A CONFORMAL INVARIANT FOR NON-VANISHING ANALYTIC FUNCTIONS AND ITS APPLICATIONS

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ABSTRACT. The quantity $V_D(\varphi) = \sup_{z\in D} \rho_D(z)^{-1}|\varphi'(z)/\varphi(z)|$ will be considered for a non-vanishing analytic function φ on a plane domain D with hyperbolic metric $\rho_D(z)|dz|$. We see that this quantity has various nice properties such as conformal invariance and monotoneity. As a special case, for a proper subdomain Ω of the punctured plane $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$, we define the domain constant $W(\Omega) = V_\Omega(\mathrm{id})$, which will be called the circular width of Ω about the origin, and we will see that $W(\Omega)$ dominates the value of $V_D(\varphi)$ if $\varphi(D) \subset \Omega$. As applications, we provide boundedness and univalence criteria for those functions f on the unit disk $\mathbb{D}$ for which $f'(\mathbb{D}) \subset \Omega$. We also compute values of circular width for typical domains.

1. Introduction

Conformal invariants play a central role in the modern theory of functions of a complex variable. One of the most important is the hyperbolic metric $\rho_D(z)|dz|$ of a hyperbolic plane domain D. Recall that a subdomain D of $\mathbb{C}$ is called hyperbolic if D admits an analytic universal covering projection p of the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ onto D. Then the hyperbolic metric is defined by the equation $\rho_D(z)|p'(\zeta)| = 1/(1-|\zeta|^2)$ for $\zeta \in p^{-1}(z)$. Note that the density $\rho_D(z)$ does not depend on the particular choice of ζ or p. The Poincaré-Koebe uniformization theorem tells us that $D \subset \mathbb{C}$ is hyperbolic if and only if D is neither the whole plane $\mathbb{C}$ nor the punctured plane $\mathbb{C}\setminus\{a\}$ for any $a \in \mathbb{C}$. The hyperbolic metric is conformally invariant in the sense that the pull-back $f^*\rho_{D'}(z) = \rho_{D'}(f(z))|f'(z)|$ of $\rho_{D'}(w)|dw|$ under a conformal map $f : D \to D'$ is equal to $\rho_D(z)$. Throughout the paper, a conformal map means a conformal homeomorphism.

In this article, we propose a sort of conformal invariants associated with a non-vanishing analytic function. This quantity proves its usefulness in estimation of the hyperbolic sup-norm of the pre-Schwarzian derivative of a locally univalent functions in various situations as in [15]. Let φ be a non-vanishing analytic function on a hyperbolic domain D, namely, $\varphi : D \to \mathbb{C}^* = \mathbb{C}\setminus\{0\}$ is holomorphic. Then we set

$$V_D(\varphi) = \sup_{z\in D} \rho_D(z)^{-1}\left|\frac{\varphi'(z)}{\varphi(z)}\right|.$$

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This quantity measures the rate of growth of φ compared with the hyperbolic metric. Note also that $V_D(\varphi)$ can be thought of the Bloch semi-norm of the (possibly multi-valued) function $\log\varphi$. The quantity $V_D(\varphi)$ does not depend on the source domain D, more precisely, $V_{D_0}(\varphi\circ f)=V_D(\varphi)$ for a conformal map $f: D_0\to D$ (see Theorem 2.2 below). On the other hand, $V_D(\varphi)$ may depend on the target domain.

One merit of this quantity is monotoneity in several respects. For instance, if ω is a holomorphic map of D_0 into D, then $V_D(\varphi)\le V_{D_0}(\varphi\circ\omega)$ holds (see Theorem 2.2). Many more properties will be discussed in Section 2.

Let Ω be a proper subdomain of the punctured plane $\mathbb{C}^*$. Then Ω admits an analytic universal covering projection p of a simply connected proper subdomain D of $\mathbb{C}$ onto it. Then the quantity $W(\Omega)=V_D(p)$ is independent of the particular choice of $p: D\to\Omega$ and will be called the *circular width* of Ω (about the origin). An important property to note is that $W(\Omega)\le W(\Omega_1)$ if $\Omega\subset\Omega_1\subset\mathbb{C}^*$. For instance, the sector $\{w\in\mathbb{C}: |\arg w|<\pi\alpha/2\}$ has circular width 2α for $0<\alpha\le 2$ (see Section 5). Fundamental properties and a geometric meaning of the circular width will be given in Section 3. Also, exact values of $W(\Omega)$ for some specific domains Ω are given in Section 5.

The circular widths (about boundary points) of a plane domain are closely related to uniform perfectness of the boundary. We will explain it in Section 4. As an application, we will give a proof of Osgood's theorem [21, Theorem 2] in a quantitative way: *∂D is uniformly perfect if and only if the hyperbolic sup-norm of univalent analytic functions on D is bounded.*

The information on $W(\Omega)$ is useful regarding univalence and boundedness criteria. For example, let us consider an analytic function f in the unit disk with $\operatorname{Re} f'>0$. In general, the function f may not be bounded (e.g., $f(z)=\log(1-z)$). As a consequence of our results, we obtain the boundedness criteria stating that if $f'(\mathbb{D})\subset\Omega$ for a subdomain Ω of the right half-plane $\mathbb{H}=\{w\in\mathbb{C}:\operatorname{Re} w>0\}$ with $W(\Omega)<2$ then f must be bounded. Note that $W(\Omega)\le 2$ holds always for $\Omega\subset\mathbb{H}$.

Applying this to the function $f=\log F$ for a non-vanishing locally univalent function F on $\mathbb{D}$ gives another approach to the problem considered by MacGregor and Rønning in [17]. In Section 6, we will give some sufficient conditions for a domain $\Omega\subset\mathbb{C}^*$ to have circular width less than 2.

We have already used some facts about $W(\Omega)$ implicitly in [15]. Moreover, some results in Section 5 were used by Ponnusamy and the second author [23] in order to deduce univalence criteria for meromorphic functions outside the unit disk. See Section 6 for more details about applications of circular width.

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2. Basic properties of the quantity $V_D(\varphi)$

In this section, basic properties of the quantity $V_D(\varphi)$ and more refined results are given. We first see that how $V_D(\varphi)$ measures the rate of growth of φ with respect to the hyperbolic metric. We denote by $d_D(z_0,z_1)$ the hyperbolic distance between two points z_0 and z_1 in D, namely,

$$d_D(z_0,z_1)=\inf_\gamma\int_\gamma \rho_D(z)|dz|,$$

where the infimum is taken over all piecewise smooth curves joining z_0 and z_1 in D. With this notation, we have the following result.

Proposition 2.1. *Let φ be a non-vanishing analytic function on a hyperbolic domain D and let c be a positive constant. Then $V_D(\varphi) \le c$ if and only if the inequality*

$$\exp\big(-c\,d_D(z_0,z_1)\big) \le \frac{|\varphi(z_1)|}{|\varphi(z_0)|} \le \exp\big(c\,d_D(z_0,z_1)\big) \tag{2.1}$$

holds for every pair of points z_0, z_1 in D.

