Chaotic composition operators on the classical holomorphic spaces

Abstract

We call the composition operator induced by a biholomorphic automorphism of the unit disc U a Möbius composition operator. Here, if ϕ has a fixed point in U, then we call ϕ elliptic. In this paper, we give an elementary proof to the fact that a Möbius composition operator is chaotic on the Hardy space H^p and on the Bergman space B^p for every $p \in (0, +\infty)$ if and only if the corresponding ϕ is non-elliptic. This result is a generalization of Hosokawa's results in [4].

1 Introduction

Let X be a topological vector space and $L : X \to X$ a continuous linear operator. We say that L is *hypercyclic* if there is an element x of X such that the orbit $\{L^k(x)\}$ is dense in X. Here L^k is the k-th iteration of L for every positive $k \in \mathbb{Z}$. On the other hand, we say that L is *chaotic* if

1) the set of periodic points of L are dense in X,

2) L is transitive, and

3) L has sensitive dependence on initial condition.

(See for instance, [7] and [9].) Here if X is (separable and) completely metrizable, it is well-known that the condition 1) and hypercyclicity of L imply that L is chaotic. Hence in the case of Banach spaces, the famous hypercyclicity criterion due to Kitai, Gethner, and Shapiro can be rewritten as a criterion for chaos.

Proposition 1 *Let X be a separable complex Banach space and $L : X \to X$ a bounded linear operator. Suppose that there are a dense subset S of X, a* *convergent series $\sum_{k=1}^{\infty} a_k$ of positive numbers, and a positive constant $A(x)$ for every $x \in S$ such that L^{-1} exists on S and that*

$$\|L^k(x)\| \leq A(x)a_{|k|}$$

for every $k \in \mathbb{Z}$, where $L^{-k} = (L^{-1})^k$ for every positive $k \in \mathbb{Z}$. Then L is chaotic.

Proof. First, L is hypercyclic by the hypercyclicity criterion (see for instance, [5] and [9]). Next, for every $x \in S$ and positive $k \in \mathbb{Z}$,

$$y_k = \sum_{m=-\infty}^{+\infty} L^{mk}(x) \in X,$$

which is periodic, i.e. $L^k(y_k) = y_k$. Also $\|y_k - x\| \leq 2A(x) \sum_{m=1}^{\infty} a_{mk}$, which tends to 0 as $k \to +\infty$. Since S is dense in X, periodic points are dense in X. ■

Now, let U be the unit disc in $\mathbb{C}$, and $\mathrm{Aut}(U)$ the group of Möbius transformations which preserve U. For every $\phi \in \mathrm{Aut}(U)$, we define the composition operator C_ϕ by setting

$$C_\phi f = f \circ \phi$$

for any function f on U, which we call the *Möbius composition operator* for ϕ.

Next, the classical holomorphic spaces are defined as follows.

Definition For every p with $0 < p < \infty$, the *Bergman space* B^p is the space consisting of all holomorphic functions f on U such that

$$\|f\|_{B^p}^p = \int_U |f|^p dA < \infty,$$

where dA is the normalized area measure $dxdy/\pi$.

The *Hardy space* H^p is the space consisting of all holomorphic functions f on U such that

$$\|f\|_{H^p}^p = \lim_{r \to 1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

Here recall that the identical embedding of H^p into B^p is continuous, that every $f \in H^p$ has the boundary function $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ almost all $\theta \in \mathbb{R}$, and that

$$\|f\|_{H^p}^p = \int_0^{2\pi} |f(e^{i\theta})|^p \frac{d\theta}{2\pi}.$$

Also, if $p \geq 1$ then H^p and B^p are Banach spaces and even if $p \in (0,1)$, they are separable completely metrizable topological vector spaces, and the following fact is well-known and easily seen.

