

# INVARIANT DIFFERENTIAL OPERATORS ASSOCIATED WITH A CONFORMAL METRIC

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ABSTRACT. Peschl defined invariant higher-order derivatives of a holomorphic or meromorphic function on the unit disk. Here, the invariance is concerned with the hyperbolic metric of the source domain and the canonical metric of the target domain. Minda and Schippers extended Peschl's invariant derivatives to the case of general conformal metrics. We introduce similar invariant derivatives for smooth functions on a Riemann surface and show a complete analogue of Faà di Bruno's formula for the composition of a smooth function with a holomorphic map with respect to the derivatives. An interpretation of these derivatives in terms of intrinsic geometry and some applications will be also given.

## 1. INTRODUCTION

The uniformization theory tells us that an arbitrary Riemann surface has the natural geometry, namely, spherical, Euclidean or hyperbolic geometry. Standard examples are the Riemann sphere  $\widehat{\mathbb{C}}$  with the spherical metric  $|dz|/(1+|z|^2)$ , the complex plane with the Euclidean metric  $|dz|$  and the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  with the hyperbolic (or the Poincaré) metric  $|dz|/(1-|z|^2)$ . For a unifying treatment, we introduce the notation  $\mathbb{C}_\varepsilon$  to designate  $\widehat{\mathbb{C}}$  for  $\varepsilon = 1$ ,  $\mathbb{C}$  for  $\varepsilon = 0$  and  $\mathbb{D}$  for  $\varepsilon = -1$ . Unless otherwise stated, we understand that  $\mathbb{C}_\varepsilon$  is equipped with the canonical metric  $\lambda_\varepsilon(z)|dz| = |dz|/(1+\varepsilon|z|^2)$ . Note that  $\lambda_\varepsilon$  has constant Gaussian curvature  $4\varepsilon$ .

For a holomorphic map  $f : \mathbb{C}_\delta \rightarrow \mathbb{C}_\varepsilon$  ( $\delta, \varepsilon = 1, 0, -1$ ), it is more natural to consider a sort of invariant derivatives of  $f(z)$  associated with  $\mathbb{C}_\delta$  and  $\mathbb{C}_\varepsilon$  rather than the usual derivatives  $f^{(n)}(z) = d^n f(z)/dz^n$ . As such, commonly used is the invariant derivative  $D^n f(z)$  due to Peschl [11], which is defined by the power series expansion

$$(1.1) \quad \frac{f\left(\frac{\zeta+z}{1-\delta\bar{z}\zeta}\right) - f(z)}{1 + \varepsilon f(z)f\left(\frac{\zeta+z}{1-\delta\bar{z}\zeta}\right)} = \sum_{n=1}^{\infty} \frac{D^n f(z)}{n!} \cdot \zeta^n$$

around  $\zeta = 0$ . Note that the group  $\text{Isom}^+(\mathbb{C}_\varepsilon)$  of sense-preserving isometries of  $\mathbb{C}_\varepsilon$  consists of the maps  $L(\zeta) = \eta(\zeta - a)/(1 + \varepsilon\bar{a}\zeta)$  for some  $a \in \mathbb{C}_\varepsilon$  and  $\eta \in \mathbb{C}$  with  $|\eta| = 1$ , where

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$L(\zeta) = -\eta/\zeta$  for  $\varepsilon = 1$  and  $a = \infty$ . For example,

$$\begin{aligned} D^1 f(z) &= \frac{(1 + \delta|z|^2)f'(z)}{1 + \varepsilon|f(z)|^2}, \\ D^2 f(z) &= \frac{(1 + \delta|z|^2)^2 f''(z)}{1 + \varepsilon|f(z)|^2} + \frac{2\delta\bar{z}(1 + \delta|z|^2)f'(z)}{1 + \varepsilon|f(z)|^2} - \frac{2\varepsilon(1 + \delta|z|^2)^2 \overline{f(z)}f'(z)^2}{(1 + \varepsilon|f(z)|^2)^2}, \\ D^3 f(z) &= \frac{(1 + \delta|z|^2)^3 f'''(z)}{1 + \varepsilon|f(z)|^2} - \frac{6\varepsilon(1 + \delta|z|^2)^3 \overline{f(z)}f'(z)f''(z)}{(1 + \varepsilon|f(z)|^2)^2} + \frac{6\delta\bar{z}(1 + \delta|z|^2)^2 f''(z)}{1 + \varepsilon|f(z)|^2} \\ &\quad + \frac{6\delta^2\bar{z}^2(1 + \delta|z|^2)f'(z)}{1 + \varepsilon|f(z)|^2} - \frac{12\delta\varepsilon\bar{z}(1 + \delta|z|^2)^2 \overline{f(z)}f'(z)^2}{(1 + \varepsilon|f(z)|^2)^2} + \frac{6\varepsilon^2(1 + \delta|z|^2)^3 \overline{f(z)}^2 f'(z)^3}{(1 + \varepsilon|f(z)|^2)^3}. \end{aligned}$$

These derivatives are invariant in the sense that  $|D^n(L \circ f \circ M)| = |D^n f| \circ M$  for  $L \in \text{Isom}^+(\mathbb{C}_\varepsilon)$  and  $M \in \text{Isom}^+(\mathbb{C}_\delta)$ . Minda [10] and Schippers [15] generalized this for arbitrary conformal metrics. We now give a generalized definition of  $D^n f$ .

In this introductory section, we consider plane domains with smooth conformal metrics for the sake of simplicity. As we will see in Section 3, the notions given here can be extended for a holomorphic map  $f$  between Riemann surfaces with smooth conformal metrics in an obvious manner. See [16] for examples of useful (but not necessarily smooth) conformal metrics on Riemann surfaces.

We define invariant differential operators  $\partial_\rho^n$  acting on the space  $C^\infty(V)$  of smooth (complex-valued) functions on a plane domain  $V$  with smooth conformal metric  $\rho = \rho(z)|dz|$  inductively by

$$\begin{aligned} \partial_\rho^1 \varphi &= \partial_\rho \varphi = \frac{1}{\rho(z)} \frac{\partial \varphi(z)}{\partial z}, \\ (1.2) \quad \partial_\rho^{n+1} \varphi &= (\partial_\rho \circ \partial_\rho^n) \varphi - n(\partial_\rho \log \rho) \cdot \partial_\rho^n \varphi, \quad n \geq 1 \end{aligned}$$

for  $\varphi \in C^\infty(V)$ . Note that the symbol  $\partial_\rho^n$  does not mean an iteration of  $\partial_\rho$ . However, when  $\rho = |dz|$  (the Euclidean metric), obviously  $\partial_\rho^n = \partial^n = (\partial/\partial z)^n$ , which is the  $n$ -th iterate of  $\partial$ .

The operator  $\partial_\rho^2$  for  $\rho = \lambda_\varepsilon$  appeared in [5] (see also [6]). Note that the quantity  $\rho^n \partial_\rho^n \varphi$  appears in some computations of the Laplacians for  $n$ -forms on Riemann surfaces with variable conformal metrics (see, for instance, [2]).

Let  $f : V \rightarrow W$  be a holomorphic map between plane domains. If  $\rho = \rho(z)|dz|$  and  $\sigma = \sigma(w)|dw|$  are smooth conformal metrics on  $V$  and  $W$ , respectively, then  $D^n f = D_{\sigma, \rho}^n f$  is defined on  $V$  inductively by

$$\begin{aligned} D^1 f &= \frac{\sigma \circ f}{\rho} f' \quad \text{and} \\ (1.3) \quad D^{n+1} f &= [\partial_\rho - n(\partial_\rho \log \rho) + (\partial_\sigma \log \sigma) \circ f \cdot D^1 f] D^n f, \quad n \geq 1. \end{aligned}$$

Here,  $D^n$  does not mean the  $n$ -th iterate of  $D^1$ . It should be noted that the chain rule

$$D_{\tau, \rho}^1(g \circ f) = (D_{\tau, \sigma}^1 g) \circ f \cdot D_{\sigma, \rho}^1 f$$

is valid for holomorphic maps  $f : V \rightarrow W$  and  $g : W \rightarrow X$  and conformal metrics  $\rho, \sigma, \tau$  on  $V, W, X$ , respectively. This definition looks different from Peschl's one, but it turns out that these are equivalent when  $V = \mathbb{C}_\delta$  and  $W = \mathbb{C}_\varepsilon$  (see Proposition 7.2).

One of the purposes of the present paper is to show that these invariant derivatives satisfy the same rule as do the ordinary derivatives of compositions of smooth functions of a real variable. It is known that higher-order derivatives of the composite function  $g \circ f$  of smooth real-valued functions  $f$  and  $g$  of a real variable are described by Faà di Bruno's formula (cf. [12, p. 36] or [1, p. 137])

$$(g \circ f)^{(n)} = \sum_{k=1}^n g^{(k)} \circ f \cdot A_{n,k}(f', \dots, f^{(n-k+1)}),$$

where  $A_{n,k} = A_{n,k}(x_1, \dots, x_{n-k+1})$  are Bell polynomials (see Section 4 for the definition). The following result is an analogue of Faà di Bruno's formula for our invariant differential operators. We note that, by virtue of transformation rules for these operators (Lemmas 3.2 and 3.6 below), the result can be extended for Riemann surfaces with smooth conformal metrics via local coordinates (see Section 3).

