THE TEICHMÜLLER SPACE OF THE IDEAL BOUNDARY

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Abstract. In this paper, we consider ideal boundaries of Riemann surfaces by themselves, and show that the set of natural equivalence classes of mutually quasiconformally related ideal boundaries admits a complex Banach manifold structure.

1. The ideal boundary

For an open Riemann surface R, we can consider various kinds of compactifications of R. In this note we consider the Royden's one (cf. [1] and [10]).

To define the Royden compactification, first we take the set $\mathbf{R}(R)$ of bounded continuous (complex) functions f on R which is differentiable in distribution sense and that the Dirichlet integral

$$D(f) = \int_R df \wedge *\overline{df}$$

of f is finite. Then

$$\|f\| = \sup_R |f| + \sqrt{D(f)}$$

is a norm on $\mathbf{R}(R)$, and $\mathbf{R}(R)$ is a Banach algebra with respect to this norm. We call this algebra the *Royden algebra* associated with R.

Now there is a compact Hausdorff space R^*, containing R as an open and dense subset, such that every element in $\mathbf{R}(R)$ can be extended to a continuous function on R^* (and hence $\mathbf{R}(R)$ can be considered as a subset of the set $C(R^*)$ of all continuous functions on R^*) and that $\mathbf{R}(R)$ separates points of R^*, i.e. for every pair of points p_1 and p_2 of R^* there is a function in $\mathbf{R}(R)$ such that $f(p_1) \neq f(p_2)$. Then such an R^* is uniquely determined up to homeomorphisms fixing R pointwise, and we call R^* the Royden compactification of R. Also the compact subset $dR = R^* - R$ is called the *Royden boundary* of R.

Here there are several ways to construct the Royden compactification canonically. One way is to consider the set X of all characters on $\mathbf{R}(R)$. Here a multiplicative linear functional χ on $\mathbf{R}(R)$ with $\chi(1) = 1$ is called a *character*. And equipped with the weak* topology, X is a compact Hausdorff space. Moreover, by considering the point evaluations, we can regard R as an open and dense subset of X and X gives a representative of the Royden compactification of R.

Remark $\mathbf{R}(R)$ is dense in $C(R^*)$ with respect to the uniform topology.

Also we recall the following fact.

1991 *Mathematics Subject Classification.* Primary 30F25, 30F60; Secondary 30C62.

Key words and phrases. Riemann surfaces, ideal boundaries, quasiconformal maps, Teichmüller spaces.

Proposition 1 ([1],[10])**.** *Every quasiconformal homeomorphism F of a Riemann surface R_1 onto another R_2 can be extended to a homeomorphism of R_1^* onto R_2^*.*

Now, we can define another smaller compactification by using, instead of $\mathbf{R}(R)$, the set $\mathbf{KS}(R)$ of continuous functions f which is a constant on every connected component of the complement of some compact set. The Kerékártó-Stoilow compactification $\hat{R}$ of R is the compact Hausdorff space uniquely determined (up to homeomorphisms fixing R point-wise) by the conditions that R is open and dense in $\hat{R}$, that every element of $\mathbf{KS}(R)$ can be extended to a continuous function on $\hat{R}$, and that $\mathbf{KS}(R)$ separates points of $\hat{R}$.

Clearly, there is the canonical projection π from R^* onto the Kerékártó-Stoilow compactification $\hat{R}$ of R such that π is the identical map on R. We call the closed set $dR_p = \pi^{-1}(p)$ a *block of dR over p* for every point $p \in \hat{R} - R$. A block dR_p is also open if p is isolated in $\hat{R} - R$.

When $p \in \hat{R} - R$ corresponds to a puncture of R, we call p a *non-essential point* of $\hat{R} - R$, and the block dR_p a *non-essential* block. Let N be the subset of $\hat{R} - R$ consisting of all non-essential points, and set

$$dR^o = dR - \bigcup_{p \in N} dR_p.$$

Then dR^o is compact, and is called the *essential part* of dR, or the *essential boundary* of R.

