# GENERALIZED OBSTACLE PROBLEM

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ABSTRACT. Fehlmann and Gardiner [3] considered the obstacle problem which asks what embedding of a Riemann surface $S$ of finite topological type minus an obstacle $E$ into another surface $R$ of the same type which induces the isomorphisms $\pi_1(S) \to \pi_1(R)$ of the fundamental groups does maximize the $L^1$-norm of the holomorphic quadratic differential on $R$ corresponding to a given one on $S$ under the heights mapping. In this paper we consider obstacles with arbitrarily many connected components while they considered the case where the obstacle has finitely many components. As an application we give a slit mapping theorem of an open Riemann surface of finite genus.

## 1. INTRODUCTION.

Since early 20th century, through the works of Koebe and de Possel and others, it has been turned out that a solution to an extremal problem often gives a conformal mapping with distinguished properties, such as parallel slit mappings. In this regard, it is important to find a good extremal problem for the investigations on complex analysis. Fehlman and Gardiner [3] studied such an extremal problem, which is called the obstacle problem. In this paper, we extend their results to the case where an obstacle may have uncountably many connected component. In order to formulate the problem, we give some explanations for necessary notions.

For a holomorphic quadratic differential $\varphi = \varphi(z)dz^2$ on a Riemann surface $S$, we define the $L^1$-norm $\|\varphi\|_{L^1(S)}$ of it by

$$\|\varphi\|_{L^1(S)} = \iint_S |\varphi(z)|dx\,dy.$$

If $\|\varphi\|_{L^1(S)} < \infty$, the quadratic differential $\varphi$ is called integrable. We denote by $A(S)$ the set of integrable holomorphic quadratic differentials on $S$.

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Suppose that $S$ is a hyperbolic Riemann surface. Then there is a holomorphic universal covering projection $P$ of $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ onto $S$. Then

$$\Gamma = \{\gamma \in \mathrm{Aut}\mathbb{D} \mid P \circ \gamma = P\}$$

is called a Fuchsian model of $S$. When $\Gamma$ is of the second kind, namely, when the limit set $\Lambda(\Gamma)$ of $\Gamma$ does not coincide with $\partial\mathbb{D}$, the Riemann surface $S^{\mathrm{d}} = (\bar{\mathbb{C}} \setminus \Lambda(\Gamma))/\Gamma$ is called the (Schottky) double of $S$, and $\partial S = (\partial\mathbb{D} \setminus \Lambda(\Gamma))/\Gamma$ is called the border of $S$. When $\Gamma$ is of the first kind, we set $S^{\mathrm{d}} = S$ and $\partial S = \emptyset$.

Let $S$ be a Riemann surface of finite topological type $(\kappa, m, l)$, that is, $S$ is obtained by removing mutually disjoint $l$ topological compact disks and $m$ points from a compact Riemann surface of genus $\kappa$. When $l = 0$, $R$ is called of finite analytic type $(\kappa, m)$. When $l > 0$, the double $S^{\mathrm{d}}$ is of finite analytic type $(2\kappa + l - 1, 2m)$. We assume that the number $6\kappa - 6 + 3l + 2m$ is positive in the sequel so that $\dim_{\mathbb{C}} A(S^{\mathrm{d}}) = 6\kappa - 6 + 3l + 2m > 0$.

A quadratic differential $\varphi = \varphi(z)dz^2$ in $A(S)$ is called *symmetric* in $\partial S$ if there is a quadratic differential $\varphi^{\mathrm{d}} \in A(S^{\mathrm{d}})$ with $\varphi^{\mathrm{d}}|_S = \varphi$ for which $\overline{j^*\varphi^{\mathrm{d}}} = \varphi^{\mathrm{d}}$ holds. Here $j : S^{\mathrm{d}} \to S^{\mathrm{d}}$ is the canonical anti-conformal involution of $S^{\mathrm{d}}$ induced by conjugation $z \mapsto \dfrac{1}{\bar{z}}$, in other words, the lift $\tilde{\varphi}$ of $\varphi^{\mathrm{d}}$ to $\bar{\mathbb{C}} \setminus \Lambda(\Gamma)$ is symmetric in $\partial\mathbb{D}$ : $\overline{\tilde{\varphi}(\frac{1}{\bar{z}}) \cdot \frac{1}{\bar{z}^4}} = \tilde{\varphi}(z)$. We denote by $A_{\mathrm{s}}(S)$ the set of quadratic differentials $\varphi \in A(S)$ symmetric in $\partial S$. Note that $\dim_{\mathbb{R}} A_{\mathrm{s}}(S) = 6\kappa - 6 + 3l + 2m$.

For a Riemann surface $S$ of finite analytic type let $\mathfrak{S}(S)$ be the set of simple closed curves $\gamma$ on $S$ which are homotopic neither to a point in $S$ nor to a puncture of $S$, and let $\mathfrak{S}[S]$ be the set of the free homotopy classes $[\gamma]$ of $\gamma \in \mathfrak{S}(S)$.

For a given $\varphi \in A(S) \setminus \{0\}$, we set

$$\mathrm{height}_\varphi(\gamma) := \int_\gamma \left| \mathrm{Im} \left( \sqrt{\varphi(z)} dz \right) \right|$$

and

$$\mathrm{height}_\varphi[\gamma] := \inf_\beta \mathrm{height}_\varphi(\beta),$$

where the infimum is taken over all closed curves $\beta \in \mathfrak{S}(S)$ freely homotopic to $\gamma$. Conversely, such a height vector $h \in \mathbb{R}^{\mathfrak{S}[S]}$ together with a conformal structure on $S$ determines a quadratic differential $\varphi \in A(S)$ with $\mathrm{height}_\varphi[\gamma] = h[\gamma]$ (see Section 2 or [5, Chap. 12] for details).

We are now ready to state the obstacle problem in the sense of Fehlmann and Gardiner [3]. They thought of a "simply connected" compact subset with finitely many connected components as an obstacle. We will consider a more general set as an obstacle.

**Definition 1.1.** A subset $E$ of a Riemann surface $S$ is said to be *allowable* if $E$ is compact and contractible in $S$ and if $S \setminus E$ is connected. For an element $\varphi \in A(S)$, if each component of an allowable subset $E$ is either a horizontal arc of $\varphi$ or the union of a finite number of horizontal arcs and critical points of $\varphi$, then $E$ is called an *allowable slit with respect to* $\varphi$.

Note that an allowable set $E$ may have uncountably many components. For detailed properties of an allowable set, see Section 2.

Let $E$ be an allowable subset of a Riemann surface $S$ of finite topological type. We denote by $\mathfrak{F}(S, E)$ the set of pairs $(g, S_g)$, where $S_g$ is a Riemann surface of the same type as $S$ and $g$ is a conformal embedding of $S \setminus E$ into $S_g$ such that $g$ maps the border $\partial S$ of $S$ to that of $S_g$ and the same applies to the punctures. We remark that $E^{\mathrm{d}} = E \cup j(E)$ and $E_g = S_g \setminus g(S \setminus E)$ are both allowable (see Lemma 2.3). For every $(g, S_g) \in \mathfrak{F}(S, E)$, the mapping $g$ naturally extends to a conformal map $g^{\mathrm{d}}$ from $S^{\mathrm{d}} \setminus E^{\mathrm{d}}$ into the double $S_g^{\mathrm{d}}$ of $S_g$. Then $g^{\mathrm{d}}$ induces an isomorphism $\iota_g$ of the fundamental group $\pi_1(S^{\mathrm{d}})$ of $S^{\mathrm{d}}$ onto that of $S_g^{\mathrm{d}}$ (see Lemma 2.4). For each $\varphi \in A_{\mathrm{s}}(S) \setminus \{0\}$, we assign the new height vector $[\gamma] \mapsto \mathrm{height}_{\varphi^{\mathrm{d}}}(\iota_g^{-1}[\gamma])$, $[\gamma] \in \mathfrak{S}[S_g^{\mathrm{d}}]$. Thus there is the unique holomorphic quadratic differential $\varphi_g^{\mathrm{d}} \in A(S_g^{\mathrm{d}}) \setminus \{0\}$ such that

$$\mathrm{height}_{\varphi_g^{\mathrm{d}}}[\gamma] = \mathrm{height}_{\varphi^{\mathrm{d}}}(\iota_g^{-1}[\gamma])$$

for every $[\gamma] \in \mathfrak{S}[S_g^{\mathrm{d}}]$. Since $\varphi_g^{\mathrm{d}}$ is symmetric in $\partial S_g$, $\varphi_g = \varphi_g^{\mathrm{d}}|_{S_g}$ belongs to $A_{\mathrm{s}}(S_g)$. (See the next section for a more detailed account.)

The *obstacle problem for* $(S, E, \varphi)$ is to find an element $(g, S_g)$ in $\mathfrak{F}(S, E)$ which maximizes the quantity

$$M_g = \|\varphi_g\|_{L^1(S_g)} = \iint_{S_g} |\varphi_g|.$$

We show existence and uniqueness of a solution to the problem and, as an application, we deduce a slit mapping theorem (see Section 5). Our first main theorem is stated as in the following, which is a generalization of a result of Fehlmann and Gardiner [3].

**Theorem 1.2** (existence). *Suppose that $S$ is a Riemann surface of finite topological type and that $\varphi \in A_{\mathrm{s}}(S) \setminus \{0\}$. Let $E$ be an allowable subset of $S$. Then there is an element $(g, S_g) \in \mathfrak{F}(S, E)$ with*

$$M_g = \sup_{(h,S_h)\in\mathfrak{F}(S,E)} M_h$$

*such that $E_g = S_g \setminus g(S \setminus E)$ is an allowable slit with respect to $\varphi_g$, and $E_g$ is of zero area.*

We call such an element $(g, S_g) \in \mathfrak{F}(S, E)$ as in Theorem 1.2 *extremal* for $(S, E, \varphi)$ and $\varphi_g \in A(S_g) \setminus \{0\}$ the *extremal differential* associated with $(g, S_g) \in \mathfrak{F}(S, E)$.

In [3], Fehlmann and Gardiner showed the above result under the additional assumption that $E$ consists of finitely many components. They also asserted in the paper that the extremal pair $(g, S_g)$ is unique in the sense that, if $(u, S_u) \in \mathfrak{F}(S, E)$ is also extremal for $(S, E, \varphi)$, then $g \circ u^{-1}$ extends to a conformal map from $S_u$ onto $S_g$. The uniqueness, however, does not necessarily hold in this sense (see [9]).