Proof. We first assume that $V_D(\varphi) \le c$, namely, $|\varphi'/\varphi| \le c\rho_D$. Since

$$\log\frac{|\varphi(z_1)|}{|\varphi(z_0)|} = \mathrm{Re}\int_\gamma \frac{\varphi'(z)}{\varphi(z)}dz$$

for γ joining z_0 and z_1 in D, we obtain the inequalities

$$\left|\log\frac{|\varphi(z_1)|}{|\varphi(z_0)|}\right| \le \int_\gamma \left|\frac{\varphi'(z)}{\varphi(z)}\right||dz| \le c\int_\gamma \rho_D(z)|dz|.$$

Thus we can now see (2.1).

We next prove the converse. Set $u(z) = \log|\varphi(z)|$ for $z \in D$. Then the condition (2.1) means that

$$|u(z') - u(z)| \le c d_D(z,z'), \quad z, z' \in D.$$

By dividing both sides by $|z'-z|$ and taking upper limits as $z' \to z$, we obtain the inequality $|\nabla u(z)| \le c\rho_D(z)$. Here ∇ denotes the gradient. Since u is the real part of the analytic function $f = \log\varphi$ (at least locally), one gets $|\nabla u| = |f'| = |\varphi'/\varphi|$. Therefore, we have proved the inequality $|\varphi'/\varphi| \le c\rho_D$. □

We next see fundamental properties of the quantity $V_D(\varphi)$. The following properties are obvious: for non-vanishing analytic functions φ and ψ on a hyperbolic domain D, the inequality

$$V_D(\varphi\cdot\psi) \le V_D(\varphi) + V_D(\psi)$$

holds and the relation

$$V_D(\varphi^\alpha) = |\alpha| V_D(\varphi) \tag{2.2}$$

holds for $\alpha \in \mathbb{C}$ as long as the power φ^α is defined as a single-valued analytic function on D. Note that φ^α is always taken to be single-valued if α is an integer or if D is simply connected.

Apart from these, we have the following important invariance properties.

Theorem 2.2. *Let D be a hyperbolic domain and let φ be a non-vanishing analytic function on D.*

(a) *Let $p : D_0 \to D$ be an analytic (unbranched and unlimited) covering projection. Then $V_{D_0}(\varphi\circ p) = V_D(\varphi)$. In particular, $V_D(\varphi)$ is conformally invariant in the sense that this does not depend on the source domain.*
(b) *$V_D(L\circ\varphi) = V_D(\varphi)$ holds for any conformal automorphism L of $\mathbb{C}^*$. In particular, $V_D(1/\varphi) = V_D(\varphi) = V_D(c\varphi)$ for any constant $c \in \mathbb{C}^*$.*
(c) *Let $\omega : D_0 \to D$ be a holomorphic map. Then $V_{D_0}(\varphi\circ\omega) \le V_D(\varphi)$.*

(d) *If* $\psi : D \to \mathbb{C}^*$ *is univalent and if* $\varphi(D) \subset \psi(D)$ *then* $V_D(\varphi) \le V_D(\psi)$.

Proof. The assertion (a) follows from the invariance property $\rho_D(p(z))|p'(z)| = \rho_{D_0}(z)$ of the hyperbolic density, and (b) is easily deduced by a straightforward computation. Next we prove property (c) when $D = D_0 = \mathbb{D}$ by the Schwarz-Pick lemma: $(1-|z|^2)|\omega'(z)| \le 1-|\omega(z)|^2,\ |z|<1$, for a holomorphic map $\omega : \mathbb{D} \to \mathbb{D}$. We now have the inequality

$$(1-|z|^2)\left|\frac{(\varphi\circ\omega)'(z)}{(\varphi\circ\omega)(z)}\right| = (1-|z|^2)|\omega'(z)|\left|\frac{\varphi'(\omega(z))}{\varphi(\omega(z))}\right| \le (1-|\omega(z)|^2)\left|\frac{\varphi'(\omega(z))}{\varphi(\omega(z))}\right|.$$

We note that equality holds in the above for some (and thus all) point $z \in \mathbb{D}$ if and only if ω is an automorphism of $\mathbb{D}$. Thus (c) has been proved for this special case. We proceed to the general case. Let $p : \mathbb{D} \to D_0$ and $q : \mathbb{D} \to D$ be holomorphic universal covering projections of $\mathbb{D}$ onto D_0 and D, respectively. We take a lift $\tilde{\omega}$ of $\omega \circ p$ via the projection q. Namely, a holomorphic map $\tilde{\omega} : \mathbb{D} \to \mathbb{D}$ satisfies $\omega \circ p = q \circ \tilde{\omega}$. Then, by using (a) and the special case of (c), we have

$$V_{D_0}(\varphi\circ\omega) = V_{\mathbb{D}}(\varphi\circ\omega\circ p) = V_{\mathbb{D}}(\varphi\circ q\circ\tilde{\omega}) \le V_{\mathbb{D}}(\varphi\circ q) = V_D(\varphi).$$

We show property (d) by applying (c) to the function $\omega = \psi^{-1}\circ\varphi : D \to D$. □

The following result ensures that the inequality $V_{\mathbb{D}}(\varphi) \le 4$ holds for any non-vanishing *univalent* function φ on $\mathbb{D}$.

Proposition 2.3. *Let* φ *be a non-vanishing univalent function in the unit disk. Then*

$$(1-|z|^2)\left|\frac{\varphi'(z)}{\varphi(z)}\right| \le 4,$$

where equality holds at $z = z_0$ *if and only if* $\mathbb{C}\setminus\varphi(\mathbb{D})$ *is a ray emanating from the origin and the value* $\varphi(z_0)$ *lies in the line containing the ray.*

Proof. By the conformal invariance of the quantity $(1-|z|^2)|\varphi'(z)/\varphi(z)|$ (see the proof of Theorem 2.2 (c)), it suffices to show the above inequality at the origin: $|\varphi'(0)/\varphi(0)| \le 4$. Then, $f(z) = (\varphi(z)-\varphi(0))/\varphi'(0)$ is a normalized univalent function in $|z|<1$. The Koebe one-quarter theorem now implies that $f(\mathbb{D})$ contains the disk $\{|w|<1/4\}$. On the other hand, by assumption, the function f omits the value $-\varphi(0)/\varphi'(0)$, therefore $|\varphi(0)/\varphi'(0)| \ge 1/4$ and equality holds if and only if f is a rotation of the Koebe function $K(z) = z/(1-z)^2$ (see [7, p. 31]). Now the assertion follows. □

Remarks. (1) Proposition 2.3 can also be deduced directly from Macintyre's inequality [18] (see also [30, p.102 and p.112]). This was pointed out to the authors by Shinji Yamashita.