Definition We say that a pair (Y, R) of a compact topological space Y and a Riemann surface R is a *primitive ideal boundary* if Y is homeomorphic to the essential part dR^o of the Royden boundary of R.

By Proposition 1, if another R' is quasiconformally equivalent to R, the Royden compactification of R' is homeomorphic to R^*. So we need to restrict such an R as in the above definition, in order to say that two ideal boundaries are the same. And, considering complex structures near the ideal boundary only, we can define a natural class of primitive ideal boundaries as follows.

Definition We say that primitive ideal boundaries (Y_1, R_1) and (Y_2, R_2) are *conformally equivalent* if there is a homeomorphism F of a neighborhood U of Y_1 in R_1^* into R_2^* such that F is conformal on $U \cap R_1$ and $F(Y_1) = Y_2$. Here we regard that Y_j is the essential boundary of R_j^*.

We call the conformal equivalence class of a primitive ideal boundary (Y, R) an *ideal boundary*, which we denote by $[Y, R]$, or simply by a representative Y if R is clear or not important. Also we call such a Riemann surface R a *supporting surface* of Y.

We say that an ideal boundary Y is of *analytically* (*in*)*finite type* if a supporting surface of Y is of an analytically (in)finite.

Here note that an ideal boundary $[Y, R]$ is determined uniquely by the complex structure of R near Y.

Proposition 2. *Suppose that (Y_1, R_1) is a primitive ideal boundary. Then if (Y_1, R_1) and (Y_2, R_2) are conformally equivalent, then we can take the same Riemann surface R as both of R_j, and hence $(Y_1, R) = (Y_2, R)$ in the sense that the identical map of R to itself can be extended to a homeomorphism of Y_1 to Y_2.*

Proof. Let $F : U \to R_2^*$ be as in the definition of the conformal equivalence between (Y_1, R_1) and (Y_2, R_2). Here we may assume that the relative boundary ∂U of $U \cap R_1$ in R_1 consists of a finite number of analytic simple closed curves. Then, there is a compact bordered Riemann surface S such that we can take $R = U \cup S$ as R_1. By identifying U and $F(U)$, we can also take R as R_2 and hence F is the identitical map on U, which implies the assertion. $\square$

Next we say that a subsurface S of a Riemann surface R is *almost compact bordered* if the closure $\overline{S}$ of S in the subsurface $\overline{R}^p$ of $\hat{R}$, obtained from R by filling all points corresponding to punctures, is compact and the relative boundary ∂S of S in R consists of a finite number of analytic simple closed curves in R. Furthermore, if every component of ∂S divides $\overline{R}^p$, then we call an open set

$$U = R^* - S \cup \partial S \cup \left(\cup_{p \in N,\, p \in \overline{S}} \, dR_p \right)$$

a *canonical neighborhood* of the ideal boundary $[Y, R]$, and call $(U \cap R)$ an *end* of R or for Y.

Definition We say that a map f of an ideal boundary $[Y_1, R_1]$ to another $[Y_2, R_2]$ is a *boundary map* (considered as a map of Y_1 to Y_2) if there are a canonical neighborhood U of Y_1 in R_1^* and a homeomorphism $F : U \to R_2^*$ such that $F = f$ on Y_1. Such a map F as above is called a *supporting map* of f.

If a boundary map f of $[Y, R]$ to itself or to another $[Y', R']$ is a surjective homeomorphism (as a map of Y_1 to itself or to Y'), then we call such an f a *boundary self-homeomorphism*, or *boundary homeomorphism*, respectively.

Further, we say that $f : Y \to Y'$ is *conformal, quasiconformal*, and *asymptotically conformal* if so is a supporting map F on $U \cap R$.

Here recall that f is *asymptotically conformal* if and only if we can find a $(1+\epsilon)$-quasiconformal supporting map of f for every $\epsilon > 0$. (For the basic facts about asymptotically conformal maps, see for instance, [5].)

2. Boundary self-homeomorphims

Let $\mathrm{BH}(Y)$ be the group of all boundary self-homeomorphisms of an ideal boundary $[Y, R]$. First we recall the following fact.

