# SOME INEQUALITIES FOR THE POINCARÉ METRIC OF PLANE DOMAINS

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Abstract. In this paper, the Poincaré (or hyperbolic) metric and the associated distance are investigated for a plane domain based on the detailed properties of those for the particular domain $\mathbb{C} \setminus \{0, 1\}$. In particular, another proof of a recent result of Gardiner and Lakic [7] is given with explicit constant. This and some other constants in this paper involve particular values of complete elliptic integrals and related special functions. A concrete estimate for the hyperbolic distance near a boundary point is also given, from which refinements of Littlewood's theorem are derived.

## 1. Introduction

Throughout this paper, unless otherwise stated, $\Omega$ will denote a hyperbolic plane domain, namely, a connected open set in the complex plane $\mathbb{C}$ whose boundary contains at least two points, where the boundary $\partial\Omega$ of $\Omega$ is taken in $\mathbb{C}$. The Poincaré-Koebe uniformization theorem states that one can take a holomorphic universal cover $p$ of $\Omega$ from the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C}; |\zeta| < 1\}$ for such a domain $\Omega$. The complete hyperbolic (Poincaré) metric $\rho_\Omega(z)|dz|$ of $\Omega$ is defined, as usual, by $\rho_\Omega(p(\zeta))|p'(\zeta)| = 1/(1 - |\zeta|^2)$ for $\zeta \in \mathbb{D}$ and the hyperbolic (Poincaré) distance between two points $z_1, z_2 \in \Omega$ is defined by

$$d_\Omega(z_1, z_2) = \inf_\gamma \int_\gamma \rho_\Omega(z)|dz|,$$

where the infimum is taken over all rectifiable paths $\gamma$ in $\Omega$ connecting $z_1$ with $z_2$. Note that the choice of $\zeta$ and $p$ does not matter in the above definition for $\rho_\Omega(z)$.

The most important feature of these quantities is probably the contraction property for holomorphic maps: for a holomorphic map $f : \Omega \to \Omega'$ one has

$$f^*\rho_{\Omega'}(z) := \rho_{\Omega'}(f(z))|f'(z)| \leq \rho_\Omega(z), \ z \in \Omega, \text{ and}$$
$$d_{\Omega'}(f(z_1), f(z_2)) \leq d_\Omega(z_1, z_2), \ z_1, z_2 \in \Omega,$$

where equality holds at some (and hence every) point $z$ in the first part if and only if $f$ is a holomorphic covering map. In particular, choosing the inclusion map, we obtain the following monotonicity properties of the hyperbolic metric for $\Omega_0 \subset \Omega$ :

$$\rho_\Omega(z) \leq \rho_{\Omega_0}(z) \quad \text{and} \quad d_\Omega(z_1, z_2) \leq d_{\Omega_0}(z_1, z_2), \quad z, z_1, z_2 \in \Omega_0. \tag{1.1}$$

As a matter of fact, it is difficult to find the exact value of $\rho_\Omega(z)$ or $d_\Omega(z_1, z_2)$ or even to estimate these in a concrete domain $\Omega$, except in some particular cases. There are two

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basic cases: a) $\Omega$ is simply-connected and b) $z$ is close to an isolated point of $\partial\Omega$. In the case a) one has a global inequality $1/4 \le \rho_\Omega(z)\delta_\Omega(z) \le 1$, where $\delta_\Omega(z) = \inf\{|z-a|; a \in \partial\Omega\}$, while in the case b), one has a local inequality near the isolated boundary point $a \in \partial\Omega$, $\rho_\Omega(z) \asymp 1/(|z-a| \log \frac{1}{|z-a|})$ (see, for example, [3]). There are many more cases between these two extreme cases where one can give explicit estimates. Perhaps the best known case is when $\partial\Omega$ is uniformly perfect (see [20]). Then one has a global inequality as in case a) but with constants depending on the parameters in the definition of uniform perfectness. The main purpose of this paper is to provide a tool which enables us to deduce further estimates for the hyperbolic density and the hyperbolic distance around a given boundary point even when $\partial\Omega$ is not uniformly perfect. These estimates involve characteristics of clustering and isolation properties of $\partial\Omega$.

Beardon and Pommerenke [5] have given an estimate of the form

$$\frac{K_1}{\delta_\Omega(z)(\beta_\Omega(z)+K_0)} \le \rho_\Omega(z) \le \frac{K_2}{\delta_\Omega(z)(\beta_\Omega(z)+K_0)}, \tag{1.2}$$

where

$$\beta_\Omega(z) = \inf\left\{\left|\log\left|\frac{z-a}{b-a}\right|\right|; a \in \partial\Omega, b \in \partial\Omega, |z-a| = \delta_\Omega(z)\right\} \tag{1.3}$$

and $K_0, K_1, K_2$ are positive universal constants, which could be given explicitly (see [5] or (4.5) below). This estimate is usually precise enough, however, it is not easy to treat the technical quantity $\beta_\Omega$ in general. Their proof of (1.2) is based on the monotonicity of the hyperbolic metric and on the concrete estimate of the hyperbolic metric of the particular domain $\mathbb{C} \setminus \{0, 1\}$.

We denote by $\lambda_{a,b}(z)|dz|$ the hyperbolic metric of the twice punctured plane $\mathbb{C} \setminus \{a, b\}$. Set $\lambda(z) = \lambda_{0,1}(z)$. A somewhat detailed account on $\lambda_{a,b}$ will be given in Section 2. If we take two points $a, b$ from $\partial\Omega$, then the monotonicity yields the inequality $\rho_\Omega(z) \ge \lambda_{a,b}(z)$ for each $z \in \Omega$. We now consider the quantity

$$\sigma_\Omega(z) = \sup_{a,b\in\partial\Omega} \lambda_{a,b}(z)$$

for $z \in \Omega$. As an immediate consequence of the above observation, we get $\sigma_\Omega(z) \le \rho_\Omega(z)$ for all $z \in \Omega$. The following theorem has recently been proved by Gardiner and Lakic.

**Theorem 1.4** (Gardiner-Lakic [7]). *There exists a universal constant $C$ such that $\sigma_\Omega(z) \le \rho_\Omega(z) \le C\sigma_\Omega(z)$ holds in $\Omega$ for every hyperbolic plane domain $\Omega$.*

The result tells us the simple principle that it is sufficient to choose a suitable pair of boundary points in order to get a right estimate of $\rho_\Omega(z)$ for a fixed $z \in \Omega$. This principle will also be used implicitly in the following sections. It may also be worth seeing that the quantity $\sigma_\Omega(z)$ varies continuously in the shape of $\Omega$ concerning the Hausdorff distance of the boundary. We give another simple proof of the theorem in Section 3 with an explicit constant. Our proof is based on the idea developed in [5] and more straightforward than in [7]. At the end of Section 3, it is observed that a similar assertion does not hold in general concerning the hyperbolic distance.

As a tool for estimation of $\rho_\Omega$ and $d_\Omega$, we define quantities for a closed set to measure the magnitude of its clustering. Let $E$ be a closed set in the complex plane $\mathbb{C}$ with $\#E \ge 2$.

Define the mappings $m_E : E \times \mathbb{R} \to [0, +\infty)$ and $m_E^* : \mathbb{R} \to [0, +\infty]$ by

$$m_E(a,t) = \inf_{b \in E} \big| t - \log|b-a| \big|,$$
$$m_E^*(t) = \sup_{a \in E} m_E(a,t),$$

respectively. We will study fundamental properties of these quantities in Section 4. The quantity $m_E(a,t)$ is closely related to $\beta_\Omega(z)$ when $E = \partial\Omega$ (see Section 4). However, we believe that $m_E(a,t)$ is easier to use. For instance, as will be seen, a closed set $E$ is uniformly perfect if and only if $m_E^*$ is bounded in $(-\infty, \log \operatorname{diam} E)$. In terms of $m_E$, we give an estimate of the hyperbolic metric of a domain $\Omega$ with $E = \mathbb{C} \setminus \Omega$. Set $h(t) = e^t \lambda(-e^t)$ for $t \in \mathbb{R}$. We will give several properties of $h$ in Lemma 2.11. Among them, we note that $h(t)$ is decreasing (increasing) for $t > 0$ ($t < 0$) and that the inequality $h(t) \ge 1/(2|t| + 2C_0)$ holds, where $C_0$ is given by (2.6) below.

**Theorem 1.5.** *Let $E$ be a closed set with $\partial\Omega \subset E \subset \mathbb{C} \setminus \Omega$ for a given hyperbolic plane domain $\Omega$. Then for an arbitrary point $a \in E$, the inequality*

$$h(m_E(a, \log|z-a|)) \le |z-a|\rho_\Omega(z) \le \frac{\pi}{4m_E(a, \log|z-a|)}, \quad z \in \Omega,$$

*holds.*

As the reader will see, the proof of this result given in Section 3 is very similar to that of Theorem 1.4. By definition, equality holds in the left-hand side above when $\Omega$ is a twice punctured plane. Note also that the above inequality is still valid even if $\Omega$ is a hyperbolic open set equipped with the hyperbolic metric component-wise. For the relation between $m_E(a,t)$ and $m_{\partial\Omega}(a,t)$, see Proposition 4.6 below. As a direct consequence, we have the following result.