(2) On the other hand, the above proof is same as that of the well-known estimate $\rho_D(z)\delta_D(z) \ge 1/4$ for a simply connected domain D, where $\delta_D(z) = \operatorname{dist}(z, \partial D)$. Actually, the quantity V_D has the following geometric meaning. Let $\hat{\rho}_D(z)|dz|$ denote the Hahn metric of the domain D. A paper [19] of D. Minda contains the following fundamental properties of the Hahn metric. i) If $f : D \to D'$ is holomorphic and injective, then $\hat{\rho}_{D'}(f(z))|f'(z)| \le \hat{\rho}_D(z)$. ii) If D is simply connected, $\hat{\rho}_D = \rho_D$. iii) For the punctured

plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, we know that $\hat{\rho}_{\mathbb{C}^*}(z) = 1/(4|z|)$. In particular, the quantity $V_D(\varphi)$ for $\varphi : D \to \mathbb{C}^*$ has the expression

$$V_D(\varphi) = 4 \sup_{z\in D} \frac{\hat{\rho}_{\mathbb{C}^*}(\varphi(z))|\varphi'(z)|}{\rho_D(z)} = 4 \sup_D \frac{\varphi^*(\hat{\rho}_{\mathbb{C}^*})}{\rho_D}.$$

Therefore, the proposition is nothing but an expression of the decreasing property of the Hahn metric under univalent maps: $\varphi^*(\hat{\rho}_{\mathbb{C}^*}) \le \hat{\rho}_D = \rho_D$, thus it may be thought of a corollary of the above-mentioned results due to Minda [19].

A holomorphic function $g : \mathbb{D} \to \mathbb{C}$ is called *Gelfer* if $g(z) + g(w) \neq 0$ for any pair of points $z, w \in \mathbb{D}$. In particular, a Gelfer function is always non-vanishing. (Note here that we drop the usual normalization condition $g(0) = 1$.)

As a corollary of Proposition 2.3, we can show the following result on Gelfer functions, which was first shown by S. A. Gelfer in his paper [9] written in Russian by means of a known result on Bieberbach-Eilenberg functions and which was effectively used by S. Yamashita in [31]. The authors could not find a short account for its proof in the literature other than the original article by Gelfer, and could not attribute anyone to the equality condition below. We thus include a simple proof of this for the convenience of the reader.

Theorem 2.4 (Gelfer [9], see also [29]). *For a Gelfer function g,*

$$(1 - |z|^2) \left| \frac{g'(z)}{g(z)} \right| \le 2$$

for $|z| < 1$, where equality holds at $z = z_0$ precisely when g maps the unit disk univalently onto a half-plane H whose boundary contains the origin and the orthogonal projection of the point $g(z_0)$ to ∂H is equal to the origin.

Proof. Let g be a Gelfer function. Without loss of generality, we may assume that $g(0) = 1$. Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. We set $f(z) = g(z)^2$, then the unique unbounded component C of $\widehat{\mathbb{C}} \setminus f(\mathbb{D})$ connects the origin and the point at infinity. Thus $D = \widehat{\mathbb{C}} \setminus C$ is a simply connected domain in $\mathbb{C}^*$ with $1 \in D$. (For this part, see also [8, Théorèm 8] or [24, Lemma].) Now let $\varphi : \mathbb{D} \to D$ be a conformall map. By the Schwarz-Pick lemma and Proposition 2.3, we can see that

$$(1 - |z|^2) \left| \frac{f'(z)}{f(z)} \right| = (1 - |z|^2)|\omega'| \left| \frac{\varphi'(\omega)}{\varphi(\omega)} \right| \le (1 - |\omega|^2) \left| \frac{\varphi'(\omega)}{\varphi(\omega)} \right| \le 4,$$

where $\omega = \varphi^{-1} \circ f$. In the above, $(1 - |z_0|^2)|f'(z_0)/f(z_0)| = 4$ holds at the point z_0 if and only if f maps $\mathbb{D}$ univalently onto the complex plane off a ray emanating from the origin and $f(z_0)$ lies in the line containing this ray. Since $g(z) = \sqrt{f(z)}$ and $f'(z)/f(z) = 2g'(z)/g(z)$, now the desired statement follows. □

In view of the above proof, we also have the next result which is a generalization of Proposition 2.3.

Proposition 2.5. *Let f be a non-vanishing holomorphic function on the unit disk $\mathbb{D}$ such that the image $f(\mathbb{D})$ does not separate the origin from the point at infinity. Then the inequality $(1 - |z|^2)|f'(z)/f(z)| \le 4$ follows and equality holds at some point if and only if f maps $\mathbb{D}$ conformally onto the complex plane off a ray emanating from the origin.*

3. Circular width of a proper subdomain of $\mathbb{C}^*$

Let Ω be a hyperbolic plane domain with $0 \in \mathbb{C} \setminus \Omega$. The quantity

$$W(\Omega) = \left(\inf_{w \in \Omega} |w| \rho_\Omega(w) \right)^{-1}$$

will be called the *circular width* of Ω (about the origin). In general, it is not easy to compute the values of the density $\rho_\Omega(w)$ of the hyperbolic metric of Ω. Therefore, another expression of $W(\Omega)$ is often useful.

Lemma 3.1. *Let Ω be a proper subdomain of the punctured plane $\mathbb{C}^*$ and let p be an analytic (unbranched) covering projection of a domain D onto Ω. Then $W(\Omega) = V_D(p)$.*

Proof. First we note that the circular width of Ω can be written in the form $W(\Omega) = V_\Omega(\mathrm{id})$. Theorem 2.2 (a) now implies the relation $V_\Omega(\mathrm{id}) = V_D(p)$. □

We now collect basic properties of the circular width. Before that, we recall the notion of circular symmetrization. For a subdomain Ω of $\mathbb{C}^*$ we define the circular symmetrization Ω^* (about the origin) by

$$\Omega^* = \{re^{i\theta} : \theta \in I(r, \Omega), 0 < r < \infty\},$$

where $I(r, \Omega)$ denotes the interval in the form $(-t/2, t/2)$ of the same length as $I_r = \{\theta \in [-\pi, \pi] : re^{i\theta} \in \Omega\}$ if $I_r \neq [-\pi, \pi]$ otherwise $I(r, \Omega) = [-\pi, \pi]$.

Theorem 3.2. *Let Ω and Ω' be proper subdomains of the punctured plane $\mathbb{C}^*$.*
(i) *$W(\Omega) = W(L(\Omega))$ for any conformal automorphism L of $\mathbb{C}^*$.*
(ii) *If $\Omega \subset \Omega'$, then $W(\Omega) \leq W(\Omega')$.*
(iii) *Circular symmetrization does not decrease circular width; $W(\Omega) \leq W(\Omega^*)$.*
(iv) *If Ω is simply connected, then $W(\Omega) \leq 4$.*

Proof. In view of the formula $W(\Omega) = V_\Omega(\mathrm{id})$, we can deduce (i) and (ii) from Theorem 2.2 (b) and (d), respectively. Part (iii) lies much deeper. We will employ Weitsman's theorem [28]: $\rho_\Omega(w) \geq \rho_{\Omega^*}(|w|)$. Then

$$\frac{1}{W(\Omega)} = \inf_{w \in \Omega} |w| \rho_\Omega(w) \geq \inf_{w \in \Omega} |w| \rho_{\Omega^*}(|w|) \geq \inf_{w \in \Omega^*} |w| \rho_{\Omega^*}(w) = \frac{1}{W(\Omega^*)},$$

which proves (iii). Part (iv) follows from the Koebe one-quarter theorem. This also follows from (ii) and (iii). Indeed, if Ω is simply connected the symmetrized domain Ω^* is contained in the slit domain $\Omega_1 = \mathbb{C} \setminus (-\infty, 0]$. A simple computation gives $W(\Omega_1) = 4$ (see Example 5.1). Thus (ii) and (iii) now yield $W(\Omega) \leq W(\Omega^*) \leq W(\Omega_1) = 4$. □

In general, the circular width may not be finite. We give here a characterization of domains with infinite circular width. In particular, if the origin is an isolated boundary point of Ω, then $W(\Omega) = \infty$.