Proposition 3 ([8], also see [9])**.** *f is an element of* $\mathrm{BH}(Y)$ *if and only if f is a quasiconformal boundary self-homeomorphism.*

Proof. Since "if"-part is clear, we assume that $f \in \mathrm{BH}(Y)$. Then there are Riemann surface R supporting Y and a homeomorphism F of a canonical neighborhood U of Y in R^* into R^* which supports f. Then by Corollary in [8], there is a quasiconformal homeomorphism of $U \cap R$ into R having the boundary value f on Y, which implies the assertion. $\square$

Also note that a boundary self-homeomorphism of Y need not necessarily the boundary map of a self-homeomorphism of R.

Theorem 4. *There are an ideal boundary Y and an $f \in \mathrm{BH}(Y)$ such that, for every supporting surface R of Y, every quasiconformal self-homeomorphism of R supports neither f nor f^{-1}.*

Proof. Set

$$R_0 = \{|\operatorname{Im} z| < 1\} - \{n \mid n \in \mathbb{Z}, n \geq 0\},$$

and let Y be the ideal boundary supported by R_0. Let f be the boundary self-homeomorphism of Y supported by $F_0(z) = z + 1$. We show that these Y and f are desired ones.

For this purpose, suppose that there were a Riemann surface R_1 supporting Y and a quasiconformal self-homeomorphism F of R_1 which, considered as a self-map of R_1^*, supports f.

Let U be a canonical neighborhood of Y in R_0^* such that $F_0(U) \subset R_0^*$. Take a smaller canonical neighborhood V in U so that $V \cap R_0$ can be considered also as a subsurface of R_1 and that $F_0(V)$ and $F(V)$ are contained in U. F_0 and F restricted to $V \cap R_0$ can be extended to quasiconformal self-homeomorphisms of $\{|\operatorname{Im} z| < 1\}$, which in turn can be identified with $\{|z| < 1\}$ by a Riemann map. Moreover, they can be extended continuously to $\{|z| \leq 1\}$, where the boundary values coincide by the assumption. Hence denoting by the same notations, we conclude that $\Phi = F^{-1} \circ F_0$ can be extended to $\{|z| \leq 1\}$ by the identical boundary values.

Now since Φ belongs to $\mathbf{R}(\{|z| < 1\})$, so is $g(z) = \Phi(z) - z$, which identically vanishes on $\{|z| = 1\}$, and hence Φ gives the identical self-map of Y. Here if there were a sequence of punctures p_n of $V \cap R_0$ (considered as a subsurface of $\{|z| < 1\}$) such that $|p_n|$ tend to 1 and $g(p_n) \neq 0$ for every n, then since $\Phi(p_n)$ also tend to $\{|z| = 1\}$, by taking a subsequence if necessary, we may further assume that

$$\Phi(p_n) \notin \{p_j\}_{j=1}^{\infty}.$$

Hence we can construct a function $P \in \mathbf{R}(R)$ such that $P(p_n) = 1$ but $P(\Phi(p_n)) = 0$ for every n, which would imply that Φ is not the identical map of Y. Indeed, take a mutually disjoint, simply connected neighborhood U_n of p_n so that $\Phi(p_n) \notin U_n$ for every n, and map U_n onto $\{|z| < 1\}$ by a Riemann map g_n so that $g_n(p_n) = 0$. Consider

$$h_n(z) = \frac{-\log(2|z|)}{n^3}$$

on $W_n = \{e^{-n^3}/2 < |z| < 1/2\}$, and set $P_n = h_n \circ g_n$ on $g^{-1}(W_n)$. Extend P_n to a continuous function by setting 0 or 1 in each connected component of $R - g_n^{-1}(W_n)$, we have a function P_n in $\mathbf{R}(R)$ such that $D(P_n) = 2\pi/n^3$. And

$$P = \sum_{n=1}^{\infty} P_n$$

is a desired function.

Thus there is a canonical neighborhood V' of Y (contained in V) such that $F_0(p) = F(p)$, for every puncture p in V'. But then the number of punctures of R_1 outside V' is smaller than that of punctures of R_1 outside $F(V')$, which is a contradiction.

