Size of the Julia set of a structurally finite transcendental entire function

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Abstract In this note, we will show that, for every structurally finite transcendental entire function, the Hausdorff dimension of its Julia set is two.

1 Introduction and main results

In [8] we defined structurally finite entire functions and proved that every such $f(z)$ of type (p, q) can be written as an indefinite integral

$$f(z) = \int^z P(t)e^{Q(t)}dt$$

with polynomials $Q(z)$ of degree q and $P(z)$ of degree p. For the definitions and details, see [8] and [9].

Since we are interested in dynamical properties of such an $f(z)$, we may assume, taking conjugation by a similarlity if necessary, that

$$f(z) = a\int_0^z P(t)e^{Q(t)}dt + b$$

with monic polynomials P and Q of degrees p and q. Here we exclude the case of linear polynomials (the case that $p = 0, q = 0$). In the sequel, we denote by $f^k(z)$ the k-th iteration of $f(z)$ and by J_f the Julia set of $f(z)$. See for instance [3] for the basic facts on dynamical properties of entire functions.

Now in this note, we give a proof of the following

Theorem 1 *For every transcendental structurally finite entire function $f(z)$, the Hausdorff dimension of J_f is two.*

Remark Compare with a theorem of Stallard ([5] II): For every transcendental entire function $f(z)$ with a bounded set of singular values, the Hausdorff dimension of J_f is greater than 1. (Here we say that α is a singlar value of $f(z)$ if every neighborbood of α is not evenly covered under $f : \mathbb{C} \to \mathbb{C}$.)

Also, we note the following

Proposition 2 *Let $f(z)$ be a (not necessarily transcendental) structurally finite entire function. If $f(z)$ is hyperbolic, namely if the union $S^+(f)$ of the orbits of all singular values is relatively compact in $\mathbb{C}$ and the distance between $S^+(f)$ and J_f is positive, then the Julia set J_f has vanishing area.*

Devaney and Keen proved in [1] that, if the Schwarzian derivative of a meromorophic $f(z)$ is a polynomial (in particular, if $f(z)$ is a structurally finite entire function) and if $f(z)$ is hyperbolic, then the Julia set J_f has vanishing area. The proof of Proposition 2 is essentially the same as those in [1] and [2]. Hence we omit the proof. Also see [6].

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2 Proof of the main theorem

Now to prove theorem 1, it suffices to show that the Hausdorff dimension of the set of escaping points of $f(z)$ in

$$\Pi = \{z = re^{i\theta} \mid r > 0, |\theta| \leq \frac{\pi}{4q} \}$$

is two, because $f(z)$ is in the Speiser class (cf. for instance, [3]).

Fix a positive constant ϵ ($< 1/4$). Next set

$$Q_\theta(t) = \operatorname{Re} Q(e^{i\theta} t) = \cos(q\theta)\, t^q + \cdots$$

and

$$R_\theta(t) = \left| \left(\frac{P'Q' - PQ''}{(Q')^3} \right)' (e^{i\theta} t) \right|^2 |e^{Q(e^{i\theta} t)}|^2$$

for every θ with $|\theta| \leq \pi/4q$. Then, unless $R_\theta(t)$ equals identically to 0, we can write $R_\theta(t)$ as

$$\frac{A_\theta(t)}{B_\theta(t)}\, e^{2Q_\theta(t)}$$

with real polynomials $A_\theta(t)$ and $B_\theta(t)$ of degree at most $2(p + 4q - 6)$ (≥ 0 unless R_θ equals identically to 0) and of degree $2(6q - 6) \geq 0$, respectively. Since the term of the highest degree of $A_\theta(t)$ and that of $B_\theta(t)$ have positive coefficients independent of θ, we can find a finite t_0 such that

1.
$$R'_\theta(t) > 0$$

unless $R_\theta(t)$ equals identically to 0, and

2.
$$R_\theta(t) \le 2c_\infty \, t^{2(p-2q)} e^{2Q_\theta(t)}$$

for every θ with $|\theta| \le \pi/4q$ and every t not less than t_0, where we set

$$c_\infty = \max\left\{1, \lim_{t\to+\infty} t^{-2(p-2q)} \frac{A_\theta(t)}{B_\theta(t)}\right\},$$

which is independent of θ as noted above. (Here the first condition is satisfied for every sufficiently large t, since

$$R'_\theta(t) = \frac{(A'_\theta B_\theta - A_\theta B'_\theta) + 2A_\theta B_\theta Q'_\theta}{(B_\theta)^2} \, e^{2Q_\theta(t)},$$

and the term of highest degree of

$$(A'_\theta B_\theta - A_\theta B'_\theta) + 2A_\theta B_\theta Q'_\theta$$

is that of $2A_\theta B_\theta Q'_\theta$ which has a positive coefficient depending only on $f(z)$ and $\cos q\theta$.)

