonical projection $p : \mathbb{D} \to \mathbb{D}/\Gamma$.

We now define two norms for $\varphi \in H_n(\mathbb{D},\Gamma)$ by

$$\|\varphi\|_{A_n(\mathbb{D},\Gamma)} = \iint_{\omega} |\varphi(z)|(1-|z|^2)^{n-2}dxdy,$$
$$\|\varphi\|_{B_n(\mathbb{D},\Gamma)} = \sup_{z\in\mathbb{D}} |\varphi(z)|(1-|z|^2)^{n},$$

where ω is a fundamental domain for Γ, that is, a subdomain of $\mathbb{D}$ such that $\omega \cap \gamma(\omega) = \emptyset$ for every $\gamma \in \Gamma$ with $\gamma \neq \mathrm{id}$, $\bigcup_{\gamma\in\Gamma}\gamma(\bar{\omega}) = \mathbb{D}$ and $\partial\omega$ is of zero area. We denote by $A_n(\mathbb{D},\Gamma)$ and $B_n(\mathbb{D},\Gamma)$ the subsets of $H_n(\mathbb{D},\Gamma)$ consisting of φ with finite norm $\|\varphi\|_{A_n(\mathbb{D},\Gamma)}$ and $\|\varphi\|_{B_n(\mathbb{D},\Gamma)}$, respectively. It is easy to see that these become complex Banach spaces. When Γ is the trivial group 1, we write $A_n(\mathbb{D})$ and $B_n(\mathbb{D})$ for $A_n(\mathbb{D},1)$ and $B_n(\mathbb{D},1)$, respectively.

The definition of the spaces $A_n(\mathbb{D})$ and $B_n(\mathbb{D})$ can be extended for hyperbolic Riemann surfaces R. Let $H_n(R)$ denote the set of holomorphic n-forms on R and set

$$\|\varphi\|_{A_n(R)} = \iint_{R} |\varphi(w)|\rho_R(w)^{2-n}dxdy,$$
$$\|\varphi\|_{B_n(R)} = \sup_{w\in R} |\varphi(w)|\rho_R(w)^{-n}$$

for $\varphi = \varphi(w)dw^n$ in $H_n(R)$. Here, we should note that $|\varphi(w)|\rho_R(w)^{-n}$ does not depend on the choice of the local coordinate w, in other words, $|\varphi|\rho_R^{-n}$ can be regarded as a *function* on R.

The Banach spaces $A_n(\mathbb{D},\Gamma)$ and $A_n(\mathbb{D}/\Gamma)$ (resp. $B_n(\mathbb{D},\Gamma)$ and $B_n(\mathbb{D}/\Gamma)$) are isometrically isomorphic through the pull-back p_n^* by the projection $p : \mathbb{D} \to \mathbb{D}/\Gamma$. Also, the following invariance property is convenient to note.

Lemma 1.3.1. *Let R and S be hyperbolic Riemann surfaces and let $p : R \to S$ be a conformal homeomorphism. Then the pullback operator $p_n^* : B_n(S) \to B_n(R)$ is a linear isometry, in other words,*

$$\|p_n^*\varphi\|_{B_n(R)} = \|\varphi\|_{B_n(S)}, \quad \varphi \in B_n(S).$$

In the theory of Teichmüller spaces, it is important to consider the spaces A_2 and B_2 as we shall see later. A 2-form $q(w)dw^2$ is traditionally called a *quadratic differential.*

1.4. Univalent functions. In connection with the universal Teichmüller space, the theory of univalent functions is of particular importance. The best textbook in this direction is [64] by O. Lehto.

We denote by $\mathscr{S}$ the set of analytic univalent functions f on the unit disk so normalized that $f(0) = 0$ and $f'(0) = 1$. An analytic function f around the origin is said to be *strongly normalized* if $f(0) = f'(0) - 1 = f''(0) = 0$. Let $\mathscr{S}_0$ be the subset of $\mathscr{S}$ consisting of strongly normalized functions. For $f \in \mathscr{S}$, the function $g = f/(1 + af)$, where $a = f''(0)/2$, is strongly normalized but not necessarily analytic in $\mathbb{D}$. It is thus natural to consider the wider class

$$\tilde{\mathscr{S}}_0 = \{f : \text{meromorphic and univalent in } \mathbb{D} \text{ and strongly normalized}\}$$

than $\mathscr{S}_0$.

The following meromorphic counterpart is also useful in the theory of univalent functions. Let Σ be the set of meromorphic univalent functions F on the exterior $\mathbb{D}^* = \{\zeta \in \widehat{\mathbb{C}} : |\zeta| > 1\}$ of the unit disk so normalized that

$$F(\zeta) = \zeta + b_0 + \frac{b_1}{\zeta} + \frac{b_2}{\zeta^2} + \dots \tag{1.4.1}$$

in $|\zeta| > 1$.

For $f \in \mathscr{S}$, the function $F(\zeta) = 1/f(1/\zeta)$ belongs to Σ and satisfies the condition $0 \notin F(\mathbb{D}^*)$, and vice versa. Let Σ' denote the set of those functions $F \in \Sigma$ that satisfy $0 \notin F(\mathbb{D}^*)$. Moreover, $b_0 = 0$ for a function $F(\zeta) = \zeta + b_0 + b_1/\zeta + \dots$ in Σ if and only if $f \in \tilde{\mathscr{S}}_0$, where $f(z) = 1/F(1/z)$. Hence, if we set

$$\Sigma_0 = \{F \in \Sigma : F(\zeta) - \zeta \to 0 \text{ as } \zeta \to \infty\},$$

the correspondence $f(z) \mapsto F(\zeta) = 1/f(1/\zeta)$ gives bijections of $\tilde{\mathscr{S}}_0$ onto Σ_0 and of $\mathscr{S}_0$ onto Σ_0', where we define $\Sigma_0' = \Sigma_0 \cap \Sigma'$.

1.5. Grunsky inequality. For a meromorphic function F near the point at infinity with an expansion of the form (1.4.1), we take a single-valued branch of $\log((F(\zeta) - F(\omega))/(\zeta - \omega))$ in $|\zeta| > R$ and $|\omega| > R$ for sufficiently large $R > 0$ and expand it in the form

$$\log \frac{F(\zeta) - F(\omega)}{\zeta - \omega} = -\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{b_{j,k}}{\zeta^j \omega^k}$$

there. The coefficients $b_{j,k}$ are called the Grunsky coefficients of F. It is easy to see that $b_{j,k} = b_{k,j}$ and $b_{1,k} = b_k$ for $j, k \geq 1$, where b_k is the coefficient in (1.4.1). The last relation is deduced in the following way. If we write $F(\zeta) = \zeta + b_0 + G(\zeta)$, then $G(\zeta) = O(|\zeta|^{-1})$ as $\zeta \to \infty$. Fix ω for a moment. Since

$$\log \frac{F(\zeta) - F(\omega)}{\zeta - \omega} = \log\left(1 + \frac{G(\zeta) - G(\omega)}{\zeta - \omega}\right) = \frac{G(\zeta) - G(\omega)}{\zeta - \omega} + O(|\zeta|^{-2}),$$

we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{b_{1,k}}{\omega^k} &= -\lim_{\zeta \to \infty} \zeta \log \frac{F(\zeta) - F(\omega)}{\zeta - \omega} \\ &= -\lim_{\zeta \to \infty} \zeta \frac{G(\zeta) - G(\omega)}{\zeta - \omega} \\ &= G(\omega), \end{aligned}$$

from which the required relation follows.

The following theorem is greatly useful in the theory of Teichmüller spaces as well as the theory of univalent functions. See [40], [27] or [82] for the proof and applications.

Theorem 1.5.1 (Grunsky). *A meromorphic function $F(\zeta)$ with expansion of the form* (1.4.1) *around $\zeta = \infty$ is analytically continued to a univalent meromorphic function in $|\zeta| > 1$ if and only if the inequality*

$$\sum_{k=1}^{\infty} k \left| \sum_{j=1}^{\infty} b_{j,k} x_j \right|^2 \le \sum_{j=1}^{\infty} \frac{|x_j|^2}{j} \tag{1.5.2}$$

holds for an arbitrary sequence of complex numbers $x_1, x_2, \dots$.

The inequality in (1.5.2) is known as the *strong Grunsky inequality.* Noting $b_{1,k} = b_k$, we take $(x_1, x_2, x_3, \dots) = (1, 0, 0, \dots)$ to obtain

$$\sum_{k=1}^{\infty} k|b_k|^2 \le 1. \tag{1.5.3}$$

This inequality is known as Gronwall's area theorem.