Proposition 2 *For every p with $0 < p < \infty$, every Möbius composition operator is continuous on B^p and on H^p.*

In the sequel, we say that $\phi \in \mathrm{Aut}(U)$ is *hyperbolic* or *parabolic*, respectively, if ϕ has exactly two or one fixed point(s) on the unit circle ∂U. If ϕ has a fixed point in U, then we call ϕ *elliptic* (including the identical map). Thus non-elliptic elements are either hyperbolic or parabolic. And it is known ([9]) that, for every non-ellipic element of $\mathrm{Aut}(U)$, the corresponding Möbius composition operator is chaotic on the complex topological vector space $H(U)$, consisting of all holomorphic functions on U with the topology induced by uniform convergence on compact sets of U.

In this paper, we give an elementary proof of the following generalization of Hosokawa's results in [4].

Theorem 3 *For every $\phi \in \mathrm{Aut}(U)$, the following conditions are equivalent.*

1) *ϕ is non-elliptic.*
2) *C_ϕ is chaotic on some Hardy space H^p with $p \in (0, +\infty)$.*
3) *C_ϕ is chaotic on every Hardy space H^p with $p \in (0, +\infty)$.*
4) *C_ϕ is chaotic on some Bergman space B^p with $p \in (0, +\infty)$.*
5) *C_ϕ is chaotic on every Bergman space B^p with $p \in (0, +\infty)$.*

Here clearly 3) implies 2) and 5) implies 4). Also it is well-known that 2) or 4) implies 1). Actually, any elliptic $\phi \in \mathrm{Aut}(U)$, C_ϕ is not hypercyclic on B^p and on H^p for every $p \in (0, +\infty)$. (cf. [1] and [8]. Indeed, fix $p \in (0, +\infty)$ and an element f in A^p or in H^p arbitrarily. Let $\alpha \in U$ be the fixed point of ϕ. Then $(C_\phi)^k f(\alpha) = f(\alpha)$ for every $k \in \mathbb{Z}$. Since the norm convergence in B^p or in H^p implies the local uniform convergence on U, the orbit of f is dense neither in B^p nor in H^p.) Thus we need to prove that 1) implies 3) and 5), which will be proved in the following sections.

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2 Möbius composition operators for hyperbolic automorphisms

In this section, we prove the following

Theorem 4 *Suppose that* $0 < p < \infty$. *Then for every hyperbolic* ϕ *in* $\mathrm{Aut}(U)$, *the composition operator* C_ϕ *on* B^p *and on* H^p *is chaotic.*

Let $H(\overline{U})$ be the set of all functions holomorphic in a neighborhood of $\overline{U}$. Then it is well known that $H(\overline{U})$ is dense in H^p and in B^p. (See for instance, [2] and [3].) In particular, H^p (as a subset) is dense in B^p. Thus we need to prove the assertion only for the Hardy spaces. Hence we fix $p \in (0, +\infty)$, and consider the Hardy space H^p only, though the same arguments give the assertion for B^p.

Lemma 5 *Let* α *be a point on the unit circle* ∂U, *and* $N(\alpha)$ *be the set of all functions holomorphic in a neighborhood of the closed unit disc* $\overline{U}$ *vanishing at* α. *Then* $N(\alpha)$ *is dense in* H^p.

Furthermore, the subsets $N(\alpha^K)$, *consisting of all functions holomorphic in a neighborhood of* $\overline{U}$ *with a zero of order not less than* $K \geq 2$ *at* α, *and* $N(\alpha, \beta) = N(\alpha) \bigcap N(\beta)$ *with another* $\beta \in \partial U$ *are also dense in* H^p.

