THE ORDER OF CONFORMAL AUTOMORPHISMS OF RIEMANN SURFACES OF INFINITE TYPE

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Abstract. Let R be a Riemann surface of infinite type such that the injectivity radius at any point in R is less than a positive constant M, and f a conformal automorphism of R fixing a compact subset in R. We show that the order of f is less than a certain constant depending on M.

1. Introduction

On a compact Riemann surface R of genus $g \geq 2$, it is known that the order of a conformal automorphism of R is not greater than $2(2g+1)$ (Wiman cf. [4, p.96]). Since the hyperbolic area of R is $4\pi(g-1)$, the injectivity radius at any point in R is not greater than a constant depending only on g. This means that the order of a conformal automorphism of R is estimated by the supremum of the injectivity radii which is taken over all points in R. We extend this result to the case of Riemann surfaces which are not necessarily of finite type. That is, for any hyperbolic Riemann surface R such that the injectivity radius at any point in R is less than a positive constant M, if a conformal automorphism f of R fixes a compact subset in R, then the order of f is estimated by M. Note that, in the case that R has the non-abelian fundamental group, a conformal automorphism f of R fixes a compact subset on R if and only if f has the finite order.

2. Main theorems

Let $\mathbf{H}$ be the upper-half plane equipped with the hyperbolic metric $d\lambda = |dz|/\mathrm{Im} z$. We say that a Riemann surface R is *hyperbolic* if it is represented by $\mathbf{H}/\Gamma$ for a torsion-free Fuchsian group Γ acting on $\mathbf{H}$. The hyperbolic distance on $\mathbf{H}$ or on R is denoted by $d(,)$. The *injectivity radius* at $p \in R$ is the supremum of radii r of embedded hyperbolic discs centered at p. The supremum of the injectivity radii is uniformly bounded from below.

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Proposition 1. ([7]) *For any hyperbolic Riemann surface R, the supremum of the injectivity radii which is taken over all points in R is not less than* $\operatorname{arcsinh}(2/\sqrt{3})$.

Before we state the main theorems, we note the following fact.

Proposition 2. *Let $R = \mathbf{H}/\Gamma$, where Γ is a Fuchsian group which is not necessarily torsion-free, and f a conformal automorphism of R with finite order $n > 1$. Then f fixes either a non-trivial simple closed geodesic, a puncture, a point or a cone point on R.*

Proof. Let $\tilde{f}$ be a lift of f to $\mathbf{H}$ which is an element of $\mathrm{PSL}_2(\mathbf{R})$. Since f has the finite order n, we see that $\tilde{f}^n$ belongs to Γ and that $\tilde{f}^m$ $(1 \leq m < n)$ does not belong to Γ. If $\tilde{f}^n$ is parabolic, then $\tilde{f}$ is parabolic. Hence f fixes a puncture on R. If $\tilde{f}^n$ is the identity, then $\tilde{f}$ is elliptic with the fixed point $\tilde{p} \in \mathbf{H}$. Hence f fixes the point on R which is the projection of $\tilde{p}$. If $\tilde{f}^n$ is elliptic, then $\tilde{f}$ is elliptic with the fixed point $\tilde{p} \in \mathbf{H}$. Hence f fixes the cone point on R which is the projection of $\tilde{p}$. Further, if $\tilde{f}^n$ is hyperbolic, then $\tilde{f}$ is hyperbolic. Hence f fixes a non-trivial closed geodesic c_* on R. In this case, we prove that f fixes either a non-trivial simple closed geodesic, a puncture, a point or a cone point on R. We consider the quotient $\hat{R} = R/\langle f \rangle$ by a cyclic group $\langle f \rangle$ and its Fuchsian model $\hat{\Gamma} = \langle \Gamma, \tilde{f} \rangle$. Then $\hat{c}_* = c_*/\langle f \rangle$ is a non-trivial closed geodesic on $\hat{R}$. There exists a subset $\hat{c}'$ of $\hat{c}_*$ such that $\hat{c}'$ is a non-trivial simple closed curve and it corresponds to a conjugacy class of an element γ in $\hat{\Gamma} - \Gamma$. Indeed, suppose that there is no such curves. That is, suppose that every non-trivial simple closed curve $\hat{c}_i \subset \hat{c}_*$ corresponds to a conjugacy class of an element γ_i in Γ. Since Γ is a normal subgroup of $\hat{\Gamma}$, the curve $\hat{c}_*$ corresponds to a conjugacy class of the composition of some elements in $\{\gamma_i, \gamma_i^{-1}\}_i$, which is in Γ. However, this is a contradiction. Let $c' \subset c_*$ be a connected component of the preimage of $\hat{c}'$. Then c' is a simple closed curve fixed by f. If γ is hyperbolic, then there exists a simple closed geodesic c'_* that is homotopic to c', and it is fixed by f. If γ is parabolic, then $\hat{c}'$ surround a puncture $\hat{p}$ on $\hat{R}$. Then c' surround a puncture p which is a lift of $\hat{p}$, and f fixes p. If γ is elliptic, then $\hat{c}'$ surround a cone point $\hat{p}$ on $\hat{R}$. In case c' is trivial, then it surrounds a point p which is a lift of $\hat{p}$, and f fixes p. In case c' is non-trivial, then it surrounds a cone point p which is a lift of $\hat{p}$, and f fixes p. □