**Theorem 1.1.** *Let  $V$  and  $W$  be plane domains with smooth conformal metrics  $\rho$  and  $\sigma$ , respectively, and let  $f : V \rightarrow W$  be holomorphic. Then, for every function  $\varphi$  in  $C^\infty(W)$ , the relation*

$$(1.4) \quad \partial_\rho^n(\varphi \circ f) = \sum_{k=1}^n (\partial_\sigma^k \varphi) \circ f \cdot A_{n,k}(D^1 f, \dots, D^{n-k+1} f)$$

holds for each  $n \geq 1$ .

We remark that these relations for  $n = 1, 2$  were previously noticed in [5] when  $V = \mathbb{C}_{-1}$  and  $W = \mathbb{C}_{+1}$ , and in [6] when  $V = W = \mathbb{C}_{-1}$ .

This theorem shows that our invariant derivatives  $D^n f$  are natural, and enables us to compute higher-order derivatives of functions more easily. For instance, when  $V = W = \mathbb{C}_{-1}$  and  $f$  maps  $\mathbb{C}_{-1}$  conformally onto a hyperbolically convex subdomain  $\Omega$  of  $\mathbb{C}_{-1}$ , several characterizations of the domain  $\Omega$  are given in terms of the invariant derivatives  $D^n f$  in [9]. Invariant derivatives of  $\mu \circ f$  can be computed and related to  $D^n f$  by using the above formula for a geometric quantity  $\mu$  on  $\Omega$ . By this way, the above theorem in this special case was utilized to simplify involved computations in [6].

The present paper is organized as follows.

In order to give a natural interpretation of our invariant derivatives on Riemann surfaces, we need a differential-geometric setup. In Section 2, we give basic concepts in differential geometry that we need and introduce necessary notation and terminology. Though the material is standard, an expository account will be given there because we could not find a convenient reference containing all the needed contents concisely.

Section 3 will be devoted to an explanation of the way that the invariant derivatives  $D^n f = D_{\sigma,\rho}^n f$  arise for a holomorphic map  $f : R \rightarrow S$  between Riemann surfaces with conformal metrics  $\rho$  and  $\sigma$ . Prior to this, we define the operators  $\partial_\rho^n$  in a natural way. Since these operators are described as tensors of specific types, it is a routine task to see that they obey certain transformation rules.

Section 4 summarizes basic properties of the (exponential) Bell polynomials as well as a principle leading to Faà di Bruno type formulae for a sort of differential operators (see Lemma 4.2). This principle plays a decisive role in the proof of Theorem 1.1.

Section 5 gives a proof of Theorem 1.1. To this end, we introduce an auxiliary differential operator. A remarkable fact is that the  $n$ -th iterate of the differential operator can

describe our differential operators  $\partial_\rho^n$  and  $D_{\sigma,\rho}^n$  in simple ways, which makes the proof of Theorem 1.1 dramatically short.

The defining recursive relations (1.3) give apparently much complicated expressions of  $D^n f$ . In Section 6, as an application of Theorem 1.1, we derive another expression of  $D^n f$  in terms of  $f^{(n)}$  and the lower order derivatives  $D^1 f, \dots, D^{n-1} f$ . Moreover, we give concrete forms for  $D^n f$  in terms of only the ordinary derivatives  $f', \dots, f^{(n)}$  and for  $f^{(n)}$  in terms of  $D^1 f, \dots, D^n f$ .

Section 7 will be devoted to see what is the outcome of the previous sections for the canonical surfaces  $\mathbb{C}_\varepsilon$  for  $\varepsilon = +1, 0, -1$ . Although some of them are known already, we believe that our approach will give a further insight even into the classical invariant derivatives.

Further applications of theorems given in the present paper to the study of Schwarzian derivatives will be supplied in forthcoming papers of the authors.

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## 2. CONNECTIONS ON VECTOR BUNDLES

We recall basic notions used in differential geometry. We refer to an excellent book [7] by Kobayashi and Nomizu for details. We give somewhat detailed exposition of the necessary material for the reader who is not familiar with differential geometry, as well as for the setup of terminology and notation.

Let  $R$  be a Riemann surface. Let  $E$  be a holomorphic vector bundle over  $R$  with projection  $\pi : E \rightarrow R$  and denote by  $\Gamma(E)$  the set of smooth cross sections of  $E$  over  $R$ . In what follows, vector bundles will always be holomorphic. The most fundamental vector bundles over  $R$  are the (complexified) tangent bundle  $T(R) = T_{\mathbb{C}}(R)$  and its dual  $T^*(R)$  (over  $\mathbb{C}$ ), the cotangent bundle. An element of  $\mathfrak{X}(R) = \Gamma(T(R))$  is called a vector field on  $R$  and an element of  $\Gamma(T^*(R))$  is called a 1-form on  $R$ .

A *connection* on  $E$  is a complex linear mapping  $\nabla : \Gamma(E) \rightarrow \Gamma(T^*(R) \otimes E)$  satisfying the Leibniz rule

$$\nabla(\varphi\xi) = d\varphi \otimes \xi + \varphi\nabla\xi, \quad \varphi \in C^\infty(R), \quad \xi \in \Gamma(E).$$

Note that the operator  $\nabla$  is local. In other words,  $\nabla$  naturally operates on  $\Gamma(E|_U)$  for an open subset  $U$  of  $R$ . (This fact enables us to consider  $\nabla$  as a sheaf homomorphism of the sheaf of local smooth sections of  $E$ , though we do not take this formalism explicitly.) Identifying  $\Gamma(T^*(R) \otimes E)$  with  $\Gamma(\text{Hom}(T(R), E))$ , we can define a linear transformation  $\nabla_X : \Gamma(E) \rightarrow \Gamma(E)$  for  $X \in \mathfrak{X}(R)$  by setting  $\nabla_X \xi = (\nabla\xi)(X)$  for  $\xi \in \Gamma(E)$ . We call  $\nabla_X \xi$  the *covariant derivative* of  $\xi$  with respect to  $X$ . Let  $\omega_k^i$  be the *connection forms* of  $\nabla$  with respect to a local frame  $(e_1, \dots, e_r)$  of  $E$ , namely,

$$\nabla e_k = \sum_{i=1}^r \omega_k^i \otimes e_i$$

and  $\omega_k^i$  are local 1-forms on  $R$ . The connection forms then reproduce  $\nabla$  by the formula

$$\nabla\xi = \sum_{i=1}^r \left( d\xi^i + \sum_{k=1}^r \xi^k \omega_k^i \right) \otimes e_i,$$

where  $\xi = \sum_k \xi^k e_k \in \Gamma(E)$ .

Let  $E$  and  $F$  be vector bundles over  $R$  with connections  $\nabla'$  and  $\nabla''$ , respectively. Then the tensor product  $E \otimes F$  admits a connection  $\nabla$  so that

$$\nabla_X(\xi \otimes \eta) = \nabla'_X \xi \otimes \eta + \xi \otimes \nabla''_X \eta, \quad X \in \mathfrak{X}(R), \quad \xi \in \Gamma(E), \quad \eta \in \Gamma(F).$$

We will write  $\nabla = \nabla' \otimes 1_F + 1_E \otimes \nabla''$  symbolically.

Let  $E$  be a vector bundle over  $R$  with connection  $\nabla$ . Then a connection, which will be denoted by the same letter  $\nabla$ , is defined on the dual vector bundle  $E^*$  by the rule

$$d\langle \xi^*, \xi \rangle = \langle \nabla \xi^*, \xi \rangle + \langle \xi^*, \nabla \xi \rangle, \quad \xi \in \Gamma(E), \quad \xi^* \in \Gamma(E^*),$$

where  $\langle \xi^*, \xi \rangle = \xi^*(\xi)$ . If  $\omega_k^i$  are the connection forms of  $\nabla$  on  $E$  with respect to a local frame  $(e_1, \dots, e_r)$  of  $E$ , the connection forms of  $\nabla$  on  $E^*$  with respect to the dual frame  $(e_1^*, \dots, e_r^*)$  of  $E^*$  are given by  $-\omega_i^k$ .

Let  $f : R \rightarrow S$  be a holomorphic map and let  $F$  be a vector bundle over  $S$  with projection  $\pi : F \rightarrow S$  and connection  $\nabla$ . Recall first that the induced bundle  $f^*F$  is realized as the fibre product  $R \times_S F = \{(p, \xi) \in R \times F : f(p) = \pi(\xi)\}$ . In particular, one can define the pullback  $f^*\xi$  of  $\xi \in \Gamma(F)$  by  $f^*\xi(p) = (p, \xi(f(p)))$ . The induced connection  $f^*\nabla$  on  $f^*F$  is defined by the connection forms  $f^*\omega_j^k$  with respect to  $(f^*e_1, \dots, f^*e_r)$ , where  $\omega_j^k$  are connection forms of  $\nabla$  with respect to a local frame  $(e_1, \dots, e_r)$ .

Let  $g$  be a smooth *conformal metric* on a Riemann surface  $R$ , namely,  $g$  is a Riemannian metric on  $R$  which is locally written in the form  $g = \rho(z)^2(dx^2 + dy^2)$ , where  $z = x + iy : U \rightarrow U'$  is a local coordinate of  $R$  and  $\rho$  is a smooth positive function on  $U'$ . It is a simple exercise to see that a Riemannian metric  $g$  on a Riemann surface is conformal if and only if it is Hermitian. Note that a smooth Hermitian metric on a Riemann surface is automatically Kählerian. A conformal metric  $g$  is sometimes written in the form  $ds = \rho(z)|dz|$  as a line element or in the form  $g = \rho(z)^2 dz d\bar{z}$  as a Hermitian metric. In what follows, we will refer to the conformal metric as  $\rho = \rho(z)|dz|$ .