It is also known that inequality (1.5.2) can be replaced in the above theorem by the (classical) Grunsky inequality:

$$\left| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} b_{j,k} x_j x_k \right| \le \sum_{j=1}^{\infty} \frac{|x_j|^2}{j}. \tag{1.5.4}$$

The symmetric matrix $(\sqrt{jk} b_{j,k})$ defines a linear operator on ℓ^2, where $b_{j,k}$ are the Grunsky coefficient of a meromorphic function $F(\zeta)$ around $\zeta = \infty$. This is sometimes called the Grunsky operator and will be denoted by $\mathcal{G}[F]$ in the following. The strong Grunsky inequality says that $F \in \Sigma$ if and only if $\mathcal{G}[F]$ is a bounded linear operator on ℓ^2 with operator norm ≤ 1. Here, the operator norm $\|\mathcal{G}[F]\|$ of $\mathcal{G}[F]$ is defined by

$$\|\mathcal{G}[F]\|^2 = \sup_{\|y\|_2 = 1} \sum_{k=1}^{\infty} \left| \sum_{j=1}^{\infty} \sqrt{jk} b_{j,k} y_j \right|^2,$$

where $\|y\|_2 = (\sum_k |y_k|^2)^{1/2}$ for $y = (y_1, y_2, \dots)$. Thus, $F \in \Sigma \Leftrightarrow \|\mathcal{G}[F]\| \le 1$. It is known (cf. [82]) that F has a quasiconformal extension to $\widehat{\mathbb{C}}$ if and only if $\|\mathcal{G}[F]\| < 1$.

1.6. Schwarzian derivative. For a non-constant meromorphic function f on a domain, we define T_f and S_f by

$$T_f = \frac{f''}{f'} = (\log f')',$$

$$S_f = (T_f)' - \frac{1}{2}(T_f)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f'}{f}\right)^2.$$

These are called the pre-Schwarzian derivative and the Schwarzian derivative of f, respectively. Note that T_f is analytic at a finite point z_0 if and only if f is analytic and injective around z_0. Similarly, S_f is analytic at z_0 if and only if f is meromorphic and injective around z_0. The following two lemmas show usefulness of these operations.

Lemma 1.6.1. *Let f be a non-constant meromorphic function on a domain D. The pre-Schwarzian derivative of f vanishes on D if and only if f is (the restriction of) a similarity. The Schwarzian derivative of f vanishes on D if and only if f is (the restriction of) a Möbius transformation.*

Lemma 1.6.2. *Let f and g be non-constant meromorphic functions for which the composition $f \circ g$ is defined. Then*

$$T_{f\circ g} = (T_f) \circ g \cdot g' + T_g = g_1^*(T_f) + T_g,$$
$$S_{f\circ g} = (S_f) \circ g \cdot (g')^2 + S_g = g_2^*(S_f) + S_g.$$

Combining these lemmas, we observe that $S_{L\circ f\circ M} = M_2^*(S_f)$ for Möbius transformations L and M. Thus the Schwarzian derivative behaves like a quadratic differential.

2. The universal Teichmüller space

We have two choices to develop the theory of the (universal) Teichmüller space; the unit disk model or the upper half-plane model. Although they can be translated into each other, in principle, via the Möbius transformation $z \mapsto (z-i)/(z+i)$, both models have their own advantage and thus can be chosen at will according to the purpose. In the present survey, we will take the unit disk model to connect with the theory of univalent functions in a direct way.

2.1. Definition 1: the quotient space of quasiconformal maps. We denote by $\mathrm{QC}(\mathbb{D})$ the set of quasiconformal automorphisms of the unit disk $\mathbb{D}$. As we will observe later, every function in $\mathrm{QC}(\mathbb{D})$ extends to a unique homeomorphism of the closed unit disk $\overline{\mathbb{D}}$. Thus, we may think that every $f \in \mathrm{QC}(\mathbb{D})$ is a self-homeomorphism of the closed unit disk $\overline{\mathbb{D}}$. Two functions f and g in $\mathrm{QC}(\mathbb{D})$ are said to be *Teichmüller equivalent* and denoted by $f \overset{\mathrm{T}}{\sim} g$ if there exists a disk automorphism $L \in \mathrm{Aut}(\mathbb{D})$ such that $g = L \circ f$ on $\partial\mathbb{D}$. The quotient space $\mathrm{QC}(\mathbb{D})/\overset{\mathrm{T}}{\sim}$ is a model of the universal Teichmüller space and will be denoted by T_1 for a moment. The equivalence class represented by $f \in \mathrm{QC}(\mathbb{D})$ will be denoted by $[f]$ below.

Let $f, g \in \mathrm{QC}(\mathbb{D})$. The *Teichmüller distance* between $p = [f]$ and $q = [g]$ is defined by

$$d_1(p,q) = \inf_{f_1 \overset{\mathrm{T}}{\sim} f,\, g_1 \overset{\mathrm{T}}{\sim} g} \frac{1}{2} \log K(g_1 \circ f_1^{-1}).$$

Recall here that $K(f)$ denotes the maximal dilatation of f. By a compactness property of quasiconformal maps, one can check that $d_1(p,q)$ is indeed a distance on T_1. In this way, T_1 becomes a metric space. It can also be shown that T_1 is a complete metric space with metric d_1 by a normality property of the set of normalized K-quasiconformal automorphisms of $\mathbb{C}$ (see [66]).

2.2. Definition 2: quasisymmetric functions. The notion of quasisymmetric functions was created by Beurling and Ahlfors [17] for functions on the real line. We give here a corresponding definition of quasisymmetric functions on the unit circle. A sense-preserving homeomorphism h of the unit circle $\partial\mathbb{D}$ is called *quasisymmetric* if

$$\frac{1}{M} \leq \frac{|h(e^{i(s+t)}) - h(e^{is})|}{|h(e^{is}) - h(e^{i(s-t)})|} \leq M, \quad s \in \mathbb{R},\ 0 < t < \frac{\pi}{2}$$

for a constant $M \geq 1$. The set of all quasisymmetric funtions on the unit circle will be denoted by $\mathrm{QS}(\partial\mathbb{D})$. The main result in [17] can be stated as follows.

Theorem 2.2.1 (Beurling-Ahlfors)**.** *The restriction of a quasiconformal automorphism of the unit disk to the unit circle is quasisymmetric. Conversely, a quasisymmetric function on the unit circle can be extended to a quasiconformal automorphism of the unit disk.*

Two functions h_1 and h_2 on the unit circle are called *Möbius equivalent* if there exists a disk automorphism $L \in \mathrm{Aut}(\mathbb{D})$ such that $h_2 = L \circ h_1$. Let T_2 denote the quotient space of $\mathrm{QS}(\partial\mathbb{D})$ by the Möbius equivalence. By the above theorem of Beurling and Ahlfors, one readily sees that T_1 can be identified with T_2 in a natural manner.

In order to get rid of taking quotient, we can define T_2 as follows. A (sense-preserving) homeomorphism h of $\partial\mathbb{D}$ is said to be normalized if h fixes the points $1, -1$ and $-i$. Since every Möbius equivalence class of quasisymmetric functions is represented by a unique normalized one, one can identify T_2 with the set of normalized quasisymmetric functions on the unit circle.

See [35] for a modern treatment of quasisymmetric functions.

2.3. Definition 3: marked quasidisks. A simply connected domain D in $\widehat{\mathbb{C}}$ is called a *quasidisk* if D is the image of the unit disk under a quasiconformal automorphism of $\widehat{\mathbb{C}}$. If D is the image under a K-quasiconformal automorphism, then D is called a K-quasidisk. Many characteristic properties of quasidisks are known. See, for instance, [38].

Let D be a quasidisk (or a Jordan domain more generally) and x_1, x_2, x_3 are positively ordered (distinct) points on ∂D. The quadruple (D, x_1, x_2, x_3) will be called a *marked quasidisk*. By the Riemann mapping theorem and the Carathéodory extension theorem, there exists a unique conformal homeomorphism $g : \mathbb{H} \to D$ with $g(0) = x_1, g(1) = x_2$ and $g(\infty) = x_3$.

We denote by Q the set of all marked quasidisks in $\widehat{\mathbb{C}}$. Two marked quasidisks (D, x_j) and (D', x'_j) are said to be *Möbius equivalent* if $D' = L(D)$ and $x'_j = L(x_j),\ j = 1, 2, 3,$ for some Möbius transformation $L \in \text{Möb} = \text{Aut}(\widehat{\mathbb{C}})$. We can define a pseudo-metric on Q by

$$d((D, x_j), (D', x'_j)) = \|S_f\|_{B_2(D)},$$

where f is a conformal homeomorphisms of D onto D' with $f(x_j) = x'_j$. It is easy to see that $d(D, D') = 0$ if and only if D and D' are Möbius equivalent.

