THE TEICHMÜLLER SPACE OF THE IDEAL BOUNDARY

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Abstract. In this paper, we consider ideal boundaries of Riemann surfaces by themselves, and show that the set of natural equivalence classes of mutually quasiconformally related ideal boundaries admits a complex Banach manifold structure.

1. The ideal boundary

For an open Riemann surface $R$, we can consider various kinds of compactifications of $R$. In this note we consider the Royden’s one (cf. [1] and [10]).

To define the Royden compactification, first we take the set $\mathbf{R}(R)$ of bounded continuous (complex) functions $f$ on $R$ which is differentiable in distribution sense and that the Dirichlet integral

$$D(f) = \int_R df \wedge \overline{df}$$

of $f$ is finite. Then

$$\|f\| = \sup_R |f| + \sqrt{D(f)}$$

is a norm on $\mathbf{R}(R)$, and $\mathbf{R}(R)$ is a Banach algebra with respect to this norm. We call this algebra the Royden algebra associated with $R$.

Now there is a compact Hausdorff space $R^*$, containing $R$ as an open and dense subset, such that every element in $\mathbf{R}(R)$ can be extended to a continuous function on $R^*$ (and hence $\mathbf{R}(R)$ can be considered as a subset of the set $C(R^*)$ of all continuous functions on $R^*$) and that $\mathbf{R}(R)$ separates points of $R^*$, i.e. for every pair of points $p_1$ and $p_2$ of $R^*$ there is a function in $\mathbf{R}(R)$ such that $f(p_1) \neq f(p_2)$. Then such an $R^*$ is uniquely determined up to homeomorphisms fixing $R$ pointwise, and we call $R^*$ the Royden compactification of $R$. Also the compact subset $dR = R^* - R$ is called the Royden boundary of $R$.

Here there are several ways to construct the Royden compactification canonically. One way is to consider the set $X$ of all characters on $\mathbf{R}(R)$. Here a multiplicative linear functional $\chi$ on $\mathbf{R}(R)$ with $\chi(1) = 1$ is called a character. And equipped with the weak* topology, $X$ is a compact Hausdorff space. Moreover, by considering the point evaluations, we can regard $R$ as an open and dense subset of $X$ and $X$ gives a representative of the Royden compactification of $R$.

Remark $\mathbf{R}(R)$ is dense in $C(R^*)$ with respect to the uniform topology.

Also we recall the following fact.

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**Proposition 1** ([1],[10]). Every quasiconformal homeomorphism \( F \) of a Riemann surface \( R_1 \) onto another \( R_2 \) can be extended to a homeomorphism of \( R_1^* \) onto \( R_2^* \).

Now, we can define another smaller compactification by using, instead of \( R(R) \), the set \( KS(R) \) of continuous functions \( f \) which is a constant on every connected component of the complement of some compact set. The Kerékártó-Stoilow compactification \( \hat{R} \) of \( R \) is the compact Hausdorff space uniquely determined (up to homeomorphisms fixing \( R \) point-wise) by the conditions that \( R \) is open and dense in \( \hat{R} \), that every element of \( KS(R) \) can be extended to a continuous function on \( \hat{R} \), and that \( KS(R) \) separates points of \( \hat{R} \).

Clearly, there is the canonical projection \( \pi \) from \( R^* \) onto the Kerékártó-Stoilow compactification \( \hat{R} \) of \( R \) such that \( \pi \) is the identical map on \( R \). We call the closed set \( dR_p = \pi^{-1}(p) \) a block of \( dR \) over \( p \) for every point \( p \in \hat{R} - R \). A block \( dR_p \) is also open if \( p \) is isolated in \( \hat{R} - R \).

When \( p \in \hat{R} - R \) corresponds to a puncture of \( R \), we call \( p \) a non-essential point of \( \hat{R} - R \), and the block \( dR_p \) a non-essential block. Let \( N \) be the subset of \( \hat{R} - R \) consisting of all non-essential points, and set

\[
dR^o = dR - \bigcup_{p \in N} dR_p.
\]

Then \( dR^o \) is compact, and is called the essential part of \( dR \), or the essential boundary of \( R \).

**Definition** We say that a pair \((Y,R)\) of a compact topological space \( Y \) and a Riemann surface \( R \) is a primitive ideal boundary if \( Y \) is homeomorphic to the essential part \( dR^o \) of the Royden boundary of \( R \).

By Proposition 1, if another \( R' \) is quasiconformally equivalent to \( R \), the Royden compactification of \( R' \) is homeomorphic to \( R^* \). So we need to restrict such an \( R \) as in the above definition, in order to say that two ideal boundaries are the same. And, considering complex structures near the ideal boundary only, we can define a natural class of primitive ideal boundaries as follows.