Let  $\nabla^\rho$  be the Levi-Civita connection (or the Riemannian connection) on  $T(R)$  associated with  $\rho$ . For a local coordinate  $z = x + iy$  of  $R$ , the (local) vector fields  $e_1 = \partial/\partial z = (1/2)(\partial/\partial x - i\partial/\partial y)$  and  $e_2 = \partial/\partial \bar{z} = (1/2)(\partial/\partial x + i\partial/\partial y)$  form a local frame  $(e_1, e_2)$  of  $T(R)$ . By using the information in [7, Vol. II, Chap. IX, §5], we obtain the connection forms of  $\nabla^\rho$  as follows:

$$(2.1) \quad \omega_1^1 = \overline{\omega_2^2} = \frac{2}{\rho} \frac{\partial \rho}{\partial z} dz = 2 \frac{\partial \log \rho}{\partial z} dz, \quad \omega_1^2 = \omega_2^1 = 0.$$

We remark that the Christoffel symbol  $\Gamma_{jk}^i$  is defined to be  $\omega_k^i(e_j)$ . Note also that the connection forms of  $\nabla^\rho$  on  $T^*(R)$  with respect to the dual frame  $(e_1^*, e_2^*) = (dz, d\bar{z})$  are given by  $-\omega_i^k$ . Thus, for instance,

$$(2.2) \quad \nabla^\rho(dz) = -2 \frac{\partial \log \rho}{\partial z} dz \otimes dz.$$

We denote by  $T_s^r(R)$  the tensor bundle of type  $(r, s)$  over  $R$ , namely,  $T_s^r(R) = T(R)^{\otimes r} \otimes T^*(R)^{\otimes s}$  and denote by  $\mathfrak{D}_s^r(R)$  the set of smooth tensor fields of type  $(r, s)$  on  $R$ , namely,  $\mathfrak{D}_s^r(R) = \Gamma(T_s^r(R))$ . By the operation explained above, the connection  $\nabla^\rho$  is defined on  $T_s^r(R)$  as well. Note that  $\mathfrak{D}_0^0(R) = C^\infty(R)$  and that  $\nabla^\rho$  acts on it as the exterior differentiation;  $\nabla^\rho \varphi = d\varphi$  for  $\varphi \in C^\infty(R)$ . The direct sum  $\mathfrak{D}(R) = \sum_{r,s=0}^\infty \mathfrak{D}_s^r(R)$  has a structure of bi-graded  $C^\infty(R)$ -algebra and is called the *mixed tensor algebra* on  $R$ .

## 3. INVARIANT HIGHER-ORDER DERIVATIVES

Let  $R$  be a Riemann surface with conformal metric  $\rho$ . We define a linear transformation  $\Lambda = \Lambda_\rho$  of  $\mathfrak{D}(R)$  by

$$\Lambda(\omega) = dz \otimes \nabla_{\partial/\partial z}(\omega), \quad \omega \in \mathfrak{D}(R),$$

where  $z : U \rightarrow U'$ ,  $U \subset R$ ,  $U' \subset \mathbb{C}$ , is a local coordinate of  $R$ . We see that  $\Lambda(\omega)$  does not depend on the choice of local coordinates. Indeed, let  $w$  be another local coordinate defined in the same domain as  $z$ . Let  $w = h(z)$  be the transition function. Since  $dw = h'dz$  and  $\partial/\partial w = (1/h')\partial/\partial z$ , we obtain

$$dw \otimes \nabla_{\partial/\partial w}(\omega) = (h'dz) \otimes \nabla_{(1/h')\partial/\partial z}(\omega) = \frac{1}{h'} \cdot h'dz \otimes \nabla_{\partial/\partial z}(\omega) = dz \otimes \nabla_{\partial/\partial z}(\omega)$$

for a tensor field  $\omega$ . We notice, by definition, that  $\Lambda(\mathfrak{D}_s^r(R)) \subset \mathfrak{D}_{s+1}^r(R)$ . We note also that  $\Lambda(\omega)$  is nothing but the projection of  $\nabla\omega$  to the  $(1, 0)$ -part.

**Lemma 3.1.** *Let  $\varphi \in C^\infty(R)$  and  $z : U \rightarrow U'$  be a local coordinate of  $R$ . The tensor field  $\Lambda^n(\varphi)$  can be written in the form*

$$\Lambda^n(\varphi) = \varphi_n(z)dz^n$$

for each  $n \geq 0$ , where  $dz^n = dz \otimes \cdots \otimes dz$  ( $n$  times) and  $\varphi_n$  is the smooth function on  $U'$  determined by the recurrence relations with initial condition:

$$(3.1) \quad \varphi_0 = \varphi \circ z^{-1} \quad \text{and} \quad \varphi_{n+1} = \frac{\partial \varphi_n}{\partial z} - 2n \frac{\partial \log \rho}{\partial z} \varphi_n, \quad n \geq 0.$$

*Proof.* We now prove by induction. When  $n = 0$ , the assertion is trivial. So, let  $n \geq 0$  and assume that the assertion holds for  $n$ , namely,  $\Lambda^n(\varphi) = \varphi_n dz^n$ . Then, we obtain

$$\Lambda^{n+1}(\varphi) = \Lambda(\varphi_n dz^n) = \Lambda(\varphi_n) \otimes dz^n + \varphi_n \Lambda(dz^n).$$

Since  $\Lambda(dz^n) = n\Lambda(dz) \otimes dz^{n-1} = -2n \frac{\partial}{\partial z}(\log \rho) dz^{n+1}$  by (2.2), we have the relation

$$\Lambda^{n+1}(\varphi) = \left\{ \frac{\partial \varphi_n}{\partial z} - 2n \frac{\partial(\log \rho)}{\partial z} \varphi_n \right\} dz^{n+1} = \varphi_{n+1} dz^{n+1},$$

which completes the induction argument.  $\square$

We now define the operator  $\partial_\rho^n$  by

$$(3.2) \quad \partial_\rho^n \varphi(z) = \rho(z)^{-n} \varphi_n(z)$$

on  $U'$  for  $\varphi \in C^\infty(R)$  and  $n \in \mathbb{N}$ , where  $\varphi_n$  is given by (3.1). We also write  $\partial_\rho$  for  $\partial_\rho^1$ . We will call  $\partial_\rho^n \varphi$  the  $n$ th  $\rho$ -derivative of  $\varphi$ .

We now see that this definition agrees with that given in Section 1 when  $R$  is a plane domain  $V$  and  $z : V \rightarrow V$  is the identity. Indeed, differentiating both sides of (3.2), we obtain

$$\begin{aligned} \frac{\partial}{\partial z}(\partial_\rho^n \varphi) &= \frac{\partial}{\partial z}(\rho^{-n} \varphi_n) = \rho^{-n} \frac{\partial \varphi_n}{\partial z} + \frac{\partial(\rho^{-n})}{\partial z} \varphi_n \\ &= \rho^{-n} \left\{ \varphi_{n+1} + 2n \frac{\partial \log \rho}{\partial z} \varphi_n \right\} - n \rho^{-n} \frac{\partial \log \rho}{\partial z} \varphi_n \\ &= \rho \partial_\rho^{n+1} \varphi + n \frac{\partial \log \rho}{\partial z} \partial_\rho^n \varphi, \end{aligned}$$

where we have used (3.1). We now divide both sides by  $\rho$  to obtain the relation (1.2).

Note that  $\partial_\rho^n \varphi$  is no longer a function on  $R$ , in general. More precisely, it should be understood as  $(\Lambda^n \varphi)/\rho^n = \varphi_n(z) dz^n / (\rho(z)^n |dz|^n)$ , which is sometimes called an  $(n/2, -n/2)$ -form on  $R$  since  $|dz| = dz^{1/2} d\bar{z}^{1/2}$  formally. At least, the modulus  $|\partial_\rho^n \varphi|$  can be regarded as a function on  $R$  and that is enough in applications.

Since a local isometry between Riemann surfaces with conformal metrics can be regarded (at least locally) as a change of local coordinates, we are able to state the invariance property of  $(\Lambda^n \varphi)/\rho^n$  as a lemma in the following way.

**Lemma 3.2.** *Let  $V$  and  $W$  be plane domains with smooth conformal metrics  $\rho$  and  $\sigma$ , respectively. Suppose that a locally univalent holomorphic map  $p : V \rightarrow W$  is locally isometric. Then the relation*

$$\partial_\sigma^n(\varphi \circ p) = \left( \frac{p'}{|p'|} \right)^n [(\partial_\rho^n \varphi) \circ p], \quad \varphi \in C^\infty(W),$$

holds for each  $n \geq 1$ .