The set T_3 of Möbius equivalence classes of all marked quasidisks constitutes another model of the universal Teichmüller space and the above-defined pseudo-metric gives a metric on T_3, which will be denoted by d_3, i.e.,

$$d_3(p, q) = \inf_{(D,x_j)\in p, (D',x'_j)\in q} d((D, x_j), (D', x'_j))$$

for $p, q \in \mathrm{T}_3$.

We can again take a suitable normalization to avoid the process of quotient and even marking. For instance, we may say that a quasidisk D is normalized if its boundary contains the points $0, 1$ and ∞ in positive order along the boundary curve. If we denote by Q_0 the set of normalized quasidisks in $\widehat{\mathbb{C}}$, then T_3 can be identified with Q_0 naturally, and the restriction of the distance d on Q_0 corresponds to the distance d_3 on T_3.

In the above, the marking is important. For two simply connected hyperbolic domains D_1 and D_2, we set

$$d(D, D') = \inf_{f: D \to D' \text{ conformal}} \|S_f\|_{B_2(D)}.$$

It is easy to see that d is a pseudo-distance. Lehto [64] posed a question whether or not $d(D, D') = 0$ implies that D and D' are Möbius equivalent. Osgood and Stowe [78] answered to this question in the negative (see also [18]).

2.4. Definition 4: Bers embedding. Let D be a hyperbolic domain in $\widehat{\mathbb{C}}$. We define a subset $T(D)$ of $B_2(D)$ to consists of those holomorphic quadratic differentials $\varphi(z)dz^2$ on D such that $\varphi = S_f$ for some univalent meromorphic function f on D which extends to a quasiconformal automorphism of the Riemann sphere. Note that $\|S_f\|_{B_2(D)} \le 12$ for every univalent meromorphic function f on D (see §5.2 and [9]). By Lemmas 1.6.1 and 1.6.2, for a Möbius transformation L, the pull-back L_2^* gives an isometoric isomorphism of $B_2(L(D))$ onto $B_2(D)$. In particular, for a circle domain Δ, that is, the interior or the exterior of a circle, or a half-plane, the space $B_2(\Delta)$ is isomorphic, say, to $B_2(\mathbb{D}^*)$. The space $\mathrm{T}_4 = T(\mathbb{D}^*)$ (or its equivalent) is a model of the universal Teichmüller space and known as the Bers embedding of the universal Teichmüller space.

Ahlfors [2] showed the following.

Theorem 2.4.1. *$T(\mathbb{D}^*)$ is a bounded, connected and open subset of $B_2(\mathbb{D}^*)$.*

Thus $\mathrm{T}_4 = T(\mathbb{D}^*)$ inherits a complex structure and a metric from $B_2(\mathbb{D}^*)$. We denote by d_4 the distance, namely, $d_4(\varphi, \psi) = \|\varphi - \psi\|_{B_2(\mathbb{D}^*)}$ for $\varphi, \psi \in \mathrm{T}_4$. Since $T(\mathbb{D}^*)$ is bounded, the distance d_4 is *not* complete.

2.5. Equivalence of T_1 through T_4. We see now that the above definitions of the universal Teichmüller space are all equivalent. Firstly, consider the restriction map $\mathrm{QC}(\mathbb{D}) \to \mathrm{QS}(\partial\mathbb{D})$ defined by $f \mapsto f|_{\partial\mathbb{D}}$. Then this map yields a bijection of T_1 onto T_2.

Secondly, we see the equivalence of T_3 and T_4. For $\varphi \in \mathrm{T}_4 = T(\mathbb{D}^*)$, by definition, there exists a quasiconformal map f of $\widehat{\mathbb{C}}$ fixing $0, 1, \infty$ such that f is conformal on $\mathbb{D}^*$ and satisfies $S_f = \varphi$. Then the image $D = f(\mathbb{D}^*)$ is a normalized quasidisk. Therefore, the correspondence $\varphi \mapsto D$ gives a map $\mathrm{T}_4 \to \mathrm{T}_3$. We next show that this map is bijective. Suppose that a normalized quasidisk D is given. By definition, $D = h(\mathbb{D}^*)$ for some quasiconformal map h of $\widehat{\mathbb{C}}$ with $h(1) = 0, h(-1) = 1$ and $h(-i) = \infty$. Let $\mu = \mu_h|_{\mathbb{D}^*}$ and set $f = h \circ (w_\mu)^{-1} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Then f is quasiconformal map of $\widehat{\mathbb{C}}$, is conformal on $w_\mu(\mathbb{D}^*) = \mathbb{D}^*$ and satifies $f(1) = 0, f(-1) = 1$ and $f(-i) = \infty$. Therefore, $\varphi = S_f$ belongs to $\mathrm{T}_4 = T(\mathbb{D}^*)$. In this way, we obtain the map of T_3 into T_4, which is obviuosly the inverse map of the previously defined map of T_4 to T_3. We have now concluded that T_3 and T_4 are equivalent by those maps.

Finally, we connect T_1 with T_4. Let $h \in \mathrm{QC}(\mathbb{D})$. We define $\mu \in \mathrm{Belt}(\mathbb{C})$ by

$$\mu = \begin{cases} \mu_h & \text{on } \mathbb{D}, \\ 0 & \text{on } \mathbb{D}^* \end{cases}$$

and define a quasiconformal map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by $f = f^\mu$, where f^μ was defined in §1.1. Since f is conformal in $\mathbb{D}^*$, the Schwarzian derivative S_f belongs to $\mathrm{T}_4 = T(\mathbb{D}^*)$. Note that $f \circ h^{-1}$ is conformal in $\mathbb{D}$ by construction. Let $h_1 \in \mathrm{QC}(\mathbb{D})$ be Teichmüller equivalent to h and define f_1 in the same way as above. We claim now that $S_{f_1} = S_f$. By assumption, $h_1 = L \circ h$ on $\partial\mathbb{D}$ for an $L \in \mathrm{Aut}(\mathbb{D})$. Define a map $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by

$$g = \begin{cases} f_1 \circ f^{-1} & \text{on } \widehat{\mathbb{C}} \setminus f(\mathbb{D}), \\ f_1 \circ h_1^{-1} \circ L \circ h \circ f^{-1} & \text{on } f(\mathbb{D}). \end{cases}$$

It is clear that g is conformal on $f(\mathbb{D})$ and $f(\mathbb{D}^*)$. Furthermore, since $h_1^{-1} \circ L \circ h = \mathrm{id}$ on $\partial\mathbb{D}$, the map g is continuous on $\widehat{\mathbb{C}}$. Since $C = f(\partial\mathbb{D})$ and $g(C) = f_1(\partial\mathbb{D})$ are both quasicircles, it turns out that g is quasiconformal in $\widehat{\mathbb{C}}$. Since $\mu_g = 0$ a.e., we conclude that g is conformal, hence, a Möbius map. Because of the relation $f_1 = g \circ f$ on $f(\mathbb{D}^*)$, $S_f = S_{f_1}$ follows as required.

In this way, we obtain the mapping of T_1 to $\mathrm{T}_4 : [h] \mapsto S_{f|_{\mathbb{D}}^*}$. It is not difficult to see that this mapping is bijective. This map is called the *Bers embedding*.

3. Analytic properties of the Bers embedding

3.1. The Teichmüller space of a Riemann surface. It is beyond the scope of the present survey to develop the theory of Teichmüller spaces of Riemann surfaces in full generality. Here, our forcus will be on the Bers embedding of the Teichmüller space of a Riemann surface. See [71], [43], [33], [34] for general properties of Teichmüller spaces. See also [1], [108] for a differential geometric approach, [90] for an algebraic approach.

For simplicity, we assume a Riemann surface R to be hyperbolic, in other words, there exists a torsion-free Fuchsian group Γ acting on $\mathbb{D}$ such that $\mathbb{D}/\Gamma$ is conformally equivalent to R. Thus, we can identify R with $\mathbb{D}/\Gamma$. We denote by $p : \mathbb{D} \to \mathbb{D}/\Gamma = R$ the canonical projection. Two quasiconformal maps $f_j : R \to S_j,\ j = 1, 2$, are called *Teichmüller equivalent* if there exists a conformal homeomorphism $g : S_1 \to S_2$ such that $f_2^{-1} \circ g \circ f_1 : R \to R$ is homotopic to the identity relative to the ideal boundary. We omit the explanation of the term "relative to the ideal boundary". See the references given above for details. Also, [31] gives several useful equivalent conditions for that.