**Corollary 1.6.** *For a hyperbolic plane domain $\Omega$ the following inequality holds:*

$$\delta_\Omega(z)\rho_\Omega(z) \ge h(m_E^*(\log \delta_\Omega(z))), \quad z \in \Omega.$$

Section 5 is devoted to an estimate of hyperbolic distance. We define the numerical function $\varphi : \mathbb{R} \to \mathbb{R}$ by (5.5) below. The behaviour of $\varphi$ will be investigated in Section 5. Here, we just mention the relation $\varphi(-t) = -\varphi(t)$ and the inequalities

$$\log\left(1 + \frac{t}{2C_0}\right) \le \varphi(t) \le \log\left(1 + \frac{t}{2\pi}\right)$$

for $t \ge 0$, where $C_0 \approx 4.37688$ is the constant given by (2.6) below (see Lemma 5.4).

**Theorem 1.7.** *Let $\Omega$ be a proper subdomain of the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and let $a_n$, $1 \le n < N$, be a (finite or infinite) sequence of points in $\mathbb{C}^* \setminus \Omega$ such that $t_1 < t_2 < \dots$ and that $\lim t_n = \infty$ when $N = \infty$, where $t_n = \log|a_n|$. Set $a_0 = 0$ and $a_N = \infty$. Then the hyperbolic distance between points $z_1$ and $z_2$ in $\Omega$ with $|z_1| \le |z_2|$ satisfies*

$$d_\Omega(z_1, z_2) \ge \frac{1}{2}\varphi\big(t_k - \log|z_1|\big) + \sum_{n=k+1}^{l} \varphi(t_n - t_{n-1}) + \frac{1}{2}\varphi\big(\log|z_2| - t_l\big), \tag{1.8}$$

*where the integers $k$ and $l$, $1 \le k \le l < N$, are determined by the relations* $\sqrt{|a_k a_{k-1}|} \le |z_1| \le \sqrt{|a_k a_{k+1}|}$ *and* $\sqrt{|a_l a_{l-1}|} \le |z_2| \le \sqrt{|a_l a_{l+1}|}$. *On the other hand, the inequality*

$$\frac{d_D(-|z_1|,-|z_2|)}{C} \le \frac{1}{2}\varphi\big(t_k - \log|z_1|\big) + \sum_{n=k+1}^{l} \varphi(t_n - t_{n-1}) + \frac{1}{2}\varphi\big(\log|z_2| - t_l\big) \tag{1.9}$$

*holds for the domain* $D = \mathbb{C}^* \setminus \{e^{t_n}; 1 \le n < N\}$, *where* $C$ *is the absolute constant appearing in Lemma 3.1.*

The first part is quite similar to Theorem I in Hayman [8]. We will prove the theorem and then apply it to some particular cases to derive simpler consequences in Section 5.

As applications of the above results, in Section 6 we obtain a couple of results analogous to Littlewood's theorem telling us the growth of analytic functions in the unit disk which omit a sequence of values.

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## 2. Hyperbolic metric of twice punctured plane

In this section, we summarize basic properties of the hyperbolic metric $\lambda_{a,b}(z)|dz|$ of the twice punctured plane $\mathbb{C} \setminus \{a,b\}$. Recall that we write $\lambda = \lambda_{0,1}$. The study of $\lambda_{a,b}$ reduces to that of $\lambda$ by virtue of the relation

$$|b-a|\lambda_{a,b}(z) = \lambda\left(\frac{z-a}{b-a}\right). \tag{2.1}$$

Sometimes, however, it is more convenient to state a property in terms of $\lambda_{a,b}$.

As is well known, the classical theorem of Schottky (and its refinements) follows from fundamental properties of $\lambda$ (see, for example, [19]). Thus, the density $\lambda(z)$ has been extensively studied by many authors. An analytic expression of $\lambda = \lambda_{0,1}$ has been obtained by Agard [1] (see (2.4) and (2.5) therein) in terms of complete elliptic integrals of the first kind:

$$\lambda(z) = \frac{\pi}{8|z(1-z)|\mathrm{Re}\{\mathcal{K}(z)\mathcal{K}(1-\bar{z})\}}, \tag{2.2}$$

where

$$\mathcal{K}(z) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-zt^2)}}.$$

Note here that the function $\mathcal{K}(z)$ should be understood as an analytic continuation of the positive function $\mathcal{K}(x)$ on $0 < x < 1$ to the upper half plane. For example, the transformation rules

$$\mathcal{K}(-x) = \frac{1}{\sqrt{1+x}}\mathcal{K}\left(\frac{x}{1+x}\right) \quad \text{and} \tag{2.3}$$

$$\mathcal{K}(1+x) = \frac{1}{\sqrt{1+x}}\left[\mathcal{K}\left(\frac{1}{1+x}\right) + i\,\mathcal{K}\left(\frac{x}{1+x}\right)\right] \tag{2.4}$$

hold for $x > 0$ (see [6, 162.02, 213.06] though our definition of $\mathcal{K}$ is different from that adopted there). Formula (2.2) can be used for the numerical computation of the values of $\lambda$. However, we need more efforts to get a mathematically rigorous estimate for $\lambda$.

Some attempts towards understanding $\lambda(z)$ have been made by Ahlfors [3, Chapter 2] and Beardon-Pommerenke [5]. The reader can find in [21] more information about $\lambda$ containing an algorithm of computing the value $\lambda(z)$ for $z \in \mathbb{C} \setminus \{0,1\}$. (The reader, however, should note that the symbol $\lambda$ is used there to designate $\lambda_{1,-1}$.)

We will require later the following inequality, which was proved by Hempel [9] and Jenkins [13] independently:

$$\frac{1}{2|z-a|\left(\left|\log|\frac{z-a}{b-a}|\right| + C_0\right)} \le \lambda_{a,b}(z), \tag{2.5}$$

where,

$$C_0 = \frac{1}{2\lambda(-1)} = \frac{2}{\lambda(1/2)} = \frac{4}{\pi}\mathcal{K}\left(\frac{1}{2}\right)^2 = \frac{\Gamma(1/4)^4}{4\pi^2} \approx 4.37688. \tag{2.6}$$

Note that this bound is not symmetric with respect to $a$ and $b$. One can swap $a$ and $b$ in (2.5) to get a better estimate if $|z-a| > |z-b|$.

Combining (2.5) with the explicit formula $\rho_{\mathbb{D}^*}(z) = 1/2|z|\log(1/|z|)$ for the punctured unit disk $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ (see [19]), we observe that

$$\frac{1}{2|z|(\log(1/|z|) + C_0)} \le \lambda(z) \le \frac{1}{2|z|\log(1/|z|)}$$

for $0 < |z| < 1$, and, in particular, $\lambda(z) \sim 1/2|z|\log(1/|z|)$ as $z \to 0$.

Note also the symmetry relations

$$\lambda(\bar{z}) = \lambda(z), \quad \lambda(1-z) = \lambda(z), \quad \text{and} \quad \lambda(1/z)/|z|^2 = \lambda(z). \tag{2.7}$$

The following inequality was first proved by Lehto, Virtanen and Väisälä [15]. Note also that this result has been substantially strengthened by Weitsman [23] to the monotonicity of the hyperbolic metric of a circularly symmetric domain (see also [21]).

**Lemma 2.8.** *The inequality $\lambda(z) \ge \lambda(-|z|)$ holds for $z \in \mathbb{C} \setminus \{0,1\}$.*

We will also need the following monotonicity of $\lambda_{a,b}(z)$ in the space parameters.

**Lemma 2.9** (Solynin-Vuorinen [21, Corollary 2.14])**.** *As a function of $b = te^{i\theta}$, the quantity $\lambda_{1,b}(0)$ is*

(0) *decreasing in $t \in \mathbb{R}_+$ for $\theta = \pi$,*
(1) *decreasing in $0 < t < 1$ for a fixed $\theta \in \mathbb{R}$, and*
(2) *increasing in $0 < \theta < \pi$ for a fixed $t \in \mathbb{R}_+$.*

Finally, we investigate the function $h(t) = e^t\lambda(-e^t)$. Note that this function can be written as $h(t) = \lambda_{1,1+e^{-t}}(0)$. The following expression of $h(t)$ can also be obtained by (2.2):

$$h(t) = \frac{\pi}{8\mathcal{K}(\frac{1}{1+e^t})\mathcal{K}(\frac{1}{1+e^{-t}})}. \tag{2.10}$$

**Lemma 2.11.**

(1) *The function $h$ is even, namely, $h(t) = h(-t)$.*
(2) *$h(t)$ is increasing in $t \le 0$ and decreasing in $t \ge 0$.*
(3) *$h(t) \ge 1/(2|t| + 2C_0)$ holds for $t \in \mathbb{R}$ and $h(t) \sim 1/2|t|$ as $|t| \to \infty$.*

*Proof.* The first assertion is immediately deduced from the last relation in (2.7) or from (2.10). The second assertion follows from the special case of Theorem 1 in [9]: $(x\lambda(-x))' > 0$ for $0 < x < 1$ and $(x\lambda(-x))' < 0$ for $x > 1$. The first inequality in (3) is a direct consequence of (2.5). The last assertion has been explained already in this section. □

On the basis of some numerical experiments we have arrived at the following conjectural results.

