ON UNIQUENESS OF OBSTACLE PROBLEM ON FINITE RIEMANN SURFACE

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Abstract. In [1], R. Fehlmann and F. P. Gardiner studied an extremal problem for a topologically finite Riemann surface and established a slit mapping theorem. In this article, we give a condition for non-uniqueness of such slit mappings, by using a deformation of a Riemann surface.

1. Introduction

Let S be a Riemann surface of finite analytic type. Let (S_ι, ι) be a pair of a Riemann surface S_ι of the same type as S and an isomorphism ι of the fundamental group $\pi_1(S)$ of S onto $\pi_1(S_\iota)$. We say that two pairs (S_{ι_1}, ι_1) and (S_{ι_2}, ι_2) are equivalent if there exists a conformal map u of S_{ι_1} onto S_{ι_2} such that

$$(u)_* \circ \iota_1 = \iota_2.$$

The family of such equivalence classes is said to be the Teichmüller space of S and denoted by $T(S)$.

Let S be a *finite bordered Riemann surface* with border Γ. In other words, the border Γ consists of finitely many mutually disjoint simple closed curves, and the double S^{d} of S with respect to the border Γ is of finite analytic type. Note that the border Γ may be empty. In that case, the double S^{d} will be interpreted as S itself. Let $A(S)$ be the set of integrable holomorphic quadratic differentials φ on S with the property that $\varphi = \varphi(z)dz^2$ is real along the border Γ (cf. [2]). Every $\varphi \in A(S)$ extends to a symmetric holomorphic quadratic differential φ^{d} on S^{d}.

Let $\mathfrak{S}(S^{\mathrm{d}})$ be the family of simple closed curves on the double S^{d}, which are homotopic neither to a point of S^{d} nor to a puncture of S^{d}. Let $\mathfrak{S}[S^{\mathrm{d}}]$ be the set of free homotopy classes of elements of $\mathfrak{S}(S^{\mathrm{d}})$. For $\varphi \in A(S)$ and $\gamma \in \mathfrak{S}(S^{\mathrm{d}})$, we denote the height of γ with respect to φ^{d} by

$$\mathrm{height}_{\varphi^{\mathrm{d}}}(\gamma) = \int_\gamma |\mathrm{Im}(\sqrt{\varphi^{\mathrm{d}}(z)}dz)|$$

and the height of the homotopy class $[\gamma]$ by

$$\mathrm{height}_{\varphi^{\mathrm{d}}}[\gamma] = \inf_\beta \mathrm{height}_{\varphi^{\mathrm{d}}}(\beta),$$

where the infimum is taken over all closed curves $\beta \in \mathfrak{S}(S^{\mathrm{d}})$ freely homotopic to γ in S^{d}.

Now we state the obstacle problem in the sense of Fehlmann and Gardiner [1]. They thought of a "simply connected" compact subset with finitely many connected components as an obstacle. We will consider a more general set as an obstacle.

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Definition 1.1. We say that E is *allowable* if E is a compact subset of the interior S° of S such that $S^\circ \setminus E$ is connected and E is contractible in S.

We further say that E is an *allowable slit with respect to* $\varphi \in A(S)\setminus\{0\}$ if E is allowable and if each component of E is a horizontal arc of φ or the union of a finite number of horizontal arcs and critical points of φ.

We remark that a compact subset $E \subset S^\circ$ is contractible if and only if there is a topological closed disk in S which contains E in its interior (see [4, Lemma 2.3]). Let E be an allowable subset of S. Set $E^{\mathrm{d}} = E \cup j(E)$, where $j : S^{\mathrm{d}} \to S^{\mathrm{d}}$ is the canonical anti-conformal involution. Let $\mathfrak{F}(S,E)$ be the family of pairs (g, S_g), where g is a conformal map of $S \setminus E$ into another Riemann surface S_g of the same type as S in such a way that g maps the border Γ onto the border of S_g and the same applies to the punctures. For every $(g,S_g) \in \mathfrak{F}(S,E)$, g extends to a conformal map g^{d} of $S^{\mathrm{d}} \setminus E^{\mathrm{d}}$ into S_g^{d} symmetrically. Then $(g,S_g) \in \mathfrak{F}(S,E)$ induces an isomorphism ι_g of the fundamental group $\pi_1(S^{\mathrm{d}})$ of S^{d} onto $\pi_1(S_g^{\mathrm{d}})$ (cf. [4, Lemma 2.5]). We denote by $[S_g^{\mathrm{d}}, \iota_g]$ the Teichmüller (equivalence) class of $(S_g^{\mathrm{d}}, \iota_g)$ in $T(S^{\mathrm{d}})$.