Let  $R$  and  $S$  be Riemann surfaces. Suppose now that a holomorphic map  $f : R \rightarrow S$  is given. The tangent map  $Tf : T(R) \rightarrow T(S)$  can be regarded as an element of  $\Gamma(\text{Hom}(T(R), f^*T(S))) = \Gamma(T^*(R) \otimes f^*T(S))$ . By using local coordinates  $z : U \rightarrow U'$  of  $R$  and  $w : V \rightarrow V'$  of  $S$  with  $f(U) \subset V$ , the section  $Tf$  is described by

$$Tf = \tilde{f}'(z) dz \otimes f^*(\partial/\partial w) + \overline{\tilde{f}'(z)} d\bar{z} \otimes f^*(\partial/\partial \bar{w}),$$

where  $\tilde{f} = w \circ f \circ z^{-1} : U' \rightarrow V' \subset \mathbb{C}$ . We now set

$$\partial f = dz \otimes Tf \left( \frac{\partial}{\partial z} \right) = \tilde{f}'(z) dz \otimes f^*(\partial/\partial w).$$

It is clear that  $\partial f$  does not depend on the choice of local coordinates and thus  $\partial f \in \Gamma(T^*(R) \otimes f^*T(S)) = \Gamma(T_1^0(R) \otimes f^*T(S))$

Suppose next that the Riemann surfaces  $R$  and  $S$  are equipped with conformal metrics  $\rho$  and  $\sigma$ , respectively. We recall that  $\nabla^\rho$  is defined on  $T_s^r(R)$  so that its action on the mixed tensor algebra  $\mathfrak{D}(R)$  satisfies the Leibnitz rule  $\nabla_X^\rho(\xi \otimes \eta) = \nabla_X^\rho \xi \otimes \eta + \xi \otimes \nabla_X^\rho \eta$  for  $\xi, \eta \in \mathfrak{D}(R)$  and  $X \in \mathfrak{X}(R)$ . For a holomorphic map  $f : R \rightarrow S$ , let

$$\nabla^{\rho, \sigma, f} : \Gamma(T_n^0(R) \otimes f^*T(S)) \rightarrow \Gamma(\text{Hom}(T(R), T_n^0(R) \otimes f^*T(S))) = \Gamma(T_{n+1}^0(R) \otimes f^*T(S))$$

be the connection given by  $\nabla^\rho \otimes 1 + 1 \otimes f^*(\nabla^\sigma)$ . Furthermore, we define a linear operator  $\Lambda_f = \Lambda_{\rho, \sigma, f} : \Gamma(T_n^0(R) \otimes f^*T(S)) \rightarrow \Gamma(T_{n+1}^0(R) \otimes f^*T(S))$  by

$$\Lambda_f \xi = dz \otimes \nabla_{\partial/\partial z}^{\rho, \sigma, f} \xi, \quad \xi \in \Gamma(T_n^0(R) \otimes f^*T(S)),$$

where  $z$  is a local coordinate of  $R$ . As before, we can check that  $\Lambda_f \xi$  does not depend on the choice of  $z$ .

Let us find a concrete expression of  $\Lambda_{\rho, \sigma, f}$ . For a pair of local coordinates  $z : U \rightarrow U'$  of  $R$  and  $w : V \rightarrow V'$  of  $S$  with  $f(U) \subset V$ , we set  $\tilde{f} = w \circ f \circ z^{-1} : U' \rightarrow V'$ . We now consider a local section  $\xi$  of  $T_n^0(R) \otimes f^*T(S)$  in the form  $\xi = \varphi(z) dz^n \otimes f^*(\partial/\partial w)$ , where  $\varphi \in C^\infty(U')$ . Recall that the connection forms  $\omega_k^i$  of  $\nabla^\sigma$  with respect to the local frame  $(e_1, e_2) = (\partial/\partial w, \partial/\partial \bar{w})$  are given by  $\omega_1^1 = 2\partial(\log \sigma)dw = \overline{\omega_2^2}$  and  $\omega_1^2 = \omega_2^1 = 0$ , see (2.1). Thus, by definition,  $(f^*\nabla^\sigma)(f^*e_1) = f^*\omega_1^1 \otimes f^*e_1$ , namely,

$$(f^*\nabla^\sigma)f^*(\partial/\partial w) = 2(\partial \log \sigma) \circ \tilde{f} \cdot \tilde{f}' dz \otimes f^*(\partial/\partial w).$$

By (2.2), we obtain

$$\begin{aligned}\nabla^{\rho,\sigma,f}\xi &= d\varphi \otimes dz^n \otimes f^*(\partial/\partial w) - 2n(\partial \log \rho)\varphi dz^{n+1} \otimes f^*(\partial/\partial w) \\ &\quad + 2(\partial \log \sigma) \circ \tilde{f} \cdot \tilde{f}'\varphi dz^{n+1} \otimes f^*(\partial/\partial w) \\ &= \left\{ [\partial\varphi - 2n(\partial \log \rho)\varphi + 2(\partial \log \sigma) \circ \tilde{f} \cdot \tilde{f}'\varphi] dz + \bar{\partial}\varphi d\bar{z} \right\} \otimes dz^n \otimes f^*(\partial/\partial w).\end{aligned}$$

Therefore,

(3.3)

$$\Lambda_{\rho,\sigma,f}[\varphi dz^n \otimes f^*(\partial/\partial w)] = \left[ \partial\varphi - 2n(\partial \log \rho)\varphi + 2(\partial \log \sigma) \circ \tilde{f} \cdot \tilde{f}'\varphi \right] dz^{n+1} \otimes f^*(\partial/\partial w).$$

For the local description  $\tilde{f}$  of  $f : R \rightarrow S$ , we define  $f_n$  inductively for  $n \geq 1$  by

$$(3.4) \quad f_1 = \tilde{f}', \quad f_{n+1} = \partial f_n - 2n(\partial \log \rho)f_n + 2(\partial \log \sigma) \circ \tilde{f} \cdot \tilde{f}'f_n, \quad n \geq 1.$$

It is easy to show a result analogous to Lemma 3.1 by induction.

**Lemma 3.3.** *For  $n \geq 1$ ,  $\Lambda_{\rho,\sigma,f}^{n-1}(\partial f) := (\Lambda_{\rho,\sigma,f} \circ \cdots \circ \Lambda_{\rho,\sigma,f})(\partial f)$  ( $n-1$  times) is of the form  $f_n dz^n \otimes f^*(\partial/\partial w)$ , where  $f_n$  is defined in (3.4).*

We define  $D^n f = D_{\sigma,\rho}^n f$  for the coordinates  $z$  and  $w$  by

$$(3.5) \quad D^n f(z) = \frac{\sigma(\tilde{f}(z))f_n(z)}{\rho(z)^n}, \quad z \in U'.$$

As in the proof of (1.2), it can be checked that this definition is same as (1.3) in Section 1 when  $R$  and  $S$  are plane domains with the identity as local coordinates.

*Remark 3.4.* When  $S = \mathbb{C}$  with the Euclidean metric  $\sigma = \lambda_0$ ,  $D_{\lambda_0,\rho}^n f$  coincides with  $\partial_\rho^n f$ .

The definition of  $D^n f$  depends on the choice of the coordinates  $z$  and  $w$ . Let us observe the effect of a change of coordinates on  $D^n f$ . Let  $\hat{z}$  and  $\hat{w}$  be other local coordinates of  $R$  and  $S$ , respectively and write  $\rho = \hat{\rho}(\hat{z})|d\hat{z}|$  and  $\sigma = \hat{\sigma}(\hat{w})|d\hat{w}|$ . We set  $\hat{f} = \hat{w} \circ f \circ \hat{z}^{-1}$  and write  $z = g(\hat{z})$ ,  $\hat{w} = h(w)$ ,  $\Lambda_{\rho,\sigma,f}^{n-1}(\partial f) = \hat{f}_n d\hat{z}^n \otimes f^*(\partial/\partial \hat{w})$  and  $\hat{D}^n f = \hat{\sigma} \circ \hat{f} \cdot \hat{f}_n / \hat{\rho}^n$ . Since

$$\Lambda_{\rho,\sigma,f}^{n-1}(\partial f) = f_n(z) \cdot (g'd\hat{z})^n \otimes (h' \circ \tilde{f})f^*(\partial/\partial \hat{w}) = (g'(\hat{z}))^n (h'(\tilde{f}(z)))f_n(z) d\hat{z}^n \otimes f^*(\partial/\partial \hat{w}),$$

we have  $\hat{f}_n(\hat{z}) = (g'(\hat{z}))^n (h'(\tilde{f}(z)))f_n(z)$ . In view of  $\rho(z)|g'(\hat{z})| = \hat{\rho}(\hat{z})$  and  $\sigma(w) = \hat{\sigma}(\hat{w})|h'(w)|$ , we obtain

$$(3.6) \quad \hat{D}^n f = \left( \frac{h'}{|h'|} \right) \circ \tilde{f} \circ g \cdot (D^n f) \circ g \cdot \left( \frac{g'}{|g'|} \right)^n.$$

In particular, it turns out that  $|D^n f|$  does not depend on the choice of local coordinates and thus it can be regarded as a global function on the Riemann surface  $R$ .

*Remark 3.5.* By the above transformation rule, we see that the quotient  $D^n f/D^m f$  is independent of the choice of the local coordinate  $w$ . Therefore, one can regard it as an  $((n-m)/2, (m-n)/2)$ -form on  $R$ .

We reformulate the above computation as an invariance property of  $D^n$  as in the following.



**Lemma 3.6.** *Let  $V, \hat{V}, W, \hat{W}$  be plane domains with smooth conformal metrics  $\rho, \hat{\rho}, \sigma, \hat{\sigma}$ , respectively. Suppose that locally isometric holomorphic maps  $g : \hat{V} \rightarrow V$  and  $h : W \rightarrow \hat{W}$  are given. Then, for a holomorphic map  $f : V \rightarrow W$ , the formula*

$$D_{\hat{\sigma}, \hat{\rho}}^n (h \circ f \circ g) = \left( \frac{h'}{|h'|} \right) \circ f \circ g \cdot (D_{\sigma, \rho}^n f) \circ g \cdot \left( \frac{g'}{|g'|} \right)^n$$

is valid on  $\hat{V}$ .