Since the case of F_0^{-1} can be treated similarly, we have the assertion. $\square$

Next, there are boundary self-homeomorphism f of Y with no fixed points. For instance, rotations gives such examples. On the other hand, the following fact seems to be non-trivial.

Proposition 5. *There is an ideal boundary Y such that every element of* $\mathrm{BH}(Y)$ *fixes the same point of Y.*

Proof. In general, the harmonic boundary d_0R of the Royden boundary is invariant under quasiconformal boundary homeomorphisms ([10] III.7.C Theorem. Also see [10] III.8.C Theorem), and hence by Proposition 3, $d_0R \cap Y$ is invariant under every $f \in BH(Y)$. On the other hand, if a supporting surface R belongs to $O_{HD} - O_G$, a theorem of Royden states that $d_0R \cap Y$ consists of a single point (cf. [10] III.F Theorem), which implies the assertion. □

Finally, eventually trivial conformal equivalence is trivial. Here we say that a conformal boundary self-homeomorphism $f : Y \to Y$ is *eventually trivial* if f is supported by a conformal homeomorphism F of a canonical neighborhood U of Y in R^* into R^* such that F on $U \cap R$ is homotopic to the identical map of $U \cap R$ in R.

Proposition 6. *Suppose that $[Y, R]$ is an ideal boundary of analytically infinite type. Let $f_1, f_2 \in BH(Y)$. If $f_1^{-1} \circ f_2$ is an eventually trivial conformal boundary self-homeomorphism, then $f_1 = f_2$.*

Proof. By a theorem of Maitani in [6], F as above is the identical map of U, and hence so is $f_1^{-1} \circ f_2$. □

3. The Teichmüller space

Similarly as before, for ideal boundaries $[Y, R]$ and $[Y', R']$, we say that a quasiconformal boundary homeomorphisms $f : Y \to Y'$ is *homotopic* to an asymptotically conformal boundary homeomorphism $g : Y \to Y'$ if there are supporting maps $F : U \to (R')^*$ of f and $G : U \to (R')^*$ of g, where U is a canonical neighborhood of Y in R^*, such that F is quasiconformal on $U \cap R$, G is asymptotically conformal on $U \cap R$, and F on $U \cap R$ is homotopic to G on $U \cap R$ in R.

In particular, if $[Y, R] = [Y', R']$ and G is the identical map, then again we say that f and F are *eventually trivial*.

Theorem 7. *For every ideal boundary Y, there is a non-identical, eventually trivial and asymptotically conformal, boundary self-homeomorphism of Y.*

Proof. Let U be a canonical neighborhood of Y in R^*, where R is a supporting surface of Y. Take a sequence of points p_n on $U \cap R$ escaping from any compact set of R, and a mutually disjoint, simply connected open neighborhood U_n of p_n for every n. Map each U_n onto $\{|z| < 1\}$ by a Riemann map g_n so that $g_n(p_n) = 0$.

Set

$$\varphi_n(z) = \frac{z + (1/n)}{1 + (1/n)\overline{z}}$$

on $\{|z| < 1\}$, then φ_n is a $(1/n)$-quasiconformal self-homeomorphism of $\{|z| < 1\}$ and $\varphi_n(z) = z$ on $\{|z| = 1\}$. Hence we can define a $(1/n)$-quasiconformal homeomorphism Φ of U into R^* by setting $g_n^{-1} \circ \varphi_n \circ g_n$ on U_n for every n, and to be the identical map outside $\cup_{n=1}^{\infty} U_n$. Then Φ gives a eventually trivial and asymptotically conformal boundary self-homeomorphism f of Y.