Proposition 3.3. *Let Ω be a proper subdomain of the punctured plane $\mathbb{C}^*$. The circular width $W(\Omega)$ is infinite if and only if there is a sequence of annuli $A_n = \{w \in \mathbb{C} : r_n < |w| < R_n\}$ with $A_n \subset \Omega$ such that $R_n / r_n \to \infty$.*

As a preparation, first we show the following.

Lemma 3.4. *Let Ω be a proper subdomain of the punctured plane $\mathbb{C}^*$ and set*
$$M(\Omega) = \sup_{w\in\Omega} \inf_{b\in\mathbb{C}\setminus\Omega} \left|\log\left|\frac{w}{b}\right|\right|.$$
Then the inequalities
$$\frac{4}{\pi}M(\Omega) \le W(\Omega) \le 2M(\Omega) + C$$
hold, where $C = \Gamma(1/4)^4/(2\pi^2) \approx 8.7538$.

Proof. For the proof, we need the following estimate (see [26, Theorem 1.5]):
$$\frac{1}{2m(w)+C} \le |w|\rho_\Omega(w) \le \frac{\pi}{4m(w)},\ \ w\in\Omega,$$
where
$$m(w) = \inf_{b\in\mathbb{C}\setminus\Omega} \left|\log\left|\frac{w}{b}\right|\right|.$$
By the definitions of $W(\Omega)$ and $M(\Omega)$, now the required inequalities follow. □

Proof of Proposition 3.3. By Lemma 3.4, $W(\Omega) = \infty$ if and only if $M(\Omega) = \infty$. Suppose that $A = \{r < |w| < R\} \subset \Omega$. Then $m(w) \ge (1/2)\log(R/r)$ for $|w| = \sqrt{rR}$. Therefore, we have $M(\Omega) \ge (1/2)\varlimsup \log(R_n/r_n) = \infty$ if $A_n = \{r_n < |w| < R_n\} \subset \Omega$ satisfies $R_n/r_n \to \infty$. Conversely, we assume that $W(\Omega) = \infty$, equivalently, $M(\Omega) = \infty$. Then, there exists a sequence w_n such that $m_n = m(w_n) \to \infty$ as $n \to \infty$. Then the annulus $A_n = \{e^{-m_n}|w_n| < |w| < e^{m_n}|w_n|\}$ does not meet $\mathbb{C}\setminus\Omega$ by the definition of the function m, therefore, $A_n \subset \Omega$. It is evident that the sequence A_n is what we wanted. □

The circular width may not behave continuously in Ω. For instance, consider the sequence of domains $\Omega_n = \{|w-1| < 1+1/n\}$. Then Ω_n converges to $\Omega_\infty = \{|w-1| < 1\}$ in the Hausdorff topology. But $W(\Omega_n) = \infty$ by Proposition 3.3 whereas $W(\Omega_\infty) \le W(\mathbb{H}) = 2$ (see Example 5.1). We can, however, show a continuity property of circular width in the following form.

Proposition 3.5. *Let Ω_n be a sequence of domains with $\Omega_n \subset \Omega_{n+1}$ such that the union $\Omega = \cup_{n=1}^\infty \Omega_n$ is a proper subdomain of $\mathbb{C}^*$. Then $W(\Omega_n) \to W(\Omega)$ as $n \to \infty$.*

Proof. By the monotoneity of circular width (Theorem 3.2 (ii)), $W(\Omega_1) \le W(\Omega_2) \le \cdots \le W(\Omega)$, therefore $\lim_{n\to\infty} W(\Omega_n) \le W(\Omega)$. On the other hand, for any number $m < W(\Omega)$, we can find a point $w_0 \in \Omega$ such that $|w_0|\rho_\Omega(w_0) < 1/m$. Since $\rho_{\Omega_n}(w_0) \to \rho_\Omega(w_0)$ (see, for example, [12, Theorem 1]), we obtain
$$\frac{1}{m} > \lim_{n\to\infty} |w_0|\rho_{\Omega_n}(w_0) \ge \lim_{n\to\infty} W(\Omega_n)^{-1}.$$
Since m was arbitrary as far as $m < W(\Omega)$, we now obtain $W(\Omega)^{-1} \ge \lim_{n\to\infty} W(\Omega_n)^{-1}$, namely, $\lim_{n\to\infty} W(\Omega_n) \ge W(\Omega)$. The proof is now complete. □

The circular width $W(\Omega)$ dominates the quantity $V_D(\varphi)$ for holomorphic maps $\varphi : D \to \Omega$.

Theorem 3.6. *Let Ω be a proper subdomain of $\mathbb{C}^*$ and let $\varphi : D \to \Omega$ be holomorphic. Then $V_D(\varphi) \le W(\Omega)$.*

Proof. By Theorem 2.2 (c), we have $V_D(\varphi) = V_D(\mathrm{id}_\Omega \circ \varphi) \le V_\Omega(\mathrm{id}_\Omega) = W(\Omega)$. □

Combining this with Proposition 2.1, we have the following, which is a slight generalization of a result of J.-H. Zheng [32].

Corollary 3.7. *Under the same hypotheses in Theorem 3.6,*

$$\exp\big(-W(\Omega)\, d_D(z_0, z_1)\big) \le \frac{|\varphi(z_1)|}{|\varphi(z_0)|} \le \exp\big(W(\Omega)\, d_D(z_0, z_1)\big), \quad z_0, z_1 \in D.$$

We remark that a similar result can be obtained by applying the Harnack inequality to the harmonic function $\log|\varphi|$ on D. The latter idea is even efficient for quasiregular mappings in higher dimensional Euclidean space (see [27, §. 13]).

4. Connection with uniform perfectness

In general, we can define the circular width $W_a(D)$ of a hyperbolic domain D about a point $a \in \mathbb{C} \setminus D$ by

$$W_a(D) = \frac{1}{\inf_{z\in D} |z-a|\rho_D(z)}.$$

Note that this can also be written as $W_a(D) = V_D(\tau_a)$, where $\tau_a(z) = z - a$. It is known that the domain constant

$$C(D) = \sup_{a\in\partial D} W_a(D) = \sup_{z\in D} \frac{1}{\delta_D(z)\rho_D(z)}$$

is finite if and only if the set ∂D is uniformly perfect (see, for example, [22] or [25]). Here we recall that $\delta_D(z) = \mathrm{dist}(z, \partial D)$. In this context, the constant $W_a(D)$ appeared essentially in a paper [32] by J.-H. Zheng. We remark that we may replace ∂D by the complement of D in the above without any essential change. The constant $C(D)$ or, equivalently, the constant $c(D) = 1/C(D)$ is studied by many authors (see, for instance, [11], [16], [25] and [30]). Note that $C(D) \ge 2$ holds for an arbitrary hyperbolic domain D with equality if and only if D is convex [11, Theorem 4].