Also fix an R so large that R satisfies the following inequalities:

1. $R > \frac{5}{2}\, q\, (32q/\pi)\, (> 8\pi q > 16q), \qquad R > \frac{1}{2}\, e^{6q/2}$
2. $R > 4q/\epsilon$,
3. $R > t_0$, and
4. for every $z \in \Pi$ with $|z| \ge R$,

   (a)
   $$\max\left\{\frac{q}{2}(|z|+1)^{q-1}, 1\right\} < |Q'(z)| < 2q(|z|-1)^{q-1}, \qquad \left|\frac{Q''(z)}{Q'(z)}\right| < \frac{2q}{|z|+1}$$

   (b)
   $$\left|\frac{P'(z)}{P(z)} + Q'(z)\right| < 2q(|z|-1)^{q-1},$$

   (c)
   $$|z|^{p-q}|e^{Q(z)}| > 1,$$

(d)

$$\frac{|a|}{2q}\,|z|^{p-q+1}|e^{Q(z)}| > \max\left\{e^{|z|^q/2},\ \frac{4|aP(z)e^{Q(z)}|}{|z|^q}\right\},$$

$$\frac{|aP(z)e^{Q(z)}|}{(|z|+1)^q} > 4\pi\,(>1).$$

Now, set

$$S = \Pi \cap \{|z| \geq R+1\},$$

and

$$T = S \cap f^{-1}(\Pi).$$

Definition We say that the image $f(B)$ of a disk B is *nearly disk-shaped* with respect to ϵ if $f'(z_B) \neq 0$ and the *distortion function*

$$V(f,B) = \sup_{z\in B, z\neq z_B}\left|\frac{f(z)-f(z_B)}{z-z_B} - f'(z_B)\right|$$

of $f(z)$ on B is bounded by $\epsilon|f'(z_B)|$. Here and in the sequel, *"disks" always mean closed disks* and z_B is the center of a disk B.

Lemma 3 *For every disk B with center $z_B \in S$ and radius $|z_B|^{-q}$, the image $f(B)$ is nearly disk-shaped (with respect to ϵ).*

In particular, $f(B)$ contains a disk with center $f(z_B)$ and radius

$$\frac{3}{4}\,|f'(z_B)||z_B|^{-q},$$

and is contained in a disk with the same center and radius

$$\frac{5}{4}\,|f'(z_B)||z_B|^{-q}.$$

Proof. First we note that, if $z \in B$, then $|z - z_B| \leq |z_B|^{-q}\,(<1)$. Hence by the condition 4-(b) for R, we have

$$|N(f)(z)|\left(=\left|\frac{f''(z)}{f'(z)}\right|\right) = \left|\frac{P'(z)}{P(z)} + Q'(z)\right| < 2q|z_B|^{q-1}$$

on B, which implies by the condition 1 for R that

$$\left|\log\frac{f'(z)}{f'(z_B)}\right| = \left|\int_{z_B}^{z}\frac{f''(t)}{f'(t)}\,dt\right| < 2q|z_B|^{q-1}\cdot|z_B|^{-q} = 2q|z_B|^{-1}(<1/8),$$

and hence that

$$\left|\frac{f'(z)}{f'(z_B)}-1\right| < 2\left|\log\frac{f'(z)}{f'(z_B)}\right| < 4q|z_B|^{-1} < \frac{1}{4}.$$

Thus we conclude that

$$f(z)-f(z_B) = f'(z_B)\left((z-z_B)+\int_{z_B}^{z}\epsilon(t)dt\right)$$

with an error function $\epsilon(t)$ such that (by the condition 2 for R)

$$|\epsilon(t)| < 4q|z_B|^{-1} < \epsilon$$

on B, which implies that the image $f(B)$ is nearly disk-shaped with respect to ϵ. In particular, since $|\epsilon(t)| < 1/4$, we conclude the second assertion. ■

Lemma 4 *For every $z \in S$, we can represent $f(z)$ as*

$$\frac{a}{q}z^{p-q+1}(1+e_f(z))e^{Q(z)}$$

with an error function $e_f(z)$ satisfying

$$|e_f(z)| \leq \frac{C_1}{|z|}$$

for every z with $|z| \geq R$, where (and in the proof below) C_k are constants depending only on $f(z)$ and R.