Proof. Set

$$g_t(z) = \frac{z - \alpha}{z - t\alpha}$$

for every real $t > 1$. Then $g_t \in H(\overline{U})$, and $\sup_{z \in \overline{U}} |g_t(z)| < 1$. Hence by the Lebesgue convergence theorem, we have

$$\lim_{t \to 1} \|f - g_t f\|_{H^p} = 0$$

for every $f \in H(\overline{U})$. Since $g_t f \in N(\alpha)$, we have the first assertion. The other cases are similar. ■

To estimate the norm of $(C_\phi)^k f$ for every $k \in \mathbb{Z}$, we divide U into the Dirichlet fundamental regions for the cyclic group G generated by ϕ. Let D_0 be the Dirichlet fundamental region for G with the center 0, i.e.

$$D_0 = \{z \in U \mid d(z, 0) < d(z, \phi^\ell(0)), \ell \in \mathbb{Z} - \{0\}\},$$

where $d(\cdot,\cdot)$ is the Poincaré distance on U. Then $D_k = \phi^k(D_0)$ is the Dirichlet fundamental region for G with the center $\phi^k(0)$.

Remark When we map U to the upper half plane H so that 0 corresponds to i and ϕ to $z \mapsto \lambda z$ with $\lambda \in (0,1)$, D_0 corrsponds to the domain

$$\{z \in H \mid \lambda^{1/2} < |z| < \lambda^{-1/2}\}.$$

In the sequel, let α and β be the attracting and the repelling fixed point of ϕ, i.e. corresonding to 0 and ∞, respectively, on $\overline{H}$.

Lemma 6 *There are constants $C > 0$ and $\lambda \in (0,1)$ such that the length $L(L_k) = \int_{L_k} d\theta/(2\pi)$ of $L_k = \overline{D_k} \cap \partial U$ is bounded from above by $C\lambda^{|k|}$ for every $k \in \mathbb{Z}$.*

Proof. There is a constant $\lambda \in (0,1)$ as in the above remark, or equivalently, such that

$$\phi^k(z) = \frac{(\alpha - \lambda^k\beta)z - (1-\lambda^k)\alpha\beta}{(1-\lambda^k)z + \alpha\lambda^k - \beta}$$

for every $k \in \mathbb{Z}$. Hence

$$(\phi^k)'(z) = \frac{\lambda^k(\alpha-\beta)^2}{((1-\lambda^k)z + \alpha\lambda^k - \beta)^2}.$$

Here,

$$(1-\lambda^k)z + \alpha\lambda^k - \beta = (z-\alpha)\left(\frac{z-\beta}{z-\alpha} - \lambda^k\right)$$

if $k > 0$ and

$$\lambda^{-k}((1-\lambda^k)z + \alpha\lambda^k - \beta) = -(z-\beta)\left(\frac{z-\alpha}{z-\beta} - \lambda^{-k}\right)$$

if $k < 0$. Since the image of $\overline{D_0}$ by $(z-\alpha)/(z-\beta)$ is disjoint from the non-negative real axis $\{x \in \mathbb{R} \mid x \geq 0\}$, $|(\phi^k)'(z)|$ is bounded by $C_0\lambda^{|k|}$ with a suitable C_0 on $\overline{D_0}$ for every $k \in \mathbb{Z}$. ∎

Lemma 7 *Fix $f \in N(\alpha, \beta)$. Then there is a constant C' such that*

$$\sup_{z \in \overline{D_k}} |f(z)| \leq C' \lambda^{|k|}$$

for every k.

Proof. Since there is a constant M such that

$$|f(z)| \leq M|z - \alpha||z - \beta|$$

on $\overline{U}$ and since

$$|\phi^k(z) - \alpha| \leq 2\lambda^k \left| \frac{z - \alpha}{z - \beta} \right|,$$

we can find a constant M_1 such that

$$|f(\phi^k(z))| \leq M_1 \lambda^k$$

on $\overline{D_0}$ for every positive k. The case that k is negative is similar by considering $|\phi^k(z) - \beta|$. ■

Lemma 8 *Under the same circumstance as above, there are constants C_0 and $c_0 \in (0,1)$ such that*

$$\|(C_\phi)^k f\|_{H^p} \leq C_0(|k|+1)^{1/p} c_0^{|k|}$$

for every $k \in \mathbb{Z}$.