In this paper we show a uniqueness result for the obstacle problem in the following form.

**Theorem 1.3** (uniqueness). *Under the same hypotheses as in Theorem 1.2, the extremal differential $\varphi_g \in A(S_g) \setminus \{0\}$ is uniquely determined. Namely, if an element $(u, S_u) \in \mathfrak{F}(S, E)$ is also extremal for $(S, E, \varphi)$, then the extremal differential $\varphi_u$ associated with $(u, S_u)$ satisfies*

$$\varphi_u = (\varphi_g \circ w)(w')^2 \text{ on } u(S \setminus E),$$

*where $w = g \circ u^{-1}$.*

We remark that, if we consider the extremal problem of finding an element minimizing $\|\varphi_g\|_{L^1(S_g)}$ instead of maximizing, then the similar result can be obtained just by replacing the term "horizontal" by "vertical".

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## 2. Preliminaries.

Let $S$ be a Riemann surface of finite analytic type. The following variant of the second minimal norm property will play an important role in this note. The assertion can be seen by analyzing the proof of Theorem 9 in [4, p. 54].

**Proposition 2.1.** *Let $S$ be a Riemann surface of finite analytic type. Suppose that $\varphi \in A(S)$ and $\psi$ is an integrable quadratic differential on $S$ such that*

$$\operatorname{height}_\varphi[\gamma] \leq \operatorname{height}_\psi(\gamma)$$

*for almost every $\varphi$-polygonal curve $\gamma \in \mathfrak{S}(S)$. Then*

$$\|\varphi\|_{L^1(S)} \leq \iint_S |\sqrt{\varphi}\sqrt{\psi}|$$

*and, in particular, $\|\varphi\|_{L^1(S)} \leq \|\psi\|_{L^1(S)}$. Moreover $\|\varphi\|_{L^1(S)} = \|\psi\|_{L^1(S)}$ only if $\varphi = \psi$ a.e.*

We now recall a definition of the Teichmüller space $T(S)$ of a Riemann surface $S$ of finite analytic type (see [6] for the compact case). Let $(R, \iota)$ be a marked Riemann surface of the same type as $S$, that is, a pair of a Riemann surface $R$ of the same type as $S$ and an orientation-preserving isomorphism $\iota$ of the fundamental group $\pi_1(S)$ onto $\pi_1(R)$. (More rigorously, we should consider the fundamental group with base point. Though we do not refer to the base point to avoid complexity, the reader can formulate it in an obvious way.) Two pairs $(R_1, \iota_1)$ and $(R_2, \iota_2)$ are called (Teichmüller) equivalent if there exists a conformal mapping $u$ of $R_1$ onto $R_2$ such that $(u)_* \circ \iota_1 = \iota_2$. The set of such equivalent classes $[R, \iota]$ is called the Teichmüller space of $S$ and denoted by $T(S)$. Every point in $T(S)$ is represented as $[R, f_*]$ by a (smooth) quasiconformal map $f : S \to R$. For the existence of such a quasiconformal map, see [7].

We next recall heights mappings (cf. [4, §11.7], [5, §12.6]). Let $|dv|$ be a measured foliation on a Riemann surface $S$ of finite analytic type, namely, there are finitely many singularities $p_1, \ldots, p_t$ in $\hat{S}$, where $\hat{S}$ is the completion of $S$ and an open cover $\{U_j\}$ of $S \setminus \{p_1, \ldots, p_t\}$ and real-valued continuous funtions $v_j$ on $U_j$ with locally $L^2$ derivatives in such a way that $v_j = \pm v_k + \text{const.}$ on $U_j \cap U_k$ and $|dv|$ behaves around each $p_s$ like the pull-back of $|\operatorname{Im}(z^{n/2}dz)|$ under a quasiconformal map with certain restriction on the integer $n$.

Let $\varphi$ be a non-zero integrable holomorphic quadratic differential on a Riemann surface $S$ of finite analytic type. We say that a curve $\gamma \in \mathfrak{S}(S)$ is *$\varphi$-polygonal* if $\gamma$ is an union of finitely many horizontal and vertical arcs of $\varphi$. For a measured foliation $|dv|$ on $S$, by abuse of language, we define the height of $\gamma$ relative to $|dv|$ by

$$\operatorname{height}_v(\gamma) = \int_\gamma |dv|.$$

Note that the height is defined for almost every $\varphi$-polygonal curves in $\mathfrak{S}(S)$. We denote by $\mathrm{height}_v[\gamma]$ the essential infimum of $\mathrm{height}_v(\gamma')$ over $\gamma' \in [\gamma]$. Measured foliations $|dv_1|$ and $|dv_2|$ on $S$ are called measure equivalent if $\mathrm{height}_{v_1}[\gamma] = \mathrm{height}_{v_2}[\gamma]$ holds for all $[\gamma] \in \mathfrak{S}[S]$. Let $\mathfrak{MF}(S)$ be the set of measure equivalence classes of measured foliations on $S$. Every measure equivalence class $[|dv|] \in \mathfrak{MF}(S)$ induces a real-valued function

$$\mathrm{height}_v : \mathfrak{S}[S] \to \mathbb{R} \quad ([\gamma] \mapsto \mathrm{height}_v[\gamma]).$$

In this way, we obtain an embedding

$$\mathfrak{MF}(S) \to \mathbb{R}^{\mathfrak{S}[S]} \quad ([|dv|] \mapsto \mathrm{height}_v).$$

Then the product topology of $\mathbb{R}^{\mathfrak{S}[S]}$ induces a topology of $\mathfrak{MF}(S)$. It is known that the mapping

$$\Psi : A(S) \setminus \{0\} \to \mathfrak{MF}(S) \quad (\varphi \mapsto |\mathrm{Im}\,(\sqrt{\varphi(z)}dz)|)$$

is a homeomorphism (see, for example, [5, p. 227]).

For a given $\varphi \in A(S)\setminus\{0\}$ and a quasiconformal map $f : S \to R$, the measured foliation $|dv| = |\mathrm{Im}\,(\sqrt{\varphi(z)}dz)|$ on $S$ induces a measured foliation $f_*(|dv|) = |d(v \circ f^{-1})|$ on $R$. The relation

$$\mathrm{height}_\varphi[\gamma] = \mathrm{height}_{f_*(|dv|)}[f_*(\gamma)], \quad [\gamma] \in \mathfrak{S}[S],$$

implies that the measure equivalent class $[f_*(|dv|)] \in \mathfrak{MF}(R)$ depends only on the Teichmüller equivalence class $\tau = [R, f_*]$. Then we obtain the unique holomorphic quadratic differential $\tau_*\varphi \in A(R) \setminus \{0\}$ as $\Psi^{-1}[f_*(|dv|)]$, namely,

$$\begin{aligned} \mathrm{height}_{\tau_*\varphi}[\gamma] &= \mathrm{height}_{(f)_*(|dv|)}[\gamma] \\ &= \mathrm{height}_\varphi[f^{-1}(\gamma)] \end{aligned}$$

for every $[\gamma] \in \mathfrak{S}[R]$. The mapping $\tau_* : A(S) \setminus \{0\} \to A(R) \setminus \{0\}$ is called the heights mapping.

We collect here basic properties of allowable subsets of a Riemann surface. First we give another characterization of allowable sets.

**Lemma 2.2.** *Let $E$ be a compact subset of a Riemann surface $S$. Then $E$ is contractible in $S$ if and only if $E$ is contained in a compact topological disk.*

*Proof.* The "if" part is trivial. We show the "only if" part when $S$ is hyperbolic. The other cases can be treated similarly. Suppose that $E$ is contractible in $S$, namely, there is a continuous map $H : E \times [0,1] \to S$ such that $H(p,1) = p$ for all $p \in E$ and that $H(p,0) = p_0$ is independent of $p \in E$. Let $P$ be the universal covering projection of $\mathbb{D}$ onto $S$ such that $P(0) = p_0$ and $\Gamma$ be the covering transformation group of $P$. Let $\tilde{H}$ be a lift of the homotopy $H$ via $P$ such that $\tilde{H}(p,0) = 0$ for all $p \in E$. The set $\tilde{E} = \{\tilde{H}(p,1) : p \in E\}$ is compact. For every $\tilde{p} \in \tilde{E}$ we can take a positive number $r_{\tilde{p}}$ with the property that a closed disk $V_{\tilde{p}}$ centered at $\tilde{p}$ with radius $r_{\tilde{p}}$ satisfies

$$\mathrm{dist}(A(V_{\tilde{p}}), \tilde{E}) \geq 2r_{\tilde{p}} \text{ for every } A \in \Gamma, A \neq \mathrm{id}.$$

By compactness of $\tilde{E}$ we may assume that there exist finitely many of points $\tilde{p}_1, \dots, \tilde{p}_k \in \tilde{E}$ so that $\tilde{E}$ is contained in the interior of the closed set $\tilde{V} = \cup_{i=1}^{k} V_{\tilde{p}_i}$. We may assume that by replacing radii for every $i \neq j$, $V_{\tilde{p}_i}$ is not tangent to $V_{\tilde{p}_j}$. We can see that $A(\tilde{V}) \cap \tilde{V} = \emptyset$ for every $A \in \Gamma, A \neq \mathrm{id}$. Indeed, if $A(\tilde{V}) \cap \tilde{V} \neq \emptyset$ for some $A \in \Gamma, A \neq \mathrm{id}$, there exist $i, j \in \{1, \dots, k\}$ and $\tilde{q}_i \in V_{\tilde{p}_i}$ and $\tilde{q}_j \in V_{\tilde{p}_j}$ such that $A(\tilde{q}_i) = \tilde{q}_j$. Because of $\mathrm{dist}(A(V_{\tilde{p}_i}), \tilde{E}) \geq 2r_{\tilde{p}_i}$, we have $r_{\tilde{p}_j} \geq \mathrm{dist}(\tilde{q}_j, \tilde{p}_j) \geq \mathrm{dist}(\tilde{q}_j, \tilde{E}) \geq \mathrm{dist}(A(V_{\tilde{p}_i}), \tilde{E}) \geq 2r_{\tilde{p}_i}$ holds. On the other hand by considering $A^{-1}$ we have $r_{\tilde{p}_i} \geq 2r_{\tilde{p}_j}$. This cannot occur. Hence $A(\tilde{V}) \cap \tilde{V} = \emptyset$ for every $A \in \Gamma, A \neq \mathrm{id}$. Let $\tilde{W}$ be a convex hall of $\tilde{V}$, that is $\tilde{W} \subset \mathbb{D}$ is the closure of the set of point $\tilde{p}$ such that there exist a Jordan curve $\gamma$ in $\tilde{V}$ whoes interior contains $\tilde{p}$. Then $\tilde{W}$ is a disjoint union of finitely many closed topological disks whose boundaries are contained in $\tilde{V}$. Then we can see that for every components $\tilde{W}_1, \tilde{W}_2$ of $\tilde{W}$ and for every $A \in \Gamma, A \neq \mathrm{id}$ intersection $A(\tilde{W}_1) \cap \tilde{W}_2$ is empty or contained in either $A(\tilde{W}_1)$ or $\tilde{W}_2$, because by the above argument $A(\partial \tilde{W}_1) \cap \partial \tilde{W}_2 = \phi$ holds. Therefore the projection $P(\tilde{W})$ also consists of mutually disjoint closed topological disks in $S$ whose union contains $E$ in its interior. Consequently, by joining these disks with suitable canals we obtain a topological closed disk in $S$ containing $E$. □

By using the above lemma, we can also show the following.