It is known (cf. [3]) that, for every $(f,S_f) \in \mathfrak{F}(S,E)$ and $\varphi \in A(S) \setminus \{0\}$, there exists the unique holomorphic quadratic differential $\varphi_f \in A(S_f) \setminus \{0\}$ such that

$$\mathrm{height}_{\varphi_f^{\mathrm{d}}}[\gamma] = \mathrm{height}_{\varphi^{\mathrm{d}}}(\iota_f^{-1}[\gamma]) \text{ for every } [\gamma] \in \mathfrak{S}[S_f^{\mathrm{d}}].$$

Fehlmann and Gardiner [1] posed an *obstacle problem for* (S,E,φ) which asks the existence of $(f,S_f) \in \mathfrak{F}(S,E)$ maximizing the quantity

$$M_f = \|\varphi_f\|_{L^1(S_f)} = \iint_{S_f} |\varphi_f|$$

in $\mathfrak{F}(S,E)$, and showed the following result.

Theorem 1.2 (Fehlmann-Gardiner)**.** *Suppose that S is a finite bordered Riemann surface, and that $\varphi \in A(S) \setminus \{0\}$. Let E be an allowable subset of S with finitely many components. Then there exists an element $(g,S_g) \in \mathfrak{F}(S,E)$ such that M_g attains the supremum*

$$M_g = \sup_{(f,S_f)\in\mathfrak{F}(S,E)} M_f.$$

Moreover, for this point $(g,S_g) \in \mathfrak{F}(S,E)$, $E_g = S_g \setminus g(S \setminus E)$ is an allowable slit with respect to φ_g.

The point $(g,S_g) \in \mathfrak{F}(S,E)$ in Theorem 1.2 is called *extremal* for (S,E,φ), and the associated differential φ_g is called the *extremal differential*.

Fehlmann and Gardiner also asserted in the paper [1] that the extremal pair (g,S_g) is unique in the sense that, if $(u,S_u) \in \mathfrak{F}(S,E)$ is also extremal for (S,E,φ), then $g \circ u^{-1}$ extends to a conformal map of S_u onto S_g. The uniqueness, however, does not necessarily hold in their sense.

We show in this note the following theorem which gives a condition for extremal, and hence extremal slit mappings, not to be unique.

Definition 1.3. Let E be an allowable slit in a finite bordered Riemann surface S with respect to a holomorphic quadratic differential $\varphi \in A(S) \setminus \{0\}$. We will call $p_0 \in E$ a *refolding point of order m for (S, E, φ)* if p_0 is a zero of φ of order m and if E contains two horizontal arcs ℓ_1 and ℓ_2 with common end point p_0 such that the angle formed by them at p_0 is greater than $2\pi/(m+2)$.

Theorem 1.4. *Let R be a finite bordered Riemann surface, and $\psi \in A(R) \setminus \{0\}$. Let E_ψ be an allowable slit of R with respect to ψ. Suppose that E has a refolding point p_0 of order $m \geq 3$ for (R, E_ψ, ψ). Then, there exist $(\tilde{u}, \tilde{R}) \in \mathfrak{F}(R, E_\psi)$ and $\tilde{\psi} \in A(\tilde{R}) \setminus \{0\}$ such that*

(i) $E_{\tilde{\psi}} = \tilde{R} \setminus \tilde{u}(R \setminus E_\psi)$ is an allowable slit of $\tilde{R}$ with respect to $\tilde{\psi}$,
(ii) $\mathrm{height}_{\tilde{\psi}^{\mathrm{d}}}[\gamma] = \mathrm{height}_{\psi^{\mathrm{d}}}(\iota_{\tilde{u}}^{-1}[\gamma])$ for every $[\gamma] \in \mathfrak{S}[\tilde{R}^{\mathrm{d}}]$, and
(iii) $[\tilde{R}^{\mathrm{d}}, \iota_{\tilde{u}}] \neq [R^{\mathrm{d}}, \mathrm{id}]$ in $T(R^{\mathrm{d}})$.