Proof. Set $\tilde{p} = \min\{p, 1\}$. Then lemmas 6 and 7 imply that

$$\begin{aligned}
& \|(C_\phi)^k f\|_{H^p}^p \\
\leq \; & \sum_{m \in \mathbb{Z}} \left(\sup_{z \in \overline{D_{k+m}}} |f(z)|^p \right) \cdot L(L_m) \leq \sum_{m \in \mathbb{Z}} C(C')^p \lambda^{|m| + p|k+m|} \\
\leq \; & C(C')^p \left(\sum_{|m+k| \leq |k|, \, |m| \leq |k|} (\lambda^{\tilde{p}})^{|m| + |k+m|} \right. \\
& \left. + \sum_{|m+k| > |k|} (\lambda^{\tilde{p}})^{|m| + |k+m|} + \sum_{|m| > |k|} (\lambda^{\tilde{p}})^{|m| + |k+m|} \right) \\
\leq \; & C(C')^p \left((|k|+1)(\lambda^{\tilde{p}})^{|k|} + \frac{4}{1 - (\lambda^{\tilde{p}})} (\lambda^{\tilde{p}})^{|k|} \right)
\end{aligned}$$

for every $k \in \mathbb{Z}$. Hence we have the assertion with $c_0 = \lambda^{\min\{1,1/p\}}$. ■

Remark We can show Theorem 4 and several corollaries such as in [4] directly by using Proposition 1. For instance, the proof of Lemma 8 implies that αC_ϕ is chaotic on H^p if $p \geq 1$ and $\lambda^{1/p} < |\alpha| < \lambda^{-1/p}$.

Finally, recall that the hypercyclicity criterion can be applied to the case of separable completely metrizable topological vector spaces. In particular, Lemma 8 implies the well-konwn fact that C_ϕ with a hyperbolic ϕ is hypercyclic on H^p. Thus Theorem 4 follows from the following lemma due to K. Matsuzaki [6].

Lemma 9 *For every $f \in N(\alpha, \beta)$ and every positive odd integer $2k+1$ with $k > 0$, set*

$$g_k = \sum_{m=-\infty}^{\infty} (C_\phi)^{(2k+1)m} f.$$

Then there is a constant C'' such that $|g_k| \leq C''$ on $\overline{U}$ for every k.

Moreover, g_k converges to f in H^p as $k \to \infty$.

Proof. First set $D_*(k) = \bigcup_{j=-k}^{k} \overline{D_j}$, then by Lemma 7, we have

$$\begin{aligned} |g_k(z)| &\leq \sum_{m=-\infty}^{\infty} |f(\phi^{(2k+1)m}(z))| \leq \sum_{m=-\infty}^{\infty} \sup_{w \in \phi^{(2k+1)m}(D_*(k))} |f(w)| \\ &\leq C' \left(1 + 2\sum_{m=1}^{\infty} (\lambda^k)^{2m-1}\right) = C' \left(1 + \frac{2\lambda^k}{1-\lambda^{2k}}\right) \end{aligned}$$

for every $z \in \overline{U}$. Hence we have the first assertion.

Next, for an arbitrary $\epsilon > 0$, choose a positive integer k_0 so that $D_*(k_0)$ contains the set

$$I = \{e^{i\theta} \mid \epsilon \leq |\theta - \theta_1| \leq \pi\} \bigcap \{e^{i\theta} \mid \epsilon \leq |\theta - \theta_2| \leq \pi\}$$

where we set $\alpha = e^{i\theta_1}$ and $\beta = e^{i\theta_2}$. If $z \in D_*(k_0)$ and if $k \geq k_0$, then we have

$$|f(z) - g_k(z)| \leq \sum_{m \in \mathbb{Z} - \{0\}} |f(\phi^{(2k+1)m}(z))| \leq \frac{2C'\lambda^k}{1-\lambda^{2k}}.$$