**Conjecture 2.12.**

(1) *For* $t \in \mathbb{R}$, $1 \le 2(|t| + C_0)h(t) < 1.25$.
(2) *The function* $t\,h(t)$ *is increasing for* $t > 0$.
(3) *The function* $H(t) = 1/h(t)$ *is convex in* $t$. *The odd function* $H'(t)$ *maps* $\mathbb{R}$ *homeomorphically onto the interval* $(-\pi/4, \pi/4)$.

## 3. Proof of Theorems 1.4 and 1.5

The proof will proceed along the similar line to that of Beardon and Pommerenke [5]. In order to prove Theorem 1.4, we introduce the technical quantity $\sigma'_\Omega(z) = \sup \lambda_{a,b}(z)$, where the supremum is taken over all the pairs of points $a, b$ in $\partial\Omega$ such that $\delta_\Omega(z) = |z-a|$ and that $b$ minimizes $|\log|(b-a)/(z-a)||$. By definition, we have $\sigma'_\Omega \le \sigma_\Omega$.

We can now derive Theorem 1.4 with $C = 2C_0 + \pi/2 \approx 10.3246$ from the following lemma, where $C_0$ is given by (2.6).

**Lemma 3.1.** *The inequality* $\rho_\Omega(z) \le C\sigma'_\Omega(z)$ *holds for all* $z \in \Omega$, *where* $C$ *is an absolute constant with* $C \le 2C_0 + \pi/2$.

*Proof.* Fix $z \in \Omega$. We take a pair of boundary points $a, b$ with $|z - a| = \delta_\Omega(z)$ so that $b$ minimizes $|\log|(b-a)/(z-a)||$ and that $\sigma'(z) = \lambda_{a,b}(z)$. By (2.5), we have

$$\sigma'_\Omega(z)\delta_\Omega(z) = |z-a|\lambda_{a,b}(z) \ge \frac{1}{2(m + C_0)},$$

where $m = |\log|(b-a)/(z-a)||$. Now we claim that the inequality $\rho_\Omega(z)\delta_\Omega(z) \le \min\{1, \pi/4m\}$ holds. Set $\delta = \delta_\Omega(z)$. First, $A = \{w \in \mathbb{C}; \delta e^{-m} < |w - z| < \delta e^m\} \subset \Omega$ implies $\rho_\Omega(z) \le \rho_A(z) = \pi/4\delta m$ (see, for example, [5]). Combining this with the well-known inequality $\delta_\Omega(z)\rho_\Omega(z) \le 1$, we get the claim. Thus, we obtain

$$\frac{\rho_\Omega(z)}{\sigma'_\Omega(z)} \le 2(m + C_0)\min\left\{1, \frac{\pi}{4m}\right\} = \min\left\{2(m + C_0), \frac{\pi}{2} + \frac{\pi C_0}{2m}\right\} \le \frac{\pi}{2} + 2C_0.$$

□

**Remark 3.2.** In the following way, we can slightly improve the above estimate. In the above, we may assume that $|(b-a)/(z-a)| \ge 1$, and thus $|b'| = e^m$, where $b' = (b-a)/(z-a)$. By Lemma 2.8, we obtain $\sigma_\Omega(z)\delta_\Omega(z) = |z-a|\lambda_{a,b}(z) = |b'|\lambda(b') \ge |b'|\lambda(-|b'|) = h(m)$. Under the hypothesis that the function $t\,h(t)$ is increasing for $t > 0$, see Conjecture 2.12 (2), we have

$$\frac{\rho_\Omega(z)}{\sigma'_\Omega(z)} \le \frac{\min\{1, \pi/4m\}}{h(m)} \le \frac{1}{h(\pi/4)} \approx 9.0157.$$

We make further remarks on this result. Let $C_2$ be the possible smallest constant for which the assertion in Theorem 1.4 holds for all $\Omega$. The above proof produces the estimate $C_2 \le 2C_0 + \pi/2 \approx 10.3246$. (As we stated in the above remark, this estimate could be improved to $C_2 \le 1/h(\pi/4) \approx 9.0157$.) On the other hand, we have the estimate $C_2 \ge C_0 \approx 4.37688$ at the moment. Indeed, we consider the case when $\Omega = \mathbb{D}$ with $z = 0$. In order to compute the value of $\sigma_{\mathbb{D}}(0)$, we take a pair of points $a, b \in \partial\mathbb{D}$. We may assume that $a = 1$. Then, by Lemma 2.9 (2), we see that $\lambda_{1,b}(0) \le \lambda_{1,-1}(0)$. Hence, we have $\sigma_{\mathbb{D}}(0) = \lambda_{1,-1}(0) = \lambda(1/2)/2 = 2\lambda(-1)$ (use (2.1) and (2.7)). Finally, we obtain $C_2 \ge \rho_{\mathbb{D}}(0)/\sigma_{\mathbb{D}}(0) = 1/2\lambda(-1) = C_0$.

The reader might think that we could take all the pairs of points $a, b$ from the complement $\mathbb{C} \setminus \Omega$ of $\Omega$ in the definition of $\sigma_\Omega(z)$. However, it turns out that this would make no difference. Indeed, the relation

$$\sigma_\Omega(z) = \sup_{a,b \in \partial\Omega} \lambda_{a,b}(z) = \sup_{a,b \in \mathbb{C}\setminus\Omega} \lambda_{a,b}(z)$$

holds. To show this, we take an arbitrary pair of points $a, b$ from $\mathbb{C} \setminus \Omega$. We may assume that $|a| \le |b|$. First, we take a point, say $a'$, from the set $[z, a] \cap \partial\Omega$, where $[z, a]$ denotes the closed line segment connecting $z$ with $a$. By Lemma 2.9 (1), we have $\lambda_{a,b}(z) \le \lambda_{a',b}(z)$. For simplicity, we further assume that $a' = 1$ and $z = 0$. Let $I = \{|b|e^{i\theta}; |\arg b| \le |\theta| \le \pi\}$ and consider the following two cases:

1. When $I \cap \partial\Omega \ne \emptyset$, we take a point, say $b'$, from $I \cap \partial\Omega$. By Lemma 2.9 (2), we have $\lambda_{a',b}(z) \le \lambda_{a',b'}(z)$.
2. When $I \cap \partial\Omega = \emptyset$, we take a point, say $b'$, from $[-|b|, 0] \cap \partial\Omega$. By assertions (2) and (0) in Lemma 2.9, we also have $\lambda_{a',b}(z) \le \lambda_{a',-|b|}(z) \le \lambda_{a',b'}(z)$.

In any case, we found a pair of points $a', b' \in \partial\Omega$ such that $\lambda_{a,b}(z) \le \lambda_{a',b}(z) \le \lambda_{a',b'}(z)$. Hence, we conclude the above relation.

We now present a proof for Theorem 1.5.

*Proof of Theorem 1.5.* Let $E$ be a closed set with $\partial\Omega \subset E \subset \mathbb{C} \setminus \Omega$. For $a \in E$ and $z \in \Omega$, we set $t = \log|z - a|$ and $m = m_E(a, t)$. Then, there exists a point $b \in \partial\Omega (\subset E)$ such that $|\log|b - a| - t| = m$. Lemma 2.8 now yields that

$$\rho_\Omega(z) \ge \lambda_{a,b}(z) = \frac{1}{|b-a|}\lambda\left(\frac{z-a}{b-a}\right) \ge \frac{1}{|b-a|}\lambda\left(-\left|\frac{z-a}{b-a}\right|\right) = \frac{h(m)}{|z-a|}.$$

Thus, the left-hand side has been shown. The right-hand side can be shown in the same way as above by using the fact that the annulus $\{w; e^{-m}|b - a| < |w - a| < e^m|b - a|\}$ is contained in $\Omega$. $\square$

Finally we observe the validity of an assertion similar to Theorem 1.4. In the rest of this section, we allow domains to be subdomains of the Riemann sphere $\widehat{\mathbb{C}}$. One may ask if the hyperbolic distance $d_\Omega(z_1, z_2)$ between two points $z_1, z_2$ in $\Omega$ is comparable with the similar quantity

$$\varepsilon'_\Omega(z_1, z_2) = \sup_{a,b \in \partial\Omega} d_{\mathbb{C}\setminus\{a,b\}}(z_1, z_2),$$

or

$$\varepsilon_\Omega(z_1, z_2) = \sup_{a,b,c \in \widehat{\mathbb{C}}\setminus\Omega} d_{\widehat{\mathbb{C}}\setminus\{a,b,c\}}(z_1, z_2).$$

We write $d_{a,b,c}(z_1,z_2)=d_{\widehat{\mathbb{C}}\setminus\{a,b,c\}}(z_1,z_2)$ for simplicity. By (1.1), we have the inequalities $\varepsilon'_\Omega(z_1,z_2)\le\varepsilon_\Omega(z_1,z_2)\le d_\Omega(z_1,z_2)$. However, the reverse inequality $d_\Omega(z_1,z_2)\le C\varepsilon_\Omega(z_1,z_2)$ does not hold in general. We show it by simple examples.