Let us introduce a variant of the quantity $V_D(\varphi)$. For a non-vanishing analytic function φ on D, we set

$$\hat{V}_D(\varphi) = \sup_{z\in D} \delta_D(z) \left|\frac{\varphi'(z)}{\varphi(z)}\right|.$$

Since $\rho_D(z)\delta_D(z) \le 1$ for $z \in D$, we have $\hat{V}_D(\varphi) \le V_D(\varphi)$. Let $N(D)$ be the least number such that

$$V_D(\varphi) \le N(D)\hat{V}_D(\varphi)$$

holds for every holomorphic map $\varphi : D \to \mathbb{C}^*$. If there is no such a number, then we set $N(D) = +\infty$. It is interesting to observe that the quantities $C(D)$ and $N(D)$ are equal.

Proposition 4.1. *Let D be a hyperbolic plane domain. Then $C(D) = N(D)$ holds. In particular, ∂D is uniformly perfect if and only if $N(D) < \infty$.*

Proof. Since $\rho_D^{-1} \le C(D)\delta_D$, the inequality $N(D) \le C(D)$ is trivial. We now show $C(D) \le N(D)$. First we note the simple fact that $\hat{V}_D(\tau_a) \le 1$ holds for each $a \in \mathbb{C} \setminus D$. We now apply the inequality $V_D(\varphi) = N(D)\hat{V}_D(\varphi)$ to the function $\varphi = \tau_a$ to obtain $W_a(D) = V_D(\tau_a) \le N(D)$ for $a \in \partial D$. Taking the supremum over a, we obtain $C(D) \le N(D)$. □

As a simple application of the above proposition, we give a proof of Osgood's theorem. In order to state it, we introduce the domain constant

$$U(D) = \sup_f V_D(f'),$$

where the supremum is taken over all univalent analytic functions f on D. Note that $V_D(f')$ is nothing but the hyperbolic sup-norm of the pre-Schwarzian derivative f''/f' of f. Osgood's theorem [21, Theorem 2] states that ∂D is uniformly perfect if and only if $U(D) < \infty$. In view of his proof, a quantitative form can be presented in the following way.

Theorem 4.2 (Osgood). *Let D be a hyperbolic plane domain. Then*

$$2C(D) \le U(D) \le 4C(D).$$

Proof. First we show the inequality $2C(D) \le U(D)$. For $a \in \partial D$, we consider the function $f_a(z) = 1/(a - z)$. It is clear that f_a is univalent analytic on D. In particular, $V_D(f_a') \le U(D)$. Since $f_a' = \tau_a^{-2}$, the relation (2.2) implies $V_D(f_a') = 2V_D(\tau_a) = 2W_a(D)$. Thus $2W_a(D) \le U(D)$, from which the required inequality follows.

We now show the inequality $U(D) \le 4C(D)$. Let $f : D \to \mathbb{C}$ be univalent and analytic. Then the sharp inequality $\hat{V}_D(f') \le 4$ holds (see [21, Lemma 1]). Thus $U(D) \le 4N(D)$ is obtained. Now we employ Proposition 4.1 to get $U(D) \le 4C(D)$. □

5. Computations of circular widths

In the present section, we give exact values of circular width for several concrete examples. These will be useful to give upper bounds of circular width for various domains. In view of Theorem 3.2 (iii), we see that circularly symmetric domains are particularly important.

Example 5.1 (sectors). For $S(\beta) = \{w : |\arg w| < \pi\beta/2\}$, $0 < \beta \le 2$, we have $W(S(\beta)) = 2\beta$.

Indeed, by Theorem 2.4, we have $W(\mathbb{H}) = 2$. Since $S(\beta) = \varphi_\beta(\mathbb{H})$ for $\varphi_\beta(z) = z^\beta$, we obtain by (2.2)

$$W(S(\beta)) = V_{\mathbb{H}}(\varphi_\beta) = |\beta| V_{\mathbb{H}}(\mathrm{id}) = \beta \cdot W(\mathbb{H}) = 2\beta.$$

We remark that the above computation remains valid even when β is a complex number. It is easy to see that φ_β is univalent in $\mathbb{H}$ if $|\beta - 1| \le 1$ and $\beta \ne 0$. Therefore, we have also $W(\varphi_\beta(\mathbb{H})) = 2|\beta|$ for such a β. Note that $\varphi_\beta(\mathbb{H})$ is a Jordan domain bounded by two logarithmic spirals ending at 0 and ∞ when $|\beta - 1| < 1$.

Example 5.2 (half-sectors). Let $S(\beta, r) = \{w : |\arg w| < \pi\beta/2, |w| < r\}$ and $S'(\beta, r) = \{w : |\arg w| < \pi\beta/2, |w| > 1/r\}$ for $0 < \beta \le 2$ and $0 < r < \infty$. Then $W(S(\beta, r)) = W(S'(\beta, r)) = 2\beta$.

Since circular width is invariant under dilations, we see that $W(S(\beta, r)) = W(S(\beta, 1))$. On the other hand, by Proposition 3.5, we have $\lim_{r\to\infty} W(S(\beta, r)) = W(S(\beta)) = 2\beta$. Thus we obtain $W(S(\beta, r)) = W(S(\beta, 1)) = 2\beta$. For the other case, the same computation works.

It is interesting to see that an arbitrary domain Ω with $S(\beta, r) \subset \Omega \subset S(\beta)$ for some $r > 0$ has circular width 2β because $2\beta = W(S(\beta, r)) \le W(\Omega) \le W(S(\beta)) = 2\beta$ by monotoneity.

Example 5.3 (annuli). For the annulus $A(r, R) = \{w : r < |w| < R\}$, $0 < r < R < \infty$, we have $W(A(r, R)) = (2/\pi)\log(R/r)$.

We may assume that $R = e^m$ and $r = e^{-m}$ for some $m > 0$ for the proof. Then the mapping $\varphi(z) = \exp((2mi/\pi)\log z) = z^{2mi/\pi}$ gives an analytic universal covering projection of the right half-plane $\mathbb{H}$ onto $A(r, R)$. Thus the same computation as in Example 5.1 gives $W(A(r, R)) = V_{\mathbb{H}}(\varphi) = 2|2mi/\pi| = 4m/\pi = (2/\pi)\log(R/r)$.

Example 5.4 (disks). Let $\mathbb{D}(a, r) = \{w : |w - a| < r\}$ for $0 < r \le a$. Then

$$W(\mathbb{D}(a, r)) = \frac{2r/a}{1 + \sqrt{1 - (r/a)^2}}.$$

Let $\varphi(z) = a + rz$. Then

$$W(\mathbb{D}(a, r)) = V_{\mathbb{D}}(\varphi) = \sup_{z\in\mathbb{D}} (1 - |z|^2)\frac{r}{|a + rz|} = \sup_{0\le x<1} \frac{r(1 - x^2)}{a - rx}.$$

Since $r(1 - x^2)/(a - rx)$ takes its maximum at $x = (a - \sqrt{a^2 - r^2})/r$, we obtain the required expression of $W(\mathbb{D}(a, r))$.