#### 4. BELL POLYNOMIALS

In this section, we give a definition and some properties of the Bell polynomials. As usual, we denote by  $\mathbb{Z}$  the ring of integers and by  $\mathbb{N}$  the set of positive integers. Consider the (commutative) polynomial ring of indeterminates  $x_j$ ,  $j \in \mathbb{N}$ , with coefficients in  $\mathbb{Z}$ :

$$P = \mathbb{Z}[x_1, x_2, \dots].$$

Let  $\mathcal{D} : P \rightarrow P$  be the derivation determined by  $\mathcal{D}x_j = x_{j+1}$  for each  $j \in \mathbb{N}$ . In other words,  $\mathcal{D}$  can be written in a formal way by

$$\mathcal{D} = \sum_{j=1}^{\infty} x_{j+1} \frac{\partial}{\partial x_j}.$$

The degree and the weight of a monomial  $x_{j_1} \cdots x_{j_k}$  are defined to be the numbers  $k$  and  $j_1 + \cdots + j_k$ , respectively. Let  $P_k$  and  $Q_n$  be the sub  $\mathbb{Z}$ -modules of  $P$  generated by monomials of degree  $k$  and by monomials of weight  $n$ , respectively. It is easy to see that  $P = \sum_{k=0}^{\infty} P_k$  becomes a graded ring as well as  $\sum_{n=0}^{\infty} Q_n$ . By definition,  $\mathcal{D}$  maps  $Q_n$  into  $Q_{n+1}$ , while  $\mathcal{D}$  preserves  $P_k$ .

We define the Bell polynomials  $A_{n,k}$ ,  $n \in \mathbb{N}$ ,  $k \in \mathbb{Z}$ , in  $P$  inductively by

$$(4.1) \quad \begin{aligned} A_{1,k} &= \delta_{1,k} x_1, \\ A_{n+1,k} &= \mathcal{D}A_{n,k} + x_1 A_{n,k-1}, \quad n \geq 1, \end{aligned}$$

where  $\delta_{1,k} = 1$  when  $k = 1$  and  $\delta_{1,k} = 0$  otherwise. By induction, we can easily check the following.

**Lemma 4.1.**

- (0) *All the coefficients of  $A_{n,k}$  are nonnegative,*
- (i)  *$A_{n,k} = 0$  unless  $1 \leq k \leq n$ ,*
- (ii)  *$A_{n,k} \in P_k$ ,*
- (iii)  *$A_{n,k} \in Q_n$ ,*
- (iv)  *$A_{n,k} \in \mathbb{Z}[x_1, \dots, x_{n-k+1}]$  for  $1 \leq k \leq n$ ,*
- (v)  *$A_{n,1} = x_n$ , and*
- (vi)  *$A_{n,n} = x_1^n$ .*

We remark that (iv) follows also from (ii) and (iii) because  $P_k \cap Q_n \subset \mathbb{Z}[x_1, \dots, x_{n-k+1}]$ . Since  $A_{n,k} \in \mathbb{Z}[x_1, \dots, x_{n-k+1}]$ , we sometimes write  $A_{n,k} = A_{n,k}(x_1, \dots, x_{n-k+1})$  for  $1 \leq k \leq n$ .

The Bell polynomials have a certain universal property, which contains Faà di Bruno's formula as a special case. In this section, let  $V$  and  $W$  be just sets and let  $\mathcal{F}(V)$  and  $\mathcal{F}(W)$  be  $\mathbb{C}$ -subalgebras of the algebra of complex-valued functions on  $V$  and  $W$ , respectively.

**Lemma 4.2.** *Let  $d_V$  and  $d_W$  are  $\mathbb{C}$ -derivations on  $\mathcal{F}(V)$  and  $\mathcal{F}(W)$ , respectively, and let  $f$  be a map of  $V$  into  $W$  such that  $\varphi \circ f \in \mathcal{F}(V)$  for every  $\varphi \in \mathcal{F}(W)$ . Suppose that there exists  $\delta f \in \mathcal{F}(V)$  satisfying*

$$d_V(\varphi \circ f) = (d_W \varphi) \circ f \cdot \delta f, \quad \varphi \in \mathcal{F}(W).$$

Then

$$d_V^n(\varphi \circ f) = \sum_{k=1}^n (d_W^k \varphi) \circ f \cdot A_{n,k}(\delta f, d_V(\delta f), \dots, d_V^{n-k}(\delta f))$$

for  $\varphi \in \mathcal{F}(W)$ . Here,  $d_V^n$  and  $d_W^k$  mean iterations of  $d_V$  and  $d_W$ , respectively.

*Proof.* We prove by induction. When  $n = 1$ , the assertion is trivial. Assume that the assertion is valid up to  $n$ . Taking the derivation  $d_V$  of the assertion for  $n$ , we obtain

$$\begin{aligned} d_V^{n+1}(\varphi \circ f) &= \sum_{k=1}^n d_V[(d_W^k \varphi) \circ f] \cdot A_{n,k}(\delta f, d_V(\delta f), \dots, d_V^{n-k}(\delta f)) \\ &\quad + \sum_{k=1}^n (d_W^k \varphi) \circ f \cdot d_V[A_{n,k}(\delta f, d_V(\delta f), \dots, d_V^{n-k}(\delta f))] \\ &= \sum_{k=1}^n (d_W^{k+1} \varphi \circ f) \cdot \delta f A_{n,k}(\delta f, d_V(\delta f), \dots, d_V^{n-k}(\delta f)) \\ &\quad + \sum_{k=1}^n (d_W^k \varphi) \circ f \cdot \sum_{j=1}^{n-k+1} \frac{\partial A_{n,k}}{\partial x_j}(\delta f, d_V(\delta f), \dots, d_V^{n-k}(\delta f)) d_V^j(\delta f) \\ &= \sum_{k=1}^{n+1} (d_W^k \varphi \circ f) \cdot [x_1 A_{n,k-1} + \mathcal{D}A_{n,k}](\delta f, d_V(\delta f), \dots, d_V^{n-k+1}(\delta f)) \\ &= \sum_{k=1}^{n+1} (d_W^k \varphi \circ f) \cdot A_{n+1,k}(\delta f, d_V(\delta f), \dots, d_V^{n-k+1}(\delta f)) \end{aligned}$$

by (4.1). Hence, the assertion is valid also for  $n + 1$ . □

The following property will be crucial in the proof of Theorem 1.1 later.

**Lemma 4.3.** *Let  $a, b \in \mathbb{C}$  and  $1 \leq k \leq n$ . Then*

$$A_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n A_{n,k}(x_1, x_2, \dots, x_{n-k+1}).$$

*Proof.* The relation  $A_{n,k}(ax_1, \dots, ax_{n-k+1}) = a^k A_{n,k}(x_1, \dots, x_{n-k+1})$  follows immediately from Lemma 4.1 (ii). Similarly,  $A_{n,k}(bx_1, \dots, b^{n-k+1}x_{n-k+1}) = b^n A_{n,k}(x_1, \dots, x_{n-k+1})$  is a consequence of Lemma 4.1 (iii). □

For instance, we have

$$\begin{aligned} A_{1,1} &= x_1, \\ A_{2,2} &= x_1^2, \quad A_{2,1} = x_2, \\ A_{3,3} &= x_1^3, \quad A_{3,2} = 3x_1x_2, \quad A_{3,1} = x_3, \\ A_{4,4} &= x_1^4, \quad A_{4,3} = 6x_1^2x_2, \quad A_{4,2} = 3x_2^2 + 4x_1x_3, \quad A_{4,1} = x_4. \end{aligned}$$

It is also known that  $A_{n,k}$  is given explicitly by

$$A_{n,k} = \sum_{\substack{j_1+2j_2+\dots+nj_n=n \\ j_1+j_2+\dots+j_n=k}} \frac{n!}{j_1! \cdots j_n!} \left(\frac{x_1}{1!}\right)^{j_1} \cdots \left(\frac{x_n}{n!}\right)^{j_n}.$$

## 5. PROOF OF THEOREM 1.1

We are now ready to prove Theorem 1.1. Let us introduce an auxiliary differential operator. Define a linear transformation  $d_\rho$  of  $C^\infty(V)$  by

$$(5.1) \quad d_\rho \varphi = \rho^{-1} \partial_\rho \varphi = \rho^{-2} \partial \varphi, \quad \varphi \in C^\infty(V).$$

Set also

$$\delta f = \frac{\sigma \circ f}{\rho} D_{\sigma,\rho}^1 f = \frac{\sigma^2 \circ f}{\rho^2} f'.$$

We note that the relation

$$(5.2) \quad d_\rho(\varphi \circ f) = (d_\sigma \varphi) \circ f \cdot \delta f$$

holds for  $\varphi \in C^\infty(W)$ . Indeed, first we have

$$(5.3) \quad \partial_\rho(\varphi \circ f) = \rho^{-1} \cdot \partial \varphi \circ f \cdot f' = [\sigma^{-1} \partial \varphi] \circ f \cdot \frac{\sigma \circ f}{\rho} f' = (\partial_\sigma \varphi) \circ f \cdot D_{\sigma,\rho}^1 f.$$

A multiplication of both sides of this with  $\rho^{-1}$  produces (5.2).

**Lemma 5.1.** *For  $n \geq 1$ ,*

$$d_\rho^n \varphi = \rho^{-n} \partial_\rho^n \varphi \quad \text{and} \quad d_\rho^{n-1}(\delta f) = \frac{\sigma \circ f}{\rho^n} D_{\sigma,\rho}^n f.$$

Here  $d_\rho^n$  means the  $n$ -th iterate of the transformation  $d_\rho$ .