Let $\Omega_t$ be the union of the two disks $\Delta=\{|z+1|<1/2\},\Delta'=\{|z-1|<1/2\}$ and the narrow canal $\{z=x+iy;|x|<1,|y|<t\}$ for $t\in(0,1/2)$. It is obvious that $d_{\Omega_t}(-1,1)\to\infty$ as $t\to 0$. On the other hand, we can show that there is an absolute constant $K$ such that $\varepsilon_{\Omega_t}(-1,1)\le K$ for every $t\in(0,1/2)$. Hence, there is no absolute constants $C$ such that $d_\Omega\le C\varepsilon_\Omega$ holds for all $\Omega$. Moreover, by joining a countably many disks by canals whose widths tend to 0, we can construct a plane domain $\Omega$ for which no constants $C$ satisfy $d_\Omega(z_1,z_2)\le C\varepsilon_\Omega(z_1,z_2)$ for all $z_1,z_2\in\Omega$.

The above claim is shown as in the following. Set $L(z)=(z-1)/(z+1)$ and $\Omega'_t=L(\Omega_t)$. One can check that $\{|w|<1/4\}\subset L(\Delta')$ and $\{|w|>4\}\subset L(\Delta)$. Let $D_j=\{w;|w|\in[0,1/4)\cup(4,\infty]\}\cup W_j$ for $j=1,2,3,4$, where $W_j=\{re^{i\theta};0<r<\infty,(j-1)\pi/2<\theta<j\pi/2\}$. Then we put $K=d_{D_1}(0,\infty)<\infty$.

By conformal invariance of the hyperbolic distance, we note that $\varepsilon_{\Omega_t}(-1,1)=\varepsilon_{\Omega'_t}(0,\infty)$. For every triple $a,b,c\in\widehat{\mathbb{C}}\setminus\Omega'_t$, we can choose at least one $j\in\{1,2,3,4\}$ such that $\{a,b,c\}\cap W_j=\emptyset$. Since $D_j\subset\widehat{\mathbb{C}}\setminus\{a,b,c\}$, by (1.1),

$$d_{a,b,c}(0,\infty)\le d_{D_j}(0,\infty)=K.$$

Thus, we have shown that $\varepsilon_{\Omega_t}(-1,1)\le K$.

## 4. Properties of $m_E$ and $m_E^*$

We see several fundamental properties of the quantities $m_E(a,t)$ and $m_E^*(t)$ for a closed set $E$ in $\mathbb{C}$ with $\#E\ge 2$. First, we note that $m_E(a,t)$ can be written as the (pointwise) infimum of the 1-Lipschitz functions $t\mapsto|t-\log|b-a||$ for a fixed $a$, where $b$ runs over $E\setminus\{a\}$. Here a real-valued function $f$ defined in $\mathbb{R}$ is called $M$-Lipschitz if $|f(s)-f(t)|\le M|s-t|$ holds for all $s,t\in\mathbb{R}$. We now recall the following elementary result.

**Lemma 4.1.** *Let $\mathcal{F}$ be a non-empty collection of $M$-Lipschitz functions on $\mathbb{R}$, where $M$ is a positive constant. Then the function $F$ defined by $F(t)=\sup\{f(t);f\in\mathcal{F}\}$ is $M$-Lipschitz unless $F\equiv+\infty$.*

*Proof.* Suppose that $F$ is not identically $+\infty$. Let $s$ be an arbitrary point in $\mathbb{R}$ for which $F(s)<+\infty$. For each $f\in\mathcal{F}$, then $f(t)\le f(s)+M|s-t|\le F(s)+M|s-t|$ holds. Hence, $F(t)\le F(s)+M|s-t|(<+\infty)$ follows for every $t\in\mathbb{R}$. Exchanging the roles of $s$ and $t$ we conclude that $|F(s)-F(t)|\le M|s-t|$ holds, namely, $F$ is $M$-Lipschitz. □

We apply the above lemma to get the following result.

**Proposition 4.2.** *The quantity $m_E(a,t)$ is a 1-Lipschitz function in $t$ for a fixed $a\in E$. The function $m_E^*(t)$ is also 1-Lipschitz unless it is identically $+\infty$.*

Here is a characterization of the property $m_E^*(t)\equiv+\infty$ for a closed set $E$.

**Proposition 4.3.** *Let $E$ be a closed set in $\mathbb{C}$ with $\#E\ge 2$. The function $m_E^*(t)$ is identically $+\infty$ if and only if there exists a sequence of annuli $A_n=\{z\in\mathbb{C};r_n<|z-a_n|<R_n\}$ in $\mathbb{C}\setminus E$ such that $R_n\to\infty$ and $r_n\to 0$ as $n\to\infty$ and that $a_n\in E$.*

*Proof.* First assume that $m_E^*(t) \equiv +\infty$. Then there exists a sequence $a_n$ in $E$ such that $m_n := m_E(a_n, 0) \to +\infty$ as $n \to \infty$. The annulus $A_n = \{z; e^{-m_n} < |z - a_n| < e^{m_n}\}$ lies in $\mathbb{C} \setminus E$, and therefore, the latter assertion follows. The converse can be handled similarly. □

We note that the last condition $a_n \in E$ in the above proposition can be relaxed to that $\{z; |z - a_n| \le r_n\} \cap E \neq \emptyset$ by an easy argument.

From this proposition, we can derive that $E$ must be unbounded when $m_E^*(t) \equiv +\infty$. On the other hand, one easily sees that $m_E^*(t) \to +\infty$ as $t \to +\infty$ when $E$ is bounded. Therefore, it may be natural to restrict $m_E^*$ on a left half line in this case.

A closed set $E$ in $\mathbb{C}$ with $\#E \ge 2$ is said to be uniformly perfect if there exists a constant $c$ with $0 < c \le 1$ such that for $a \in E$ and $0 < r < \operatorname{diam} E$ there is a point $b \in E$ with $cr \le |b - a| \le r$ (cf. [20]). This condition can be restated in our terms: $m_E(a,t) \le -(\log c)/2$ for $t < \log \operatorname{diam} E + (\log c)/2$. Since $m_E^*(t)$ is 1-Lipschitz, this implies that $m_E^*(t) \le -\log c$ for $t < \log \operatorname{diam} E$. Based on this observation, we get the following result.

**Proposition 4.4.** *Let $E$ be a closed set in $\mathbb{C}$ with $\#E \ge 2$. Then $E$ is uniformly perfect if and only if $m_E^*(t)$ is bounded in $t < \log \operatorname{diam} E$.*

More precisely, if $E$ is uniformly perfect with constant $c$ then $m_E^*(t) \le \log(1/c)$ for $t < \log \operatorname{diam} E$. Conversely, if $m_E^*(t) \le m$ for $t < \log \operatorname{diam} E$, then $E$ is uniformly perfect with constant $c = e^{-2m}$. For a survey on uniformly perfect sets, see also [22].

We next state a relation between our $m_{\partial\Omega}(a,t)$ and the quantity $\beta_\Omega(z)$ (see (1.3)) introduced in [5]. By definition, we observe

$$\beta_\Omega(z) = \min_{a \in \partial\Omega;\, \delta_\Omega(z) = |z-a|} m_{\partial\Omega}(a, \log \delta_\Omega(z)).$$

In particular, setting $t = \log \delta_\Omega(z)$, we can immediately derive the following inequality from Theorem 1.5:

$$h(\beta_\Omega(z)) \le \delta_\Omega(z)\rho_\Omega(z) \le \frac{\pi}{4\beta_\Omega(z)}.$$

Together with the inequality $\delta_\Omega(z)\rho_\Omega(z) \le 1$, we obtain also the upper estimate $\delta_\Omega(z)\rho_\Omega(z) \le \min\{1, \pi/4\beta_\Omega(z)\} \le (C_0 + \pi/4)/(\beta_\Omega(z) + C_0)$. Since $h(t) \ge 1/(2|t| + 2C_0)$, finally we obtain

$$\frac{1}{2(\beta_\Omega(z) + C_0)} \le \delta_\Omega(z)\rho_\Omega(z) \le \frac{C}{2(\beta_\Omega(z) + C_0)}, \tag{4.5}$$

where $C = 2C_0 + \pi/2 \approx 10.3246$. This inequality is essentially same as in [5].

We end this section with a comment on the relation between $m_{\partial\Omega}(a,t)$ and $m_{\mathbb{C}\setminus\Omega}(a,t)$. More generally, we can see the following.

**Proposition 4.6.** *Let $E$ be a closed set satisfying $\partial\Omega \subset E \subset \mathbb{C} \setminus \Omega$ for a hyperbolic open set $\Omega \subset \mathbb{C}$. Then $m_E(a,t) = m_{\partial\Omega}(a,t)$ holds for every $a \in \partial\Omega$ and for $t = \log|z - a|$ with some $z \in \Omega$.*

*Proof.* Let $a \in \partial\Omega$ and $t = \log|z_0 - a|$ for some $z_0 \in \Omega$. Set $m = m_E(a,t)$. By definition of $m$, there exists a point $b \in E$ with $m = |t - \log|b - a||$. Note that $m \le m_{\partial\Omega}(a,t)$. We first assume that $m > 0$. Since $A = \{w; e^{-m}|b - a| < |w - a| < e^m|b - a|\}$ does not intersect $E$, the annulus $A$ is contained in $\mathbb{C} \setminus \partial\Omega$, and therefore, $A \subset \Omega$ or $A \cap \Omega = \emptyset$. On the other hand, since the point $z_0 \in \Omega$ lies in $\partial A$, the annulus $A$ intersects $\Omega$. Hence, $A$ must be entirely contained in $\Omega$. Therefore we conclude that $b \in \partial\Omega$. This implies $m_{\partial\Omega}(a,t) \le |t - \log|b-a|| = m$, and hence, $m = m_{\partial\Omega}(a,t)$. Next we consider the case when $m = 0$. Then the circle $\{w; |w-a| = |b-a|\}$ contains the point $z_0$ in $\Omega$ and the point $b$ in $\mathbb{C} \setminus \Omega$. Therefore, there is a point $c \in \partial\Omega$ on that circle. We now conclude that $m_{\partial\Omega}(a,t) = 0 = m$. □

## 5. Estimates of hyperbolic distance

In this section, we investigate the hyperbolic distance by using results given in the preceding sections.