**Definition** We say that primitive ideal boundaries \((Y_1,R_1)\) and \((Y_2,R_2)\) are conformally equivalent if there is a homeomorphism \( F \) of a neighborhood \( U \) of \( Y_1 \) in \( R_1^* \) into \( R_2^* \) such that \( F \) is conformal on \( U \cap R_1 \) and \( F(Y_1) = Y_2 \). Here we regard \( Y_j \) as the essential boundary of \( R_j^* \).

We call the conformal equivalence class of a primitive ideal boundary \((Y,R)\) an ideal boundary, which we denote by \([Y,R]\), or simply by a representative \( Y \) if \( R \) is clear or not important. Also we call such a Riemann surface \( R \) a supporting surface of \( Y \).

We say that an ideal boundary \( Y \) is of analytically (in)finite type if a supporting surface of \( Y \) is of an analytically (in)finite.

Here note that an ideal boundary \([Y,R]\) is determined uniquely by the complex structure of \( R \) near \( Y \).

**Proposition 2.** Suppose that \((Y_1,R_1)\) is a primitive ideal boundary. Then if \((Y_2,R_2)\) and \((Y_2,R_2)\) are conformally equivalent, then we can take the same Riemann surface \( R \) as both of \( R_j \), and hence \((Y_1,R) = (Y_2,R)\) in the sense that the identical map of \( R \) to itself can be extended to a homeomorphism of \( Y_1 \) to \( Y_2 \).
Proof. Let \( F : U \to R^*_2 \) be as in the definition of the conformal equivalence between \((Y_1, R_1)\) and \((Y_2, R_2)\). Here we may assume that the relative boundary \( \partial U \) of \( U \cap R_1 \) in \( R_1 \) consists of a finite number of analytic simple closed curves. Then, there is a compact bordered Riemann surface \( S \) such that we can take \( R = U \cup S \) as \( R_1 \). By identifying \( U \) and \( F(U) \), we can also take \( R \) as \( R_2 \) and hence \( F \) is the identical map on \( U \), which implies the assertion.

Next we say that a subsurface \( S \) of a Riemann surface \( R \) is almost compact bordered if the closure \( \overline{S} \) of \( S \) in the subsurface \( \overline{R} \) of \( R \), obtained from \( R \) by filling all points corresponding to punctures, is compact and the relative boundary \( \partial S \) of \( S \) in \( R \) consists of a finite number of analytic simple closed curves in \( R \). Furthermore, if every component of \( \partial S \) divides \( \overline{R} \), then we call an open set

\[
U = R^* - S \cup \partial S \cup \left( \bigcup_{p \in N \cup \overline{S}} dR_p \right)
\]

a canonical neighborhood of the ideal boundary \([Y, R]\), and call \((U \cap R)\) an end of \( R \) or for \( Y \).

**Definition** We say that a map \( f \) of an ideal boundary \([Y_1, R_1]\) to another \([Y_2, R_2]\) is a boundary map (considered as a map of \( Y_1 \) to \( Y_2 \)) if there are a canonical neighborhood \( U \) of \( Y_1 \) in \( R_1 \) and a homeomorphism \( F : U \to R^*_2 \) such that \( F = f \) on \( Y_1 \). Such a map \( F \) as above is called a supporting map of \( f \).

If a boundary map \( f \) of \([Y, R]\) to itself or to another \([Y', R']\) is a surjective homeomorphism (as a map of \( Y_1 \) to itself or to \( Y' \)), then we call such an \( f \) a boundary self-homeomorphism, or boundary homeomorphism, respectively.

Further, we say that \( f : Y \to Y' \) is conformal, quasiconformal, and asymptotically conformal if so is a supporting map \( F \) on \( U \cap R \).

Here recall that \( f \) is asymptotically conformal if and only if we can find a \((1+\epsilon)\)-quasiconformal supporting map of \( f \) for every \( \epsilon > 0 \). (For the basic facts about asymptotically conformal maps, see for instance, [5].)

## 2. Boundary self-homeomorphisms

Let \( BH(Y) \) be the group of all boundary self-homeomorphisms of an ideal boundary \([Y, R]\). First we recall the following fact.

**Proposition 3** ([8], also see [9]). \( f \) is an element of \( BH(Y) \) if and only if \( f \) is a quasiconformal boundary self-homeomorphism.