Corollary 1.5. *Suppose that S is a finite bordered Riemann surface and that $\varphi \in A(S) \setminus \{0\}$. Let E be an allowable subset of S, and $(g, S_g) \in \mathfrak{F}(S, E)$ be extremal for (S, E, φ). If the allowable slit E_g of S_g with respect to the extremal differential φ_g has a refolding point of order at least three, then there exists another extremal element for (S, E, φ) which induces a point in $T(S^{\mathrm{d}})$ different from $[S_g^{\mathrm{d}}, \iota_g]$.*

Proof. Take the triple (S_g, E_g, φ_g) as the triple (R, E_ψ, ψ) in Theorem 1.4. Then we obtain $(\tilde{u}, \tilde{R}) \in \mathfrak{F}(S_g, E_g)$, and $\tilde{\psi} \in A(\tilde{R}) \setminus \{0\}$ satisfying (i) (ii) (iii) in the theorem. Then from [1] we see that (i) and (ii) implies that the point $(\tilde{u} \circ g, \tilde{R}) \in \mathfrak{F}(S, E)$ is extremal for (S, E, φ). Moreover we can see, by (iii), $[\tilde{R}^{\mathrm{d}}, \iota_{\tilde{u} \circ g}] \neq [S_g^{\mathrm{d}}, \iota_g]$ in $T(S^{\mathrm{d}})$. Thus we have the assertion. □

Remark In the proof, we will actually construct a continuous family of extremal elements $(\tilde{u}_t, \tilde{R}_t) \in \mathfrak{F}(R, E_\psi)$ for the same obstacle problem for (R, E_ψ, ψ) in such a way that the marked Riemann surface $\tau_t = [\tilde{R}_t^{\mathrm{d}}, \iota_{\tilde{u}_t}]$ varies continuously in $T(R^{\mathrm{d}}) \setminus \{[R^{\mathrm{d}}, \mathrm{id}]\}$ and that τ_t approaches $[R^{\mathrm{d}}, \mathrm{id}]$ as $t \to 0$. Since the Teichmüller modular group acts on $T(R^{\mathrm{d}})$ discontinuously, this implies that the Riemann surface $\tilde{R}_t^{\mathrm{d}}$ is not conformally equivalent to $\tilde{R}^{\mathrm{d}}$ for sufficiently small $t > 0$.

In [4], the auther showed a uniqueness result in the weaker form: *Let (g, S_g) and (u, S_u) be both extremal for (S, E, φ). Then the extremal differentials φ_g and φ_u satisfy the relation $\varphi_u = (\varphi_g \circ w)(w')^2$ on $u(S \setminus E)$, where $w = g \circ u^{-1}$.*

2. Example

In this section we give an example of the triple (S, E, φ) which satisfies the assumptions of Corollary 1.5.

First make three copies M_1, M_2, M_3 of the rectangle

$$M = \{z = x + iy \in \mathbb{C} \mid |x| \leq 2, |y| \leq 1\},$$

and let z_j be the coordinate corresponding to z on each M_j. Next on each M_j, identify the two pairs of parallel sides under the translations

$$z_j \to z_j + 4, \qquad z_j \to z_j + 2i.$$

Then we obtain three copies T_1, T_2, T_3 of a torus T. The quadratic differential dz^2 on M induces a holomorphic quadatic differential φ_0 on T.

Cut T_j along the segment

$$I_j = \{z_j = x_j + iy_j \mid -1 \leq x_j \leq 0, y_j = 0\},$$

and glue them cyclically. More precisely, we paste the upper edge I_1^+ of the slit I_1 to the lower edge I_2^- of the slit I_2, the upper edge I_2^+ of the slit I_2 to the lower edge I_3^- of the slit I_3, and the upper edge I_3^+ of the slit I_3 to the lower edge I_1^- of the slit I_1. Then we obtain a compact Riemann surface S of genus three.