*Proof.* We only show the second relation by induction. The first one can be shown similarly (see also Remark 3.4).

The relation is trivial for  $n = 1$ . Assume that the relation holds for  $n$ . Taking the  $d_\rho$ -derivative of the logarithm of both sides of it, we obtain

$$\frac{d_\rho^n(\delta f)}{d_\rho^{n-1}(\delta f)} = \rho^{-1} \left[ (\partial_\sigma \log \sigma) \circ f \cdot D^1 f - n \partial_\rho \log \rho + \frac{\partial_\rho D^n f}{D^n f} \right] = \frac{\rho^{-1} D^{n+1} f}{D^n f},$$

where we used the relations (5.3) and (1.3). Hence

$$d_\rho^n(\delta f) = \frac{\sigma \circ f}{\rho^{n+1}} D_{\sigma,\rho}^{n+1} f,$$

which completes the induction. □

*Proof of Theorem 1.1.*

Since the above  $\delta f$  satisfies the relation (5.2) for the  $\mathbb{C}$ -derivations  $d_\rho$  and  $d_\sigma$  on  $C^\infty(V)$  and  $C^\infty(W)$ , Lemma 4.2 yields the formula

$$d_\rho^n(\varphi \circ f) = \sum_{k=1}^n (d_\sigma^k \varphi) \circ f \cdot A_{n,k}(\delta f, d_\rho^1(\delta f), \dots, d_\rho^{n-k}(\delta f))$$

for  $\varphi \in C^\infty(W)$  and  $n \geq 1$ , where  $A_{n,k}$  are Bell polynomials.

By Lemma 5.1, we can rewrite this in the form

$$\rho^{-n} \partial_\rho^n(\varphi \circ f) = \sum_{k=1}^n (\sigma^{-k} \partial_\sigma^k \varphi) \circ f \cdot A_{n,k} \left( \frac{\sigma \circ f}{\rho} D^1 f, \dots, \frac{\sigma \circ f}{\rho^{n-k+1}} D^{n-k+1} f \right).$$

We now make use of Lemma 4.3 to conclude the validity of (1.4).  $\square$

## 6. ANOTHER EXPRESSION OF $D^n f$

Let  $V$  and  $W$  be plane domains with smooth conformal metrics  $\rho$  and  $\sigma$ , respectively, and let  $f : V \rightarrow W$  be holomorphic. The invariant derivative  $D^n f$  was defined in terms of  $D^{n-1} f$  and its  $\rho$ -derivative in (1.3) recursively. It is, however, natural to find out an expression of  $D^n f$  in terms of the ordinary derivative  $f^{(k)}$  without  $\rho$ -derivative of  $D^{n-1} f$ . To this end, we introduce a double sequence of auxiliary functions associated with metric  $\rho$ .

Define the double sequence  $a_{n,k} = a_{n,k}^\rho$ ,  $n = 1, 2, \dots$ ,  $k = 0, \pm 1, \pm 2, \dots$ , of functions in  $C^\infty(V)$  inductively by

$$(6.1) \quad a_{1,k} = \delta_{1,k} \quad \text{and} \quad a_{n+1,k} = a_{n,k-1} + \partial a_{n,k} - 2n(\partial \log \rho) \cdot a_{n,k}, \quad n \geq 1.$$

Note that  $a_{n,k} = 0$  unless  $1 \leq k \leq n$ . Here is a small table of  $a_{n,k}$ 's:

$$\begin{aligned} a_{1,1} &= 1, \\ a_{2,2} &= 1, \quad a_{2,1} = -2\partial \log \rho, \\ a_{3,3} &= 1, \quad a_{3,2} = -6\partial \log \rho, \quad a_{3,1} = -2\partial^2 \log \rho + 8(\partial \log \rho)^2, \\ a_{4,4} &= 1, \quad a_{4,3} = -12\partial \log \rho, \quad a_{4,2} = -8\partial^2 \log \rho + 44(\partial \log \rho)^2, \\ a_{4,1} &= -2\partial^3 \log \rho + 28\partial \log \rho \cdot \partial^2 \log \rho - 48(\partial \log \rho)^3. \end{aligned}$$

Let  $\varphi_n$  be the function given in (3.1) for  $\varphi \in C^\infty(V)$ . We are now able to express  $\varphi_n$  in terms of  $a_{n,k}$  as follows.

**Lemma 6.1.**

$$\varphi_n = \sum_{k=1}^n a_{n,k} \partial^k \varphi = \sum_{k=-\infty}^{\infty} a_{n,k} \partial^k \varphi, \quad n \geq 1.$$

*Proof.* We show the assertion by induction. For  $n = 1$ , it is trivial. Suppose that the assertion is valid for  $n$ . By definition, we compute

$$\begin{aligned}
\varphi_{n+1} &= \frac{\partial}{\partial z} \sum_{k=-\infty}^{\infty} a_{n,k} \partial^k \varphi - 2n(\partial \log \rho) \sum_{k=-\infty}^{\infty} a_{n,k} \partial^k \varphi \\
&= \sum_{k=-\infty}^{\infty} (\partial a_{n,k} \cdot \partial^k \varphi + a_{n,k} \partial^{k+1} \varphi) - 2n(\partial \log \rho) \sum_{k=-\infty}^{\infty} a_{n,k} \partial^k \varphi \\
&= \sum_{k=-\infty}^{\infty} (\partial a_{n,k} + a_{n,k-1} - 2n(\partial \log \rho) \cdot a_{n,k}) \partial^k \varphi \\
&= \sum_{k=-\infty}^{\infty} a_{n+1,k} \partial^k \varphi.
\end{aligned}$$

Thus the assertion is valid also for  $n + 1$ .  $\square$

We define double sequence  $B_{n,k}$ ,  $n \in \mathbb{N}, k \in \mathbb{Z}$ , of polynomials in  $P$  inductively by

$$\begin{aligned}
B_{1,k} &= \delta_{1,k} \\
B_{n+1,k} &= B_{n,k-1} + \mathcal{D}B_{n,k} - 2nx_1 B_{n,k}, \quad n \geq 1.
\end{aligned}$$

By definition,  $B_{n,k} = 0$  unless  $1 \leq k \leq n$ . For instance,

$$\begin{aligned}
B_{1,1} &= 1, \\
B_{2,2} &= 1, \quad B_{2,1} = -2x_1, \\
B_{3,3} &= 1, \quad B_{3,2} = -6x_1, \quad B_{3,1} = -2x_2 + 8x_1^2, \\
B_{4,4} &= 1, \quad B_{4,3} = -12x_1, \quad B_{4,2} = -8x_2 + 44x_1^2, \quad B_{4,1} = -2x_3 + 28x_1x_2 - 48x_1^3.
\end{aligned}$$

We summarize properties of  $B_{n,k}$  and the relation with  $a_{n,k}^\rho$  in the next lemma.

**Lemma 6.2.**

- (i)  $B_{n,k} \in Q_{n-k}$  for  $1 \leq k \leq n$  and, in particular,  $B_{n,k} \in \mathbb{Z}[x_1, \dots, x_{n-k}]$ ,
- (ii)  $B_{n,n} = 1$ ,
- (iii)  $a_{n,k}^\rho = B_{n,k}(\partial \log \rho, \dots, \partial^{n-k} \log \rho)$ .

As a consequence of Theorem 1.1, we obtain the following expression of  $D^n f$ .

**Proposition 6.3.** *Let  $f : V \rightarrow W$  be a holomorphic map between plane domains  $V$  and  $W$  with smooth conformal metrics  $\rho$  and  $\sigma$ . The invariant derivative  $D^n f = D_{\sigma,\rho}^n f$  can be expressed by*

$$(6.2) \quad D^n f = \sigma \circ f \cdot \rho^{-n} \sum_{k=1}^n a_{n,k}^\rho f^{(k)} - \sum_{k=2}^n \{\sigma^{1-k} a_{k,1}^\sigma\} \circ f \cdot A_{n,k}(D^1 f, \dots, D^{n-k+1} f),$$

where  $A_{n,k}$  is the Bell polynomial given in (4.1) and  $a_{n,k}^\rho$  and  $a_{n,k}^\sigma$  are defined in (6.1) for  $\rho$  and  $\sigma$ , respectively.