This proposition immediately gives the following well known result.

Corollary 1. *Let R be a compact Riemann surface, and f a conformal automorphism of R. If f is irreducible, then f has a fixed point on R.*

Assume that R has the non-abelian fundamental group. Then the action of $\mathrm{Aut}(R)$ is properly discontinuous (cf. [6, Theorem X.48]). Thus, if a conformal automorphism f of R fixes either a non-trivial simple closed geodesic, a puncture, a point or a cone point on R, then f has the finite order. Hence, by Proposition 2, f has the finite order if and only if f fixes either a non-trivial simple closed geodesic, a puncture, a point or a cone point on R. In each case, we estimate the order of f concretely in terms of the injectivity radius on R.

Theorem 1. (hyperbolic case) *Let R be a hyperbolic Riemann surface. Suppose that there exists a positive constant M such that the injectivity radius at any point in R is less than M. Let c be a non-trivial simple closed geodesic on R whose length is ℓ. Further, let f be a conformal automorphism of R that satisfies $f(c) = c$, and n the order of f. Then*

$$n < (e^{2M} - 1)\cosh(\ell/2).$$

Theorem 2. (parabolic case) *Let R be a hyperbolic Riemann surface with a puncture p. Suppose that there exists a positive constant M such that the injectivity radius at any point in R is less than M. Let f be a conformal automorphism of R that satisfies $f(p) = p$, and n the order of f. Then*

$$n < e^{2M} - 1.$$

Theorem 3. (elliptic case) (i) *Let R be a hyperbolic Riemann surface. For a positive constant M, let p be a point in R at which the injectivity radius is less than M. Let f be a conformal automorphism of R that satisfies $f(p) = p$, and n the order of f. Then*

$$n < 2\pi \cosh M.$$

(ii) *Let Γ be a Fuchsian group which contains an elliptic element γ with order $m > 1$. Let $\tilde{p}$ be the fixed point of γ, and p its projection to the orbifold $R = \mathbf{H}/\Gamma$. Suppose that there exists a positive constant M such that the injectivity radius at any point in R is less than M. Let f be a conformal automorphism of R that satisfies $f(p) = p$, and n the order of f. Then*

$$n < (e^{2M} - 1)\frac{\pi}{m}\left(\frac{1}{\sin^2 \frac{\pi}{m}} + \frac{1}{\sinh^2 M}\right)^{\frac{1}{2}}.$$

Remark 1. In the assumptions of Theorems 1, 2 and 3 (ii), the injectivity radius at any point in R is uniformly bounded from above. Then R must have the non-abelian fundamental group. Further, in the assumption of Theorem 3 (i), the conformal automorphism f fixes a point on R at which the injectivity radius is bounded. Thus, if the fundamental group of R is abelian, then the order of f is not greater than 2. Hence we may assume that R has the non-abelian fundamental group.