Now let Π be the natural projection of S onto the torus T, and φ be the pull-back of φ_0 by Π. Finally, let E be the subset of S consisting of ℓ_1 and ℓ_2, where ℓ_i is the arc on T_i corresponding to $\{z \mid 0 \leq x \leq 1, y = 0\}$.

Now we consider the obstacle problem for (S, E, φ). Then the set E is an allowable slit of S with respect to φ. Hence we know that the identy mapping of S gives an extremal slit map associated with the extremal problem for this triple. Moreover, we can easily see that $\{p_0\} = \Pi^{-1}(0) \subset S$ consists of the refolding point for (S, E, φ).

Thus the assumptions in Corollary 1.5 are satisfied and, as a consequence, the points in $T(S^{\mathrm{d}})$ which are induced by the extremals for (S, E, φ) are not uniquely determined.

3. Proof of Theorem 1.4

Assume that a component J of E_ψ contains a refolding point p_0 of ψ of order $m \geq 3$ and horizontal arcs ℓ_1 and ℓ_2 with common end point p_0 and that the angle formed by ℓ_1 and ℓ_2 at p_0 is

$$\frac{2k\pi}{m+2} \qquad \left(2 \leq k \leq \frac{m+2}{2}\right).$$

Note that the arcs ℓ_1, ℓ_2 are segments on the real axis with endpoint at the origin with respect to the natural parameter

$$\zeta_\psi = \int_{z_0}^{z} \sqrt{\psi(z)}dz,$$

where z is a local chart near p_0 and $z_0 = z(p_0)$.

We take closed subarcs $\kappa_j \subset \ell_j (j = 1, 2)$ with the same ψ-length such that p_0 is an endpoint of each κ_j and that ψ has no zeros on $\kappa_j \setminus \{p_0\}$. Let p_j be the other endpoint of κ_j for each j. Also set $K = \kappa_1 \cup \kappa_2$.

Now, cut R along κ_1 and κ_2. For each j, let κ_j^+ and κ_j^-, respectively, be the right-side and the left-side edges of the slit κ_j, with respect to the orientation which corresponds to the move along the slit from p_0 to p_j. Assume that κ_1^- and κ_2^+, κ_1^+ and κ_2^- form the angles

$$\frac{2k\pi}{m+2} \quad \text{and} \quad \frac{2\pi(m+2-k)}{m+2}.$$

at p_0, respectively.

Paste κ_1^- and κ_2^+ so that points having the same absolute value with respect to ζ_ψ are identified. In the same way, paste κ_1^+ and κ_2^-. Let $\tilde{K}$ be the union of the pasted segments. Then we obtain a new finite bordered Riemann surface $\tilde{R}$ and the natural conformal embedding $\tilde{u} : R \setminus K \to \tilde{R}$. The pair $(\tilde{u}, \tilde{R})$ is an element of the family $\mathfrak{F}(R, K) \subset \mathfrak{F}(R, E_\psi)$.

Moreover, from the construction we can extend $(\tilde{u}^{-1})^*\psi$ naturally to a holomorphic quadratic differential $\tilde{\psi}$ on $\tilde{R}$, and $E_{\tilde{\psi}} = \tilde{R} \setminus \tilde{u}(R \setminus E_\psi)$ is an allowable slit of $\tilde{R}$ with respect to $\tilde{\psi}$.

Lemma 3.1.

$$\mathrm{height}_{\tilde{\psi}^{\mathrm{d}}}[\tilde{\gamma}] = \mathrm{height}_{\psi^{\mathrm{d}}}(\iota_{\tilde{u}}^{-1}[\tilde{\gamma}])$$

for every $[\tilde{\gamma}] \in \mathfrak{S}[\tilde{R}^{\mathrm{d}}]$.