**Lemma 2.3.** *Let $E$ be an allowable subset of a Riemann surface $S$ of finite topological type. Then $E^{\mathrm{d}}$ is an allowable subset of the double $S^{\mathrm{d}}$. Also, for any $(g, S_g) \in \mathfrak{F}(S, E)$, the set $E_g = S_g \setminus g(S \setminus E)$ is allowable in $S_g$.*

*Proof.* By joining a compact topological disk $\Delta$ containing $E$ and its reflection in $\partial S$ with a suitable canal, we obtain a compact topological disk $\Delta^{\mathrm{d}}$ containing $E^{\mathrm{d}}$. Thus $E^{\mathrm{d}}$

is allowable by Lemma 2.2. We may take the compact topological disk $\Delta$ so that $E$ is contained in the interior of $\Delta$. If $S$ is of finite topological type $(\kappa, m, l)$, then $S \setminus \Delta$ and $g(S \setminus \Delta)$ are of type $(\kappa, m, l+1)$. The set $\Delta_g = S_g \setminus g(S \setminus \Delta) \subset S_g$ is of finite topological type, say, $(\kappa', m', l')$. Since $S_g$ is reconstructed from $g(S \setminus \Delta)$ and $\Delta_g$ by glewing along a single Joldan curve, $(\kappa, m, l) = (\kappa, m, l+1) + (\kappa', m', l') - (0, 0, 2)$, i.e., $(\kappa', m', l') = (0, 0, 1)$. Then the set $\Delta_g$ which contain $E_g$ in the interior is a compact topological disk in $S_g$. Thus the latter assertion has been proved. □

The following lemma now easily follows.

**Lemma 2.4.** *Let $E$ be an allowable subset of a Riemann surface $S$ of finite topological type and $(g, S_g) \in \mathfrak{F}(S, E)$. Then the natural isomorphism $g_*^{\mathrm{d}} : \pi_1(S^{\mathrm{d}} \setminus E^{\mathrm{d}}) \to \pi_1(S_g^{\mathrm{d}} \setminus E_g^{\mathrm{d}})$ induces the isomorphism $\iota_g : \pi_1(S^{\mathrm{d}}) \to \pi_1(S_g^{\mathrm{d}})$.*

*Proof.* It is sufficient to show that if analytic closed curves $\gamma_1, \gamma_2 \subset S^{\mathrm{d}} \setminus E^{\mathrm{d}}$ are homotopic in $S^{\mathrm{d}}$, then $g^{\mathrm{d}}(\gamma_1)$ and $g^{\mathrm{d}}(\gamma_2)$ are also homotopic in $S_g^{\mathrm{d}}$. Let $\Delta^{\mathrm{d}}$ be a compact topological closed disk which contains $E^{\mathrm{d}}$ in the interior. Since the number of components of $\gamma_i \cap \partial\Delta^{\mathrm{d}} (i = 1, 2)$ is finite, we can decompose $\Delta^{\mathrm{d}}$ to finitely many components each of which does not intersect to $\gamma_1 \cup \gamma_2$. Then by suitable composition of components of the boundary $\partial\Delta^{\mathrm{d}}$ of the decomposed topological disk $\Delta^{\mathrm{d}}$ we can modify $\gamma_1$ to $\gamma_1'$ so that $\gamma_1'$ is homotopic to $\gamma_2$ in $S^{\mathrm{d}} \setminus E^{\mathrm{d}}$. Then $g^{\mathrm{d}}(\gamma_1')$ is homotopic to $g^{\mathrm{d}}(\gamma_2)$ in $S_g^{\mathrm{d}} \setminus E_g^{\mathrm{d}}$. We know by Lemma 2.3 $g^{\mathrm{d}}(\gamma_1')$ is homotopic to $g^{\mathrm{d}}(\gamma_1)$ in $S_g^{\mathrm{d}}$. Hence $g^{\mathrm{d}}(\gamma_2)$ is homotopic to $g^{\mathrm{d}}(\gamma_1)$ in $S_g^{\mathrm{d}}$. □

Following [3], we now introduce a few lemmas. For a simple closed curve $\beta$ on a Riemann surface $S$, we denote by $\lambda([\beta], S)$ the extremal length of the family $[\beta]$ of all closed curves on $S$ freely homotopic to $\beta$. For a point $\tau = [R, \iota]$ in $T(S)$, let $K_0(\tau)$ be the dilatation of the unique extremal quasiconformal map from $S$ onto $R$ which induces the isomorphism $\iota : \pi_1(S) \to \pi_1(R)$. In other words, $\log K_0(\tau)$ is the Teichmüller distance between $\tau$ and the base point $[S, \mathrm{id}]$.

**Lemma 2.5** ([5, p. 247])**.** *Let $S$ be a Riemann surface of finite analytic type. Then there exists a positive constant $c$ and finitely many simple closed curves $\beta_1, \ldots, \beta_N$ on $S$ such that the inequality*

$$K_0(\tau) \leq c \max_{1 \leq l \leq N} \lambda(\iota[\beta_l], R)$$

*holds for every point $\tau = [R, \iota]$ in $T(S)$.*

**Lemma 2.6** ([4, p. 218]). *Let $S$ be a Riemann surface of finite analytic type and $\varphi \in A(S) \setminus \{0\}$. Then for every $\tau \in T(S)$*

$$K_0(\tau)^{-1}\|\varphi\|_{L^1(S)} \leq \|\tau_*\varphi\|_{L^1(S_\tau)} \leq K_0(\tau)\|\varphi\|_{L^1(S)}.$$

Let $E$ be an allowable subset of a Riemann surface $S$ of finite topological type. By Lemma 2.4, every pair $(g, S_g) \in \mathfrak{F}(S, E)$ corresponds to a point $\tau = [S_g^{\mathrm{d}}, \iota_g] \in T(S^{\mathrm{d}})$. Note that the point $\tau$ is represented as $[S_g^{\mathrm{d}}, f_*]$ by a quasiconformal map $f$ of $S^{\mathrm{d}}$ onto $S_g^{\mathrm{d}}$ which is symmetric in the sense that $f \circ j = j_g \circ f$ holds, where $j : S^{\mathrm{d}} \to S^{\mathrm{d}}$ and $j_g : S_g^{\mathrm{d}} \to S_g^{\mathrm{d}}$ are the canonical anti-conformal involutions of $S^{\mathrm{d}}$ and $S_g^{\mathrm{d}}$, respectively. Recall that the holomorphic quadratic differential $\tau_*\varphi \in A(S_g^{\mathrm{d}}) \setminus \{0\}$ is determined by

$$\mathrm{height}_{\tau_*\varphi}[\gamma] = \mathrm{height}_{\varphi^{\mathrm{d}}}[f^{-1}(\gamma)]$$

for every $[\gamma] \in \mathfrak{S}[S_g^{\mathrm{d}}]$. We now see that the quadratic differential $\varphi_g = \tau_*\varphi|_{S_g}$ is symmetric in $\partial S_g$. Indeed,

$$\begin{aligned} \mathrm{height}_{\overline{j_g^*\tau_*\varphi}}[\gamma] &= \mathrm{height}_{\tau_*\varphi}[j_g(\gamma)] \\ &= \mathrm{height}_{\varphi^{\mathrm{d}}}[f^{-1} \circ j_g(\gamma)] \\ &= \mathrm{height}_{\varphi^{\mathrm{d}}}[j \circ f^{-1}(\gamma)] \\ &= \mathrm{height}_{\varphi^{\mathrm{d}}}[f^{-1}(\gamma)] \\ &= \mathrm{height}_{\tau_*\varphi}[\gamma] \end{aligned}$$

for every $[\gamma] \in \mathfrak{S}[S_g^{\mathrm{d}}]$ and Proposition 2.1 implies

$$\overline{j_g^*\tau_*\varphi} = \tau_*\varphi,$$

that is, $\varphi_g$ is symmetric in $\partial S_g$.

Let $S$ be a Riemann surface of finite topological type, $\varphi \in A_{\mathrm{s}}(S) \setminus \{0\}$ and $E$ be an allowable subset of $S$. We now show that

$$\{M_h : (h, S_h) \in \mathfrak{F}(S, E)\}$$

is bounded above and away from zero. Moreover we see that the set

$$\mathfrak{T}(S, E) = \{[S_g^{\mathrm{d}}, \iota_g] : (g, S_g) \in \mathfrak{F}(S, E)\}$$

is relatively compact in $T(S^{\mathrm{d}})$.