Hence we conclude that

$$\begin{aligned}
& \|f-g_k\|_{H^p}^p \\
\leq\ & \sum_{j=1}^{2} \int_{\{|\theta-\theta_j|<\epsilon\}} |f(e^{i\theta})-g_k(e^{i\theta})|^p \frac{d\theta}{2\pi} + \int_I |f(e^{i\theta})-g_k(e^{i\theta})|^p \frac{d\theta}{2\pi} \\
\leq\ & 2\frac{\epsilon}{\pi}(C'+C'')^p + \left(\frac{2C'\lambda^k}{1-\lambda^{2k}}\right)^p,
\end{aligned}$$

which shows the second assertion. ■

3 Möbius composition operators for parabolic automorphisms

In this section, we prove the following

Theorem 10 *Suppose that* $0 < p < \infty$. *Then for every parabolic* ϕ *in* $\mathrm{Aut}(U)$, *the composition operator* C_ϕ *on* B^p *and on* H^p *is chaotic.*

As before, we need to prove Theorem 10 for H^p only. Fix $p \in (0, +\infty)$ and let α be the fixed point of ϕ. Let D_0 be the Dirichlet fundamental region for the group generated by ϕ with the center 0. Then again, $D_k = \phi^k(D_0)$ is the Dirichlet fundamental region for G with the center $\phi^k(0)$ for every $k \in \mathbb{Z}$.

Remark When we map U to the upper half plane H so that 0 corresponds to i and ϕ to $z \mapsto z+1$, D_0 corrsponds to the domain

$$\{z \in H \mid |\mathrm{Re}\, z| < 1/2\}.$$

Lemma 11 *There is a constant* C *such that*

$$L(L_k) \leq C(\max\{1, |k|\})^{-2}$$

for every $k \in \mathbb{Z}$.

Next, fix $f \in N(\alpha^K)$ *with* $K \geq 2$. *Then there is a constant* $C' > 0$ *such that*

$$\sup_{z \in \overline{D_k}} |f(z)| \leq C'(\max\{1, |k|\})^{-K}$$

for every $k \in \mathbb{Z}$.

Proof. First, since

$$\phi^k(z) = \alpha + \frac{z-\alpha}{1+k\gamma(z-\alpha)}$$

with a suitable $\gamma \in \mathbb{C}$,

$$(\phi^k)'(z) = \frac{1}{(1+k\gamma(z-\alpha))^2},$$

and we have the first assertion.

Next, since

$$|\phi^k(z) - \alpha| = \frac{|z-\alpha|}{|1+k\gamma(z-\alpha)|},$$

we see that there is a constant M such that D_k is contained in a disc with center α and radius $M(\max\{1,|k|\})^{-1}$ for every $k \in \mathbb{Z}$. Also as before, there is a constant M' such that

$$|f(z)| \leq M'|z-\alpha|^K$$

on $\overline{U}$, and hence

$$|f(\phi^k(z))| \leq M' \sup_{z\in\overline{D_0}} |\phi^k(z)-\alpha|^K \leq M'M(\max\{1,|k|\})^{-K}$$

on $\overline{D_0}$ for every k. Thus we have the second assertion. ■

Hence as in §2, we have the following estimates.

Lemma 12 *Under the same circumstance as above, fix a positive integer K such that $Kp \geq 2$, and $f \in N(\alpha^K)$. Then there is a constant C_1 such that*

$$\|(C_\phi)^k f\|_{H^p} \leq C_1(\max\{1,|k|\})^{-2/p}$$

for every $k \in \mathbb{Z}$.

Proof. As before, we have

$$\begin{aligned}\|(C_\phi)^k f\|_{H^p}^p &\leq \sum_{m\in\mathbb{Z}} CC'(\max\{1,|m|\})^{-2}(\max\{1,|k+m|\})^{-Kp} \\ \leq\ & CC'\Bigg(2(\max\{1,|k|\})^{-2} + \sum_{|k+m|<|k|,\,|m|<|k|} |m|^{-2}|k+m|^{-2} \\ &+ \sum_{|k+m|>|k|} |m|^{-2}|k+m|^{-2} + \sum_{|m|>|k|} |m|^{-2}|k+m|^{-2}\Bigg).\end{aligned}$$