Remark 2. By Proposition 1, the constant M in Theorems 1 and 2 satisfies $M \geq \operatorname{arcsinh}(2/\sqrt{3})$.

Remark 3. The upper bound of the order of f obtained in Theorem 2 is the limiting case of that in Theorem 1 as $\ell \to 0$. It is also the limiting case of that in Theorem 3 (ii) as $m \to \infty$.

3. The collar, cusp and cone lemmas

The proofs of the theorems are based on the collar, cusp and cone lemmas (cf. [3] and [5]).

Definition 1. A subset $S \subset \mathbf{H}$ is said to be *precisely invariant* under a subgroup Γ_S of a Fuchsian group Γ, if $\gamma(S) = S$ for all $\gamma \in \Gamma_S$ and $\gamma(S) \cap S = \emptyset$ for all $\gamma \in \Gamma - \Gamma_S$.

Collar Lemma. Let Γ be a Fuchsian group (which is not necessarily torsion-free) acting on $\mathbf{H}$, and L an axis of a hyperbolic element $\gamma \in \Gamma$ whose translation length is less than ℓ. Assume that there is no fixed points of elements in Γ on L. Then a collar

$$C(L) = \{z \in \mathbf{H} \mid d(z, L) \leq \omega(\ell)\}$$

is precisely invariant under the cyclic subgroup $\langle\gamma\rangle$ generated by γ, where $\sinh \omega(\ell) = (2\sinh(\ell/2))^{-1}$. Equivalently, the boundaries $\partial C(L)$ of $C(L)$ and the real axis make an angle θ, where $\tan\theta = 2\sinh(\ell/2)$.

Cusp Lemma. Let Γ be a Fuchsian group (which is not necessarily torsion-free) acting on $\mathbf{H}$. Suppose that Γ contains a parabolic element γ with the fixed point ζ. Then there exists a horoball $C(\zeta)$ tangent at ζ such that $C(\zeta)$ is precisely invariant under the cyclic subgroup $\langle\gamma\rangle$ generated by γ, and that the area of cusp neighborhood $C(\zeta)/\langle\gamma\rangle$ is 1.

Cone Lemma. Let Γ be a Fuchsian group acting on $\mathbf{H}$. Suppose that a point $p \in \mathbf{H}$ is fixed by an elliptic element $\gamma \in \Gamma$ whose order is $n > 2$. Then a hyperbolic disc

$$C(p) = \{z \in \mathbf{H} \mid d(z, p) < \rho(n)\}$$

is precisely invariant under the cyclic subgroup $\langle\gamma\rangle$ generated by γ, where for $2 < n < 7$, $\rho(n)$ is a constant $\mu \approx .075$, and for $n \geq 7$, $\cosh\rho(n) = (2\sin(\pi/n))^{-1}$.

4. Proofs of the theorems

In this section, we prove the theorems. First we give a proof of Theorem 1 which is based on Collar Lemma. The proof follows from the fact that there exists a wider collar of the non-trivial simple closed geodesic c, in proportion to the order of a conformal automorphism f that fixes c.