The hyperbolic distance in the disk or the half-plane is well known. For example, one can compute the hyperbolic distance between $z_1$ and $z_2$ in the right half-plane $\mathbb{H}$ by

$$d_{\mathbb{H}}(z_1, z_2) = \frac{1}{2}\log\frac{|z_1 + \bar{z}_2| + |z_1 - z_2|}{|z_1 + \bar{z}_2| - |z_1 - z_2|}.$$

By using a conformal mapping, one may compute the hyperbolic distance for some simply connected domains as well. Since the universal covering is explicitly given for the annulus $1 < |z| < R$, $(1 < R \le \infty)$, the hyperbolic distance of some ring domains can also be given (see, for instance, [11]). However, very little is known about exact values of the hyperbolic distance for domains of the other type.

As we have seen several times, the other extreme case $\mathbb{C} \setminus \{a, b\}$ is very important. We write $d_{a,b}(z_1, z_2) = d_{\mathbb{C}\setminus\{a,b\}}(z_1, z_2)$. The following is one of the known cases when the hyperbolic distance can be computed exactly. For another case, see also [2].

**Lemma 5.1** (cf. [21, Lemma 3.10]). *The hyperbolic distance between $-x$ and $-y$ in $\mathbb{C} \setminus \{0,1\}$ is given by $d_{0,1}(-x,-y) = |\Phi(x) - \Phi(y)|$ for $x, y > 0$, where the function $\Phi : \mathbb{R} \to \mathbb{R}$ is given by*

$$\Phi(x) = \frac{1}{2}\log\frac{\mathcal{K}(x/(1+x))}{\mathcal{K}(1/(1+x))}. \tag{5.2}$$

*Proof.* For convenience, we reproduce the proof. Let $p$ be the elliptic modular function which maps the hyperbolic triangle $\Delta = \{z; 0 < \operatorname{Re} z < 1, |z - 1/2| > 1/2, \operatorname{Im} z > 0\}$ conformally onto the upper half plane in such a way that the vertices $1, \infty, 0$ of the triangle correspond to $\infty, 0, 1$, respectively. It is known that the inverse function of $p|_\Delta$ is given by $p^{-1}(z) = i\mathcal{K}(1-z)/\mathcal{K}(z)$ (see [18, pp. 318,319]). Note that $\mathcal{K}(1-z)$ is analytically continued through the lower half plane while so is $K(z)$ through the upper half plane. In particular, we should apply (2.4) for $\mathcal{K}(1-z)$ by replacing the plus sign by the minus. By (2.3) and (2.4) with the above convention of the sign, we obtain $p^{-1}(-x) = 1 + iu(x)$ for $x > 0$, where $u(x) = \mathcal{K}(1/(1+x))/\mathcal{K}(x/(1+x))$. Let $0 < x < y$. Since $\lambda(p(z))|p'(z)| = 1/2\operatorname{Im} z$ for $\operatorname{Im} z > 0$, the relation

$$d_{0,1}(-x,-y) = \int_{-y}^{-x} \lambda(-\xi)d\xi = \int_{u(y)}^{u(x)} \frac{ds}{2s} = \frac{1}{2}\log\left(\frac{u(x)}{u(y)}\right) = \Phi(y) - \Phi(x)$$

is obtained. $\square$

It may be useful to note that the function $u(x) = e^{-2\Phi(x)}$ can be expressed in terms of the modulus of the Grötzsch ring. Indeed, $u(x) = (2/\pi)\mu(\sqrt{x/(1+x)})$, where $\mu(r) = (\pi/2)\mathcal{K}(1-r^2)/\mathcal{K}(r^2)$ denotes the modulus of the Grötzsch ring $\mathbb{D}\setminus[0,r]$. Moreover, the quantity $\pi u(x)$ is exactly same as the modulus of the Teichmüller ring $\mathbb{C}\setminus([-x,0]\cup[1,+\infty))$ (see, for instance, [14, II §1]). Note also the relation $u(1/x) = 1/u(x)$ for $x > 0$. The quantity $\Phi(x)$ can be regarded as the signed distance function from $-1$ to $-x$ in $\mathbb{C}\setminus\{0,1\}$. We remark that the function $\Phi(x)$ plays an important role in Schottky's theorem and in distortion theorems of quasiconformal mappings (see [10] or [17]).

Hempel gave nice estimates for the quantity $u(x)$ in [10, Lemma (ii)]:

$$\frac{1}{\pi}\log\left(\frac{16}{x}\right) < u(x) = e^{-2\Phi(x)} \le \frac{1}{\pi}\log\left(\frac{e^\pi}{x}\right), \quad 0 < x \le 1. \tag{5.3}$$

In order to compare with $\Phi(x)$, we consider the function

$$\Psi_K(x) = \frac{1}{2}\log\left(1 + \frac{\log x}{K}\right)$$

for $x \ge 1$, where $K > 0$ is a given constant. Then we have the following.

**Lemma 5.4.** *The function $\Phi(x)$ is (strictly) increasing for $x > 0$ and the inequalities $\Psi_{C_0} \le \Phi \le \Psi_\pi$ hold on $[1,\infty)$, namely,*

$$\frac{1}{2}\log\left(1 + \frac{\log x}{C_0}\right) \le \Phi(x) \le \frac{1}{2}\log\left(1 + \frac{\log x}{\pi}\right), \quad x \ge 1,$$

*where $C_0 \approx 4.37688$ is the constant given by* (2.6).

*Proof.* The monotonicity of $\Phi(x) = -(1/2)\log u(x)$ is obvious by the geometric meaning of $u(x)$. The inequality $\Phi \le \Psi_\pi$ is a direct consequence of Hempel's estimate (5.3). We now prove the other part. The inequality $\Psi_{C_0}(x) \le \Phi(x),\ x \ge 1$, is equivalent to

$$1 + \frac{1}{C_0}\log\left(\frac{1-t}{t}\right) \le \frac{\mathcal{K}(1-t)}{\mathcal{K}(t)}, \quad 0 < t \le \frac{1}{2},$$

where we have used the transformation $t = 1/(1+x)$. For convenience, we use the notation $\mu(r) = (\pi/2)\mathcal{K}(1-r^2)/\mathcal{K}(r^2),\ 0 < r < 1$. Then the above inequality can be expressed in the form

$$g(r) := \mu(r) + \frac{\pi}{C_0}\log(r/r') - \frac{\pi}{2} \ge 0, \quad 0 < r \le \frac{1}{\sqrt{2}},$$

where we write $r' = \sqrt{1-r^2}$. In the same way as in the proof of Theorem 5.13 (3) in [4], we compute

$$g'(r) = \frac{\pi}{C_0 r r'^2 \mathcal{K}(r^2)^2}\left(\mathcal{K}(r^2)^2 - \frac{\pi C_0}{4}\right).$$

(Note that the definition of $\mathcal{K}$ is slightly different from the one in [4].) Since $\mathcal{K}(t)$ is strictly increasing and $\mathcal{K}(1/2)^2 = \pi C_0/4$ (see (2.6)), we see that $g'(r) < 0$ in $0 < r < 1/\sqrt{2}$. Hence, we have $g(r) > g(1/\sqrt{2}) = \mu(1/\sqrt{2}) - \pi/2 = 0$ for $0 < r < 1/\sqrt{2}$. $\square$

For later use, we now define the functions $\varphi$ and $\psi_K$, $K \in (0,\infty)$, on $\mathbb{R}$ by $\varphi(t) = 2\Phi(e^{t/2})$, namely,

$$\varphi(t) = \log\left(\frac{\mathcal{K}(1/(1+e^{-t/2}))}{\mathcal{K}(1/(1+e^{t/2}))}\right), \tag{5.5}$$

and by $\psi_K(t) = 2\Psi_K(e^{t/2})$ for $t \geq 0$ and $\psi_K(t) = -2\Psi_K(e^{-t/2})$ for $t \leq 0$, namely,

$$\psi_K(t) = \begin{cases} \log\left(1 + \frac{t}{2K}\right), & t \geq 0, \\ -\log\left(1 - \frac{t}{2K}\right), & t \leq 0. \end{cases} \tag{5.6}$$

Note that $\varphi$ and $\psi_K$ are odd functions. We immediately obtain the following.