The Teichmüller space $\mathrm{Teich}(R)$ of R is defined as the set of all the Teichmüller equivalence classes of such quasiconformal maps of R onto another surface.

Suppose that $f_1 : R \to S_1$ and $f_2 : R \to S_2$ are quasiconformal maps. Let Γ_j be a Fuchsian model of S_j acting on $\mathbb{D}$ and $h_j : \mathbb{D} \to \mathbb{D}$ be a lift of f_j, namely, $p_j \circ h_j = f_j \circ p$, where $p_j : \mathbb{D} \to \mathbb{D}/\Gamma_j = S_j$ is the canonical projection. Then, it is known that f_1 and f_2 are Teichmüller equivalent if and only if h_1 and h_2 are Teichmüller equivalent in the sense of §2.1. Note that $h_j \circ \gamma \circ h_j^{-1} \in \Gamma_j$ for each $\gamma \in \Gamma$, namely, $h_j \Gamma h_j^{-1} = \Gamma_j$.

Set

$$\mathrm{QC}(\mathbb{D}, \Gamma) = \{h \in \mathrm{QC}(\mathbb{D}) : h\Gamma h^{-1} \text{ is Fuchsian}\}$$

and denote by $\mathrm{Teich}(\Gamma)$ the quotient space $\mathrm{QC}(\mathbb{D}, \Gamma)/\overset{\mathrm{T}}{\sim}$. As we have seen, $\mathrm{Teich}(R)$ and $\mathrm{Teich}(\Gamma)$ are canonically isomorphic through the universal covering projection $p : \mathbb{D} \to \mathbb{D}/\Gamma = R$. Also, $\mathrm{Teich}(\Gamma)$ is naturally contained in $\mathrm{Teich}(1) = \mathrm{T}_1$. In this sense, the universal Teichmüller space $T(\mathbb{D}^*)$ contains all the Teichmüller space of an arbitrary hyperbolic Riemann surface.

By using (1.1.1), the complex dilatation of $f \in \mathrm{QC}(\mathbb{D}, \Gamma)$ is seen to be contained in

$$\mathrm{Belt}(\mathbb{D}, \Gamma) = \{\mu \in \mathrm{Belt}(\mathbb{D}) : (\mu \circ \gamma)\overline{\gamma'}/\gamma' = \mu\ \forall \gamma \in \Gamma\}.$$

Furthermore, for $h \in \mathrm{QC}(\mathbb{D}, \Gamma)$, let f be the function constructed in §2.5 and let $\gamma \in \Gamma$. Since f and $\gamma \circ f \circ \gamma^{-1}$ has the same complex dilatation, $\gamma \circ f \circ \gamma^{-1} = L \circ f$ for an $L \in \mathrm{Aut}(\widehat{\mathbb{C}}) = \text{Möb}$ by Lemma 1.1.2. Lemma 1.6.2 now implies that $\gamma_2^*(S_f) = S_f$. Therefore, S_f is contained in the closed subspace $B_2(\mathbb{D}^*, \Gamma)$ of $B_2(\mathbb{D}^*)$ defined in §1.3. As in the previous section, we see that S_f depends only on the Teichmüller equivalence class of h in $\mathrm{QC}(\mathbb{D}, \Gamma)$ and the corresponding $h \mapsto S_f$ is one-to-one, we obtain an embedding $\beta_\Gamma : \mathrm{Teich}(\mathbb{D}, \Gamma) \to B_2(\mathbb{D}^*, \Gamma)$, which is called the Bers embedding of $\mathrm{Teich}(\mathbb{D}, \Gamma)$. We set $T(\mathbb{D}^*, \Gamma) = \beta_\Gamma(\mathrm{Teich}(\mathbb{D}, \Gamma))$.

Bers [14] showed that $T(\mathbb{D}^*, \Gamma)$ is a bounded domain in $B_2(\mathbb{D}^*, \Gamma)$. It is obvious that $T(\mathbb{D}^*, \Gamma)$ is contained in $T(\mathbb{D}^*)$ by definition. Indeed, by using the Douady-Earle extension [26], it can be seen that

$$T(\mathbb{D}^*, \Gamma) = T(\mathbb{D}^*) \cap B_2(\mathbb{D}^*, \Gamma)$$

and that $T(\mathbb{D}^*, \Gamma)$ is contractible.

3.2. Relationship with quasi-Teichmüller spaces. In view of the description of the set $T(\mathbb{D}^*, \Gamma)$, it may be natural to consider the following sets more generally. Let D be a hyperbolic domain in $\widehat{\mathbb{C}}$ and let G be a subgroup of $\mathrm{Aut}(D)$. Typically, G is a Kleinian group and D is a connected component of its region of discontinuity. Then we set (cf. [94])

$$S(D, G) = \{\varphi \in B_2(D, G) : \exists f : D \to \widehat{\mathbb{C}} \text{ s.t. } \varphi = S_f \text{ and } f \text{ is univalent in } D\},$$

$$T(D, G) = \{\varphi \in B_2(D, G) : \exists f : D \to \widehat{\mathbb{C}} \text{ s.t. } \varphi = S_f \text{ and } f \text{ extends to a qc map of } \widehat{\mathbb{C}}\},$$

For a circle domain Δ and a Fuchsian group Γ acting on Δ, the set $S(\Delta, \Gamma)$ sometimes called the quasi-Teichmüller space of Γ. (But, note that this terminology is not popular.) Clearly, $T(D, G) \subset S(D, G)$. It is easy to see that $S(\mathbb{D}^*)$ is closed while, as Ahlfors showed, $T(\mathbb{D}^*)$ is open in $B_2(\mathbb{D}^*)$. The boundary of $T(\Delta, \Gamma)$ in $B_2(\Delta, \Gamma)$ is called the Bers boundary and is important in relation with the deformation theory of Kleinian groups (see [15]).

When G is the trivial group 1, we write $S(D), T(D)$ for $S(D, 1), T(D, 1)$, respectively. Note that under the mapping $f \mapsto S_f$, the sets $\tilde{\mathscr{S}_0}$ and Σ_0 correspond to $S(\mathbb{D})$ and $S(\mathbb{D}^*)$, respectively, in one-to-one fashion. It is a challenging problem to characterize those functions f in $\mathscr{S}$ whose Schwarzian derivatives lie on $\partial T(\mathbb{D})$. See [7] and [41] for some attempts.

In 1970's, it had been a conjecture of Bers [15] that the closure of $T(\mathbb{D}^*)$ in $B_2(\mathbb{D}^*)$ is $S(\mathbb{D}^*)$. In 1978, Gehring [37] disproved it. Prior to it, Gehring [36] proved the weaker assertion that the interior of $S(\mathbb{D}^*)$ in $B_2(\mathbb{D}^*)$ coincides with $T(\mathbb{D}^*)$. See [32] for a relevant result. Thurston [106] proved the more striking result that $S(\mathbb{D}^*)$ even has an isolated point in $B_2(\mathbb{D}^*)$ (see also [5]). After that, the Bers conjecture was reformulated in the form that the closure of $T(\mathbb{D}^*, \Gamma)$ is equal to $S(\mathbb{D}^*, \Gamma)$ for a cofinite Fuchain group Γ, that is, a finitely generalted Fuchsian group of the first kind. (This is nowadays generalized to the Bers-Thurston density conjecture.) Shiga [91] proved a weaker version of it: the interior of $S(\mathbb{D}^*, \Gamma)$ in $B_2(\mathbb{D}^*, \Gamma)$ coincides with $T(\mathbb{D}^*, \Gamma)$ for a cofinite Γ. In the line of these studies, the author showed that $S(\mathbb{D}^*, \Gamma) \setminus \overline{T(\mathbb{D}^*)} \neq \emptyset$ for a Fuchsian group Γ of the second kind ([95]) and that the interior of $S(\mathbb{D}^*, \Gamma)$ in $B_2(\mathbb{D}^*, \Gamma)$ coincides with $T(\mathbb{D}^*, \Gamma)$ for a finitely generated, purely hyperbolic Fuchsian group Γ of the second kind ([96]). Matsuzaki [68] generalized the former to the case of a certain kind of inifinitely generated Fuchsian groups of the first kind. In recent years, a huge amount of progress has been made in the theory of Kleinian groups, which enabled to prove the Bers-Thurston conjecture partially. See, for instance, [19] and [76] for the recent progress.