Proof of Theorem 1: Let Γ be a Fuchsian model of R, and $\tilde{f}$ a lift of f which is a hyperbolic element in $\mathrm{PSL}_2(\mathbf{R})$. Note that $\tilde{f}^n$ is a hyperbolic element in Γ which is corresponding to c. We consider the quotient $\hat{R} = R/\langle f\rangle$ by the cyclic group $\langle f\rangle$ and its Fuchsian model $\hat{\Gamma} = \langle\Gamma, \tilde{f}\rangle$. Then $\hat{c} = c/\langle f\rangle$ is a non-trivial simple closed geodesic on $\hat{R}$ whose length is ℓ/n. Since $\tilde{f}$ is corresponding to $\hat{c}$, we may assume that $\tilde{f}(z) = \exp(\ell/n)z$ with the axis $L = \{iy \mid y > 0\}$. Applying Collar Lemma for $\hat{\Gamma}$ and $\gamma = \tilde{f}$, we can take a collar

$$\tilde{C}(L) = \{re^{i\theta} \in \mathbf{H} \mid 0 < r,\ \theta_0 < \theta < \pi - \theta_0\}$$

so that it is precisely invariant under $\langle\tilde{f}\rangle \subset \hat{\Gamma}$, where

$$\tan\theta_0 = 2\sinh(\ell/2n).$$

In particular, $\gamma(\tilde{C}(L)) \cap \tilde{C}(L) = \emptyset$ for any $\gamma \in \Gamma - \langle\tilde{f}^n\rangle$. Then we can take a tubular neighborhood $C(c) = \tilde{C}(L)/\langle\tilde{f}^n\rangle$ of c on R whose fundamental region is

$$A = \{re^{i\theta} \in \mathbf{H} \mid 1 < r < e^{\ell},\ \theta_0 < \theta < \pi - \theta_0\}.$$

We may assume that $d(c, \partial C(c)) = \omega(\ell/n) > M$. Indeed, if

$$\omega(\ell/n) = \operatorname{arcsinh}\left(2\sinh\frac{\ell}{2n}\right)^{-1} \leq M,$$

then we have

$$n \leq \ell\{2\ \operatorname{arcsinh}(2\sinh M)^{-1}\}^{-1} = N(M, \ell).$$

By Remark 2, we have $M \geq \operatorname{arcsinh}(2/\sqrt{3}) > 0.987$. Then

$$\{\operatorname{arcsinh}(2\sinh M)^{-1}\}^{-1} < e^{2M} - 1.$$

Indeed, this inequality is equivalent to

$$2\sinh(e^{2M} - 1)^{-1}\sinh M < 1.$$

For $x < 1$, we have $\sinh x < 3x/2$. Since $M > 0.987$, we have $(e^{2M} - 1)^{-1} < 1$. Thus,

$$\sinh(e^{2M}-1)^{-1} < 3\{2(e^{2M}-1)\}^{-1}.$$

Further, $\sinh M < e^M/2$. Hence, for $M > 0.987$,

$$\begin{aligned} 2\sinh(e^{2M}-1)^{-1}\sinh M &< 3e^M\{2(e^{2M}-1)\}^{-1} \\ &= 3/(4\sinh M) \\ &< 1. \end{aligned}$$

Furthermore, $\ell/2 < \cosh(\ell/2)$ for $\ell > 0$. Hence

$$N(M,\ell) < (e^{2M}-1)\cosh(\ell/2),$$

and we have nothing to prove.

We take a point p in $C(c)$ which satisfies $d(p, \partial C(c)) = M$. Here $\partial C(c)$ is a boundary curve of $C(c)$. From the assumption, the injectivity radius at p is less than M. That is, the length r_p of the shortest non-trivial simple closed curve γ passing through p is less than $2M$. Since $d(p, \partial C(c)) = M$, the curve γ is in $C(c)$. Let $\tilde{p} = re^{i\theta} \in A$ $(\theta_0 < \theta < \pi/2)$ be a lift of p. Setting $z_1(t) = re^{it}$ for $t \geq 0$, we have

$$M = d(\tilde{p}, \partial\tilde{C}(L)) = \int_{\theta_0}^{\theta} \frac{|z_1'(t)|}{\mathrm{Im} z_1(t)} dt = \int_{\theta_0}^{\theta} \frac{1}{\sin t} dt \geq \int_{\theta_0}^{\theta} \frac{1}{t} dt = \log \frac{\theta}{\theta_0}.$$