**Corollary 5.7.** *The function $\varphi(t)$ is increasing for $t \in \mathbb{R}$ and the inequalities $\psi_{C_0}(t) \leq \varphi(t) \leq \psi_\pi(t)$ hold for $t \geq 0$, where $C_0$ is the constant given by* (2.6).

From (5.3) we can also deduce the estimate

$$\varphi(t) > \log\left(c_0 + \frac{t}{2\pi}\right), \quad t \geq 0,$$

where $c_0 = (\log 16)/\pi \approx 0.88254$. The bounds $\psi_K(t)$ have the following advantage.

**Lemma 5.8.** *The function $\psi = \psi_K$ defined by* (5.6) *for some $K > 0$ satisfies the following properties:*

(i) *$\psi(0) = 0$ and the function $\psi(t)/t$ is decreasing in $t > 0$,*
(ii) *$\psi$ is subadditive on $[0,\infty)$, namely, $\psi(t_1+t_2) \leq \psi(t_1)+\psi(t_2)$ holds for $t_1, t_2 \in [0,\infty)$.*

*Proof.* It is routine to see property (i). Property (ii) is known to follow from (i) [4, Lemma 1.24] (or can be shown directly). □

In view of Lemma 5.8, the following statement is plausible.

**Conjecture 5.9.** *The function $\varphi(t)$ defined by* (5.5) *has decreasing quotient $\varphi(t)/t$, and hence, it is subadditive on $0 \leq t < \infty$.*

When we are given an estimate for the hyperbolic density, the following elementary method can be used. Note that we would lose nothing as far as the hyperbolic distance is concerned if we assume a domain to be in the punctured plane $\mathbb{C}^*$.

**Lemma 5.10.** *Let $\Omega$ be a proper subdomain of $\mathbb{C}^*$. Suppose that a non-negative measurable function $\eta(t)$, $t \in \mathbb{R}$, is given in such a way that $|z|\rho_\Omega(z) \geq \eta(\log|z|)$ for all $z \in \Omega$. Then*

$$d_\Omega(z_1, z_2) \geq \int_{t_1}^{t_2} \eta(t)|dt| \tag{5.11}$$

*holds for $z_1, z_2 \in \Omega$ with $t_j = \log|z_j|$, $j = 1, 2$.*

*Proof.* Let $\gamma$ be an arbitrary rectifiable curve joining $z_1$ and $z_2$ in $\Omega$. Then

$$\int_\gamma \rho_\Omega(z)|dz| \geq \int_\gamma \frac{\eta(\log|z|)|dz|}{|z|} \geq \int_{|z_1|}^{|z_2|} \frac{\eta(\log r)|dr|}{r} = \int_{t_1}^{t_2} \eta(t)|dt|.$$

By definition of the hyperbolic distance, the required inequality follows. □

As an application of the above lemma, we derive the following result which will be utilized in the proof of Littlewood-type theorems in Section 6.

**Theorem 5.12.** *Let $\Omega$ be a subdomain of $\mathbb{C}^*$ and let $a_n$ be an infinite sequence of points in $\mathbb{C} \setminus \Omega$ satisfying the following properties:*

(i) $0 = |a_0| < |a_1| \le |a_2| \le \dots,$

(ii) $|a_{n+1}| \le e^c |a_n|$ *for* $n = 1, 2, \dots,$ *where* $c > 0$ *is a constant, and*

(iii) $a_n \to \infty$ *as* $n \to \infty$.

*Then*

$$d_\Omega(z_1, z_2) \ge h(c/2)\big(\log|z_2| - \log|z_1|\big)$$

*for* $z_1, z_2 \in \Omega$ *with* $|z_2| \ge |z_1| \ge e^{-c/2}|a_1|$, *where* $h$ *is the function given by* (2.10).

We remark that the order of $|z_1|$ and $|z_2|$ in the above inequality is strong enough. Observe that $d_{\mathbb{H}}(x_1, x_2) = (1/2)(\log x_2 - \log x_1)$ for $0 < x_1 < x_2$, where $\mathbb{H}$ denotes the right half plane.

*Proof.* By the monotonicity of the hyperbolic distance (1.1), we may assume that $\Omega = \mathbb{C} \backslash E$, where $E = \{a_0, a_1, \dots\}$. Then it is easy to see that $m_E(0, t) \le c/2$ if $t \ge \log|a_1| - c/2$. Theorem 1.5 yields the estimate $|z|\rho_\Omega(z) \ge h(c/2)$ for $z \in \Omega$ with $|z| \ge e^{-c/2}|a_1|$. We now apply Lemma 5.10 to deduce the conclusion. □

The most typical case is when $E = \mathbb{C}^* \setminus \Omega$ consists of the geometric series $r^n,\ n \in \mathbb{Z}$, for some $r > 1$. In this case, Hayman (see Lemma 3 in p. 169 of [8]) has given a surprisingly accurate estimate if in addition $\log r \ge \pi^2/2$:

$$\frac{1}{2}\log\left(\frac{4\log r}{\pi^2}\right) < d_\Omega(-1, -r) < \frac{1}{2}\log\left(\frac{4\log r}{\pi^2}\right) + \frac{7\pi^2}{12\log r}.$$

What can we say when the sequence is more scarce? For instance, we consider the situation in Theorem 5.12 with condition (ii) being replaced by

$$1 < |a_1| \quad \text{and} \quad |a_{n+1}| \le e^c |a_n|^\alpha, \quad n = 1, 2, \dots, \tag{5.13}$$

where $\alpha > 1$ and $c \in \mathbb{R}$ are constants.

In this case, we can also establish an upper estimate for $m_E(0, t)$, where $E = \{a_0, a_1, \dots\}$. Indeed, we can show the inequality

$$m_E(0, t) \le \frac{(\alpha - 1)t + c}{\alpha + 1} \tag{5.14}$$

for $t \ge \log|a_1|$. We set $t_n = \log|a_n|$ for $n = 1, 2, \dots$. The hypothesis means that

$$0 < t_n \le t_{n+1} \le \alpha t_n + c \tag{5.15}$$

for $n = 1, 2, \dots$. Choosing $n$ so that $t_n \le t \le t_{n+1}$, we see that $m := m_E(0, t) = \min\{t - t_n, t_{n+1} - t\}$. Since $m = t - t_n$ when $t \le (t_n + t_{n+1})/2$, we have

$$m - \frac{\alpha - 1}{\alpha + 1}t = \frac{2t}{\alpha + 1} - t_n \le \frac{t_n + t_{n+1}}{\alpha + 1} - t_n = \frac{t_{n+1} - \alpha t_n}{\alpha + 1}.$$

We now use (5.15) to get (5.14). We can handle similarly with the case when $t \ge (t_n + t_{n+1})/2$. Thus, (5.14) has been shown.

At this stage, one could apply Theorem 1.5 and Lemma 5.10 to obtain a lower bound for the hyperbolic distance of the domain $\Omega = \mathbb{C}\setminus E$. Unfortunately, however, the resulting estimate would not be very sharp because the effect of oscillation of $m_E(0,t)$ could not be neglected (observe that $m_E(0,t_j) = 0$ by definition). We now prove Theorem 1.7 for the possible application to this case. As a preparation, we show the following simple estimate.

**Lemma 5.16.** *For the twice punctured plane $\mathbb{C}\setminus\{0,1\}$, the inequality*

$$d_{0,1}(z_1,z_2) \ge d_{0,1}(-|z_1|,-|z_2|) = \Phi(|z_2|) - \Phi(|z_1|)$$

*holds for $z_1, z_2 \in \mathbb{C}\setminus\{0,1\}$ with $|z_1| \le |z_2|$.*

*Proof.* Let $\gamma$ be an arbitrary rectifiable curve joining $z_1$ and $z_2$ in $\Omega$. Then, by Lemma 2.8, we have

$$\int_\gamma \lambda(z)|dz| \ge \int_\gamma \lambda(-|z|)|dz| \ge \int_{|z_1|}^{|z_2|} \lambda(-x)dx = d_{0,1}(-|z_1|,-|z_2|).$$

Since $\gamma$ is arbitrary, we obtain the required inequality. The last relation follows from Lemma 5.1. □

Note that the above lemma equally holds in the general case when $\Omega$ is a circularly symmetric domain. We, however, do not need this level of generality in the sequel.