For the double $S^{\mathrm{d}}$ we take a family of curves $\beta_1, \ldots, \beta_N$ and a positive constant $c$ as in Lemma 2.5. We may assume that each $\beta_k$ is contained in $S^{\mathrm{d}} \setminus E^{\mathrm{d}}$. For every $(h, S_h) \in \mathfrak{F}(S, E)$ set $\tau = [S_h^{\mathrm{d}}, \iota_h] \in T(S^{\mathrm{d}})$. Lemma 2.6, together with $\|\varphi^{\mathrm{d}}\|_{L^1(S^{\mathrm{d}})} = 2\|\varphi\|_{L^1(S)}$ and $\|\varphi_h^{\mathrm{d}}\|_{L^1(S_h^{\mathrm{d}})} = 2\|\varphi_h\|_{L^1(S_h)}$, implies

$$\frac{\|\varphi\|_{L^1(S)}}{K_0(\tau)} \leq \|\varphi_h\|_{L^1(S_h)} \leq K_0(\tau)\|\varphi\|_{L^1(S)}.$$

Moreover from Lemma 2.5, we obtain

$$\begin{aligned} K_0(\tau) &\leq c \max_{1 \leq l \leq N} \lambda(\iota_h[\beta_l]), S_h^{\mathrm{d}}) \\ &= c \max_{1 \leq l \leq N} \lambda([h^{\mathrm{d}}(\beta_l)], S_h^{\mathrm{d}}) \\ &\leq c \max_{1 \leq l \leq N} \lambda([h^{\mathrm{d}}(\beta_l)], h^{\mathrm{d}}(S^{\mathrm{d}} \setminus E^{\mathrm{d}})) \\ &= c \max_{1 \leq l \leq N} \lambda([\beta_l], S^{\mathrm{d}} \setminus E^{\mathrm{d}}) < \infty. \end{aligned}$$

The last constant is independent of $(h, S_h) \in \mathfrak{F}(S, E)$. This means that $\mathfrak{T}(S, E)$ is relatively compact in $T(S^{\mathrm{d}})$. Moreover, letting

$$c_0 = c \max_{1 \leq l \leq N} \lambda([\beta_l], S^{\mathrm{d}} \setminus E^{\mathrm{d}}),$$

we have

$$c_0^{-1}\|\varphi\|_{L^1(S)} \leq M_h = \|\varphi_h\|_{L^1(S_h)} \leq c_0\|\varphi\|_{L^1(S)}$$

for every $(h, S_h) \in \mathfrak{F}(S, E)$. We have shown

**Lemma 2.7.** *Let $E$ be an allowable subset of a Riemann surface $S$ of finite topological type and $\varphi \in A_{\mathrm{s}}(S) \setminus \{0\}$. Then, the set $\mathfrak{T}(S, E)$ is relatively compact in $T(S^{\mathrm{d}})$ and there is a positive constant $c_0$ such that*

$$c_0^{-1}\|\varphi\|_{L^1(S)} \leq M_h \leq c_0\|\varphi\|_{L^1(S)}$$

*for every $(h, S_h) \in \mathfrak{F}(S, E)$.*

## 3. Proof of Theorem 1.2

Take a sequence $\{(g_n, S_{g_n})\}_n \subset \mathfrak{F}(S, E)$ such that

$$M_{g_n} \nearrow \sup_{(h, S_h) \in \mathfrak{F}(S, E)} M_h.$$

Set $\tau_n = [S^{\mathrm{d}}_{g_n}, \iota_{g_n}] \in T(S^{\mathrm{d}})$. Let $\Gamma$ and $\Gamma_n$ $(n \in \mathbb{N})$ be the normalized Fuchsian models ([6, p. 59]) of $S$ and $S_{g_n}$, respectively. Since $\{[S^{\mathrm{d}}_{g_n}, \iota_{g_n}]\}_n$ is relatively compact in $T(S^{\mathrm{d}})$ by Lemma 2.7, we may assume that $[S^{\mathrm{d}}_{g_n}, \iota_{g_n}]$ converge in $T(S^{\mathrm{d}})$. Then the isomorphisms $\chi_n : \Gamma \to \Gamma_n$ induced by $\iota_{g_n}$ algebraically converge to an isomorphism $\chi_\infty : \Gamma \to \Gamma_\infty$. We denote by $S_\infty$ the quotient space $\mathbb{D}/\Gamma_\infty$. Let $G$ be a Fuchsian model of $S \setminus E$ and $\rho : G \to \Gamma$ be the surjective homomorphism induced by the natural homomorphism $\pi_1(S \setminus E) \to \pi_1(S)$. For each $n \in \mathbb{N}$, set

$$\rho_n = \chi_n \circ \rho : G \to \Gamma_n.$$

Since $\chi_n$ converge to $\chi_\infty$, the homomorphisms $\rho_n$ converge to the surjective homomorphism

$$\rho_\infty = \chi_\infty \circ \rho : G \to \Gamma_\infty.$$

Let $\tilde{g}_n : \mathbb{D} \to \mathbb{D}$ $(n \in \mathbb{N})$ be the lift of $g_n$ associated with $\rho_n$. Then

$$\tilde{g}_n \circ A = \rho_n(A) \circ \tilde{g}_n$$

for every $A \in G$. Since $\{\tilde{g}_n\}_n$ is normal, there is a subsequence which converges to a holomorphic map $\tilde{g}$ on $\mathbb{D}$ uniformly on any compact subset of $\mathbb{D}$. Then we obtain

$$\tilde{g} \circ A = \rho_\infty(A) \circ \tilde{g}$$

for every $A \in G$. If $\tilde{g}$ is a constant $c$, then $\Gamma_\infty$ has the common fixed point $c$, and thus $\Gamma_\infty$ is a cyclic group, which is not the case. So $\tilde{g}$ is not constant and thus a holomorphic map into $\mathbb{D}$. Therefore $\tilde{g}$ descends to an injective holomorphic map $g$ of $S \setminus E$ into $S_\infty$ such that the homomorphism $(g)_*$ corresponds to $\rho_\infty$. We denote $S_\infty$ by $S_g$. Then it is easy to see that $(g, S_g) \in \mathfrak{F}(S, E)$.

Let $\hat{\Gamma}$, $\hat{\Gamma}_n$ $(n \in \mathbb{N})$ and $\hat{\Gamma}_\infty$ be the normalized Fuchsian models of the compact Riemann surface $\hat{S}$, $\hat{S}_{g_n}$ and $\hat{S}_g$, which are the completion of the doubles $S^{\mathrm{d}}$, $S^{\mathrm{d}}_{g_n}$ and $S^{\mathrm{d}}_g$, respectively. (In the case when $\hat{S}$ is not hyperbolic, the proof will be simpler.) Let $f_n$ be a quasiconformal map of $S^{\mathrm{d}}$ onto $S^{\mathrm{d}}_{g_n}$ which represents $[S^{\mathrm{d}}_{g_n}, \iota_{g_n}] \in T(S^{\mathrm{d}})$ and $f$ be a quasiconformal map of $S^{\mathrm{d}}$ onto $S^{\mathrm{d}}_g$ representing the element $[S^{\mathrm{d}}_g, \iota_g] \in T(S^{\mathrm{d}})$. We denote by $\hat{f}_n$ and $\hat{f}$ the lifts of $f_n$ and $f$ fixing $\pm 1, -i$ to the covering space $\mathbb{D}$ over $\hat{S}_{g_n}$ and $\hat{S}_g$, respectively. Let $\hat{\Omega}$ be a relatively compact fundamental domain of $\hat{\Gamma}$. Set $\hat{\Omega}_n = \hat{f}_n(\hat{\Omega})$ $(n \in \mathbb{N})$

and $\hat{\Omega}_\infty = \hat{f}(\hat{\Omega})$. Then $\hat{\Omega}_n$ and $\hat{\Omega}_\infty$ are the fundamental domains of $\hat{\Gamma}_n$ and $\hat{\Gamma}_\infty$, respectively. Let $\hat{\varphi}_{g_n}$ be a lift of $\varphi^{\mathrm{d}}_{g_n}$ to the universal covering space $\mathbb{D}$ over $\hat{S}_{g_n}$. For each $n \in \mathbb{N}$, $\hat{\varphi}_{g_n}$ is a meromorphic quadratic differential on $\mathbb{D}$ with at most simple poles. Let $D$ be a subdomain of $\mathbb{D}$, which is obtained from $\mathbb{D}$ by removing the points corresponding to the punctures of $S^{\mathrm{d}}_g$.

**Lemma 3.1.** *The family* $\{\hat{\varphi}_{g_n}\}_n$ *is normal on* $D$.

*Proof.* It is sufficient to show that $\{\|\hat{\varphi}_{g_n}\|_{L^1(K)}\}_n$ is bounded for any compact subset $K$ of $\mathbb{D}$. Let

$$\{A \in \hat{\Gamma}_\infty : A(\hat{\Omega}_\infty \cup \partial\hat{\Omega}_\infty) \cap K \neq \emptyset\} = \{A_1, \dots, A_{m_0}\}.$$

Then, the euclidean distance $d_\infty$ between $K$ and $(\cup_{i=1}^{m_0} A_i(\hat{\Omega}_\infty))^c$ is positive. Since $\hat{f}_n$ converge to $\hat{f}$ uniformly on any compact subset of $\mathbb{D}$, for each $A_i$ $(1 \leq i \leq m_0)$ an element

$$A_{i,n} = (\hat{f}_n \circ \hat{f}^{-1}) \circ A_i \circ (\hat{f}_n \circ \hat{f}^{-1})^{-1} \in \hat{\Gamma}_n$$

converge to $A_i$ as $n \to \infty$. Then the distance $d_n$ between $K$ and $(\cup_{i=1}^{m_0} A_{i,n}(\hat{\Omega}_n))^c$ converge to $d_\infty$. Hence $d_n > 0$ for sufficiently large $n$. This implies

$$K \subset \bigcup_{i=1}^{m_0} A_{i,n}(\hat{\Omega}_n).$$

Then

$$\#\{A \in \hat{\Gamma}_n : A(\hat{\Omega}_n) \cap K \neq \emptyset\} \leq m_0$$

for sufficiently large $n$. This result together with boundness of $\{M_{g_n}\}_n$ (Lemma 2.7) implies that $\{\|\hat{\varphi}_{g_n}\|_{L^1(K)}\}_n$ is bounded. □

**Lemma 3.2.** $M_g = \sup\{M_h : (h, S_h) \in \mathfrak{F}(S, E)\}$.

*Proof.* By Lemma 3.1, there is a subsequence of $\{\hat{\varphi}_{g_n}\}_n$ which uniformly converges on any compact subset of $D$ to a holomorphic function $\hat{\varphi}_\infty$ . For each $n \in \mathbb{N}$, $\hat{\varphi}_{g_n}$ is $\hat{\Gamma}_n$-invariant, that is,

$$(\hat{\varphi}_{g_n} \circ A_n)(A_n')^2 = \hat{\varphi}_{g_n}$$

for every $A_n \in \hat{\Gamma}_n$. Then we see that $\hat{\varphi}_\infty$ is $\hat{\Gamma}_\infty$-invariant, so we can project $\hat{\varphi}_\infty$ to a holomorphic quadratic differential $\varphi^*_\infty$ on $S^{\mathrm{d}}_g$. Moreover for any compact subset $K$ of $D$,

$$\|\hat{\varphi}_\infty\|_{L^1(K)} = \lim_{n\to\infty} \|\hat{\varphi}_{g_n}\|_{L^1(K)} \leq 2c_0\|\varphi\|_{L^1(S)}$$

holds, where $c_0$ is the positive number obtained in Lemma 2.7. Hence we can see that $\hat{\varphi}_\infty$ is integrable on $\hat{\Omega}_\infty$.