**Proof.** Since "if"-part is clear, we assume that \( f \in BH(Y) \). Then there are Riemann surface \( R \) supporting \( Y \) and a homeomorphism \( F \) of a canonical neighborhood \( U \) of \( Y \) in \( R^* \) into \( R^* \) which supports \( f \). Then by Corollary in [8], there is a quasiconformal homeomorphism of \( U \cap R \) into \( R \) having the boundary value \( f \) on \( Y \), which implies the assertion.

Also note that a boundary self-homeomorphism of \( Y \) need not necessarily the boundary map of a self-homeomorphism of \( R \).

**Theorem 4.** There are an ideal boundary \( Y \) and an \( f \in BH(Y) \) such that, for every supporting surface \( R \) of \( Y \), every quasiconformal self-homeomorphism of \( R \) supports neither \( f \) nor \( f^{-1} \).
Proof. Set
\[ R_0 = \{ \{ \text{Im } z \} < 1 \} - \{ n \mid n \in \mathbb{Z}, n \geq 0 \}, \]
and let \( Y \) be the ideal boundary supported by \( R_0 \). Let \( f \) be the boundary self-homeomorphism of \( Y \) supported by \( F_0(z) = z + 1 \). We show that these \( Y \) and \( f \) are desired ones.

For this purpose, suppose that there were a Riemann surface \( R_1 \) supporting \( Y \) and a quasiconformal self-homeomorphism \( F \) of \( R_1 \) which, considered as a self-map of \( R_1^* \), supports \( f \).

Let \( U \) be a canonical neighborhood of \( Y \) in \( R_0^* \) such that \( F_0(U) \subset R_0^* \). Take a smaller canonical neighborhood \( V \) in \( U \) so that \( V \cap R_0 \) can be considered also as a subsurface of \( R_1 \) and that \( F_0(V) \) and \( F(V) \) are contained in \( U \). \( F_0 \) and \( F \) restricted to \( V \cap R_0 \) can be extended to quasiconformal self-homeomorphisms of \( \{ \{ \text{Im } z \} < 1 \} \), which in turn can be identified with \( \{ \{ z \} < 1 \} \) by a Riemann map. Moreover, they can be extended continuously to \( \{ \{ z \} \leq 1 \} \), where the boundary values coincide by the assumption. Hence denoting by the same notations, we conclude that \( \Phi = F^{-1} \circ F_0 \) can be extended to \( \{ \{ z \} < 1 \} \) by the identical boundary values.

Now since \( \Phi \) belongs to \( R(\{ \{ z \} < 1 \}) \), so is \( g(z) = \Phi(z) - z \), which identically vanishes on \( \{ \{ z \} = 1 \} \), and hence \( \Phi \) gives the identical self-map of \( Y \). Here if there were a sequence of punctures \( p_n \) of \( V \cap R_0 \) (considered as a subsurface of \( \{ \{ z \} < 1 \} \) such that \( |p_n| \) tend to \( 1 \) and \( g(p_n) \neq 0 \) for every \( n \), then since \( \Phi(p_n) \) also tend to \( \{ \{ z \} = 1 \} \), by taking a subsequence if necessary, we may further assume that
\[ \Phi(p_n) \notin \{ p_j \}_{j=1}^{\infty}. \]
Hence we can construct a function \( P \in R(R) \) such that \( P(p_n) = 1 \) but \( P(\Phi(p_n)) = 0 \) for every \( n \), which would imply that \( \Phi \) is not the identical map of \( Y \). Indeed, take a mutually disjoint, simply connected neighborhood \( U_n \) of \( p_n \) so that \( \Phi(p_n) \notin U_n \) for every \( n \), and map \( U_n \) onto \( \{ \{ z \} < 1 \} \) by a Riemann map \( g_n \) so that \( g_n(p_n) = 0 \). Consider
\[ h_n(z) = \frac{-\log(2|z|)}{n^3}, \]
on \( W_n = \{ e^{-n^3}/2 < |z| < 1/2 \} \), and set \( P_n = h_n \circ g_n \) on \( g^{-1}(W_n) \). Extend \( P_n \) to a continuous function by setting \( 0 \) or \( 1 \) in each connected component of \( R - g_n^{-1}(W_n) \), we have a function \( P_n \) in \( R(R) \) such that \( D(P_n) = 2\pi/n^3 \). And
\[ P = \sum_{n=1}^{\infty} P_n \]
is a desired function.

Thus there is a canonical neighborhood \( V' \) of \( Y \) (contained in \( V \) such that \( F_0(p) = F(p) \), for every puncture \( p \) in \( V' \). But then the number of punctures of \( R_1 \) outside \( V' \) is smaller than that of punctures of \( R_1 \) outside \( F(V') \), which is a contradiction.