We end the subsection with the following conjecture.

Conjecture 3.2.1. *The interior of quasi-Teichmüller space* $S(\mathbb{D}^*, \Gamma)$ *in* $B_2(\mathbb{D}^*, \Gamma)$ *is equal to the Bers embedding* $T(\mathbb{D}^*, \Gamma)$ *of the Teichmüller space of a Fuchsian group* Γ *acting on* $\mathbb{D}^*$.

Note that Zhuravlev (Žuravlev) [112] proved that $T(D^*, \Gamma)$ is the connected component of the interior of $S(\mathbb{D}^*, \Gamma)$ which contains the origin for an arbitrary Fuchsian group Γ (see also [94]). Thus the conjecture is equivalent to connectedness of the interior of $S(\mathbb{D}^*, \Gamma)$.

3.3. The Bers projection. Let D be a hyperbolic domain in $\widehat{\mathbb{C}}$ and denote by E its complement in $\widehat{\mathbb{C}}$. We define the map $\Phi : \mathrm{Belt}(E) \to B_2(D)$ by $\Phi(\mu) = S_{f^\mu|_D}$, where f^μ is defined as in §1.1 for μ which is extended to $\mathbb{C}$ by setting $\mu = 0$ on D. Recall here that $\mathrm{Belt}(E)$ is the open unit ball of the complex Banach space $L^\infty(E)$ with norm $\|\cdot\|_\infty$. It is clear by definition that $\Phi(\mathrm{Belt}(E)) = T(D)$. The map $\Phi : \mathrm{Belt}(E) \to T(D)$ is called the (generalized) Bers projection. It is known that $\Phi : \mathrm{Belt}(E) \to B_2(D)$ is holomorphic (cf. [94]) and that the Fréchet derivative $d_0\Phi : L^\infty(E) \to B_2(D)$ of Φ at the origin is described by

$$d_0\Phi[\nu](z) = -\frac{6}{\pi}\iint_E \frac{\nu(\zeta)}{(\zeta - z)^4} d\xi d\eta \quad (\zeta = \xi + i\eta)$$

for $\nu \in L^\infty(E)$. Bers [14] strengthened Ahlfors' theorem (Theorem 2.4.1) to the following form.

Theorem 3.3.1. *The Bers projection* $\Phi : \mathrm{Belt}(\mathbb{D}) \to T(\mathbb{D}^*)$ *is a holomorphic split submersion, in other words, the Fréchet derivative of* Φ *at every point exists and has a (bounded) left inverse.*

Indeed, Bers showed the above theorem for the projection $\Phi : \mathrm{Belt}(\mathbb{D}, \Gamma) \to T(\mathbb{D}^*, \Gamma)$ for an arbitrary Fuchsian group Γ. In particular, $T(\mathbb{D}^*, \Gamma)$ is shown to be an open subset of $B_2(\mathbb{D}^*, \Gamma)$.

3.4. Convexity. Krushkal [54] proved that the Bers embedding $T(\mathbb{D}^*)$ of the universal Teichmüller space is not starlike with respect to any point, and hence, not convex in $B_2(\mathbb{D}^*)$. For non-starlikeness of general Teichmüller spaces, see Krushkal [57] and Toki [107].

In spite of the above fact, the (Bers embededing of the) Teichmüller spaces enjoy many kinds of convexity properties. We briefly list some of them in this subsection.

The most useful is perhaps the following "disk convexity" due to Zhuravlev [112], which is shown as an application of the Grunsky inequality. A weaker version can be proved also by the λ-lemma (see [98]).

Theorem 3.4.1 (Zhuravlev). *Let* Γ *be a Fuchsian group acting on* $\mathbb{D}^*$. *Suppose that a continuous map* $\alpha : \overline{\mathbb{D}} \to B_2(\mathbb{D}^*, \Gamma)$ *is holomorphic in* $\mathbb{D}$ *and satisfies*

$\alpha(\partial\mathbb{D}) \subset S(\mathbb{D}^*)$. *Then* $\alpha(\overline{\mathbb{D}}) \subset S(\mathbb{D}^*, \Gamma)$. *Furthermore, if* $\alpha(\overline{\mathbb{D}}) \cap T(\mathbb{D}^*)$ *is nonempty, then* $\alpha(\mathbb{D}) \subset T(\mathbb{D}^*, \Gamma)$.

Outline of the proof. For each $z \in \overline{\mathbb{D}}$, there exists a unique $F_z \in \Sigma_0$ such that $S_{F_z} = \alpha(z)$. Let $\mathscr{B}(\ell^2)$ denote the complex Hilbert space consisting of bounded linear operators on ℓ^2. Then the map $\beta : \mathbb{D} \to \mathscr{B}(\ell^2)$ defined by $z \mapsto \mathcal{G}[F_z]$ turns to be holomorphic. Then the (generalized) maximum principle implies that

$$\sup_{z\in\mathbb{D}} \|\beta(z)\| = \sup_{z\in\partial\mathbb{D}} \|\beta(z)\| \leq 1$$

and that either $\|\beta(z)\| < 1$ for all $z \in \mathbb{D}$ or else $\|\beta(z)\| = 1$ for all $z \in \overline{\mathbb{D}}$. Theorem 1.5.1 now yields that $\alpha(\mathbb{D}) \subset S(\mathbb{D}^*)$. If we assume that $\alpha(z_0) \in T(\mathbb{D}^*)$ for some point $z_0 \in \overline{\mathbb{D}}$ in addition, then $\|\beta(z_0)\| < 1$ and thus $\|\beta(z)\| < 1$ for all $z \in \mathbb{D}$. This means that $\alpha(\mathbb{D}) \subset T(\mathbb{D}^*) \cap B_2(\mathbb{D}^*, \Gamma) = T(\mathbb{D}^*, \Gamma)$. □

We remark that the above argument is a variant of Lehto's principle (see [12] or [64]).

A more sophisticated application of Grunsky inequality to Teichmüller spaces can be found in [92].

Bers and Ehrenpreis [16] proved that finite dimensional Teichmüller spaces are holomorphically convex. Krushkal [55] strengthened it by showing that the Teichmüller space of an arbitrary Riemann surface R is complex hyperconvex, that is to say, there exists a negative plurisubharmonic function $u(x)$ on $\mathrm{Teich}(R)$ such that $u(x) \to 0$ when x tends to ∞. He proved it by pointing out that the function $\log\tanh(d(x,y))$ gives the Green function on $\mathrm{Teich}(R)$, where $d(x,y)$ denotes the Teichmüller distance of $\mathrm{Teich}(R)$. Krushkal [56] also proved that finite dimensional Teichmüller spaces are polynomially convex.

3.5. Teichmüller distance and other natural distances (metrics). In §2, we defined two kinds of distances on the universal Teichmüller space; the Teichmüller distance and the distance induced by the Bers embedding. These distances can be defined for the Teichmüller space of an arbitrary Riemann surface. On the other hand, since Teichmüller spaces have complex structure, it carries natural invariant distances for holomorphic maps (see [44] as a general reference).

Let X be a complex (Banach) manifold. The Kobayashi pseudo-distance $d_{\mathrm{K}}(x,y)$ is defined as

$$\inf \sum_{j=1}^{N} d_{\mathbb{D}}(z_{j-1}, z_j),$$

where the infimum is taken over all finitely many holomorphic maps $f_j : \mathbb{D} \to X$ $(j = 1, \ldots, N)$ which satisfy $f_j(z_j) = f_{j+1}(z_j)(1 < j < N), f_1(z_0) = x$, and $f_N(z_N) = y$. Here, $d_{\mathbb{D}}(z,w)$ denotes the Poincaré distance of $\mathbb{D}$:

$$d_{\mathbb{D}}(z,w) = \operatorname{arctanh} \left| \frac{w - z}{1 - \bar{z}w} \right|.$$

The following theorem was proved by Royden [87] for finite dimensional case and by Gardiner (see [33] or [34]) for general case. (For a simple proof using the λ-lemma, see [30].)

Theorem 3.5.1. *The Kobayashi pseudo-distance of the Teichmüller space of a Riemann surface is equal to the Teichmüller distance.*

For other invariant metrics on Teichmüller spaces, see [71, Appendix 6].

Earle [29] proved that the Carathéodory (pseudo)distance of the Teichmüller space of an arbitrary Fuchsian group is complete.