*Proof.* For brevity, we write  $A^f = A(f', \dots, f^{(n)})$  and  $A^{Df} = A(D^1 f, \dots, D^n f)$  for  $A \in \mathbb{Z}[x_1, \dots, x_n]$ . We will express both sides of (1.4) in terms of  $\partial^k \varphi$ ,  $k = 1, 2, \dots, n$  with the aid of Lemma 6.1. We begin with the left-hand side:

$$\partial_\rho^n(\varphi \circ f) = \sum_{k=1}^n \rho^{-n} a_{n,k}^\rho \partial^k(\varphi \circ f) = \sum_{k=1}^n \rho^{-n} a_{n,k}^\rho \sum_{l=1}^k (\partial^l \varphi) \circ f \cdot A_{k,l}^f,$$

where we used Lemma 4.2 with  $d_V = d_W = \partial$ . On the other hand, the right-hand side can be written by

$$\sum_{k=1}^n (\partial_\sigma^k \varphi) \circ f \cdot A_{n,k}^{Df} = \sum_{k=1}^n \sum_{l=1}^k (\sigma^{-k} a_{k,l}^\sigma \partial^l \varphi) \circ f \cdot A_{n,k}^{Df}.$$

Equating both sides, we obtain the relation

$$\sum_{l=1}^n (\partial^l \varphi) \circ f \sum_{k=l}^n \left[ \rho^{-n} a_{n,k}^\rho \cdot A_{k,l}^f - (\sigma^{-k} a_{k,l}^\sigma) \circ f \cdot A_{n,k}^{Df} \right] = 0.$$

Since, for each  $l$ ,  $1 \leq l \leq n$ , we can choose  $\varphi \in C^\infty(W)$  so that  $\partial^l \varphi \neq 0$  but  $\partial^m \varphi = 0$  for all  $m$  with  $1 \leq m < l$  at a given point in  $W$ , we conclude that the relation

$$(6.3) \quad \sum_{k=l}^n \left[ \rho^{-n} a_{n,k}^\rho \cdot A_{k,l}^f - (\sigma^{-k} a_{k,l}^\sigma) \circ f \cdot A_{n,k}^{Df} \right] = 0$$

holds for every  $l = 1, 2, \dots, n$ . In particular, by letting  $l = 1$ , we obtain the required relation.  $\square$

As an application of the last proposition, we give the first three of the invariant derivatives of  $f$ :

$$\begin{aligned} D^1 f &= \frac{\sigma \circ f}{\rho} f', \\ D^2 f &= \frac{\sigma \circ f}{\rho^2} f'' - 2\partial_\rho \log \rho \cdot D^1 f + 2(\partial_\sigma \log \sigma) \circ f \cdot (D^1 f)^2, \\ D^3 f &= \sigma \circ f \left[ \rho^{-3} f''' - 6\partial_\rho \log \rho \cdot \rho^{-2} f'' - 2(\partial_\rho^2 \log \rho - 2(\partial_\rho \log \rho)^2) \rho^{-1} f' \right] \\ &\quad + 6(\partial_\sigma \log \sigma) \circ f \cdot D^1 f D^2 f + 2(\partial_\sigma^2 \log \sigma - 2(\partial_\sigma \log \sigma)^2) \circ f \cdot (D^1 f)^3. \end{aligned}$$

Observing (6.3), we obtain an expression of the invariant derivative  $D^n f$  in terms of only the ordinary derivatives  $f^{(k)}$ ,  $1 \leq k \leq n$  as follows.

**Corollary 6.4.** *Let  $f : V \rightarrow W$  be a holomorphic map between plane domains  $V$  and  $W$  with smooth conformal metrics  $\rho$  and  $\sigma$ . Then,*

$$D^n f = (\sigma \circ f) \cdot \begin{pmatrix} b_1 & a_{2,1}^\sigma \circ f & a_{3,1}^\sigma \circ f & a_{4,1}^\sigma \circ f & \dots & a_{n,1}^\sigma \circ f \\ b_2 & 1 & a_{3,2}^\sigma \circ f & a_{4,2}^\sigma \circ f & \dots & a_{n,2}^\sigma \circ f \\ b_3 & 0 & 1 & a_{4,3}^\sigma \circ f & \dots & a_{n,3}^\sigma \circ f \\ b_4 & 0 & 0 & 1 & \dots & a_{n,4}^\sigma \circ f \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_n & 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

where

$$b_l = \sum_{k=l}^n \rho^{-n} a_{n,k}^\rho \cdot A_{k,l}(f', f'', \dots, f^{(k-l+1)})$$

for  $l = 1, 2, \dots, n$ .

*Proof.* The equation (6.3) can be interpreted as a system of equations for  $A_k = \sigma^{-k} \circ f \cdot A_{n,k}^{Df}$ ,  $1 \leq k \leq n$ ;

$$\sum_{k=l}^n (a_{k,l}^\sigma \circ f) \cdot A_k = b_l, \quad l = 1, 2, \dots, n.$$

Therefore,  $D^n f = (\sigma \circ f) \cdot A_1$  is obtained as above.  $\square$

Conversely, by using (6.2), we can also express the ordinary  $n$ -th order derivative  $f^{(n)}$  of  $f$  in terms of the invariant derivatives  $D^1 f, \dots, D^n f$  as in the following.

**Corollary 6.5.** *Let  $f : V \rightarrow W$  be a holomorphic map between plane domains  $V$  and  $W$  with smooth conformal metrics  $\rho$  and  $\sigma$ . Then*

$$f^{(n)} = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & p_1 \\ a_{2,1}^\rho & 1 & 0 & \dots & 0 & p_2 \\ a_{3,1}^\rho & a_{3,2}^\rho & 1 & \dots & 0 & p_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n-1,1}^\rho & a_{n-1,2}^\rho & a_{n-1,3}^\rho & \dots & 1 & p_{n-1} \\ a_{n,1}^\rho & a_{n,2}^\rho & a_{n,3}^\rho & \dots & a_{n,n-1}^\rho & p_n \end{vmatrix},$$

where

$$p_l = \rho^l \cdot \sum_{k=1}^l (\sigma^{-k} a_{k,1}^\sigma) \circ f \cdot A_{l,k}(D^1 f, \dots, D^{l-k+1} f)$$

for  $l = 1, 2, \dots, n$ .

*Proof.* From the equation (6.2), we obtain the following system of equations:

$$\sum_{k=1}^l a_{l,k}^\rho \cdot f^{(k)} = p_l, \quad l = 1, 2, \dots, n.$$

By solving it,  $f^{(n)}$  is given as above.  $\square$

## 7. CASE OF CANONICAL METRIC

In this section, we consider the special case when  $R = \mathbb{C}_\delta, S = \mathbb{C}_\varepsilon$  and  $\rho$  and  $\sigma$  are their canonical metrics  $\lambda_\delta$  and  $\lambda_\varepsilon$ , where  $\delta, \varepsilon = +1, 0, -1$ . First of all, we observe a special nature of the metric  $\lambda_\varepsilon$ . Since

$$(7.1) \quad \partial_{\lambda_\varepsilon} \log \lambda_\varepsilon(z) = -\varepsilon \bar{z}, \quad z \in \mathbb{C}_\varepsilon,$$

many computations become simple. For instance, we have the following.

**Lemma 7.1.** Define  $\alpha_{n,k}$ ,  $n \in \mathbb{N}$ ,  $k \in \mathbb{Z}$ , by

$$\alpha_{n,k} = (-1)^{n-k} \frac{n!(n-1)!}{k!(k-1)!(n-k)!} = (-1)^{n-k} \frac{n!}{k!} \binom{n-1}{k-1}$$

if  $1 \leq k \leq n$  and by  $\alpha_{n,k} = 0$  otherwise. Then

$$a_{n,k}^{\lambda_\varepsilon} = \alpha_{n,k} (\partial \log \lambda_\varepsilon)^{n-k}, \quad n \in \mathbb{N}, \quad k \in \mathbb{Z}.$$

*Proof.* We prove by induction on  $n$ . The assertion is trivial for  $n = 1$ . Assuming the assertion for  $n$ , we prove it for  $n + 1$ . First, by taking the  $\partial$ -derivative of both sides of (7.1), we have

$$(7.2) \quad \partial^2 \log \lambda_\varepsilon = (\partial \log \lambda_\varepsilon)^2.$$

By the defining relation (6.1) of  $a_{n,k} = a_{n,k}^{\lambda_\varepsilon}$ , we compute

$$\begin{aligned} a_{n+1,k} &= \alpha_{n,k-1} (\partial \log \lambda_\varepsilon)^{n-k+1} + \alpha_{n,k} \partial (\partial \log \lambda_\varepsilon)^{n-k} - 2n \partial \log \lambda_\varepsilon \cdot \alpha_{n,k} (\partial \log \lambda_\varepsilon)^{n-k} \\ &= [\alpha_{n,k-1} + (n-k)\alpha_{n,k} - 2n\alpha_{n,k}] (\partial \log \lambda_\varepsilon)^{n-k+1} \\ &= \alpha_{n+1,k} (\partial \log \lambda_\varepsilon)^{n-k+1}. \end{aligned}$$

Thus the assertion is valid for  $n + 1$ , too.  $\square$

Concerning the invariant derivative  $D^n f = D_{\lambda_\varepsilon, \lambda_\delta}^n f$ , the definition (1.1) due to Peschl looks different from the present definition (1.3). Schippers [15] gives a brief explanation for the coincidence based on the recurrence relations. For convenience of the reader, we give another explanation for it as an application of Proposition 6.3.

**Proposition 7.2.** Let  $f : \mathbb{C}_\delta \rightarrow \mathbb{C}_\varepsilon$  be a holomorphic map. Then Peschl's invariant derivative  $D^n f$  defined by (1.1) satisfies the recurrence relations (1.3) for  $\rho = \lambda_\delta$  and  $\sigma = \lambda_\varepsilon$ .

*Proof.* For clarity, only in this proof, we write  $D_n f$  for  $D_{\lambda_\varepsilon, \lambda_\delta}^n f$ , and  $D^n f$  for Peschl's derivative. Now, we show that  $D^n f = D_n f$ . It is straightforward to check that  $D^n$  obeys the same transformation rule for isometries as  $D_n$  does (Lemma 3.6). Since  $\text{Isom}^+(\mathbb{C}_\varepsilon)$  acts on  $\mathbb{C}_\varepsilon$  transitively, it is enough to show that  $D^n f(0) = D_n f(0)$  for a holomorphic map  $f : \mathbb{C}_\delta \rightarrow \mathbb{C}_\varepsilon$  with  $f(0) = 0$ . Then, by (1.1), we have  $D^n f(0) = f^{(n)}(0)$ . We now compute  $D_n f(0)$ . Apply Lemma 7.1 to see

$$a_{n,k}^{\lambda_\varepsilon}(0) = \alpha_{n,n} \delta_{n,k} = \delta_{n,k}.$$

Substituting  $z = 0$  and  $f(0) = 0$  into the expression of  $D_n f$  in Proposition 6.3, we find  $D_n f(0) = f^{(n)}(0) = D^n f(0)$ .  $\square$

Our definition of  $D^n$  gives, in turn, the following relations for Peschl's invariant derivatives.