*Proof of Theorem 1.7.* Let $a_n$, $0 \le n < N$, be a sequence as in the statement of the theorem and set $t_n = \log|a_n|$ and $s_n = (t_{n-1}+t_n)/2$. Note that $t_0 = -\infty$ and $t_N = \infty$. By the monotonicity property (1.1), it is enough to prove the theorem in the case when $\Omega = \mathbb{C}\setminus E$, where $E = \{a_0, a_1, \dots\}$. Let $z_1$ and $z_2$ be points in $\Omega$ with $|z_1| < |z_2|$ and take integers $k$ and $l$ as in the theorem. Let $\gamma$ be a hyperbolic geodesic joining $z_1$ and $z_2$ in $\Omega$ whose hyperbolic length equals $d_\Omega(z_1,z_2)$ and let $w_n$ be the last point on $\gamma$ satisfying $\log|w_n| = s_n$ when we go along $\gamma$ from $z_1$ to $z_2$ for $k < n \le l$. Set $w_k = z_1$ and $w_{l+1} = z_2$. Then we see

$$d_\Omega(z_1,z_2) = \sum_{n=k}^{l} d_\Omega(w_n, w_{n+1}).$$

Since $\Omega \subset \mathbb{C}\setminus\{0,a_n\}$, by (1.1) and Lemma 5.16, we further obtain

$$\begin{aligned} d_\Omega(w_n,w_{n+1}) &\ge d_{0,a_n}(w_n,w_{n+1}) = d_{0,1}(w_n/a_n, w_{n+1}/a_n) \\ &\ge \Phi(|w_{n+1}/a_n|) - \Phi(|w_n/a_n|) = \Phi(|w_{n+1}/a_j|) + \Phi(|a_n/w_n|) \\ &= \frac{1}{2}\left(\varphi(t_{n+1}-t_n) + \varphi(t_n - t_{n-1})\right) \end{aligned}$$

for $n = k+1, \dots, l-1$. In the similar way, we also obtain

$$\begin{aligned} d_\Omega(w_k,w_{k+1}) &\ge \frac{1}{2}(\varphi(t_{k+1}-t_k) + \varphi(t_k - \log|z_1|)) \quad \text{and} \\ d_\Omega(w_l,w_{l+1}) &\ge \frac{1}{2}(\varphi(\log|z_2| - t_l) + \varphi(t_l - t_{l-1})). \end{aligned}$$

Summing up these inequalities, we finally get

$$d_\Omega(z_1,z_2)\ge \frac{1}{2}\varphi(t_k-\log|z_1|)+\sum_{n=k+1}^{l}\varphi(t_n-t_{n-1})+\frac{1}{2}\varphi(\log|z_2|-t_l).$$

Next, in order to show the inequality in (1.9), we restrict ourselves to the particular case when $a_n>0$ for all $1\le n<N$. Thus $E=\{e^{t_n};0\le n<N\}$ and $\Omega=\mathbb{C}\setminus E=D$. Since the nearest point of $E$ to $-x$ is the origin, we have $\sigma_D'(-x)=\lambda_{0,a_n}(-x)$ for $r_n<x<r_{n+1}$, where $\sigma_D'$ is the quantity introduced in Section 3 and $r_n=e^{s_n}$. Therefore, we obtain

$$\int_{x_1}^{x_2}\sigma_D'(-x)dx=d_{0,a_n}(-x_1,-x_2)=d_{0,1}(-x_1/a_n,-x_2/a_n)=\Phi(x_2/a_n)-\Phi(x_1/a_n)$$

for $r_n\le x_1<x_2\le r_{n+1}$. Hence, in the same way as the first part, we observe

$$\int_{|z_1|}^{|z_2|}\sigma_D'(-x)dx=\frac{1}{2}\varphi(t_k-\log|z_1|)+\sum_{n=k+1}^{l}\varphi(t_n-t_{n-1})+\frac{1}{2}\varphi(\log|z_2|-t_l).$$

Now (1.9) follows from Lemma 3.1. □

As a slightly different approach to a result similar to Theorem 5.12, we shall use Theorem 1.7. We make the same hypothesis as in Theorem 5.12 and we take integers $k$ and $l$ as above for a given pair of points $z_1$ and $z_2$. Since $\omega(t):=\psi_{C_0}(t)/t$ is decreasing in $t>0$ and $t_n-t_{n-1}\le c$, we observe that $\varphi(t_n-t_{n-1})\ge\psi_{C_0}(t_n-t_{n-1})\ge\omega(c)(t_n-t_{n-1})$ by Corollary 5.7. Therefore, by (1.8), we obtain

$$\begin{aligned}d_\Omega(z_1,z_2)\ge&\frac{1}{2}\varphi\big(t_k-\log|z_1|\big)+\omega(c)(t_l-t_k)+\frac{1}{2}\varphi\big(\log|z_2|-t_l\big)\\ =&\omega(c)(\log|z_2|-\log|z_1|)+\left(\frac{1}{2}\varphi\big(t_k-\log|z_1|\big)-\omega(c)(t_k-\log|z_1|)\right)\\ &+\left(\frac{1}{2}\varphi\big(\log|z_2|-t_l\big)-\omega(c)(\log|z_2|-t_l)\right).\end{aligned}$$

Let $|t|\le c/2$ and consider the quantity $\Delta=(1/2)\varphi(t)-\omega(c)t$. If $t\ge 0$, then

$$\Delta\ge-\frac{\omega(c)t}{2}\ge-\frac{\psi_{C_0}(c)}{4}\ge-\frac{c}{8C_0}.$$

If $t<0$, by Corollary 5.7,

$$\Delta\ge-\frac{1}{2}\psi_\pi(|t|)+\omega(c)t\ \ge-\left(\frac{1}{4\pi}-\omega(c)\right)|t|\ge-\frac{c}{8\pi}.$$

At any event, we obtain $\Delta\ge-c/8\pi$. Finally, we have the inequality

$$d_\Omega(z_1,z_2)\ge\frac{1}{c}\log\left(1+\frac{c}{2C_0}\right)(\log|z_2|-\log|z_1|)-\frac{c}{4\pi}. \tag{5.17}$$

We return to the scarce case with (5.13). Put $T=\{t_1,t_2,\dots\}$. Let $\beta$ be an arbitrary number satisfying $\beta>\alpha$ and consider the sequence $u_n=\beta^{n-1}(t_1+c/(\beta-1))-c/(\beta-1)$. Note that $u_n\ge u_1=t_1>0$ and $u_n/\beta^n\to((\beta-1)t_1+c)/\beta(\beta-1)>0$ as $n\to\infty$. By assumption, $[u_n,\alpha u_n+c]\cap T\ne\emptyset$. Thus, by passing to subsequence, we may assume that $t_n\in[u_n,\alpha u_n+c]$. On the other hand, $t_{n+1}-t_n\ge u_{n+1}-(\alpha u_n+c)=(\beta-\alpha)u_n$. Hence, we will not lose great generality if we assume that the sequence $t_n = \log|a_n|$ satisfies the condition

$$A\alpha^n \le t_n - t_{n-1} \quad \text{and} \quad A\alpha^n \le t_n \le B\alpha^n, \quad n = 1, 2, \dots, \tag{5.18}$$

for some constants $A, B \in (0, \infty)$ and $\alpha \in (1, \infty)$. Recall that we always define $t_0 = -\infty$. For instance, the sequence $t_n = C\gamma^n,\ n = 1, 2, \dots,$ where $C > 0$ and $\gamma > 1$ are constants, satisfies (5.18) with $A = (1 - 1/\gamma)C,\ B = C$ and $\alpha = \gamma$. Based on Theorem 1.7, we are now able to deduce the following result.

**Theorem 5.19.** *Suppose that a sequence $a_0 = 0, a_1, a_2, \dots$ satisfies* (5.18). *If a domain $\Omega \subset \mathbb{C}$ meets none of $a_n$, then the hyperbolic distance between two points $z_1$ and $z_2$ in $\Omega$ with $|a_1| \le |z_1| \le |z_2|$ which satisfy*

$$\log|z_1| \ge \frac{2AC_0}{B\sqrt{\alpha}} \quad \textit{and} \quad \log|z_2| \ge \frac{2BC_0\sqrt{\alpha}}{A} \tag{5.20}$$

*is estimated from below by*

$$d_\Omega(z_1, z_2) \ge \frac{{X_2}^2 - {X_1}^2}{2\log\alpha} + \left(\frac{1}{2} + \frac{\log(A/2C_0)}{\log\alpha}\right)(X_2 - X_1), \tag{5.21}$$

*where $X_1 = \log(\log|z_1|) - \log(A/\alpha)$ and $X_2 = \log(\log|z_2|) - \log(B\,\alpha)$.*

*Proof.* We choose integers $k$ and $l$ with $2 \le k \le l$ so that $t_{k-1} \le \log|z_1| \le t_k$ and that $t_l \le \log|z_2| \le t_{l+1}$. Take intermediate points $w_1$ and $w_2$ from the hyperbolic geodesic joining $z_1$ and $z_2$ in $\Omega$ in such a way that $\log|w_1| = t_k, \log|w_2| = t_l$ and that $d_\Omega(w_1, w_2) \le d_\Omega(z_1, z_2)$. Applying Theorem 1.7 to $w_1$ and $w_2$ and using Lemma 5.4, we obtain

$$d_\Omega(w_1, w_2) \ge \sum_{n=k+1}^{l} \psi_{C_0}(t_n - t_{n-1}).$$

From (5.18), we deduce

$$\psi_{C_0}(t_n - t_{n-1}) \ge \psi_{C_0}(A\alpha^n) \ge \log(A\alpha^n/2C_0),$$

so that

$$\begin{aligned} d_\Omega(w_1, w_2) &\ge \sum_{n=k+1}^{l} \big(n\log\alpha + \log(A/2C_0)\big) \\ &= \frac{\log\alpha}{2}\big(l(l+1) - k(k+1)\big) + (l-k)\log(A/2C_0) \\ &= \frac{\log\alpha}{2}\Big\{(l^2 - k^2) + (l-k)\Big\} + (l-k)\log(A/2C_0). \end{aligned}$$

On the other hand, since $\log|z_1| \ge t_{k-1}$ and $\log|z_2| \le t_{l+1}$, we deduce from (5.18) the estimates

$$\begin{aligned} k\log\alpha &\le \log(\log|z_1|) - \log(A/\alpha) = X_1 \quad \text{and} \\ l\log\alpha &\ge \log(\log|z_2|) - \log(B\,\alpha) = X_2. \end{aligned}$$

So far, we have not used the assumptions in (5.20). We need them when we combine all the estimates above in order to get the final conclusion. The quadratic polynomial $P(x) =$

$(\log \alpha)(x^2+x)/2 + x\log(A/2C_0)$ is increasing for $x \geq x_0 := -1/2 - (\log(A/2C_0))/\log\alpha$. Therefore, it is enough to check that $k \geq x_0$ and $X_2/\log\alpha \geq x_0$ in order to guarantee a decreasing effect by replacement of $k$ and $l$ by $X_1/\log\alpha$ and $X_2/\log\alpha$, respectively.