Proof. We say that a simple closed curve $\tilde{\beta}$ on $\tilde{R}^{\mathrm{d}}$ is a $\tilde{\psi}^{\mathrm{d}}$-polygon, if $\tilde{\beta}$ is the union of finitely many horizontal arcs and vertical arcs of $\tilde{\psi}^{\mathrm{d}}$. Note that for every $[\tilde{\gamma}] \in \mathfrak{S}[\tilde{R}^{\mathrm{d}}]$

$$\mathrm{height}_{\tilde{\psi}^{\mathrm{d}}}[\tilde{\gamma}] = \inf_{\tilde{\beta}} \mathrm{height}_{\tilde{\psi}^{\mathrm{d}}}(\tilde{\beta}),$$

where the infimum is taken over all $\tilde{\psi}^{\mathrm{d}}$-polygons $\tilde{\beta}$ freely homotopic to $\tilde{\gamma}$ in $\tilde{R}^{\mathrm{d}}$.

We can now add horizontal segments contained in K to the pre-image $(\tilde{u}^{\mathrm{d}})^{-1}(\tilde{\beta})$ of such a $\tilde{\psi}^{\mathrm{d}}$-polygon $\tilde{\beta}$ so that the resulting ψ^{d}-polygon β is a closed curve in the class $\iota_{\tilde{u}}^{-1}[\tilde{\gamma}]$. Then, by construction, we obtain

$$\mathrm{height}_{\psi^{\mathrm{d}}}(\beta) = \mathrm{height}_{\tilde{\psi}^{\mathrm{d}}}(\tilde{\beta}).$$

Hence we conclude that

$$\mathrm{height}_{\psi^{\mathrm{d}}}(\iota_{\tilde{u}}^{-1}[\tilde{\gamma}]) \le \mathrm{height}_{\psi^{\mathrm{d}}}(\beta) = \mathrm{height}_{\tilde{\psi}^{\mathrm{d}}}(\tilde{\beta})$$

for every $\tilde{\psi}^{\mathrm{d}}$-polygon $\tilde{\beta}$ freely homotopic to $\tilde{\gamma}$, which in turn implies that

$$\mathrm{height}_{\psi^{\mathrm{d}}}(\iota_{\tilde{u}}^{-1}[\tilde{\gamma}]) \le \mathrm{height}_{\tilde{\psi}^{\mathrm{d}}}[\tilde{\gamma}]$$

for every $[\tilde{\gamma}] \in \mathfrak{S}[\tilde{R}^{\mathrm{d}}]$.

On the other hand, we can similarly see as above that

$$\mathrm{height}_{\tilde{\psi}^{\mathrm{d}}}(\iota_{\tilde{u}}[\gamma]) \le \mathrm{height}_{\psi^{\mathrm{d}}}[\gamma]$$

for every $[\gamma] \in \mathfrak{S}[R^{\mathrm{d}}]$. Thus we have proved the assertion. □

By Lemma 3.1, we see that the holomorphic quadratic differential $\tilde{\psi}^{\mathrm{d}} \in A(\tilde{R}^{\mathrm{d}}) \setminus \{0\}$ satisfies the condition (ii) in Theorem 1.4. Moreover, by definition, $E_{\tilde{\psi}}$ is an allowable slit of $\tilde{R}$ with respect to $\tilde{\psi}$, and

$$(\tilde{\psi} \circ \tilde{u})(\tilde{u}')^2 = \psi \text{ on } R \setminus E_\psi. \quad (*)$$

Lemma 3.2. $[\tilde{R}^{\mathrm{d}}, \iota_{\tilde{u}}] \neq [R^{\mathrm{d}}, id]$ *in* $T(R^{\mathrm{d}})$.

Proof. Suppose that $[\tilde{R}^{\mathrm{d}}, \iota_{\tilde{u}}] = [R^{\mathrm{d}}, \mathrm{id}]$ in $T(R^{\mathrm{d}})$. Then there exists a conformal map $f : R^{\mathrm{d}} \to \tilde{R}^{\mathrm{d}}$ with $(f)_* = \iota_{\tilde{u}}$ which is symmetric in the border Γ.