Since the case of \( F_0^{-1} \) can be treated similarly, we have the assertion.

Next, there are boundary self-homeomorphism \( f \) of \( Y \) with no fixed points. For instance, rotations gives such examples. On the other hand, the following fact seems to be non-trivial.
Proposition 5. There is an ideal boundary $Y$ such that every element of $BH(Y)$ fixes the same point of $Y$.

Proof. In general, the harmonic boundary $d_0R$ of the Royden boundary is invariant under quasiconformal boundary homeomorphisms ([10] III.7.C Theorem). Also see [10] III.8.C Theorem), and hence by Proposition 3, $d_0R \cap Y$ is invariant under every $f \in BH(Y)$. On the other hand, if a supporting surface $R$ belongs to $O_{HD} - O_G$, a theorem of Royden states that $d_0R \cap Y$ consists of a single point (cf. [10] III.F Theorem), which implies the assertion. 

Finally, eventually trivial conformal equivalence is trivial. Here we say that a conformal boundary self-homeomorphism $f : Y \to Y$ is eventually trivial if $f$ is supported by a conformal homeomorphism $F$ of a canonical neighborhood $U$ of $Y$ in $R^*$ into $R^*$ such that $F$ on $U \cap R$ is homotopic to the identical map of $U \cap R$ in $R$.

Proposition 6. Suppose that $[Y, R]$ is an ideal boundary of analytically infinite type. Let $f_1, f_2 \in BH(Y)$. If $f_1^{-1} \circ f_2$ is an eventually trivial conformal boundary self-homeomorphism, then $f_1 = f_2$.

Proof. By a theorem of Maitani in [6], $F$ as above is the identical map of $U$, and hence so is $f_1^{-1} \circ f_2$. 

3. The Teichmüller Space

Similarly as before, for ideal boundaries $[Y, R]$ and $[Y', R']$, we say that a quasiconformal boundary homeomorphisms $f : Y \to Y'$ is homotopic to an asymptotically conformal boundary homeomorphism $g : Y \to Y'$ if there are supporting maps $F : U \to (R')^*$ of $f$ and $G : U \to (R')^*$ of $g$, where $U$ is a canonical neighborhood of $Y$ in $R^*$, such that $F$ is quasiconformal on $U \cap R$, $G$ is asymptotically conformal on $U \cap R$, and $F$ on $U \cap R$ is homotopic to $G$ on $U \cap R$ in $R$.

In particular, if $[Y, R] = [Y', R']$ and $G$ is the identical map, then again we say that $f$ and $F$ are eventually trivial.

Theorem 7. For every ideal boundary $Y$, there is a non-identical, eventually trivial and asymptotically conformal, boundary self-homeomorphism of $Y$.

Proof. Let $U$ be a canonical neighborhood of $Y$ in $R^*$, where $R$ is a supporting surface of $Y$. Take a sequence of points $p_n$ on $U \cap R$ escaping from any compact set of $R$, and a mutually disjoint, simply connected open neighborhood $U_n$ of $p_n$ for every $n$. Map each $U_n$ onto $\{|z| < 1\}$ by a Riemann map $g_n$ so that $g_n(p_n) = 0$.

Set

$$\varphi_n(z) = \frac{z + (1/n)}{1 + (1/n)z}$$

on $\{|z| < 1\}$, then $\varphi_n$ is a $(1/n)$-quasiconformal self-homeomorphism of $\{|z| < 1\}$ and $\varphi_n(z) = z$ on $\{|z| = 1\}$. Hence we can define a $(1/n)$-quasiconformal homeomorphism $\Phi$ of $U$ into $R^*$ by setting $g_n^{-1} \circ \varphi_n \circ g_n$ on $U_n$ for every $n$, and to be the identical map outside $\bigcup_{n=1}^{\infty} U_n$. Then $\Phi$ gives a eventually trivial and asymptotically conformal boundary self-homeomorphism $f$ of $Y$.