Here the second term in the right hand side is 0 if $k = 0, \pm 1$, and is

$$\begin{aligned} & \sum_{m=1}^{|k|-1} m^{-2}(|k|-m)^{-2} \\ \leq\ & \sum_{m=1}^{[|k|/2]} m^{-2}(|k|/2)^{-2} + \sum_{m=[|k|/2]+1}^{|k|-1} (|k|/2)^{-2}(|k|-m)^{-2} \\ \leq\ & 16|k|^{-2} \sum_{m=1}^{\infty} m^{-2} \end{aligned}$$

if $k \notin \{-1, 0, 1\}$. Hence the right hand side is bounded from above by

$$CC'(\max\{1, |k|\})^{-2} \left(2 + 16 \sum_{m=1}^{\infty} |m|^{-2} + 4 \sum_{m=1}^{\infty} |m|^{-2} \right),$$

which implies the assertion. ■

Remark For $f \in N(\alpha^K)$ with $Kp \geq 3$, we can show similarly as above that there is a constant C_1' such that

$$\|(C_\phi)^k f\|_{B^p} \leq C_1'(\max\{1, |k|\})^{-3/p}$$

for every $k \in \mathbb{Z}$. Hence in the parabolic case, we can show Theorem 10 directly by using Proposition 1 for H^p with $1 \leq p < 2$ and B^p with $1 \leq p < 3$.

Finally, Lemma 12 and the hypercyclicity criterion imply that C_ϕ with a parabolic ϕ is hypercyclic on H^p. Hence Theorem 10 follows from the following lemma.

Lemma 13 *For every $f \in N(\alpha^K)$ with $Kp \geq 2$ and every positive odd integer $2k+1$ with $k > 0$, set*

$$g_k = \sum_{m=-\infty}^{\infty} (C_\phi)^{(2k+1)m} f.$$

Then there is a constant C'' such that $|g_k| \leq C''$ on $\overline{U}$ for every k.

Moreover, g_k converges to f in H^p as $k \to \infty$.

Proof. First set $D_*(k) = \bigcup_{j=-k}^{k} \overline{D_j}$, then by Lemma 11, we have

$$\begin{aligned} |g_k(z)| &\leq \sum_{m=-\infty}^{\infty} |f(\phi^{(2k+1)m}(z))| \leq \sum_{m=-\infty}^{\infty} \sup_{w\in\phi^{(2k+1)m}(D_*(k))} |f(w)| \\ &\leq C'\left(1 + 2\sum_{m=1}^{\infty} \frac{1}{k^2(2m-1)^2}\right) \end{aligned}$$

for every $z \in \overline{U}$. Hence we have the first assertion.

Next, for an arbitrary $\epsilon > 0$, choose a positive odd integer $2k_0 + 1$ so that $D_*(k_0)$ contains the arc $I = \{e^{i\theta} \mid \epsilon \leq |\theta - \theta_0| \leq \pi\}$, where we set $\alpha = e^{i\theta_0}$. If $z \in D_*(k_0)$ and if $k \geq k_0$, then we have

$$|f(z) - g_k(z)| \leq \sum_{m\in\mathbb{Z}-\{0\}} |f(\phi^{(2k+1)m}(z))| \leq \frac{2C'}{k^2} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2}.$$

Hence we conclude that

$$\begin{aligned} &\|f - g_k\|_{H^p}^p \\ = &\int_{\{|\theta-\theta_0|<\epsilon\}} |f(e^{i\theta}) - g_k(e^{i\theta})|^p \frac{d\theta}{2\pi} + \int_I |f(e^{i\theta}) - g_k(e^{i\theta})|^p \frac{d\theta}{2\pi} \\ \leq &\frac{\epsilon}{\pi}(C' + C'')^p + \left(\frac{2C'}{k^2} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2}\right)^p, \end{aligned}$$

which shows the second assertion. ■

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