The Weil-Petersson metric is another important (Riemannian) metric on finite dimensional Teichmüller spaces. Since the complex structure of the Teichmüller space of a general Riemann surface is modelled on a complex Banach space which may not be reflexive, this metric cannot be defined on general Teichmüller spaces unless the structure of the space is changed. However, some attempts were made to construct analogs of the Weil-Petersson metric on the universal Teichmüller space, see [72], [73], [103], [104].

4. Pre-Schwarzian models

The Schwarzian derivative plays an important role in the definition of the Teichmüller space. But, it is not easy to treat with Schwarzian derivative, in general, because of its complicated form. Therefore, some attempts of replacing Schwarzian by pre-Schwarzian have been made. See [111] and [6]. Though the pre-Schwarzian model is sometimes called "poor man's model" (cf. [41]) since it does not have much invariance, this model is interesting in connection with geometric function theory.

When dealing with pre-Schwarzian derivative, the point at infinity plays a special role. Therefore, we have to consider the case $\infty \in D$ separately.

4.1. The models $\hat{T}(\mathbb{D})$ and $\hat{T}(\mathbb{H})$. Let Δ be a disk or a half-plane in $\mathbb{C}$. Set

$$\hat{S}(\Delta) = \{T_f : f : \Delta \to \mathbb{C} \text{ is holomorphic and univalent}\}$$

and

$$\hat{T}(\Delta) = \{T_f : f : \Delta \to \mathbb{C} \text{ is holomorphic and extends to a qc map of } \widehat{\mathbb{C}}\}.$$

Here, T_f denotes the pre-Schwarzian derivative of f (see §1.6). By definition, $\hat{T}(\Delta) \subset \hat{S}(\Delta)$.

We recall that the pre-Schwarzian derivative vanishes only when the function is affine. Since each circle domain in $\mathbb{C}$ is similar (affinely equivalent) to either the unit disk $\mathbb{D}$ or the half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Therefore, we may restrict ourselves on the two cases $\Delta = \mathbb{D}$ and $\mathbb{H}$. First let $f \in \mathscr{S}$. By the well-known inequality (cf. [27])

$$\left|(1-|z|^2)T_f(z) - 2\bar{z}\right| \le 4, \qquad (4.1.1)$$

we obtain $\|T_f\|_{B_1(\mathbb{D})} \le 6$. In particular, $\hat{S}(\mathbb{D}) \subset B_1(\mathbb{D})$. Note also that the constant 6 is sharp as the Koebe function $K(z) = z/(1-z)^2$ shows. It is easy to see that $\hat{S}(\mathbb{D})$ is closed in $B_1(\mathbb{D})$.

Let $L(z) = (z - \mathrm{i})/(z + \mathrm{i})$. Note that $\|T_L\|_{B_1(\mathbb{H})} = 4$ and hence $T_L \in \hat{T}(\mathbb{H})$. Since $L_1^* : B_1(\mathbb{D}) \to B_1(\mathbb{H})$ is a linear isometry and $T_{f\circ L} = L_1^*(T_f) + T_L$, the space $\hat{T}(\mathbb{H})$ is contained in $B_1(\mathbb{H})$ and it is isometrically equivalent to $\hat{T}(\mathbb{D})$. In this sense, it is enough to consider only $\hat{T}(\mathbb{D})$.

We define the map $\pi : B_1(\mathbb{D}) \to B_2(\mathbb{D})$ by $\pi(\psi) = \psi' - \psi^2/2$. By definition, $\pi(\hat{S}(\mathbb{D})) = S(\mathbb{D})$ and $\pi(\hat{T}(\mathbb{D})) = T(\mathbb{D})$. Duren, Shapiro and Shields [28] noticed that this map is continuous (see also §5.3).

Astala and Gehring [6] proved an analogous result to the case of Schwarzian derivative.

Theorem 4.1.2. *The interior of $\hat{S}(\mathbb{D})$ in $B_1(\mathbb{D})$ is equal to $\hat{T}(\mathbb{D})$, while the closure of $\hat{T}(\mathbb{D})$ in $B_1(\mathbb{D})$ is not equal to $\hat{S}(\mathbb{D})$. Moreover, $\partial T(\mathbb{D}) \setminus \pi(\partial\hat{T}(\mathbb{D}))$ is not empty.*

Zhuravlev [111] revealed the following remarkable property of $\hat{T}(\mathbb{D})$.

Theorem 4.1.3 (Zhuravlev)**.** *The space $\hat{T}(\mathbb{D})$ decomposes into the uncountably many connected components $\hat{T}_0$ and $\hat{T}_\omega$, $\omega \in \partial\mathbb{D}$, where*

$$\hat{T}_0 = \{T_f \in \hat{T}(\mathbb{D}) : f(\mathbb{D}) \text{ is bounded }\} \quad \text{and} \quad \hat{T}_\omega = \{T_f \in \hat{T}(\mathbb{D}) : f(z) \to \infty \text{ as } z \to \omega\}.$$

Moreover, $\{\psi \in B_1(\mathbb{D}) : \|\psi - \psi_\omega\|_{B_1(\mathbb{D})} < 1\} \subset \hat{T}_\omega$ holds for each $\omega \in \partial\mathbb{D}$, where $\psi_\omega(z) = 2\bar{\omega}/(1-\bar{\omega}z)$ is the pre-Schwarzian derivative of the function $z/(1-\bar{\omega}z)$.

Note that the map π is not injective even in each connected component of $\hat{T}(\mathbb{D})$. Therefore, we should note that this model of the universal Teichmüller space has some redundancy.

4.2. The model $\hat{T}(\mathbb{D}^*)$. There is some subtlety in consideration of the pre-Schwarzian model of the universal Teichmüller space $\hat{T}(\mathbb{D}^*)$ on the exterior $\mathbb{D}^*$ of the unit circle. The first thing to note is the fact that the Banach space $B_1(\mathbb{D}^*)$ is not the right space on which $\hat{T}(\mathbb{D}^*)$ is modeled. We define

$$\hat{S}(\mathbb{D}^*) = \{T_F : F \in \Sigma\}$$

and

$$\hat{T}(\mathbb{D}^*) = \{T_F : F \in \Sigma \text{ extends to a quasiconformal map of } \widehat{\mathbb{C}}\}.$$

If $F(\zeta) = \zeta + b_0 + b_1/\zeta + b_2/\zeta^2 + \dots$, then $T_F(\zeta) = 2b_1/\zeta^3 + \dots = O(\zeta^{-3})$ as $\zeta \to \infty$. Therefore, the norm

$$\mathrm{B}(\psi) = \sup_{\zeta\in\mathbb{D}^*}(|\zeta|^2 - 1)|\zeta\psi(\zeta)| \tag{4.2.1}$$

is more natural. Indeed, Becker's univalence criterion [11] and Avhadiev's inequality [8]

$$(4.2.2)\qquad (|\zeta|^2-1)\left|\frac{\zeta F''(\zeta)}{F'(\zeta)}\right| \le 6$$

imply the following result.

Theorem 4.2.3. *If a meromorphic function* $F(\zeta)=\zeta+b_0+b_1/\zeta+\dots$ *in* $|\zeta|>1$ *satisfies* $\mathrm{B}(T_F)\le 1$, *then* $F\in\Sigma$. *Conversely, every function* F *in* Σ *satisfies* $\mathrm{B}(T_F)\le 6$.

We set

$$B_1'(\mathbb{D}^*)=\{\psi\in B_1(\mathbb{D}^*) : \lim_{\zeta\to\infty}\zeta^2\psi(\zeta)=0\}.$$

Then, it is easy to see that $B_1'(\mathbb{D}^*)=\{\psi:\mathbb{D}^*\to\mathbb{C}\ \text{ holomorphic and }\ \mathrm{B}(\psi)<\infty\}$. The above theorem now yields that $\hat{S}(\mathbb{D}^*)$ is a bounded subset of $B_1'(\mathbb{D}^*)$.

We define $\pi: B_1'(\mathbb{D}^*)\to B_2(\mathbb{D}^*)$ as before by $\pi(\psi)=\psi'-\psi^2/2$. Then π is continuous [12, Lemma 6.1]. By definition, $\pi(\hat{S}(\mathbb{D}^*))=S(\mathbb{D}^*)$ and $\pi(\hat{T}(\mathbb{D}^*))=T(\mathbb{D}^*)$. Since $T(\mathbb{D}^*)$ is an open set and $\hat{T}(\mathbb{D}^*)=\pi^{-1}(T(\mathbb{D}^*))$, the set $\hat{T}(\mathbb{D}^*)$ is also open in $B_1'(\mathbb{D}^*)$. In this way, we see that the space $\hat{T}(\mathbb{D}^*)$ is a complex Banach manifold modeled on $B_1'(\mathbb{D}^*)$. We remark that π does not map $B_1(\mathbb{D}^*)$ into $B_2(\mathbb{D}^*)$.