Next we show that the integrable holomorphic quadratic differential $\varphi^*_\infty$ on $S^{\mathrm{d}}_g$ is symmetric in $\partial S_g$. Let $\Lambda_n (n \in \mathbb{N})$ and $\Lambda_\infty$ be the limit sets of $\Gamma_n$ and $\Gamma_\infty$ respectively. Let $\tilde{\varphi}_{g_n} (n \in \mathbb{N})$ be the lift of $\varphi^{\mathrm{d}}_{g_n}$ to the covering space $\bar{\mathbb{C}} \setminus \Lambda_n$ over $S^{\mathrm{d}}_{g_n}$ and let $\tilde{\varphi}_\infty$ be the representation of $\varphi^*_\infty$ on the double $\bar{\mathbb{C}} \setminus \Lambda_\infty$. By assumption, for each $n \in \mathbb{N}$, $\tilde{\varphi}_{g_n}$ satisfies

$$\overline{\tilde{\varphi}_{g_n}(\frac{1}{\bar{z}^4})} \cdot \frac{1}{z^4} = \tilde{\varphi}_{g_n} \quad \text{on } \bar{\mathbb{C}} \setminus \Lambda_n.$$

Since $\tilde{\varphi}_{g_n}$ converge to $\tilde{\varphi}_\infty$ uniformly on any compact subset of $\bar{\mathbb{C}} \setminus \Lambda_\infty$, we have

$$\overline{\tilde{\varphi}_{\infty}(\frac{1}{\bar{z}^4})} \cdot \frac{1}{z^4} = \tilde{\varphi}_{\infty} \quad \text{on } \bar{\mathbb{C}} \setminus \Lambda_\infty.$$

Hence $\tilde{\varphi}_\infty$ is symmetric for $\partial\mathbb{D}$, that is, the projection $\varphi^*_\infty$ of $\tilde{\varphi}_\infty$ to $S^{\mathrm{d}}_g$ is symmetric in $\partial S_g$. We denote by $\varphi_\infty$ the restriction of $\varphi^*_\infty$ to $S_g$.

Now we show

$$\lim_{n\to\infty} \|\varphi_{g_n}\|_{L^1(S_{g_n})} = \|\varphi_\infty\|_{L^1(S_g)}.$$

Let $p_1, \ldots, p_k \in \hat{\Omega}_\infty$ be the points which correspond to the punctures of $S^{\mathrm{d}}_g$. We may assume that all $p_1, \ldots, p_k$ are interior points of $\hat{\Omega}_\infty \cup \partial\hat{\Omega}_\infty$. Set

$$p_{i,n} = \hat{f}_n \circ \hat{f}^{-1}(p_i) \quad (n \in \mathbb{N},\ 1 \leq i \leq k).$$

Then $p_{i,n} \to p_i$ as $n \to \infty$.

For sufficiently small $\varepsilon > 0$, let $V_i \subset \hat{\Omega}_\infty$ $(1 \leq i \leq k)$ be the open disk of radius $\varepsilon$ centered at $p_i$ such that $V_i \cap V_j = \phi$ if $i \neq j$. Since holomorphic functions $(z - p_{i,n})\hat{\varphi}_{g_n}$ $(n \in \mathbb{N})$ on $V_i$ converge to $(z - p_i)\hat{\varphi}_\infty$ uniformly on $V_i$, the residues $c_{i,n} = \mathrm{Res}(\hat{\varphi}_{g_n}, p_{i,n})$ converge to $c_i = \mathrm{Res}(\hat{\varphi}_\infty, p_i)$ as $n \to \infty$. Moreover we can also see that holomorphic functions

$$\hat{\varphi}_{g_n} - \frac{c_{i,n}}{z - p_{i,n}}$$

converge to

$$\hat{\varphi}_\infty - \frac{c_i}{z - p_i}$$

uniformly on $V_i$. We take the union $\bigcup_{n \geq n_0} \hat{\Omega}_n \cup \partial \hat{\Omega}_n$ for sufficiently large $n_0 \in \mathbb{N}$. Then $\hat{\varphi}_{g_n}$ converge to $\hat{\varphi}_\infty$ uniformly on the compact set

$$\left( \bigcup_{n \geq n_0} (\hat{\Omega}_n \cup \partial \hat{\Omega}_n) \right) \bigcup (\hat{\Omega}_\infty \cup \partial \hat{\Omega}_\infty) \setminus \bigcup_{i=1}^{k} V_i.$$

Set

$$C = \max_{1 \leq i \leq k} |c_i|.$$

Then we have

$$\begin{aligned}
& \limsup_{n \to \infty} \left| \|\hat{\varphi}_{g_n}\|_{L^1(\hat{\Omega}_n)} - \|\hat{\varphi}_\infty\|_{L^1(\hat{\Omega}_\infty)} \right| \\
\leq \; & \limsup \sum_{i=1}^{k} \left( \left\| \frac{c_i}{z - p_i} \right\|_{L^1(V_i)} + \left\| \frac{c_{i,n}}{z - p_{i,n}} \right\|_{L^1(V_i)} \right) \\
& + \limsup \sum_{i=1}^{k} \left\| \left( \hat{\varphi}_{g_n} - \frac{c_{i,n}}{z - p_{i,n}} \right) - \left( \hat{\varphi}_\infty - \frac{c_i}{z - p_i} \right) \right\|_{L^1(V_i)} \\
& + \limsup \left| \|\hat{\varphi}_{g_n}\|_{L^1(\hat{\Omega}_n \setminus \cup_{i=1}^{k} V_i)} - \|\hat{\varphi}_\infty\|_{L^1(\hat{\Omega}_\infty \setminus \cup_{i=1}^{k} V_i)} \right| \\
\leq \; & Ck6\pi\varepsilon.
\end{aligned}$$

Thus

$$\lim_{n \to \infty} \|\varphi_{g_n}\|_{L^1(S_{g_n})} = \|\varphi_\infty\|_{L^1(S_g)},$$

and we obtain $\|\varphi_\infty\|_{L^1(S_g)} = \sup_{(h, S_h) \in \mathfrak{F}(S,E)} M_h$. Moreover, by Lemma 2.7, we have

$$\|\varphi_\infty\|_{L^1(S_g)} = \lim_{n \to \infty} \|\varphi_{g_n}\|_{L^1(S_{g_n})} \geq c_0^{-1} \|\varphi\|_{L^1(S)}.$$

Therefore $\varphi_\infty \in A_{\mathrm{s}}(S_g) \setminus \{0\}$.

Next we show $\varphi_\infty = \varphi_g$ on $S_g$. Take any curve $\gamma \in \mathfrak{S}(S_g^{\mathrm{d}})$ and sufficiently small $\varepsilon > 0$. Set $\gamma_n = f_n \circ f^{-1}(\gamma)$ ($n \in \mathbb{N}$). Assume that $A_\infty \in \hat{\Gamma}_\infty$ corresponds to the homotopy class $[\gamma]$ on $\hat{S}_g$. Then

$$A_n = \left( \hat{f}_n \circ \hat{f}^{-1} \right) \circ A_\infty \circ \left( \hat{f}_n \circ f^{-1} \right)^{-1}$$

corresponds to the homotopy class $[\gamma_n]$ on $\hat{S}_{g_n}$.

Let $\alpha_n \in \mathfrak{S}(S^{\mathrm{d}}_{g_n})$ be a $\varphi^{\mathrm{d}}_{g_n}$-geodesic curve freely homotopic on $S^{\mathrm{d}}_{g_n}$ to $\gamma_n$, that is, $\alpha_n$ satisfies $[\alpha_n] = [\gamma_n]$ on $S^{\mathrm{d}}_{g_n}$ and $\ell_{\varphi^{\mathrm{d}}_{g_n}}[\gamma_n] = \ell_{\varphi^{\mathrm{d}}_{g_n}}(\alpha_n)$, where $\ell_{\varphi^{\mathrm{d}}_{g_n}}(\alpha_n)$ is the length

$$\int_{\alpha_n} |\sqrt{\varphi^{\mathrm{d}}_{g_n}(z)}dz|$$

of $\alpha_n$ associated with $\varphi^{\mathrm{d}}_{g_n}$ and $\ell_{\varphi^{\mathrm{d}}_{g_n}}[\gamma_n]$ is that of homotopy class $[\gamma_n]$ defined by