The condition $k \geq x_0$ is equivalent to the inequality $B\alpha^k \geq 2BC_0/A\sqrt{\alpha}$, which is implied by the first condition in (5.20) because $\log|z_1| \leq B\alpha^k$. The second condition can be dealt with similarly. $\square$

## 6. Littlewood's theorem and its generalization

First of all, we give a simple principle which leads to Schottky's theorem and Littlewood-type theorems. That is more or less standard, see [19] and [9], for example. Let $f : \mathbb{D} \to \Omega$ be a holomorphic function from the unit disk into a given hyperbolic domain $\Omega \subset \mathbb{C}$. Then, by (1.1), we obtain

$$d_\Omega(f(0), f(z)) \leq d_{\mathbb{D}}(0, z) = \frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right) = \operatorname{arctanh}|z|. \tag{6.1}$$

If, in addition, we have a lower estimate for $d_\Omega(f(0), f(z))$ in terms of $|f(z)|$, then we would deduce a growth estimate for $|f(z)|$ from (6.1).

The simplest case is Schottky's theorem. Indeed, we assume that $\Omega = \mathbb{C}\setminus\{0,1\}$ and $f : \mathbb{D} \to \Omega$ satisfies $|f(0)| = a$. Let $M = M(a,r)$ be the best possible constant so that $|f(z)| \leq M$ holds if $|z| \leq r$ for any such $f$. Then, by Lemma 5.16, we obtain the inequality $\Phi(|f(z)|) - \Phi(a) \leq \operatorname{arctanh}|z| \leq \arctan r$. Since the universal covering map of $\Omega$ attains equality, we see the relation $\Phi(M) - \Phi(a) = \operatorname{arctanh} r$. This sharp form of Schottky's theorem was obtained by Hempel [9]. See also [10] for more concrete forms of Schottky's theorem.

As direct applications of lower estimates for the hyperbolic distance obtained in the previous section, we derive a few results analogous to Littlewood's theorem. Here, Littlewood's theorem refers to the following result.

**Theorem 6.2** (Littlewood's theorem [16])**.** *Let $k \geq 0$ be an integer and $c > 1$ be a constant. Suppose that a sequence $a_1, a_2, \ldots$ satisfies the conditions same as in Theorem 5.12 with constant $c > 0$. If a function $f(z) = b_0 + b_1 z + \ldots$ is holomorphic and takes no value $a_n$ more than $k$ times in the unit disk, then*

$$|f(z)| \leq K_1\mu(1-|z|)^{-\gamma},$$

*where $\gamma = c(|a_1|+1)K_2$, $\mu = \max\{1, |b_0|, |b_1|, \ldots, |b_k|\}$ and $K_1$ and $K_2$ are constants depending only on $k$.*

Hayman [8] proved the above result in a sharper form when $k = 0$ with $\mu = \{|b_0|, |a_1|\}$, $\gamma = \alpha(c)$, $K_1 = e^{2c+6\alpha(c)}$ and $\alpha(c) = c/2(\log(2c/\pi^2) - 30/c)$ for $c \geq c_0 \approx 115.9$, $\alpha(c) = 20$ for $c \leq c_0$. Our result below improves theirs in the case $k = 0$. Järvi and Vuorinen [12] proved a counterpart of Littlewood's theorem for quasiregular mappings.

**Theorem 6.3.** *Let $a_1, a_2, \ldots$ be a sequence which satisfies the same conditions as in Theorem 5.12 with constant $c > 0$. Suppose that $f$ is a holomorphic function in the unit* *disk $\mathbb{D}$ which omits all the values $a_n,\ n = 0, 1, 2, \dots$ above. Then the inequality*

$$|f(z)| \le \max\{|f(0)|, e^{-c/2}|a_1|\}\left(\frac{1+|z|}{1-|z|}\right)^{1/2h(c/2)}$$

*holds for all $z \in \mathbb{D}$, where $h$ is the function described in* (2.10).

*Proof.* We set $\mu = \max\{|f(0)|, e^{-c/2}|a_1|\}$ and may assume that $|f(z)| \ge \mu$. If $|f(0)| < e^{-c/2}|a_1|$, we pick up an intermediate point $z_0$ from the line segment $[0, z]$ so that $|f(z_0)| = e^{-c/2}|a_1|$. Otherwise, we set $z_0 = 0$. By Theorem 5.12 and (1.1), we obtain

$$\begin{aligned} h(c/2)(\log|f(z)| - \log|f(z_0)|) \le d_\Omega(f(z), f(z_0)) &\le d_{\mathbb{D}}(z, z_0) \\ \le d_{\mathbb{D}}(z, 0) &= \frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right). \end{aligned}$$

Since $|f(z_0)| = \mu$, we obtain the required inequality. □

In the same way, we could produce growth estimates when we are given lower estimates for the hyperbolic density $\rho_\Omega$ by using Lemma 5.10. We remark that a related growth estimate has been obtained by Zheng [24] under the assumption that $|w|\rho_\Omega(w) \ge c$ for all $w \in \Omega$.

We conclude this article with an application of Theorem 5.19 to this direction.

**Theorem 6.4.** *Suppose that a sequence $a_0 = 0, a_1, a_2, \dots$ satisfy* (5.18) *with constants $A, B, \alpha$. If a function $f$ holomorphic in $\mathbb{D}$ omits all the values $a_n,\ n = 0, 1, \dots,$ then*

$$\log^+(\log^+|f(z)|) \le M + \sqrt{(\log\alpha)\log\left(\frac{1+|z|}{1-|z|}\right)},$$

*where $M$ is a constant depending only on $A$, $B$, $\alpha$ and $|f(0)|$.*

Hayman gave a similar assertion in [8, p. 166] without details.

*Proof.* We may assume that $A \ge 2C_0/\sqrt{\alpha}$, and therefore, $1/2 + \log(A/2C_0)/\log\alpha \ge 0$. Otherwise, taking the minimal $k$ such that $A\alpha^k \ge 2C_0/\sqrt{\alpha}$, we discard $a_1, \dots, a_k$ and renumber $a_{n+k}$ by $a_n$. Then we can replace $A$ by $A\alpha^k$ in (5.18).

Let $\Omega = \mathbb{C} \setminus \{a_0, a_1, \dots\}$. Fix $\zeta_2 \in \mathbb{D}$ and set $z_2 = f(\zeta_2)$. Put

$$\mu = \max\{|f(0)|, \exp(2AC_0/B\alpha), \exp(B\alpha)\}.$$

Letting $M \ge \mu$, we may assume that $|z_2| \ge \mu$. Take a point $\zeta_1$ from the line segment $[0, \zeta_2]$ so that $|f(\zeta_1)| = \mu$ if $\mu > |f(0)|$. Otherwise, set $\zeta_1 = 0$. We put $z_1 = f(\zeta_1)$, then we have $|z_1| \ge \mu \ge |a_1|$. We may further assume that

$$\log|z_2| \ge \max\left\{\frac{2BC_0\sqrt{\alpha}}{A}, \frac{B\alpha^2}{A}\log|z_1|\right\}. \tag{6.5}$$

We now apply Theorem 5.19 to obtain

$$d := d_\Omega(z_1, z_2) \ge \frac{{X_2}^2 - {X_1}^2}{2\log\alpha}$$

where we have used the fact that the second term in the right-hand side in (5.21) is non-negative because (6.5) implies $X_2 - X_1 \geq 0$. Hence,

$$\log(\log|z_2|) - \log(A/\alpha) = X_2 \leq \sqrt{2d\log\alpha + {X_1}^2}.$$

Using $d \leq d_{\mathbb{D}}(\zeta_1, \zeta_2) \leq d_{\mathbb{D}}(0, \zeta_2) = \operatorname{arctanh}|\zeta_2|$ and the estimate $\sqrt{x+y} \leq \sqrt{x} + y/2\sqrt{x}$, we obtain the inequality

$$\log(\log|f(\zeta_2)|) \leq \sqrt{(\log\alpha)\log\left(\frac{1+|\zeta_2|}{1-|\zeta_2|}\right)} + \frac{\big(\log(\log|f(\zeta_1)|) - \log(B\alpha)\big)^2}{2\sqrt{(\log\alpha)\log\left(\frac{1+|\zeta_2|}{1-|\zeta_2|}\right)}} + \log(A/\alpha).$$

In this way, the expected form has been obtained. □

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