**Corollary 7.3.** Let  $f : \mathbb{C}_\delta \rightarrow \mathbb{C}_\varepsilon$  be holomorphic. Then

$$(1 + \delta|z|^2) \partial(D^n f)(z) = D^{n+1} f(z) - \delta n \bar{z} D^n f(z) + \varepsilon \overline{f(z)} D^1 f(z) D^n f(z)$$

for  $n \geq 1$ .



*Remark 7.4.* These relations appeared in [17, p. 7] for the cases  $n = 1, 2$ . The above relations also follow from the identity

$$(1 - \delta\bar{z}\zeta)\frac{\partial W}{\partial \zeta} - (1 + \delta|z|^2)\frac{\partial W}{\partial z} = (1 - \varepsilon\overline{f(z)}W)D^1f(z),$$

where  $W$  is the left-hand side of (1.1).

Since  $\alpha_{k,1}^{\lambda_\varepsilon} = (-1)^{k-1}k!$  for  $k \geq 1$ , Proposition 6.3 now gives the following result.

**Corollary 7.5.** *Let  $f : \mathbb{C}_\delta \rightarrow \mathbb{C}_\varepsilon$  be holomorphic. Then*

$$\begin{aligned} D^n f &= \lambda_\varepsilon \circ f \cdot \lambda_\delta^{-n} \sum_{k=1}^n a_{n,k}^{\lambda_\delta} f^{(k)} - \sum_{k=2}^n \{ \lambda_\varepsilon^{1-k} \cdot a_{k,1}^{\lambda_\varepsilon} \} \circ f \cdot A_{n,k}(D^1 f, \dots, D^{n-k+1} f) \\ &= \sum_{k=1}^n \alpha_{n,k} \frac{(-\delta\bar{z})^{n-k} (1 + \delta|z|^2)^k f^{(k)}(z)}{1 + \varepsilon|f(z)|^2} - \sum_{k=2}^n k! \left( \varepsilon \overline{f(z)} \right)^{k-1} A_{n,k}(D^1 f, \dots, D^{n-k+1} f), \end{aligned}$$

where  $A_{n,k}$  is the Bell polynomial given in (4.1),  $a_{n,k}^{\lambda_\varepsilon}$  is defined in (6.1) for the canonical metric  $\lambda_\varepsilon$  and  $\alpha_{n,k}$  is given in Lemma 7.1.

The special case when  $\varepsilon = 0$  was previously proved by Sheng Gong [3] (see also [4, p. 133]). Using this Corollary, we can express the ordinary derivative  $f^{(n)}$  in terms of the invariant derivatives  $D^k f$ ,  $k = 1, 2, \dots, n$  as follows. Here and hereafter, we set  $z^0 = 1$  whatever a complex number  $z$  is.

**Corollary 7.6.** *Let  $f : \mathbb{C}_\delta \rightarrow \mathbb{C}_\varepsilon$  be holomorphic. Then*

$$\frac{(1 + \delta|z|^2)^n}{(1 + \varepsilon|f(z)|^2)} \frac{f^{(n)}(z)}{n!} = \sum_{k=1}^n \binom{n-1}{k-1} (-\delta\bar{z})^{n-k} \cdot c_k,$$

where

$$c_k = \sum_{l=1}^k \frac{l!}{k!} \left( \varepsilon \overline{f(z)} \right)^{l-1} A_{k,l}(D^1 f, \dots, D^{k-l+1} f).$$

*Proof.* Let  $f_k = (1 + \delta|z|^2)^k f^{(k)}(z) / (k!(1 + \varepsilon|f(z)|^2))$ . Then, putting the explicit form of  $\alpha_{n,k}$  given in Lemma 7.1 into the formula in Corollary 7.5 yields the relations

$$\sum_{k=1}^n \binom{n-1}{k-1} (\delta\bar{z})^{n-k} \cdot f_k = c_n.$$

It is not difficult to invert this relation to express  $f_n$  in terms of  $c_k$ ,  $1 \leq k \leq n$ , as given in the assertion.  $\square$

For the case when  $\delta = -1$ , and  $\varepsilon = 0$ , Ruscheweyh [14] gave the relation in this corollary. We end this section with an application to a rough estimate of higher-order derivatives of a holomorphic map  $f : \mathbb{C}_\delta \rightarrow \mathbb{C}_\varepsilon$ .

**Theorem 7.7.** *Let  $\delta, \varepsilon \in \{0, 1, -1\}$  and write  $D^n = D_{\lambda_\varepsilon, \lambda_\delta}^n$ . Suppose that a holomorphic map  $f : \mathbb{C}_\delta \rightarrow \mathbb{C}_\varepsilon$  satisfies the inequality  $|D^1 f| \leq M$  on  $\mathbb{C}_\delta$  for a positive constant  $M$ .*

Then, for each  $n \in \mathbb{N}$ , there exists a positive constant  $M_n$  depending only on  $n$  and  $M$  such that  $|D^n f| \leq M_n$  on  $\mathbb{C}_\delta$ . Moreover, the inequality

$$(7.3) \quad \frac{(1 + \delta|z|^2)^n |f^{(n)}(z)|}{(1 + \varepsilon|f(z)|^2) n!} \leq \sum_{k=1}^n \binom{n-1}{k-1} |\delta z|^{n-k} \cdot C_k$$

holds for  $z \in \mathbb{C}_\delta$ , where

$$C_k = \sum_{l=1}^k \frac{l!}{k!} |\varepsilon f(z)|^{l-1} A_{k,l}(M_1, M_2, \dots, M_{k-l+1}).$$

*Remark 7.8.* It is well known that a holomorphic map  $f : \mathbb{C}_\delta \rightarrow \mathbb{C}_\varepsilon$  must be constant whenever  $\delta > \varepsilon$ . The Schwarz-Pick lemma implies the inequality  $|D^1 f| \leq 1$  for any holomorphic map  $f : \mathbb{C}_{-1} \rightarrow \mathbb{C}_{-1}$ . On the other hand, a meromorphic function  $f : \mathbb{C}_{-1} \rightarrow \mathbb{C}_{+1}$  is called *normal* if  $D^1 f$  is bounded (cf. Lehto and Virtanen [8]).

*Proof.* We first show that  $|D^n f| \leq M_n$  holds for a constant  $M_n$  depending only on  $n$  and  $M$ . Since the relevant conditions are invariant under isometries, it suffices to show that  $|D^n f(0)| = |f^{(n)}(0)| \leq M_n$  for  $f$  with  $f(0) = 0$ . Let  $d_\varepsilon(z, w)$  denote the distance induced by the metric  $\lambda_\varepsilon$ . Note that  $d_1(z, 0) = \arctan|z|$ ,  $d_0(z, 0) = |z|$  and  $d_{-1}(z, 0) = \operatorname{arctanh}|z|$ . Since

$$f^* \lambda_\varepsilon(z) = \frac{|f'(z)|}{1 + \varepsilon|f(z)|^2} \leq \frac{M}{1 + \delta|z|^2} = M \lambda_\delta(z),$$

we obtain the estimate

$$(7.4) \quad d_\varepsilon(f(z), 0) \leq M d_\delta(z, 0).$$

In particular, for a fixed positive number  $r$  in  $\mathbb{C}_\delta$ , there exists a number  $N$  depending only on  $M, \delta$  and  $\varepsilon$  such that  $|f(z)| \leq N$  for  $|z| \leq r$ . Cauchy's theorem now yields

$$\begin{aligned} |f^{(n)}(0)| &= \left| \frac{(n-1)!}{2\pi i} \int_{|z|=r} \frac{f'(z) dz}{z^n} \right| \\ &\leq \frac{(n-1)!}{2\pi r^n} \int_{|z|=r} \frac{M(1 + \varepsilon|f(z)|^2) |dz|}{(1 + \delta r^2)} \\ &\leq \frac{M(n-1)!(1 + \max\{\varepsilon N^2, 0\})}{(1 + \delta r^2)r^{n-1}}. \end{aligned}$$

Since the last term depends only on  $n$  and  $M$  (and  $\delta, \varepsilon$ ), we have shown an inequality of the form  $|D^n f| \leq M_n$ . Now the inequality (7.3) follows from Corollary 7.6 together with  $|D^n f| \leq M_n$  and Lemma 4.1 (0).  $\square$

*Remark 7.9.* When  $\varepsilon \leq 0$ , in the proof, we can choose  $M_n$  by

$$M_n = \inf_{r \in \mathbb{C}_\delta, r > 0} \frac{M(n-1)!}{(1 + \delta r^2)r^{n-1}}.$$

When  $\delta = \varepsilon = -1$ , by the last theorem, we obtain an estimate of the form

$$\frac{(1 - |z|^2)^n}{(1 - |f(z)|^2)} |f^{(n)}(z)| \leq K_n$$

for a holomorphic map  $f : \mathbb{C}_{-1} \rightarrow \mathbb{C}_{-1}$  with some constant  $K_n$ . Note that Ruscheweyh [13] showed the same inequality with the sharp constant  $K_n = 2^{n-1}n!$ .

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