$$\inf_{\beta} \ell_{\varphi^{\mathrm{d}}_{g_n}}(\beta),$$

where the infimum is taken over all closed curves $\beta$ freely homotopic to $\gamma_n$ on $S^{\mathrm{d}}_{g_n}$. It is known [11, Theorem 24.1] that each $\alpha_n$ satisfies $\mathrm{height}_{\varphi^{\mathrm{d}}_{g_n}}[\gamma_n] = \mathrm{height}_{\varphi^{\mathrm{d}}_{g_n}}(\alpha_n)$. Let $\hat{\alpha}_n$ $(n \in \mathbb{N})$ be a lift of $\alpha_n$ to the universal covering space $\mathbb{D}$ over $\hat{S}_{g_n}$ such that $\hat{\alpha}_n$ is a closed arc starting from a point on $\partial\hat{\Omega}_n$. We parametrize each $\hat{\alpha}_n$ by the $\hat{\varphi}_{g_n}$-length $s$ and we set $l(n) = \ell_{\hat{\varphi}_{g_n}}(\hat{\alpha}_n)$. Then $\{l(n)\}_n$ is bounded because the set of $\{[S^{\mathrm{d}}_{g_n}, \iota_{g_n}]\}_n$ is relatively compact in $T(S^{\mathrm{d}})$ by Lemma 2.7, so we may assume that $l(n)$ converge to a positive constant $l_\infty$. For each $n \in \mathbb{N}$ by re-parametrizing $\hat{\alpha}_n$ by $\hat{\alpha}_n(l(n)s)$, we may assume that for every $n \in \mathbb{N}$, $\hat{\alpha}_n$ is defined on $[0,1]$. The curve $\alpha_n$ $(n \in \mathbb{N})$ is analytic except for finitely many points where $\varphi^{\mathrm{d}}_{g_n}$ vanishes. Since the number of zeros of $\varphi^{\mathrm{d}}_{g_n}$ on $S^{\mathrm{d}}_{g_n}$ is uniformly bounded for $n \in \mathbb{N}$ and $\{l(n)\}_n$ is also bounded, the orders of singularities of $\hat{\alpha}_n$ are uniformly bounded. Then we can assume that there exists a natural number $N_0$ and real numbers $t_1, t_2, \ldots, t_{N_0} \in [0,1]$ $(t_1 \leq t_2 \leq \ldots \leq t_{N_0})$ such that the number of critical points of $\hat{\alpha}_n$ is $N_0$ for all $n \in \mathbb{N}$ and for each $i = 1, \ldots, N_0$ critical point $t_{n,i}$ of $\hat{\alpha}_n$ converge to $t_i$ as $n \to \infty$. We can see by reduction to absurdity that the family $\{\hat{\alpha}_n\}_{n\in\mathbb{N}}$ is equicontinuous. Then, together with uniform boundedness of $\{\hat{\alpha}_n\}_{n\in\mathbb{N}}$, we may assume that $\hat{\alpha}_n$ uniformly converge to a continuous function $\hat{\alpha}_\infty$ on $[0,1]$. The image $\hat{\alpha}_\infty[0,1]$ is a lift of a closed curve on $\hat{S}_g$.

Now we can show that $\mathrm{height}_{\hat{\varphi}_{g_n}}(\hat{\alpha}_n)$ converge to $\mathrm{height}_{\hat{\varphi}_\infty}(\hat{\alpha}_\infty)$. Let $s_0$ be any point on $(0,1) \setminus \{t_1, \ldots, t_{N_0}\}$ such that $\hat{\varphi}_\infty(\hat{\alpha}_\infty(s_0)) \neq 0$ and $\hat{\alpha}_\infty(s_0) \in D$. We can take a neighborhood $U$ of $s_0$ in $(0,1)$ such that for every $n \in \mathbb{N}$ $\hat{\varphi}_{g_n}(\hat{\alpha}_n(s)) \neq 0$ on $U \cup \partial U$ and $\hat{\alpha}_n(U \cup \partial U)$ does not contain any points corresponding to punctures of $S^{\mathrm{d}}_{g_n}$. Let $\hat{\zeta}_n$ $(n \in \mathbb{N})$

and $\hat{\zeta}_\infty$ be the natural parameter near $\hat{\alpha}_n(s_0)$ and $\hat{\alpha}_\infty(s_0)$, which are represented by

$$\hat{\zeta}_n = \int_{\hat{\alpha}_n(s_0)}^{z} \sqrt{\hat{\varphi}_{g_n}(z)}dz$$

and

$$\hat{\zeta}_\infty = \int_{\hat{\alpha}_\infty(s_0)}^{z} \sqrt{\hat{\varphi}_\infty(z)}dz,$$

respectively. For every $n \in \mathbb{N}$, $\alpha_n$ is geodesic relative to $\varphi_{g_n}^{\mathrm{d}}$-length, so we see that

$$\frac{d\hat{\zeta}_n}{|d\hat{\zeta}_n|} = e^{i\theta_n} \text{ on } U$$

for a constant $\theta_n \in [0, 2\pi)$. For any subsequence $\{\theta_{n_k}\}_k$ which is converging to some $\theta_\infty \in [0, 2\pi]$

$$\begin{aligned} \frac{d\hat{\alpha}_{n_k}}{ds} &= \frac{l(n_k)d\hat{\alpha}_{n_k}}{|d\hat{\zeta}_{n_k}|} \\ &= \frac{l(n_k)d\hat{\alpha}_{n_k}}{d\hat{\zeta}_{n_k}} \cdot \frac{d\hat{\zeta}_{n_k}}{|d\hat{\zeta}_{n_k}|} \\ &= \frac{l(n_k)}{\sqrt{\hat{\varphi}_{g_{n_k}}(\hat{\alpha}_{n_k})}} \cdot e^{i\theta_{n_k}} \\ &\xrightarrow{k\to\infty} \frac{l_\infty}{\sqrt{\hat{\varphi}_\infty(\hat{\alpha}_\infty)}} \cdot e^{i\theta_\infty}. \end{aligned}$$

The convergence is uniform on $U$, so we can see that $\hat{\alpha}_\infty$ is of $C^1$-class on $U$ and

$$\frac{d\hat{\alpha}_\infty}{ds} = \lim_{k\to\infty} \frac{d\hat{\alpha}_{n_k}}{ds} = \frac{l_\infty}{\sqrt{\hat{\varphi}_\infty(\hat{\alpha}_\infty)}} \cdot e^{i\theta_\infty}.$$

Since the argument $\theta_\infty$ is independent of the choice of subsequence, the sequence $\{\theta_n\}_n$ is converging itself and we obtain

$$\frac{d\hat{\alpha}_\infty}{ds} = \lim_{n\to\infty} \frac{d\hat{\alpha}_n}{ds} \text{ on } U.$$

Consequently the curve $\hat{\alpha}_\infty$ satisfies

$$\frac{d\hat{\alpha}_\infty}{ds} = \lim_{n\to\infty} \frac{d\hat{\alpha}_n}{ds} \text{ on } (0,1)$$

except for finitely many of points. Then we obtain

$$\lim_{n\to\infty} \mathrm{height}_{\hat{\varphi}_{g_n}}(\hat{\alpha}_n) = \mathrm{height}_{\hat{\varphi}_\infty}(\hat{\alpha}_\infty)$$

because of

$$\left|\mathrm{Im}\left(\sqrt{\hat{\varphi}_{g_n}(\hat{\alpha}_n)}\frac{d\hat{\alpha}_n}{ds}\right)\right| \le \left|\sqrt{\hat{\varphi}_{g_n}(\hat{\alpha}_n)}\frac{d\hat{\alpha}_n}{ds}\right| = l(n) \le \sup_{n\in\mathbb{N}} l(n) < \infty$$

for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty}\left|\mathrm{Im}\left(\sqrt{\hat{\varphi}_{g_n}(\hat{\alpha}_n(s))}\frac{d\hat{\alpha}_n}{ds}\right)\right| = \left|\mathrm{Im}\left(\sqrt{\hat{\varphi}_\infty(\hat{\alpha}_\infty(s))}\frac{d\hat{\alpha}_\infty}{ds}\right)\right| \text{ a.e. on } [0,1].$$

If there exists $s_1 \in [0,1]$ such that $\hat{\alpha}_\infty(s_1)$ corresponds to a puncture of $S_g^{\mathrm{d}}$, we modify the curve $\hat{\alpha}_\infty$ locally in a neighborhood of $s_1$ to a curve $\hat{\alpha}'_\infty$ so that the projection on $S_g^{\mathrm{d}}$ is homotopic to $\gamma$ on $S_g^{\mathrm{d}}$. It is possible to choose $\hat{\alpha}'_\infty$ so that $\ell_{\hat{\varphi}_\infty}(\hat{\alpha}'_\infty)$ is as close to $\ell_{\hat{\varphi}_\infty}(\hat{\alpha}_\infty)$ as we want. Therefore, for any $\varepsilon > 0$, there is a curve $\hat{\alpha}'_\infty$ so that

$$\begin{aligned}
\mathrm{height}_{\varphi_\infty^{\mathrm{d}}}[\gamma] &\le \mathrm{height}_{\hat{\varphi}_\infty}(\hat{\alpha}'_\infty) \\
&< \mathrm{height}_{\hat{\varphi}_\infty}(\hat{\alpha}_\infty) + \varepsilon \\
&= \lim_{n\to\infty} \mathrm{height}_{\hat{\varphi}_{g_n}}(\hat{\alpha}_n) + \varepsilon \\
&= \lim_{n\to\infty} \mathrm{height}_{\varphi_{g_n}^{\mathrm{d}}}[\gamma_n] + \varepsilon \\
&= \mathrm{height}_{\varphi_g^{\mathrm{d}}}[\gamma] + \varepsilon.
\end{aligned}$$

Then we have the inequality $\mathrm{height}_{\varphi_\infty^{\mathrm{d}}}[\gamma] \le \mathrm{height}_{\varphi_g^{\mathrm{d}}}[\gamma]$. Particularly, by Proposition 2.1,

$$\|\varphi_\infty^{\mathrm{d}}\|_{L^1(S_g^{\mathrm{d}})} \le \|\varphi_g^{\mathrm{d}}\|_{L^1(S_g^{\mathrm{d}})}.$$

On the other hand, from maximality of $\|\varphi_\infty\|_{L^1(S_g)}$ in $\{M_h; (h, S_h) \in \mathfrak{F}(S,E)\}$ we have

$$\|\varphi_g^{\mathrm{d}}\|_{L^1(S_g^{\mathrm{d}})} \le \|\varphi_\infty^{\mathrm{d}}\|_{L^1(S_g^{\mathrm{d}})}.$$

Therefore $\|\varphi_\infty^{\mathrm{d}}\|_{L^1(S_g^{\mathrm{d}})} = \|\varphi_g^{\mathrm{d}}\|_{L^1(S_g^{\mathrm{d}})}$ holds, and by Proposition 2.1, we have

$$\varphi_\infty^{\mathrm{d}} = \varphi_g^{\mathrm{d}} \text{ on } S_g^{\mathrm{d}}.$$

□

**Lemma 3.3.** *$E_g = S_g \setminus g(S \setminus E)$ has measure zero.*

*Proof.* If $E_g$ has positive measure, we can define a Beltrami coefficient $\nu$ on $S_g^{\mathrm{d}}$ so that $\nu = 0$ on $S_g^{\mathrm{d}} \setminus E_g^{\mathrm{d}}$ and $\iint_{S_g^{\mathrm{d}}} \nu\varphi_g^{\mathrm{d}} > 0$.