Hence $\theta \leq e^M \theta_0$. We put $a = e^{i\theta}$ and $b = e^{\ell + i\theta}$. Then $r_p = d(a,b)$. From Theorem 7.2.1 in [1], we have

$$\begin{aligned} \sinh\frac{1}{2}d(a,b) &= \frac{|a-b|}{2(\mathrm{Im} a\,\mathrm{Im} b)^{\frac{1}{2}}} = \frac{e^\ell - 1}{2e^{\frac{\ell}{2}}\sin\theta} = \frac{\sinh\frac{\ell}{2}}{\sin\theta} \geq \frac{\sinh\frac{\ell}{2}}{\theta} \\ &\geq \frac{\sinh\frac{\ell}{2}}{e^M\theta_0} = \frac{\sinh\frac{\ell}{2}}{e^M \arctan(2\sinh\frac{\ell}{2n})} \geq \frac{\sinh\frac{\ell}{2}}{2e^M\sinh\frac{\ell}{2n}} \\ &= \frac{n\sinh\frac{\ell}{2}}{e^M\ell}\frac{\frac{\ell}{2n}}{\sinh\frac{\ell}{2n}} \geq \frac{n\sinh\frac{\ell}{2}}{e^M\ell}\frac{\ell}{\sinh\ell} = \frac{n\sinh\frac{\ell}{2}}{e^M\sinh\ell} \\ &= \frac{n}{2e^M\cosh\frac{\ell}{2}}. \end{aligned}$$

For the last inequality, we have used the fact that $x(\sinh x)^{-1}$ is a monotone decreasing function for $x > 0$. Since $r_p < 2M$, this implies that

$$\begin{aligned} n &< 2e^M \sinh M \cosh(\ell/2) \\ &= (e^{2M}-1)\cosh(\ell/2). \end{aligned}$$

□

Next, we give a proof of Theorem 2 which is based on Cusp Lemma. The idea is similar to that in Theorem 1.

Proof of Theorem 2: Let Γ be a Fuchsian model of R, and $\tilde{f}$ a lift of f to $\mathbf{H}$ which is a parabolic element in $\mathrm{PSL}_2(\mathbf{R})$. We may assume that $\tilde{f}(z) = z+1$. Note that $\tilde{f}^n$ belongs to Γ. We set $\hat{\Gamma} = \langle \Gamma, \tilde{f} \rangle$, which is a Fuchsian model of $\hat{R} = R/\langle f \rangle$. Applying Cusp Lemma for $\hat{\Gamma}$ and $\gamma = \tilde{f}$, we can take a horoball

$$\tilde{C}(\infty) = \{z \in \mathbf{H} \mid 1 < \mathrm{Im} z\}$$

so that it is precisely invariant under $\langle \tilde{f} \rangle \subset \hat{\Gamma}$. In particular, we have $\gamma(\tilde{C}(\infty)) \cap \tilde{C}(\infty) = \emptyset$ for any $\gamma \in \Gamma - \langle \tilde{f}^n \rangle$. We set $C(p) = \tilde{C}(\infty)/\langle \tilde{f}^n \rangle$, whose fundamental region is

$$\{z \in \mathbf{H} \mid 0 < \mathrm{Re} z < n,\ 1 < \mathrm{Im} z\}.$$

We take a point q in $C(p)$ so that $d(q, \partial C(p)) = M$. Here $\partial C(p)$ is the boundary curve of $C(p)$. From the assumption, the injectivity radius at q is less than M. That is, the length r_q of the shortest non-trivial simple closed curve γ passing through q is less than $2M$. Since $d(q, \partial C(p)) = M$, the curve γ is in $C(p)$. We put $a = e^M i$ and $b = n + e^M i$. Then $r_q = d(a,b)$. From Theorem 7.2.1 in [1], we have

$$\sinh \frac{1}{2} d(a,b) = \frac{|a-b|}{2(\mathrm{Im} a \, \mathrm{Im} b)^{\frac{1}{2}}} = \frac{n}{2e^M}.$$