Next similarly as before, set

$$h_n(z) = \frac{-\log(n|z|)}{n^3}$$

on $\{|z| < 1\}$. Hence we can define a $(1/n)$-quasiconformal homeomorphism $\Phi$ of $U$ into $R^*$ by setting $g_n^{-1} \circ h_n \circ g_n$ on $U_n$ for every $n$, and to be the identical map outside $\bigcup_{n=1}^{\infty} U_n$. Then $\Phi$ gives a eventually trivial and asymptotically conformal boundary self-homeomorphism $f$ of $Y$.
on $W_n = \{(1/n)e^{-n^2} < |z| < (1/n)\}$. Then we have an element $P_n$ of $R(R)$ by setting $P_n = h_n \circ g_n$ on $g_n^{-1}(W_n)$ and extending it by a constant 0 or 1 on each component of $R - g_n^{-1}(W_n)$. Since $D(P_n) = 2\pi/n^3$, $P = \sum_{n=1}^{\infty} P_n$ also belongs to $R(R)$, and $P(p_n) = 1$ and $P(\Phi(p_n)) = 0$ for every $n$. Thus $f$ is not the identical map. \hfill \Box

We say that two ideal boundaries $Y_1$ and $Y_2$ are quasiconformally related if there is a quasiconformal boundary homeomorphism of $Y_1$ onto $Y_2$. Then we can define the Teichmüller space of quasiconformally related ideal boundaries.

**Definition** For a given ideal boundary $Y_0$, consider pairs $(Y, f)$ of an ideal boundary $Y$ and a quasiconformal boundary homeomorphism $f : Y_0 \to Y$, which is called a marking of $Y$.

We say that two pairs $(Y_1, f_1)$ and $(Y_2, f_2)$ are Teichmüller equivalent if there is an asymptotically conformal boundary homeomorphism of $Y_1$ to $Y_2$ which is homotopic to $f_2 \circ f_1^{-1}$.

We call the set of all Teichmüller equivalence classes $[Y, f]$ (or more precisely $[(Y, R), f]$ of marked ideal boundaries $(Y, f)$ the Teichmüller space of the ideal boundary $Y_0$, which is denoted by $T(Y_0)$. A point of $T(Y_0)$ is called a marked ideal boundary.

Here note that if $Y_0$ is an ideal boundary of analytically finite type, then $Y_0$ is empty, and hence $T(Y_0)$ consists of a single point (which can be compared with results in [2],[4]). It is remarkable that the Teichmüller space of every ideal boundary admits a natural complex structure.

**Theorem 8.** Let $Y_0$ be an ideal boundary. Then the Teichmüller space $T(Y_0)$ of $Y_0$ has a canonical complex Banach manifold structure.

**Proof.** A theorem of Miyaji in [7] implies that the asymptotic Teichmüller spaces $AT(R_0)$ of $R_0$ are mutually biholomorphic for all supporting surfaces $R_0$ of $Y_0$. Indeed, if $R_1$ and $R_2$ are such surfaces, then there is another supporting surface $R_3$ of $Y_0$ and analytically finite Riemann surfaces $S_1$ and $S_2$ such that $R_3$ and $S_j$ are obtained from $R_j$ by applying a conformal 2-surgery along a dividing simple closed curve for each $j$. And Reducing Theorem in [7] states that the asymptotic Teichmüller space $AT(R_j)$ is biholomorphic to the product $AT(S_j) \times AT(R_3)$ for each $j$. Here since $AT(S_j)$ are trivial, we have a canonical biholomorphism between $AT(R_j)$. (For the details of the asymptotic Teichmüller theory, see [5],[2], and [3].)

Next fix a supporting surface $R_0$ of $Y_0$. Then we can construct a natural bijection from $T(Y_0)$ onto $AT(R_0)$ as follows. Take any element $[Y, f]$ of $T(Y_0)$. Then there is a quasiconformal homeomorphism $F$ of $U \cap R_0$ of $Y$ in $R_0$ into $R$ where $U$ is a canonical neighborhood of $Y_0$ in $R_0$ and $R$ is a supporting surface of $Y$. Then $F$ can be extended to a quasiconformal map of $R_0$ onto another supporting surface $R'$ of $Y$ (possibly different from $R$), which gives a point in $AT(R_0)$. By the definitions, we can easily see that this map induces a bijection of $T(Y_0)$ to $AT(R_0)$. Thus we have proved the assertion. \hfill \Box

**Remark** We say that two boundary self-homeomorphisms $f_1$ and $f_2$ in $BH(Y_0)$ are AC-equivalent if $f_2 \circ f_1^{-1}$ is homotopic to a asymptotically conformal self-homeomorphism of $Y$. The equivalence class of $f$ is called an AC-mapping class, and denoted by $[f]$. 
Now every element $f$ of $\text{BH}(Y_0)$ naturally induces an automorphism $f^*$ of $T(Y_0)$, by setting
\[ f^*([Y, g]) = [(Y, g \circ f^{-1})]. \]
Then it is clear from the definition that $f_1^* = f_2^*$ if and only if $[f_1] = [f_2]$.

References