The set $\hat{T}(\mathbb{D}^*)$ seems to be less investigated, but could be more useful. For instance, the mapping $F\mapsto T_F$ sends Σ_0 to $\hat{S}(\mathbb{D}^*)$ bijectively. Recall that the mapping $F\mapsto S_F$ sends Σ_0 to $S(\mathbb{D}^*)$ bijectively. Therefore, the mapping π sends $\hat{S}(\mathbb{D}^*)$ to $S(\mathbb{D}^*)$ bijectively.

4.3. Loci of typical subclasses of $\mathscr{S}$. Since the differential operator T_f is closely related with geometric function theory, many classical subclasses of univalent functions correspond to sets with nice properties in $\hat{S}(\mathbb{D})$.

We recall several fundamental classes in univalent function theory. We denote by $\mathscr{A}$ the set of analytic functions f in the unit disk $\mathbb{D}$ so normalized that $f(0)=0$ and $f'(0)=1$. A function $f\in\mathscr{A}$ is called *starlike* (*convex*) if f is univalent and if $f(\mathbb{D})$ is starlike with respect to the origin (convex). We denote by $\mathscr{S}^*$ and $\mathscr{K}$ the sets of starlike and convex functions in $\mathscr{A}$, respectively. A function $f\in\mathscr{A}$ is called *close-to-convex* if $e^{i\alpha}f'/g'$ has positive real part in $\mathbb{D}$ for a convex function g and for a real constant α. Denote by $\mathscr{C}$ the set of close-to-convex functions in $\mathscr{A}$. It is known that $\mathscr{C}\subset\mathscr{S}$ (cf. [27]).

It is interesting to see how pre-Schwarzians of those functions are located in the space $\hat{S}(\mathbb{D})$. The following result gives an answer to this question.

Theorem 4.3.1 ([25], [49]). $\{T_f : f\in\mathscr{K}\}$ *and* $\{T_f : f\in\mathscr{C}\}$ *are both convex subsets of* $\hat{S}(\mathbb{D})$.

It may be natural to conjecture the following.

Conjecture 4.3.2 ([46]). *The subset* $\{T_f : f \in \mathscr{S}^*\}$ *of* $\hat{S}(\mathbb{D})$ *is starlike with respect to the origin.*

Note that the vector operations in $B_1(\mathbb{D})$ is translated to the Hornich operations in the space of uniformly locally univalent functions (see, for example, [46]).

5. Univalence criteria

As is well developed in Lehto's textbook [64], univalence criteria are closely connected with the universal Teichmüller space. The present section will be devoted to this topic.

5.1. Univalence criteria due to Nehari and Ahlfors-Weill. Nehari [74] proved the following result, which is fundamental in the Teichmüller spaces.

Theorem 5.1.1. *Every meromorphic univalent function* f *on the unit disk satisfies the inequality* $\|S_f\|_{B_2(\mathbb{D})} \leq 6$. *Conversely, if a meromorphic function* f *on the unit disk satisfies the inequality* $\|S_f\|_{B_2(\mathbb{D})} \leq 2$, *then* f *must be univalent.*

The constants 6 and 2 are sharp since the Koebe function $K(z) = z/(1-z)^2$ satisfies $\|S_K\|_{B_2(\mathbb{D})} = 6$ and since the function $f(z) = ((1+z)/(1-z))^{\mathrm{i}\epsilon}$, $\epsilon > 0$, is never univalent but $\|S_f\|_{B_2(\mathbb{D})} = 2(1+\varepsilon^2)$ can approach 2 (Hille [42]). The former assertion was first proved by Kraus [53] and reproved by Nehari. Therefore, it is called nowadays the Kraus-Nehari theorem. The Kraus-Nehari theorem is a consequence of the Bieberbach theorem. By the Möbius invariance of $(1-|z|^2)^2|S_f(z)|$, it is enough to show the inequality only at the origin, namely, $|S_f(0)| \leq 6$ for $f \in \mathscr{S}$. A straightforward computation gives $S_f(0) = 6(a_3 - a_2^2)$ for $f(z) = z + a_2z^2 + a_3z^3 + \dots$. If we set $F(\zeta) = 1/f(1/\zeta) = \zeta + b_0 + b_1/\zeta + \dots$, then $b_1 = a_2^2 - a_3$, and thus the inequality $|b_1| \leq 1$ (see (1.5.3)) implies the required one.

The class $N = \{f \in \mathscr{A} : \|S_f\|_{B_2(\mathbb{D})} \leq 2\}$ is sometimes called the Nehari class. Gehring and Pommerenke [39] showed that $f \in N$ maps the unit disk conformally onto a Jordan domain unless $f(\mathbb{D})$ is Möbius equivalent to the parallel strip $\{z : |\mathrm{Im}\, z| < \pi/4\}$. For further development, see [22], [23] and [24].

In connection with Nehari's theorem, Ahlfors and Weill established the following quasiconformal extension criterion. For $\varphi \in B_2(\mathbb{D}^*)$ with $\|\varphi\|_{B_2(\mathbb{D}^*)} < 2$, we set $\alpha[\varphi] \in \mathrm{Belt}(\mathbb{D})$ by $\alpha[\varphi](z) = -\rho_{\mathbb{D}}(z)^{-2}\varphi(1/\bar{z})\bar{z}^{-4}/2$. Note that the map α is the restriction of a bounded linear operator which maps $B_2(\mathbb{D}^*, \Gamma)_2 = \{\varphi \in B_2(\mathbb{D}^*, \Gamma) : \|\varphi\|_{B_2(\mathbb{D})} < 2\}$ into $\mathrm{Belt}(\mathbb{D}^*, \Gamma)$ for every Fuchsian group Γ.

Theorem 5.1.2 (Ahlfors-Weill). *The map* $\alpha : B_2(\mathbb{D}^*)_2 \to \mathrm{Belt}(\mathbb{D})$ *is the local inverse of the Bers projection* $\Phi : \mathrm{Belt}(\mathbb{D}) \to T(\mathbb{D}^*)$, *in other words,* $\Phi(\alpha[\varphi]) = \varphi$ *for* $\varphi \in B_2(\mathbb{D}^*)$ *with* $\|\varphi\|_{B_2(\mathbb{D}^*)} < 2$.

Corollary 5.1.3. *The universal Teichmüller space* $\mathrm{T}(\mathbb{D}^*)$ *contains the open ball centered at the origin with radius* 2 *in* $B_2(\mathbb{D}^*)$.

The map $\alpha : B_2(\mathbb{D}^*)_2 \to \mathrm{Belt}(\mathbb{D})$ is sometimes called the Ahlfors-Weill section.

5.2. Inner radius and outer radius. Let D be a hyperbolic domain in $\widehat{\mathbb{C}}$. The inner radius $\sigma_{\mathrm{I}}(D)$ and the outer radius $\sigma_{\mathrm{O}}(D)$ of univalence is defined respectively by

$$\sigma_{\mathrm{I}}(D) = \sup\{\sigma \geq 0 : \|S_f\|_{B_2(D)} \leq \sigma \Rightarrow f \text{ is univalent in } D\},$$
$$\sigma_{\mathrm{O}}(D) = \sup\{\|S_f\|_{B_2(D)} : f : D \to \widehat{\mathbb{C}} \text{ is univalent}\}.$$

We also define the number $\tau(D) \in [0, +\infty]$ as $\|S_p\|_{B_2(\mathbb{D})}$, where p is a holomorphic universal covering projection of $\mathbb{D}$ onto D. The quantity $\tau(D)$ is independent of the choice of p and thus well defined. Note that $\tau(D) < \infty$ if and only if ∂D is uniformly perfect (cf. [83] or [99]).

Summarizing theorems of Ahlfors [2], Gehring [36], Nehari [74], we obtain the following.

Theorem 5.2.1. $\sigma_{\mathrm{I}}(\Delta) = 2, \sigma_{\mathrm{O}}(\Delta) = 6, \tau(\Delta) = 0$ *hold for a circle domain* Δ. *Let* D *be a simply connected hyperbolic domain. Then* $\sigma_{\mathrm{O}}(\Delta) \leq 12$ *and* $\tau(D) \leq 6$. *Moreover,* D *is a quasidisk if and only if* $\sigma_{\mathrm{I}}(D) > 0$.