Note that $W(\mathbb{D}(a, a)) = 2$ for $a > 0$. Since circular width is invariant under the inversion $z \mapsto 1/z$ (see Theorem 3.2 (i)), we also obtain $W(H) = 2$ for the half-plane $H = \{w : \operatorname{Re} w > b\}$ for $b = 1/(2a) > 0$.

Example 5.5 (parallel strips). Let $P(a, b) = \{w : a < \operatorname{Re} w < b\}$ for $0 \le a < b < \infty$. Then

$$W(P(a, b)) = \max_{0\le\theta\le\pi/2} \frac{2t\cos\theta}{1 - t\theta},$$

where t is a number with $0 < t \le 2/\pi$ determined by

$$\frac{\pi t}{2} = \frac{b - a}{b + a}.$$

Note that the function $\varphi(z) = 1 + it\log z$ maps the right half-plane $\mathbb{H}$ onto the parallel strip $P(1 - \pi t/2, 1 + \pi t/2)$. Therefore, if we choose t as above, then this strip is similar to $P(a,b)$ and thus they have the same circular width. If we write $z = re^{i\theta}$, we compute

$$\begin{aligned} W(P(a,b)) = V_{\mathbb{H}}(\varphi) &= \sup_{z\in\mathbb{H}} 2\mathrm{Re}\, z\frac{t/|z|}{|1+it\log z|} \\ &= \sup_{0<r<\infty, -\pi/2<\theta<\pi/2} \frac{2t\cos\theta}{|1-t\theta+it\log r|} \\ &= \sup_{-\pi/2<\theta<\pi/2} \frac{2t\cos\theta}{1-t\theta}. \end{aligned}$$

Clearly we can discard the case $\theta < 0$, and thus we have the required form.

We remark that these parallel strips are *not* circularly symmetric.

Example 5.6 (truncated wedges). Let $S(\beta, r, R) = \{w : |\arg w| < \pi\beta/2, r < |w| < R\},\ 0 < \beta \le 2, 0 < r < R < \infty$. Then

$$W(\Omega) = \frac{\log(R/r)}{(1+t)\mathbf{K}(t)},$$

where

$$\mathbf{K}(t) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-t^2x^2)}}$$

is the complete elliptic integral of the first kind and $0 < t < 1$ is a number such that

$$\frac{\mathbf{K}(\sqrt{1-t^2})}{\mathbf{K}(t)} = \frac{2\pi\beta}{\log(R/r)}.$$

Note that the quantity $\mu(t) = (\pi/2)\mathbf{K}(\sqrt{1-t^2})/\mathbf{K}(t)$ is the modulus of the Grötzsch ring $\mathbb{D}\backslash[0,t]$ for $0 < t < 1$ and decreasing from $+\infty$ to 0 (see, for example, [1]). Therefore, we can always take such a t satisfying the above relation.

We set $K = \mathbf{K}(t)$ and $K' = \mathbf{K}(\sqrt{1-t^2})$. Since the rectangles $Q_1 = (-K, K) \times (0, K')$ and $Q_2 = (\log r, \log R) \times (-\pi\beta/2, \pi\beta/2)$ are similar by the choice of t, there is a linear function $L(z) = az + b$ with $a > 0$ such that $L(Q_1) = Q_2$. Note that

$$a = \frac{\log(R/r)}{2K} = \frac{\pi\beta}{K'}. \tag{5.1}$$

It is well known that the function

$$F(z) = \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-t^2\zeta^2)}}$$

maps the upper half-plane H onto the rectangle Q_1. Therefore, the composed function $\varphi(z) = \exp(aF(z) + b)$ is a conformal map of H onto $S(\beta, r, R)$. We now have

$$W(S(\beta,r,R)) = V_H(\varphi) = \sup_{z\in H} 2\mathrm{Im}\, z \cdot a|F'(z)| = \sup_{z\in H} \frac{2a\mathrm{Im}\, z}{|(1-z^2)(1-t^2z^2)|^{1/2}}.$$

We write $z = x + iy$ with $x \in \mathbb{R}, y > 0$. Then we have

$$\begin{aligned} &|(1-z^2)(1-t^2z^2)|^2 - (1+t)^4y^4 \\ &= (1-x^2-t^2x^2+t^2x^4-t^2y^4)^2 + 2(1+t^2)y^2(tx^2+ty^2-1)^2 \\ &\quad + 2x^2y^2\left[2t(1-t)^2 + (t^2x^2+t^2y^2-1)^2 + t^4(x^2+y^2-1)^2\right] \end{aligned}$$

and thus

$$|(1-z^2)(1-t^2z^2)|^{1/2} \ge (1+t)y,$$

where equality holds when $x = 0$ and $ty^2 = 1$. Therefore, in view of (5.1), we obtain

$$W(S(\beta, r, R)) = \frac{2a}{1+t} = \frac{\log(R/r)}{(1+t)K} = \frac{2\pi\beta}{(1+t)K'}.$$

Observe that the limiting case $S(\beta, 0, \infty) = S(\beta)$ corresponds to $t = 1^-$. Since $K' \to \pi/2$ as $t \to 1^-$, we reproduce the relation $W(S(\beta)) = 2\beta$. We also see that the other limiting case $S(\infty, r, R) = A(r, R)$ corresponds to $t = 0^+$. (We have to regard $S(\beta, r, R)$ as an overlapped domain when $\beta > 2$.) Since $K \to \pi/2$ as $t \to 0^+$, we reproduce the relation $W(A(r, R)) = (2/\pi)\log(R/r)$.

We remark that the essentially same observations were made by Avhadiev and Aksent'ev [2] (see also Corollary 6.9 below) though they did not make systematic use of circular width.

We end the present section with a criterion for a subdomain of the right half-plane $\mathbb{H}$ to have circular width 2.

Proposition 5.1. *Let Ω be a subdomain of $\mathbb{H}$. Suppose that for each number $\beta \in (0, 1)$ there is a number $\delta > 0$ such that $S(\beta, \delta) \subset \Omega$. Then $W(\Omega) = 2$.*

Proof. Since $\Omega \subset \mathbb{H}$, we have $W(\Omega) \le W(\mathbb{H}) = 2$. On the other hand, by assumption, $W(\Omega) \ge W(S(\beta, \delta)) = 2\beta$ for each $\beta < 1$ (see Example 5.2). Thus we conclude that $W(\Omega) = 2$. □

Obviously, we may replace $S(\beta, \delta)$ by $S'(\beta, \delta)$ in the assertion of the last proposition.

For instance, if Ω contains a disk whose boundary contains the origin, then $W(\Omega) \ge 2$ (see also Example 5.4).

6. Applications

In this section, we give a few applications of circular width. More concrete applications can be found in [15] and [23].

Let us introduce some notation. For a locally univalent function f on $\mathbb{D}$, the quantity $T_f = f''/f'$ is called the pre-Schwarzian derivative of f and measured by the norm

$$\|T_f\|_{\mathbb{D}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)|T_f(z)|.$$

Note that this can be described by $\|T_f\|_{\mathbb{D}} = V_{\mathbb{D}}(f')$. Let $\mathcal{A}$ denote the class of holomorphic functions f on $\mathbb{D}$ normalized by $f(0) = 0, f'(0) = 1$.