Since $r_q < 2M$, this implies that

$$\begin{aligned} n &< 2e^M \sinh M \\ &= e^{2M} - 1. \end{aligned}$$

□

Finally, we prove Theorem 3. The proof is based on Cusp Lemma.

Proof of Theorem 3 (i): Since $2\pi \cosh M > 6$ for $M > 0$, we may assume that $n \geq 7$. Let $\tilde{p}$ be a lift of p to $\mathbf{H}$, and $\tilde{f}$ a lift of f to $\mathbf{H}$ fixing the point $\tilde{p}$, which is an elliptic element in $\mathrm{PSL}_2(\mathbf{R})$. Note that $\tilde{f}^n$ is the identity. We set $\hat{\Gamma} = \langle \Gamma, \tilde{f} \rangle$, which is a discrete group. Applying Cone Lemma for $\hat{\Gamma}$ and $\gamma = \tilde{f}$, we can take a hyperbolic disc

$$\tilde{C}(\tilde{p}) = \{z \in \mathbf{H} \mid d(z, \tilde{p}) < \rho(n)\}$$

so that it is precisely invariant under $\langle \tilde{f} \rangle \subset \hat{\Gamma}$, where

$$\cosh \rho(n) = (2 \sin(\pi/n))^{-1}.$$

In particular, $\gamma(\tilde{C}(\tilde{p})) \cap \tilde{C}(\tilde{p}) = \emptyset$ for any $\gamma \in \Gamma - \{\mathrm{id}\}$. Then there exists a hyperbolic disc $C(p)$ centered at p with radius $\rho(n)$. Thus the length of any non-trivial simple closed curve passing through p is greater than $2\rho(n)$. On the other hand, the injectivity radius at p is less than M by the assumption. That is, there exists a non-trivial simple closed curve passing through p whose length is less than $2M$. Hence we have $\rho(n) < M$, and this implies that

$$\begin{aligned} n &< \pi\{\arcsin(2\cosh M)^{-1}\}^{-1} \\ &< 2\pi\cosh M. \end{aligned}$$

□

Proof of Theorem 3 (ii): Let $\tilde{f}$ be a lift of f to $\mathbf{H}$, which is an elliptic element in $\mathrm{PSL}_2(\mathbf{R})$. Note that the order of $\tilde{f}$ is mn, and that $\tilde{f}^n \in \Gamma$. Applying Cone Lemma for $\hat{\Gamma} = \langle \Gamma, \tilde{f} \rangle$, we can take a hyperbolic disc

$$\tilde{C}(\tilde{p}) = \{z \in \mathbf{H} \mid d(z, \tilde{p}) < \rho(mn)\}$$

so that it is precisely invariant under $\langle \tilde{f} \rangle \subset \hat{\Gamma}$, where

$$\cosh\rho(mn) = (2\sin(\pi/mn))^{-1}.$$

In particular, $g(\tilde{C}(\tilde{p})) \cap \tilde{C}(\tilde{p}) = \emptyset$ for any $g \in \Gamma - \langle \tilde{f}^n \rangle$. Then the fundamental region of $C(p) = \tilde{C}(\tilde{p})/\Gamma$ is conformally equivalent to

$$\{re^{i\theta} \in \mathbf{C} \mid 0 \le r < \rho(mn),\ 0 < \theta < 2\pi/m\}.$$