For each $t \in (0,1)$, let $f^{t\nu}$ be a quasiconformal map of $S_g^{\mathrm{d}}$ onto the double $S_{t\nu}^{\mathrm{d}}$ of a Riemann surface $S_{t\nu}$ with Beltrami coefficient $t\nu$. Let $\tau(t) = [S_{t\nu}^{\mathrm{d}}, (f^{t\nu})_*] \in T(S_g^{\mathrm{d}})$. For $\varphi_g \in A_{\mathrm{s}}(S_g) \setminus \{0\}$ we set $\varphi_{t\nu} = \tau(t)_*\varphi_g|_{S_{t\nu}} \in A_{\mathrm{s}}(S_{t\nu}) \setminus \{0\}$ whose height on $S_{t\nu}^{\mathrm{d}}$ is equal to that of $\varphi_g^{\mathrm{d}}$ on $S_g^{\mathrm{d}}$. By the variational formura [4, p. 217], we have

$$\log \|\varphi_{t\nu}^{\mathrm{d}}\|_{L^1(S_{t\nu}^{\mathrm{d}})} = \log \|\varphi_g^{\mathrm{d}}\|_{L^1(S_g^{\mathrm{d}})} + \frac{2}{\|\varphi_g^{\mathrm{d}}\|_{L^1(S_g^{\mathrm{d}})}} \mathrm{Re} \iint_{S_g^{\mathrm{d}}} t\nu\varphi_g^{\mathrm{d}}\, dxdy + o(\|t\nu\|_\infty).$$

Then we can see

$$\|\varphi_{t\nu}\|_{L^1(S_{t\nu})} > \|\varphi_g\|_{L^1(S_g)}$$

for sufficiently small $t > 0$. On the other hand, $(f^{t\nu} \circ g, S_{t\nu}) \in \mathfrak{F}(S, E)$ for every $t \in (0,1)$, and the maximality of $\|\varphi_g\|_{L^1(S_g)}$ yields the inequality

$$\|\varphi_{t\nu}\|_{L^1(S_{t\nu})} \leq \|\varphi_g\|_{L^1(S_g)},$$

which is a contradiction. □

**Lemma 3.4.** *A component of $E_g$ is either*

*(i) a horizontal arc of $\varphi_g$ or,*

*(ii) a connected union of finitely many horizontal arcs and critical points of $\varphi_g$.*

*In particular, $E_g$ is an allowable slit with respect to $\varphi_g$.*

*Proof.* Fix any component $J$ of $E_g$. We consider the obstacle problem for $(S_g, J, \varphi_g)$. Then there is a solution $(h, S_h) \in \mathfrak{F}(S_g, J)$ and $\varphi_h \in A_{\mathrm{s}}(S_h) \setminus \{0\}$ satisfying

$$\|\varphi_h\|_{L^1(S_h)} = \sup\{\|\varphi_f\|_{L^1(S_f)} : (f, S_f) \in \mathfrak{F}(S_g, J)\}.$$

Since $(h \circ g, S_h) \in \mathfrak{F}(S, E)$, we have $\|\varphi_h\|_{L^1(S_h)} \leq \|\varphi_g\|_{L^1(S_g)}$. On the other hand, $(\mathrm{id}, S_g) \in \mathfrak{F}(S_g, J)$ yields the equality $\|\varphi_g\|_{L^1(S_g)} \leq \|\varphi_h\|_{L^1(S_h)}$. Thus $(\mathrm{id}, S_g) \in \mathfrak{F}(S_g, J)$ attains the extremum. By the same argument as in [3], we conclude that the component $J$ satisfies either (i) or (ii) in the lemma. □

We have proved Theorem 1.2 completely.

## 4. Proof of Theorem 1.3

Let $E_n \subset S$ $(n \in \mathbb{N})$ be an allowable subset of $S$ such that each $E_n$ is a disjoint union of finitely many of closed analytic disks and satisfying $E_n \supset E_{n+1} \supset \cdots$ and $\cap_{n=1}^{\infty} E_n = E$. Set $U_n = S \setminus E_n$ $(n \in \mathbb{N})$. (Consider a regular exhaustion $\{R_n\}_{n\in\mathbb{N}}$ of the Riemann surface $\hat{S} \setminus E^{\mathrm{d}}$ and set $U_n = S \cap R_n$ and $E_n = S \setminus U_n$ for sufficiently large $n$'s; cf. [1, p. 144].) First we show that we can obtain a solution $(h_\infty, S_{h_\infty}) \in \mathfrak{F}(S, E)$ of the obstacle problem for $(S, E, \varphi)$ from the family of solutions $(h_n, S_{h_n}) \in \mathfrak{F}(S, E_n)$ for $(S, E_n, \varphi)$ $(n \in \mathbb{N})$. Next we show that any other extremal element $(u, S_u) \in \mathfrak{F}(S, E)$ for $(S, E, \varphi)$ satisfies

$$\varphi_u = (\varphi_{h_\infty} \circ w_\infty)(w_\infty')^2 \quad \text{on } u(S \setminus E),$$

where $w_\infty = h_\infty \circ u^{-1}$. The solution $(g, S_g) \in \mathfrak{F}(S, E)$ in the Theorem 1.2 also satisfies

$$\varphi_g = (\varphi_{h_\infty} \circ w_0)(w_0')^2 \quad \text{on } g(S \setminus E),$$

where $w_0 = h_\infty \circ g^{-1}$, so we have

$$\varphi_u = (\varphi_g \circ w)(w')^2 \quad \text{on } u(S \setminus E),$$

where $w = g \circ u^{-1}$.

We now proceed to the proof. For each $n \in \mathbb{N}$, by considering the obstacle problem for $(S, E_n, \varphi)$, we obtain a solution $(h_n, S_{h_n}) \in \mathfrak{F}(S, E_n)$. Let $\tau_n = [S_{h_n}^{\mathrm{d}}, \iota_{h_n}] \in T(S^{\mathrm{d}})$. We can see in a similar manner to Section 2 that the set $\{\tau_n\}_n \subset T(S^{\mathrm{d}})$ is relatively compact in $T(S^{\mathrm{d}})$. In fact, we may assume that the curve family $\beta_1, \ldots, \beta_N$ in Lemma 2.5 is contained in $S^{\mathrm{d}} \setminus U_1^{\mathrm{d}}$. Then the dilatation $K_0(\tau_n)$ of the extremal quasiconformal map of the Teichmüller class $\tau_n$ satisfies

$$K_0(\tau_n) \leq c \max_{1 \leq l \leq N} \lambda([\beta_l], U_1^{\mathrm{d}}) =: c_1$$

for every $n \in \mathbb{N}$. Since the constant $c_1$ is independent of $n \in \mathbb{N}$, $\{\tau_n\}_n$ is relatively compact in $T(S^{\mathrm{d}})$. We can also see

$$c_1^{-1} \|\varphi\|_{L^1(S)} \leq \|\varphi_{h_n}\|_{L^1(S_{h_n})} \leq c_1 \|\varphi\|_{L^1(S)}$$

for every $n \in \mathbb{N}$. By the same argument as in the proof of Theorem 1.2, we obtain a subsequence $\{\tau_{n_k}\}_k$ and a point $\tau_\infty = [S_{h_\infty}, \iota_{h_\infty}] \in T(S^{\mathrm{d}})$ associated with an element $(h_\infty, S_{h_\infty}) \in \mathfrak{F}(S, E)$ such that $\tau_{n_k}$ converge to $\tau_\infty$ in $T(S^{\mathrm{d}})$, and the lift of $h_{n_k}^{\mathrm{d}}$ to the covering space $\mathbb{D}$ over $S^{\mathrm{d}} \setminus E^{\mathrm{d}}$ converge to that of $h_\infty^{\mathrm{d}}$ uniformly on any compact subset of $\mathbb{D}$. For brevity, we renumber $n_k$ by $k$. For every $(h, S_h) \in \mathfrak{F}(S, E)$ and every $n \in \mathbb{N}$, by considering the restriction of $h$ to $U_n$, we have

$$\|\varphi_h\|_{L^1(S_h)} \leq \|\varphi_{h_n}\|_{L^1(S_{h_n})}.$$

Since $\|\varphi_{h_n}\|_{L^1(S_{h_n})}$ converge to $\|\varphi_{h_\infty}\|_{L^1(S_{h_\infty})}$, we have

$$\|\varphi_h\|_{L^1(S_h)} \leq \|\varphi_{h_\infty}\|_{L^1(S_{h_\infty})}$$

for every $(h, S_h) \in \mathfrak{F}(S, E)$. Hence, $(h_\infty, S_{h_\infty}) \in \mathfrak{F}(S, E)$ is extremal for $(S, E, \varphi)$ ;

$$M_{h_\infty} = \sup_{(h, S_h) \in \mathfrak{F}(S,E)} M_h.$$

Next we assume that an element $(u, S_u) \in \mathfrak{F}(S, E)$ also attains the supremum. Set

$$\begin{aligned} w_n = h_n \circ u^{-1} \quad &, \quad \psi_n = (\varphi_{h_n} \circ w_n)(w_n')^2 \ (n \in \mathbb{N}) \\ w_\infty = h_\infty \circ u^{-1} \quad &, \quad \psi_\infty = (\varphi_{h_\infty} \circ w_\infty)(w_\infty')^2. \end{aligned}$$

We show that $\varphi_u = \psi_\infty$ on $u(S \setminus E)$. For the sake of convenience, we extend $\psi_n$ to $S_u$ so that $\psi_n = 0$ on $S_u \setminus u(U_n)$, similarly $\psi_\infty = 0$ on $S_u \setminus u(S \setminus E)$. Then $\psi_n$ $(n \in \mathbb{N})$ and $\psi_\infty$ are integrable quadratic differentials on $S_u$.