The inequality $\sigma_{\mathrm{O}}(\Delta) \leq 12$ is shown as follows. Let $f : D \to \widehat{\mathbb{C}}$ be univalent and set $\Omega = f(D)$. Take a conformal map $g : \mathbb{D}^* \to D$ and set $h = f \circ g$. Then, by Lemmas 1.3.1, 1.6.2 and the Kraus-Nehari theorem, we obtain

$$\|S_f\|_{B_2(D)} = \|g_2^*(S_f)\|_{B_2(\mathbb{D}^*)} = \|S_h - S_g\|_{B_2(\mathbb{D}^*)} \leq \|S_h\|_{B_2(\mathbb{D}^*)} + \|S_g\|_{B_2(\mathbb{D}^*)} \leq 12.$$

It is a remarkable fact due to Beardon and Gehring [9] that $\sigma_{\mathrm{O}}(D) \leq 12$ holds even for an arbitrary hyperbolic domain D.

The inner and outer radii of univalence are better understood in the context of (quasi-) Teichmüller space.

Lemma 5.2.2. *Let* $g : \mathbb{D}^* \to D$ *be a conformal homeomorphism of* $\mathbb{D}^*$ *onto a simply connected hyperbolic domain* D. *Then* $\{\varphi \in S(\mathbb{D}^*) : \|\varphi - S_g\| < \sigma_{\mathrm{I}}(D)\}$ *is the maximal open ball centered at* S_g *contained in* $T(\mathbb{D}^*)$. *On the other hand,* $\sigma_{\mathrm{O}}(D) = \max\{\|\varphi - S_g\|_{B_2(\mathbb{D}^*)} : \varphi \in S(\mathbb{D}^*)\}$.

Lehto [62] proved the following relations.

Theorem 5.2.3. *The relation* $\sigma_{\mathrm{O}}(D) = \tau(D) + 6$ *holds for a simply connected hyperbolic domain* D. *Furthermore,* $2 - \tau(D) \leq \sigma_{\mathrm{I}}(R) \leq \min\{2, 6 - \tau(D)\}$.

As for the quantity $\tau(D)$, the following are known. For a convex domain D, we have $\tau(D) \leq 2$. This result is repeatedly re-discovered by many mathematicians;

[81], [86], [**?**], [75], [62]. Suita [102] refined this result by showing the sharp inequality

$$\tau(f(\mathbb{D})) \le \begin{cases} 2, & 0 \le \alpha \le 1/2, \\ 8\alpha(1-\alpha), & 1/2 \le \alpha \le 1 \end{cases}$$

for a convex function $f \in \mathscr{K}$ of order α, namely, when $\operatorname{Re}(1+zf''(z)/f'(z)) > \alpha$.

It is known that $\tau(D) \le 6(K^2-1)/(K^2+1)$ for a K-quasidisk D (see [64]). See also [65], [50], [21] for further information.

It is not easy to determine, or even to estimate from below, the value of $\sigma_{\mathrm{I}}(D)$, in general. Known examples are sectors [59], triangles [61], the interiors and the exteriors of regular polygons [20], [61], some other polygonal domains [69], [70], the exteriors of hyperbolas [60].

For a general method of estimating $\sigma_{\mathrm{I}}(D)$ from below, see [63], [64] and [100]. See also [101].

5.3. Pre-Schwarzian counterpart. One can define quantities similarly as in the previous section with respect to pre-Schwarzian derivative. We add the symbol ˆ to indicate it. For instance,

$$\hat{\sigma}_{\mathrm{I}}(D) = \sup\{\sigma \ge 0 : \|T_f\|_{B_1(D)} \le \sigma \Rightarrow f \text{ is univalent in } D\}$$

for a hyperbolic domain D in $\mathbb{C}$. In the case when $D = \mathbb{D}^*$, we adopt the norm $\mathrm{B}(\psi)$:

$$\hat{\sigma}_{\mathrm{I}}(\mathbb{D}^*) = \sup\{\sigma \ge 0 : \mathrm{B}(T_F) \le \sigma \Rightarrow f \text{ is univalent in } \mathbb{D}^*\}.$$

Duren, Shapiro and Shields [28] proved that $\hat{\sigma}_{\mathrm{I}}(\mathbb{D}) \ge 2(\sqrt{5}-2) = 0.472\cdots$ by observing that $\|\psi'\|_{B_2(\mathbb{D})} \le 4\|\psi\|_{B_1(\mathbb{D})}$ and thus $\pi(\psi) = \psi' - \psi^2/2$ is a continuous map of $B_1(\mathbb{D})$ into $B_2(\mathbb{D})$. Note that Wirths [109] found the sharp constant $C = (13\sqrt{3} + 55\sqrt{11})/64 = 3.20204\ldots$ for the estimate $\|\psi'\|_{B_2(\mathbb{D})} \le C\|\psi\|_{B_1(\mathbb{D})}$. Nowadays, the best value for this univalence criterion is known.

Theorem 5.3.1. $\hat{\sigma}_{\mathrm{I}}(\Delta) = 1$ *and* $\hat{\sigma}_{\mathrm{O}}(\Delta) = 6$ *for* $\Delta = \mathbb{D}, \mathbb{H}$ *and* $\mathbb{D}^*$.

Becker [10], [11] showed that $\hat{\sigma}_{\mathrm{I}}(\mathbb{D}) \ge 1$ and $\hat{\sigma}_{\mathrm{I}}(\mathbb{D}^*) \ge 1$ and Becker-Pommerenke [13] showed that equality hold for $\Delta = \mathbb{D}$ and that $\hat{\sigma}_{\mathrm{I}}(\mathbb{H}) = 1$. Pommerenke [84] showed the sharpness for $\Delta = \mathbb{D}^*$.

By (4.1.1) and the fact that the Koebe function K satisfies $\|T_K\|_{B_1(\mathbb{D})} = 6$, we see that $\hat{\sigma}_{\mathrm{O}}(\mathbb{D}) = 6$. $\hat{\sigma}_{\mathrm{O}}(\mathbb{H}) = 6$ can be seen by noting the relation

$$\|\psi\|_{B_1(\mathbb{H})} = \lim_{r \to 1-} \|\psi\|_{B_1(\Delta_r)}$$

for $\psi \in B_1(\mathbb{H})$, where $\Delta_r = \{z : |z - \mathrm{i}(1+r^2)/(1-r^2)| < 2r/(1-r^2)\}$. The formula $\hat{\sigma}_{\mathrm{O}}(\mathbb{D}^*) = 6$ follows from the fact that the inequality in (4.2.2) is sharp for each ζ.

For concrete estimates of $\hat{\tau}(D)$ for several geometric classes of domains, see [110], [97], [77], [47], [48].

In spite of relative simplicity of the operation T_f, very little is known for quantities $\hat{\sigma}_{\mathrm{I}}(D)$ and $\hat{\sigma}_{\mathrm{O}}(D)$. Stowe [93] gave non-trivial examples of domains D for which $\hat{\sigma}_{\mathrm{I}}(D) \geq 1$.

5.4. Directions of further investigation. The Bers embedding of Teichmüller spaces is still mysterious. We know very little about the shape of it. Pictures of one-dimensional Teichmüller spaces were recently given in [51] and [52]. Note that the first attempt towards it was done by Porter [85] as early as in 1970's.

It is an interesting and important problem to describe the intersection of $T(\Delta)$ or $\hat{T}(\Delta)$ with a (complex) one-dimensional vector subspace of $B_2(\Delta)$ or $B_1(\Delta)$ for a circle domain. Completely known examples are essentially, as far as the author knows, the linear hull of $1/(1-z)$ in $B_1(\Delta)$ [88] and the linear hull of z^{-2} in $B_2(\mathbb{H})$ in [45], only.

The results presented above could be generalized to various directions. We end this survey with remarks on possible ways to study furthermore.

In this section, we considered mainly the case when the domain is simply connected. When the domain is multiply connected, the problem will become much more difficult. See [79] and [80] for fundamental information.

We were concerned here with only pre-Schwarzian and Schwarzian derivatives. On the other hand, several definitions of higher-order Schwarzian derivatives have been proposed (e.g., [105], [89]). Thus, we may develop the theory for those higher-order Schwarzian derivatives.

Of course, we may consider domains in $\mathbb{C}^n$ or $\mathbb{R}^n$ but with great difficulty caused by the lack of canonical metrics such as hyperbolic metric, the lack of Riemann mapping theorem and so on. Note that Martio and Sarvas [67] gave some injectivity conditions even in higher dimensions.

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