We may assume that $\rho(mn) > M$. Indeed, if

$$\rho(mn) = \operatorname{arccosh}(2\sin(\pi/mn))^{-1} \le M,$$

then

$$\begin{aligned} n &\le \frac{\pi}{m}\left(\arcsin(2\cosh M)^{-1}\right)^{-1} < \frac{2\pi}{m}\cosh M < \frac{2\pi e^M}{m} \\ &< \frac{2\pi e^M}{m}\left\{\left(\frac{\sinh M}{\sin\frac{\pi}{m}}\right)^2 + 1\right\}^{\frac{1}{2}} \\ &= (e^{2M} - 1)\frac{\pi}{m}\left(\frac{1}{\sin^2\frac{\pi}{m}} + \frac{1}{\sinh^2 M}\right)^{\frac{1}{2}}. \end{aligned}$$

and we have nothing to prove.

We take a point q in $C(p)$ that satisfies $d(q, \partial C(p)) = M$. Here $\partial C(p)$ is the boundary curve of $C(p)$. From the assumption, the injectivity radius at q is less than M. That is, the length r_q of the shortest non-trivial simple closed curve γ passing through q is less than $2M$. Since

$d(q, \partial C(p)) = M$, the curve γ is in $C(p)$. From the cosine rule of triangles (cf. [1, p.148]), we have

$$\begin{aligned}\cosh r_q &= \cosh^2(\rho(mn) - M) - \sinh^2(\rho(mn) - M)\cos(2\pi/m) \\ &= \sinh^2(\rho(mn) - M)(1 - \cos(2\pi/m)) + 1.\end{aligned}$$

Then $r_q < 2M$ implies that

$$\begin{aligned}\rho(mn) &< \operatorname{arcsinh}\left(\frac{\cosh 2M - 1}{1 - \cos\frac{2\pi}{m}}\right)^{\frac{1}{2}} + M \\ &= \operatorname{arcsinh}\left(\frac{\sinh M}{\sin\frac{\pi}{m}}\right) + M \\ &=: M_1.\end{aligned} \tag{1}$$

To obtain (1), we have used the fact that the inverse function of $y = \sinh^2 x$ for $x > 0$ is $\operatorname{arcsinh}\sqrt{y}$. Since

$$\rho(mn) = \operatorname{arccosh}(2\sin(\pi/mn))^{-1},$$

the inequality (1) implies that

$$\begin{aligned}n &< (\pi/m)\left\{\arcsin\left(2\cosh M_1\right)^{-1}\right\}^{-1} \\ &< (2\pi/m)\cosh M_1 \\ &= (2\pi/m)\cosh\left\{\operatorname{arcsinh}\left(\frac{\sinh M}{\sin\frac{\pi}{m}}\right) + M\right\}.\end{aligned}$$

We set

$$X = \frac{\sinh M}{\sin\frac{\pi}{m}}.$$

Using the hyperbolic cosine formula and the fact that $\cosh(\operatorname{arcsinh} x) = \sqrt{x^2+1}$ for $x > 0$, we have

$$\begin{aligned}n &< (2\pi/m)\cosh\{\operatorname{arcsinh}X + M\} \\ &= (2\pi/m)\{\cosh(\operatorname{arcsinh}X)\cosh M + \sinh(\operatorname{arcsinh}X)\sinh M\} \\ &< (2\pi/m)\{\cosh(\operatorname{arcsinh}X)\cosh M + \cosh(\operatorname{arcsinh}X)\sinh M\} \\ &= (2\pi/m)\,e^M\cosh(\operatorname{arcsinh}X) \\ &= (2\pi/m)\,e^M(X^2+1)^{\frac{1}{2}} \\ &= \frac{2\pi e^M}{m}\left\{\left(\frac{\sinh M}{\sin\frac{\pi}{m}}\right)^2 + 1\right\}^{\frac{1}{2}} \\ &= (e^{2M} - 1)\frac{\pi}{m}\left(\frac{1}{\sin^2\frac{\pi}{m}} + \frac{1}{\sinh^2 M}\right)^{\frac{1}{2}}.\end{aligned}$$

□

5. Application

In this section, we apply our main theorem to investigating a certain property on the hyperbolic geometry on Riemann surfaces. The property we observe is as follows.