Take any $[\gamma] \in \mathfrak{S}[R^{\mathrm{d}}]$. Then Lemma 3.1 gives

$$\mathrm{height}_{\tilde{\psi}^{\mathrm{d}}}[f(\gamma)] = \mathrm{height}_{\psi^{\mathrm{d}}}[\gamma].$$

Since $\mathrm{height}_{\tilde{\psi}^{\mathrm{d}} \circ f (f')^2}[\gamma] = \mathrm{height}_{\tilde{\psi}^{\mathrm{d}}}[f(\gamma)]$, we obtain

$$\mathrm{height}_{\tilde{\psi}^{\mathrm{d}} \circ f (f')^2}[\gamma] = \mathrm{height}_{\psi^{\mathrm{d}}}[\gamma]$$

for every $[\gamma] \in \mathfrak{S}[R^{\mathrm{d}}]$. Hence the heights mapping theorem [2] implies that

$$(\tilde{\psi}^{\mathrm{d}} \circ f)(f')^2 = \psi^{\mathrm{d}} \text{ on } R^{\mathrm{d}}. \quad (**)$$

In particular, the map f sends the zeros of ψ^{d} to those of $\tilde{\psi}^{\mathrm{d}}$ while keeping multiplicities.

Now from the construction, the zero p_0 of orders $m \geq 3$ breaks into two zeros $\tilde{q}_1$ and $\tilde{q}_2$ of $\tilde{\psi}^{\mathrm{d}}$ of orders $k-2$ and $m-k$, respectively, with $2 \leq k \leq (m+2)/2$. Also the endpoints p_1 of κ_1 and p_2 of κ_2 gather into a zero $\tilde{q}$ of $\tilde{\psi}^{\mathrm{d}}$ on $\tilde{R}^{\mathrm{d}}$ of order 2.

Set $\tilde{K} = \tilde{R} \setminus \tilde{u}(R \setminus K)$. Then the zeros $\tilde{q}, \tilde{q}_1$ and $\tilde{q}_2$ of $\tilde{\psi}$ on $\tilde{K}$ have orders less than m. Hence we see that

$$f(p_0) \in \tilde{R} \setminus \tilde{K}.$$

Since the conformal embedding $\tilde{u}$ maps $R \setminus K$ onto $\tilde{R} \setminus \tilde{K}$, $(\tilde{u}^{\mathrm{d}})^{-1} \circ f(p_0)$ is well defined and $(\tilde{u}^{\mathrm{d}})^{-1} \circ f(p_0) \notin K$. In particular,

$$(\tilde{u}^{\mathrm{d}})^{-1} \circ f(p_0) \neq p_0.$$

Next assume that, for a positive integer n,

$$((\tilde{u}^{\mathrm{d}})^{-1} \circ f)^n(p_0) \neq ((\tilde{u}^{\mathrm{d}})^{-1} \circ f)^k(p_0)$$

for every k with $0 \leq k \leq n-1$. Then, $f \circ ((\tilde{u}^{\mathrm{d}})^{-1} \circ f)^n(p_0) \notin \tilde{K}$, for $f \circ ((\tilde{u}^{\mathrm{d}})^{-1} \circ f)^n(p_0)$ is a zero of $\tilde{\psi}$ of order m. Hence, similarly as above, $((\tilde{u}^{\mathrm{d}})^{-1} \circ f)^{n+1}(p_0) \notin K$. In particular,

$$((\tilde{u}^{\mathrm{d}})^{-1} \circ f)^{n+1}(p_0) \neq p_0.$$

Also by the assumption,

$$((\tilde{u}^{\mathrm{d}})^{-1} \circ f)^{n+1}(p_0) \neq ((\tilde{u}^{\mathrm{d}})^{-1} \circ f)^k(p_0)$$

for every k with $1 \leq k \leq n$.

Thus by induction, we conclude that, for every positive integer n,

$$((\tilde{u}^{\mathrm{d}})^{-1} \circ f)^n(p_0) \neq ((\tilde{u}^{\mathrm{d}})^{-1} \circ f)^k(p_0)$$

for every k with $0 \leq k \leq n-1$. Therefore ψ has infinitely many zeros, which is impossible. So we have shown that

$$[\tilde{R}^{\mathrm{d}}, \iota_{\tilde{u}}] \neq [R^{\mathrm{d}}, \mathrm{id}]$$

in $T(R^{\mathrm{d}})$. □

Theorem 1.3 follows immediately from Lemma 3.2.

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