Theorem 6.1. *Let Ω be a proper subdomain of the punctured plane $\mathbb{C}^*$ with $W(\Omega) < 2$. If $f \in \mathcal{A}$ satisfies $f'(\mathbb{D}) \subset \Omega$, then $|f(z)| < M,\ z \in \mathbb{D}$. Here M is a constant depending only on $W(\Omega)$.*

The assumption implies $\|T_f\|_{\mathbb{D}} = V_{\mathbb{D}}(f') \le W(\Omega) < 2$ by Theorem 3.6. Though it is known that the condition $\|T_f\|_{\mathbb{D}} < 2$ implies boundedness of f (see [14]), we will give a proof for completeness.

Proof. Set $\lambda = W(\Omega)/2$. By Corollary 3.7, we have $|f'(z)| \le \exp(2\lambda d_{\mathbb{D}}(z,0))$. Since $d_{\mathbb{D}}(z,0) = \operatorname{arctanh}(|z|) = (1/2)\log((1+|z|)/(1-|z|))$, this inequality is equivalent to

$$|f'(z)| \le \left(\frac{1+|z|}{1-|z|}\right)^{\lambda}.$$

Since $\lambda < 1$, the function $((1+x)/(1-x))^{\lambda}$ is integrable over $(0,1)$. Thus we have

$$|f(z)| \le \int_0^1 \left(\frac{1+x}{1-x}\right)^{\lambda} dx < 2^{\lambda}\int_0^1 (1-x)^{-\lambda} dx = \frac{2^{\lambda}}{1-\lambda} = \frac{2^{W(\Omega)/2}}{1-W(\Omega)/2}.$$

□

We remark that the above integral can be expressed by

$$\int_0^1 \left(\frac{1+x}{1-x}\right)^{\lambda} dx = \lambda\Big[\psi\big(-\tfrac{\lambda}{2}\big) - \psi\big(\tfrac{1-\lambda}{2}\big)\Big] - 1,$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function (see [14]).

If $W(\Omega) \ge 2$, there is no guarantee that f is bounded. For instance, consider the function $f(z) = -2\log(1-z) - z$. Though $f'(\mathbb{D}) \subset \mathbb{H}$ and $W(\mathbb{H}) = 2$, the function f is unbounded.

It may be interesting to find a characterization of such subdomains Ω of $\mathbb{H}$ that $f'(\mathbb{D}) \subset \Omega$ implies boundedness of $f \in \mathcal{A}$. The last theorem gives the sufficient condition $W(\Omega) < 2$ for that. A similar problem was considered by MacGregor and Rønning [17]. They tried to find conditions for subdomains Ω of $\mathbb{H}$ to have the property that $g'(z)/g(z) \in \Omega,\ z \in \mathbb{D}$, implies boundedness of $\log|g(z)|$ for non-vanishing locally univalent function g on $\mathbb{D}$. Letting $f = \log g$, we see that the latter conclusion is weaker than the former. In particular, the condition $W(\Omega) < 2$ is sufficient for MacGregor-Rønning's problem. Their conditions, however, are more refined because they cover even cases where $W(\Omega) = 2$.

Note also that the condition $f'(\mathbb{D}) \subset \mathbb{H}$ implies univalence of f (Noshiro-Warschawski theorem). Recently, Chuaqui and Gevirtz [6] gave a characterization of such subdomains Ω of $\mathbb{H}$ that $f'(\mathbb{D}) \subset \Omega$ implies quasiconformal extensibility of $f \in \mathcal{A}$.

We now consider sufficient conditions for proper subdomains Ω of $\mathbb{C}^*$ to satisfy $W(\Omega) < 2$, which implies boundedness criterion by Theorem 6.1. Note that if $\Omega \subset \mathbb{H}$ then $W(\Omega) \le 2$. Thus, the following result gives a sufficient condition for such Ω to have circular width less than 2. We also remind the reader that we gave a sufficient condition for subdomains Ω of $\mathbb{H}$ to have circular width 2 (see Proposition 5.1).

Let $\tau_\Omega(r)$ denote the half of the length of the set $\{\theta \in [-\pi,\pi] : re^{i\theta} \in \Omega\}$. By Theorem 3.2, we see that $W(\Omega) \le 2$ if $\tau_\Omega(r) \le \pi/2$ for every $r > 0$. Furthermore, we have the following result.

Theorem 6.2. *Let Ω be a proper subdomain of the punctured plane $\mathbb{C}^*$ with the property that $\tau_\Omega \le \pi/2$ on $(0,\infty)$. If $\overline{\lim}_{r\to 0}\tau_\Omega(r) < \pi/2$ and if $\overline{\lim}_{r\to\infty}\tau_\Omega(r) < \pi/2$, then $W(\Omega) < 2$.*

Proof. Let Ω^* be the circular symmetrization of Ω. By Theorem 3.2 (iii), we have $W(\Omega) \le W(\Omega^*)$. Note that Ω^* is contained in the right half-plane $\mathbb{H}$ by assumption.

For positive constants m and R, we define the domains $\Omega_\infty(m,R)$ and $\Omega_0(m,R)$ by $\{w=u+iv : u>0, |v|<mu+R\}$ and $\{1/w : w\in\Omega_\infty(m,R)\}$, respectively.

By assumption, Ω^* is contained in the domain $\Omega'=\Omega_0(m,R)\cap\Omega_\infty(m,R)$ for sufficiently large m and R. Since $W(\Omega)\le W(\Omega^*)\le W(\Omega')$, it suffices to show $W(\Omega')<2$.

Let $\psi:\mathbb{H}\to\Omega'$ be a conformal homeomorphism. Since Ω' is a Jordan domain, by the Carathéodory extension theorem, ψ extends uniquely to a homeomorphism from $\overline{\mathbb{H}}$ onto $\overline{\Omega'}$. We may take ψ so that $\psi(0)=0$ and $\psi(\infty)=\infty$. Consider the function

$$\Phi(z)=\operatorname{Re} z\left|\frac{\psi'(z)}{\psi(z)}\right|$$

in $\mathbb{H}$. Since $\psi((1+z)/(1-z))$ is Gelfer, Theorem 2.4 implies that $\Phi(z)<2$ for every $z\in\mathbb{H}$. Therefore, in order to show $W(\Omega')<2$, it is enough to show that $\overline{\lim}_{z\to\zeta}\Phi(z)<2$ for each $\zeta\in\partial\Omega'$. Since ψ is symmetric, we may further assume that $\operatorname{Im}\zeta\ge 0$. Let ia and ib be the inverse images of i/R and iR, respectively, under the mapping ψ. We can see that the function $\psi'(z)/\psi(z)$ analytically extends to a holomorphic function across the boundary point iy for $y>0$ except for $y=a,b$. Therefore, $\overline{\lim}_{z\to iy}\Phi(z)=0$ for such y.