Next similarly as before, set

$$h_n(z) = \frac{-\log(n|z|)}{n^3}$$

on $W_n = \{(1/n)e^{-n^3} < |z| < (1/n)\}$. Then we have an element P_n of $\mathbf{R}(R)$ by setting $P_n = h_n \circ g_n$ on $g_n^{-1}(W_n)$ and extending it by a constant 0 or 1 on each component of $R - g_n^{-1}(W_n)$. Since $D(P_n) = 2\pi/n^3$, $P = \sum_{n=1}^{\infty} P_n$ also belongs to $\mathbf{R}(R)$, and $P(p_n) = 1$ and $P(\Phi(p_n)) = 0$ for every n. Thus f is not the identical map. □

We say that two ideal boundaries Y_1 and Y_2 are *quasiconformally related* if there is a quasiconformal boundary homeomorphism of Y_1 onto Y_2. Then we can define the Teichmüller space of quasiconformally related ideal boundaries.

Definition For a given ideal boundary Y_0, consider pairs (Y, f) of an ideal boundary Y and a quasiconformal boundary homeomorphism $f : Y_0 \to Y$, which is called a *marking* of Y.

We say that two pairs (Y_1, f_1) and (Y_2, f_2) are *Teichmüller equivalent* if there is asymptotically conformal boundary homeomorphism of Y_1 to Y_2 which is homotopic to $f_2 \circ f_1^{-1}$.

We call the set of all Teichmüller equivalence classes $[Y, f]$ (or more precisely $[[Y, R], f]$ of marked ideal boundaries (Y, f) the *Teichmüller space* of the ideal boundary Y_0, which is denoted by $T(Y_0)$. A point of $T(Y_0)$ is called a *marked ideal boundary*.

Here note that if Y_0 is an ideal boundary of analytically finite type, then Y_0 is empty, and hence $T(Y_0)$ consists of a single point (, which can be compared with results in [2],[4]). It is remarkable that the Teichmüller space of every ideal boundary admits a natural complex structure.

Theorem 8. *Let Y_0 be an ideal boundary. Then the Teichmüller space $T(Y_0)$ of Y_0 has a canonical complex Banach manifold structure.*

Proof. A theorem of Miyaji in [7] implies that the asymptotic Teichmüller spaces $AT(R_0)$ of R_0 are mutually biholomorphic for all supporting surfaces R_0 of Y_0. Indeed, if R_1 and R_2 are such surfaces, then there is another supporting surface R_3 of Y_0 and analytically finite Riemann surfaces S_1 and S_2 such that R_3 and S_j are obtained from R_j by applying a conformal 2-surgery along a dividing simple closed curve for each j. And Reducing Theorem in [7] states that the asymptotic Teichmüller space $AT(R_j)$ is biholomorphic to the product $AT(S_j) \times AT(R_3)$ for each j. Here since $AT(S_j)$ are trivial, we have a canonical biholomorphism between $AT(R_j)$. (For the details of the asymptotic Teichmüller theory, see [5],[2], and [3].)

Next fix a supporting surface R_0 of Y_0. Then we can construct a natural bijection from $T(Y_0)$ onto $AT(R_0)$ as follows. Take any element $[Y, f]$ of $T(Y_0)$. Then there is a quasiconformal homeomorphism F of $U \cap R_0$ of Y in R_0 into R where U is a canonical neighborhood of Y_0 in R_0 and R is a supporting surface of Y. Then F can be extended to a quasiconformal map of R_0 onto another supporting surface R' of Y (possibly different from R). which gives a point in $AT(R_0)$. By the definitions, we can easily see that this map induces a bijection of $T(Y_0)$ to $AT(R_0)$. Thus we have prove the assertion. □

Remark We say that two boundary self-homeomorpihsms f_1 and f_2 in $\mathrm{BH}(Y_0)$ are *AC-equivalent* if $f_2 \circ f_1^{-1}$ is homotopic to a asymptotically conformal self-homeomorphism of Y. The equivalence class of f is called an *AC-mapping class*, and denoted by $[f]$.

Now every element f of $\mathrm{BH}(Y_0)$ naturally induces an automoprhism f^* of $T(Y_0)$, by setting

$$f^*([Y,g]) = [(Y, g \circ f^{-1})].$$

Then it is clear from the definition that $f_1^* = f_2^*$ if and only if $[f_1] = [f_2]$.

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