Fix any point $p_0 \in S_u^{\mathrm{d}}$ with $\varphi_u^{\mathrm{d}}(p_0) \neq 0$. Let $\zeta_u = \xi_u + i\eta_u$ be the natural parameter of $\varphi_u^{\mathrm{d}}$, which is defined about $p_0$ by

$$\zeta_u = \int_{z_0}^{z} \sqrt{\varphi_u^{\mathrm{d}}(z)}dz,$$

where $z$ is a local chart near $p_0$ and $z(p_0) = z_0$. Let $N_{p_0}$ be a sufficiently small closed neighborhood of $p_0$ such that $N_{p_0}$ corresponds to, by the parameter $\zeta_u$, a square centered at $p_0$ and each side of which is parallel to the axis. Since for each $n \in \mathbb{N}$ $\psi_n^{\mathrm{d}}$ is integrable on $S_u^{\mathrm{d}}$, the height relative to $\psi_n^{\mathrm{d}}$ of almost every vertical segment of $\varphi_u^{\mathrm{d}}$ on $S_u^{\mathrm{d}}$, particularly on $N_{p_0}$, is well defined and the same holds also to that relative to $\psi_\infty^{\mathrm{d}}$. Then we can see that for almost every vertical segment of $\varphi_u^{\mathrm{d}}$ on $S_u^{\mathrm{d}}$, particularly on $N_{p_0}$, the heights relative to $\psi_n^{\mathrm{d}}$ $(n \in \mathbb{N})$ and $\psi_\infty^{\mathrm{d}}$ are all well defined. We denote by $\mathcal{V}_{p_0}$ the set of such vertical segments of $\varphi_u^{\mathrm{d}}$ in $N_{p_0}$.

For each $\beta \in \mathcal{V}_{p_0}$ we set $\beta_n = \beta \cap u^{\mathrm{d}}(U_n^{\mathrm{d}})$ $(n \in \mathbb{N})$. Fix any $\varepsilon > 0$ and any 1-dimensional measurable set $A \subset N_{p_0}$ on a horizontal trajectory of $\varphi_u^{\mathrm{d}}$. Since $\varphi^{\mathrm{d}}$ and $\varphi_u^{\mathrm{d}}$

are integrable on $S^{\mathrm{d}}$ and $S_u^{\mathrm{d}}$, respectively, by Fubini's theorem and Schwarz's inequality, for every $n, m \in \mathbb{N}$ $(n \geq m)$ we have

$$\begin{aligned}
\int_A |\mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta) - \mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta_m)| d\xi_u \quad &\leq \quad \int_A \int_{\beta_n \setminus \beta_m} |\sqrt{\psi_n^{\mathrm{d}}(\zeta_u)}| d\eta_u \, d\xi_u \\
&= \quad \iint_{V(n,m,A)} |\sqrt{\psi_n^{\mathrm{d}}(\zeta_u)}| d\xi_u \, d\eta_u \\
&\leq \quad \|\psi_n^{\mathrm{d}}\|_{L^1(V(n,m,A))}^{\frac{1}{2}} \cdot \|\varphi_u^{\mathrm{d}}\|_{L^1(V(n,m,A))}^{\frac{1}{2}} \\
&\leq \quad \left(c_1 \|\varphi^{\mathrm{d}}\|_{L^1(S^{\mathrm{d}})}\right)^{\frac{1}{2}} \cdot \|\varphi_u^{\mathrm{d}}\|_{L^1(V(n,m,A))}^{\frac{1}{2}} \\
&\overset{m\to\infty}{\longrightarrow} \quad 0,
\end{aligned}$$

where $V(n,m,A) = \{p \in N_{p_0} \cap u^{\mathrm{d}}(U_n^{\mathrm{d}} \setminus U_m^{\mathrm{d}}) \mid \mathrm{Re}\,\zeta_u(p) \in A\}$. Then there exists a natural number $N_1$ such that for every $n \geq N_1$

$$\int_A |\mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta) - \mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta_{N_1})| d\xi_u < \varepsilon.$$

Furthermore we can see that

$$\int_A |\mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta_{N_1}) - \mathrm{height}_{\psi_\infty^{\mathrm{d}}}(\beta_{N_1})| d\xi_u \to 0 \quad (n \to \infty),$$

because $\psi_n^{\mathrm{d}}$ uniformly converge to $\psi_\infty^{\mathrm{d}}$ on $N_{p_0} \cap u^{\mathrm{d}}(U_{N_1}^{\mathrm{d}})$. Then, by Fatou's lemma, we have

$$\begin{aligned}
&\int_A \liminf_{n\to\infty} \mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta) d\xi_u \\
\leq\ & \liminf_{n\to\infty} \int_A \mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta) d\xi_u \\
\leq\ & \liminf_{n\to\infty} \left\{ \int_A \left(\mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta) - \mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta_{N_1})\right) d\xi_u \right. \\
& \left. + \int_A \left(\mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta_{N_1}) - \mathrm{height}_{\psi_\infty^{\mathrm{d}}}(\beta_{N_1})\right) d\xi_u + \int_A \mathrm{height}_{\psi_\infty^{\mathrm{d}}}(\beta_{N_1}) d\xi_u \right\} \\
=\ & \liminf_{n\to\infty} \int_A \left(\mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta) - \mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta_{N_1})\right) d\xi_u \\
& + \lim_{n\to\infty} \int_A \left(\mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta_{N_1}) - \mathrm{height}_{\psi_\infty^{\mathrm{d}}}(\beta_{N_1})\right) d\xi_u + \int_A \mathrm{height}_{\psi_\infty^{\mathrm{d}}}(\beta_{N_1}) d\xi_u \\
\leq\ & \varepsilon + \int_A \mathrm{height}_{\psi_\infty^{\mathrm{d}}}(\beta) d\xi_u.
\end{aligned}$$

Therefore almost every $\beta \in \mathcal{V}_{p_0}$ satisfies

$$\liminf_{n\to\infty} \mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta) \leq \mathrm{height}_{\psi_\infty^{\mathrm{d}}}(\beta).$$

The same holds for almost every vertical segments on $S_u^{\mathrm{d}}$. Then we can see that almost every $\varphi_u^{\mathrm{d}}$-polygonal curve $\gamma$, which is a union of finitely many horizontal and vertical segments, satisfies

$$\liminf_{n\to\infty} \mathrm{height}_{\psi_n^{\mathrm{d}}}(\gamma) \leq \mathrm{height}_{\psi_\infty^{\mathrm{d}}}(\gamma).$$

Let $\tilde{w}_n(\gamma)$ be a simple closed curve on $S_{h_n}^{\mathrm{d}}$ which is obtained from $w_n(\gamma)$ by supplying finitely many horizontal segments of $\varphi_{h_n}^{\mathrm{d}}$. We have

$$\begin{aligned}
\mathrm{height}_{\varphi_u^{\mathrm{d}}}[\gamma] &= \mathrm{height}_{\varphi_{h_n}^{\mathrm{d}}}[\tilde{w}_n(\gamma)] \\
&\leq \mathrm{height}_{\varphi_{h_n}^{\mathrm{d}}}(\tilde{w}_n(\gamma)) \\
&= \mathrm{height}_{\varphi_{h_n}^{\mathrm{d}}}(w_n(\gamma)) \\
&= \mathrm{height}_{\psi_n^{\mathrm{d}}}(\gamma)
\end{aligned}$$

for every $n \in \mathbb{N}$. Therefore $\mathrm{height}_{\varphi_u^{\mathrm{d}}}[\gamma] \leq \liminf_{n\to\infty} \mathrm{height}_{\psi_n^{\mathrm{d}}}(\gamma)$ holds, and we obtain $\mathrm{height}_{\varphi_u^{\mathrm{d}}}[\gamma] \leq \mathrm{height}_{\psi_\infty^{\mathrm{d}}}(\gamma)$ and, in particular, $\|\varphi_u^{\mathrm{d}}\|_{L^1(S_u^{\mathrm{d}})} \leq \|\psi_\infty^{\mathrm{d}}\|_{L^1(S_u^{\mathrm{d}})}$. Then the maximality of $M_u = \|\varphi_u^{\mathrm{d}}\|_{L^1(S_u^{\mathrm{d}})}$ implies, together with Proposition 2.1,

$$\varphi_u^{\mathrm{d}} = \psi_\infty^{\mathrm{d}} \text{ on } u^{\mathrm{d}}(S^{\mathrm{d}} \setminus E^{\mathrm{d}}).$$

Consequently we have

$$\varphi_u = (\varphi_{h_\infty} \circ w_\infty)(w'_\infty)^2 \text{ on } u(S \setminus E).$$

We have proved Theorem 1.3. □

## 5. Slit mapping theorem of open Riemann surface of finite genus.

Let $R$ be an open Riemann surface of finite genus. As was observed by Bochner, $R$ can be embedded into a compact Riemann surface $S$ of the same genus. Let $w$ be a conformal embedding of $R$ into $S$ such that $(w)_* : \pi_1(R) \to \pi_1(S)$ is surjective. We fix such a compact Riemann surface $S$, an embedding $w$ and an element $\varphi \in A(S) \setminus \{0\}$. We remark that $E = S \setminus w(R)$ is allowable in $S$. Let $\mathfrak{M}$ be the family of pairs $(g, S_g)$, where $g$ is a conformal map from $R$ into a compact Riemann surface $S_g$ such that $(g)_*$ is surjective. For every $(g, S_g) \in \mathfrak{M}$, $g \circ w^{-1} : S \setminus E \hookrightarrow S_g$ induces an isomorphism from $\pi_1(S)$ onto $\pi_1(S_g)$. We denote by $\tau = [S_g, (g \circ w^{-1})_*]$ the Teichmüller class in $T(S)$. Then for this $\tau \in T(S)$ and for $\varphi \in A(S) \setminus \{0\}$ we obtain a unique element $\varphi_g \in A(S_g) \setminus \{0\}$ whose height on $S_g$ is equal to that of $\varphi$ on $S$. Consider the extremal problem to find an element $(g, S_g) \in \mathfrak{M}$ maximizing $\|\varphi_g\|_{L^1(S_g)}$. By Theorem 1.2, we obtain an extremal element $(g, S_g) \in \mathfrak{M}$. The set $S_g \setminus g(R)$ is of zero area on $S_g$ and each component of $S_g \setminus g(R)$ is either

(i) a horizontal arc of $\varphi_g$ or,

(ii) a connected union of finitely many horizontal arcs and critical points of $\varphi_g$.

We have shown

**Corollary 5.1.** *Let $R$ be an open Riemann surface of finite genus. Then there are a compact Riemann surface $S$ and a conformal embedding $g : R \to S$ and a holomorphic quadratic differential $\varphi_g$ on $S$ such that $S \setminus g(R)$ has measure zero and each component of $S \setminus g(R)$ is a possibly branched arc on horizontal trajectories of $\varphi_g$.*

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