When $y=b$ or $y=a$, we need more efforts. First we note that the opening angle of Ω' at iR is $\pi\beta=\arctan m+\pi/2$. Therefore, $\varphi=(\psi-iR)^{1/\beta}$ extends to a conformal map around ib. In particular, $\varphi(z)=c(z-ib)(1+o(1))$ and $\varphi'(z)=c(1+o(1))$ as $z\to ib$, where $c=\varphi'(ib)\ne 0$. Since $\psi=iR+\varphi^\beta$, we see

$$\begin{aligned}\Phi(z)&=\operatorname{Re}(z-ib)\frac{\beta|\varphi(z)|^{\beta-1}|\varphi'(z)|}{|iR+\varphi(z)^\beta|}\\ &\le(1+o(1))|z-ib|\frac{\beta|c|^\beta|z-ib|^{\beta-1}}{R}\\ &=(R^{-1}+o(1))\beta|c|^\beta|z-ib|^\beta=o(1)\end{aligned}$$

as $z\to ib$. Considering $1/\psi$ instead of ψ, we can also see that $\Phi(z)=o(1)$ as $z\to ia$.

Finally, we consider the cases where $\zeta=0$ and $\zeta=\infty$. We first claim that

$$\overline{\lim_{\mathbb{H}\ni z\to 0}}\,\Phi(z)\le\frac{4}{\pi}\arctan m(<2).$$

To show this, letting $\alpha>(2/\pi)\arctan m$, we consider the function $h(z)=\psi(\delta z)$ in $D_0=\{z\in\mathbb{D}:\operatorname{Re} z>0\}$ for $\delta>0$. We can choose δ so small that $h(D_0)$ is contained in the sector $S=\{w:|\arg w|<\pi\alpha/2\}$. As we saw in Example 5.1, $W(S)=2\alpha$. Therefore, by Theorem 3.6, $V_{D_0}(h)\le W(S)=2\alpha$. Note that the function $f(z)=(\frac{1+iz}{1-iz})^2$ maps the right half-disk D_0 conformally onto the upper half-plane. A direct computation shows that the hyperbolic density $\rho_{D_0}(z)=|f'(z)|/2\operatorname{Im} f(z)$ satisfies that $\rho_{D_0}(z)^{-1}=2\operatorname{Re} z+O(|z|^2)$ as $z\to 0$ in D_0. Hence, we see

$$\overline{\lim_{z\to 0}}\,\Phi(z)=\overline{\lim_{z\to 0}}\operatorname{Re}\delta z\left|\frac{\psi'(\delta z)}{\psi(\delta z)}\right|=\overline{\lim_{z\to 0}}\,\frac{1}{2}\rho_{D_0}(z)^{-1}\left|\frac{h'(z)}{h(z)}\right|\le\frac{1}{2}V_{D_0}(h)\le\alpha.$$

Since α was arbitrary as far as $\alpha>(2/\pi)\arctan m$ is satisfied, we have now proved the above claim. By considering $1/\psi$, we have the same inequality when $z\to\infty$ in $\mathbb{H}$. □

We next apply Theorem 3.6 to the problem of quasiconformal extensibility. Our result is based on the following theorem due to J. Becker. See, for sharpness, Becker and Pommerenke [5].

Theorem 6.3 (Becker [4]). *Let $f \in \mathcal{A}$ be locally univalent. If $\|T_f\|_{\mathbb{D}} \le 1$, then f is univalent. Furthermore, if $\|T_f\| \le k$ for $k \in [0,1)$, then f has a K-quasiconformal extension to the whole plane, where $K = (1+k)/(1-k)$.*

We are now in a position to show the following result.

Theorem 6.4. *Suppose that a proper subdomain Ω of the punctured plane $\mathbb{C}^*$ satisfies $W(\Omega) \le k$ for some $k \le 1$. If $f'(\mathbb{D}) \subset \Omega$ for $f \in \mathcal{A}$, then f is univalent and, moreover, f has a K-quasiconformal extension to the whole plane when $K = (1+k)/(1-k) < \infty$.*

See [23] for a counterpart of the theorem for meromorphic functions.

Proof. As we noted, the condition $f'(\mathbb{D}) \subset \Omega$ implies that $\|T_f\|_{\mathbb{D}} \le W(\Omega) \le k$. We now apply Theorem 6.3 to deduce the assertions. □

Combining this with examples presented in the previous section, we obtain a series of corollaries. (Remember the fact that circular width is invariant under rotations.) Note that since most domains are contained in half-planes, univalence assertion is implied by the Noshiro-Warschawski theorem in those cases.

The first corollary was noted by Avhadiev and Aksent'ev [3, pp. 33–34] at least when $\gamma = 0$.

Corollary 6.5. *Let $0 < k \le 1$ and $f \in \mathcal{A}$. If $|\arg f'(z) - \gamma| < \pi k/4$ in $|z| < 1$ for some real constant γ, then f is univalent and, moreover, it extends to a K-quasiconformal mapping of the whole plane when $K = (1+k)/(1-k) < \infty$.*

Note that the condition $|\arg f'(z)| < M,\ |z| < 1$, implies quasiconformal extensibility of f when $M < \pi/2$ (see [6]).

Corollary 6.6. *Let k, r, R be positive numbers with $0 < \log(R/r) \le \pi k/2, k \le 1$ and let $f \in \mathcal{A}$. If $r < |f'(z)| < M$ for $|z| < 1$, then f is univalent and, moreover, it extends to a K-quasiconformal mapping of the whole plane when $K = (1+k)/(1-k) < \infty$.*

This sort of univalence criterion was first given by John [13]. The greatest number $\gamma > 1$ so that $1 < |f'(z)| < \gamma$ for $|z| < 1$ implies univalence of f is called the John constant. He proved that $\log \gamma \ge \pi/2$, while Gevirtz [10] showed that $\log \gamma < 0.6279\pi$.

Corollary 6.7. *Let $k \in (0,1), a \in \mathbb{C}$ and $r > 0$ with $r \le |a|$ and $2r \le k(|a| + \sqrt{|a|^2 - r^2})$. If $f \in \mathcal{A}$ satisfies $|f'(z) - a| < r$ in $|z| < 1$, then f is univalent and extends to a K-quasiconformal mapping of the whole plane when $K = (1+k)/(1-k) < \infty$.*

Note that the inequality $|a - 1| < r$ must be satisfied under the assumptions in the last corollary because $f'(0) = 1$.

Corollary 6.8. *Let a, b, k be positive numbers with $0 \le k \le 1$ such that $2t \cos\theta \le k(1 - t\theta)$ for all $0 \le \theta \le \pi/2$, where $t = (2/\pi)(b-a)/(b+a)$. If $f \in \mathcal{A}$ admits the inequality $a < \mathrm{Re}\,(e^{i\gamma} f'(z)) < b$ in $|z| < 1$ for some real constant γ, then f is univalent and extends to a K-quasiconformal mapping of the whole plane when $K = (1+k)/(1-k) < \infty$.*

Corollary 6.9. *Let k, r, R, β be positive numbers with $0 < \log(R/r)/((1+t)\mathbf{K}(t)) \le k \le 1$, where t is as in Example 5.6 and let $f \in \mathcal{A}$. If $|\arg f'(z) - \gamma| < \pi\beta/2$ and $r < |f'(z)| < R$ in $|z| < 1$ for some real constant γ, then f is univalent and extends to a K-quasiconformal mapping of the whole plane when $K = (1+k)/(1-k) < \infty$.*

Note that the last result was first shown by Avhadiev and Aksent'ev [2] (see also [3, Theorem 34]) for the case $\gamma = 0$. Related results are also given by Minda and Wright [20].

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