Definition 2. We say that a Riemann surface R satisfies the *lower bound condition* if there exists a positive constant ϵ such that the ϵ-thin part of R consists only of cusp neighborhoods. Further, we say that R satisfies the *upper bound condition* if there exists a positive constant M such that the injectivity radius at any point in R is less than M.

In [2], we have defined the (generalized) upper bound condition, and have shown the following.

Proposition 3. ([2]) *Let R be a hyperbolic Riemann surface of analytically finite type, and $\tilde{R}$ a normal covering surface of R which is not a universal cover. Then $\tilde{R}$ satisfies the lower and (generalized) upper bound conditions.*

In connection with this result, we show that Riemann surfaces inherit the lower and upper bound conditions from its normal covering surface.

Proposition 4. *Let R be a hyperbolic Riemann surface, and $\tilde{R}$ a normal covering surface of R. If $\tilde{R}$ satisfies the lower and upper bound conditions, then R also satisfies these conditions.*

Proof. It is clear that R satisfies the upper bound condition. Suppose that R does not satisfy the lower bound condition. Then R has a sequence $\{c_n\}$ of disjoint simple closed geodesics with $\ell_n = \ell(c_n) \to 0$ $(n \to \infty)$. Here $\ell(\cdot)$ means the hyperbolic length of a curve. Let $\tilde{c}_n \subset \tilde{R}$ be a connected component of the preimage of c_n. Since $\tilde{R}$ satisfies the lower bound conditions, there exists a positive constant ϵ such that $\ell(\tilde{c}_n) > \epsilon$ for all n. We take a positive constant M so that $\tilde{R}$ satisfies the upper bound condition for M. Assume that $\ell(\tilde{c}_n) \leq 2M$ for infinitely many n. Then, by Theorem 1, the order of a conformal automorphism $\tilde{f}_n$ of $\tilde{R}$ that fixes $\tilde{c}_n$ is less than $N = (e^{2M} - 1)\cosh M$. Then we have $\ell(c_n) > \epsilon/N$. However, this contradicts $\ell(c_n) \to 0$ $(n \to \infty)$. Next, we assume that $\ell(\tilde{c}_n) > 2M$ (including the case that $\tilde{c}_n$ is not closed) for infinitely many n. By Collar Lemma, there exists a tubular neighborhood $C(c_n)$ of c_n with width $\omega(\ell_n)$, where $\sinh \omega(\ell_n) = (2\sinh(\ell_n/2))^{-1}$. From the proof of Theorem 1, there exists a tubular neighborhood of $\tilde{c}_n$ with width $\omega(\ell_n)$. Since $\tilde{R}$ satisfies the upper bound condition for the constant M, there exists a non-trivial simple closed curve passing through $\tilde{p}_n \in \tilde{c}_n$ whose length is less than $2M$. However, since

$\ell(\tilde{c}_n) > 2M$ and since $\omega(\ell_n) \to \infty$ as $n \to \infty$, we have a contradiction. □

The following example shows that, in Proposition 4, if the normal covering surface $\tilde{R}$ of R satisfies only one of the two conditions, then R does not necessarily satisfy the two conditions.

Example 1. Let

$$\tilde{R} = \mathbf{C} - \bigcup_{n=1}^{\infty} \bigcup_{m \in \mathbf{Z}} \left\{ \frac{m}{n} \pm n^2 \sqrt{-1} \right\},$$

and set $R = \tilde{R}/\langle f \rangle$, where $f(z) = z + 1$. This Riemann surface R satisfies the upper bound condition but does not satisfy the lower bound condition. On the other hand, the normal covering surface $\tilde{R}$ of R satisfies the lower bound condition but does not satisfy the upper bound condition.

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