Proof. First, fix a $z \in S$. By the condition 4-(a) for R, $|Q'(z)| > 1$ on the ray

$$\ell_z = \{te^{i\theta} \mid R \leq t \leq |z|,\, \theta = \arg z\}.$$

Integrating on this ray, we have

$$\begin{aligned} f(z) &= a\int_{\ell_z} P(t)e^{Q(t)}dt + C(\theta) + b \\ &= a\left[\frac{P(t)}{Q'(t)}e^{Q(t)}\right]_R^z - a\int_{\ell_z}\left(\frac{P}{Q'}\right)'(t)e^{Q(t)}dt + C(\theta) + b \\ &= a\frac{P(z)}{Q'(z)}e^{Q(z)} - a\left(\frac{P'Q'-PQ''}{(Q')^3}\right)(z)e^{Q(z)} + I(z) \\ &\quad +C(\theta) + \tilde{C}(\theta) + b, \end{aligned}$$

where

$$I(z) = a \int_{\ell_z} \left(\frac{P'Q' - PQ''}{(Q')^3} \right)' (t) e^{Q(t)} dt,$$

$$C(\theta) = a \int_0^R P(re^{i\theta}) e^{Q(re^{i\theta})} e^{i\theta} dr,$$

and

$$\tilde{C}(\theta) = -a \frac{P(Re^{i\theta})}{Q'(Re^{i\theta})} e^{Q(Re^{i\theta})} + a \left(\frac{P'Q' - PQ''}{(Q')^3} \right) (Re^{i\theta}) e^{Q(Re^{i\theta})}.$$

Now for the first term in the right hand side, we have an estimate

$$\left| a \frac{P(z)}{Q'(z)} - \frac{a}{q} z^{p-q+1} \right| \leq C_2 |z|^{p-q}$$

and similarly the second term is bounded by $C_3 |z^{p-2q+1} e^{Q(z)}|$, on S with suitable C_2 and C_3. (Here we assume that $C_3 = 0$ if $p = 0$ and $q = 1$.)

Next, by the definition of t_0 and the condition 3 for R, $R_\theta(t)$ is strictly increasing on $[t_0, +\infty)$ for every θ with $|\theta| \leq \pi/4q$. Hence, we conclude that

$$|I(z)| \leq |a| \int_R^{|z|} \sqrt{R_\theta(t)} dt \leq C_4 |z^{p-2q+1} e^{Q(z)}|$$

with a suitable C_4. (We assume that $C_4 = 0$ if $p \leq 1$ and $q = 1$.) Finally, $|C(\theta)|$ and $|\tilde{C}(\theta)|$ are continuous on $[-\pi/(4q), \pi/(4q)]$, and hence are bounded by a constant C_5 for every such θ. Thus, by noting the condition 4-(c) for R, we have a desired estimate by setting $C_1 = C_2 + C_3 + C_4 + 2C_5 + |b|$. ■

Now we take another $R_1 > R + 2\pi$ such that

1. $$\frac{C_1}{R_1} < \sin \frac{\pi}{8q} \left(< \frac{1}{2} \right)$$

   and, if $p - q + 1 \neq 0$, then

2. $$R_1 \sin \frac{\pi}{8|p-q+1|} > 4\pi.$$

In particular, Lemma 4 and the condition 1 for R_1 give that

$$\frac{|a|}{2q} |z|^{p-q+1} |e^{Q(z)}| < |f(z)| < \frac{2|a|}{q} |z|^{p-q+1} |e^{Q(z)}|$$

for every z with $|z| \geq R_1$. Furthermore, fixing a value A of $\arg a$, we have by Lemma 4 a continuous branch of the $\arg f(z)$ on S such that

$$|\arg\, f(z) - [(p-q+1)\theta + \operatorname{Im} Q(z) + A]| \leq \sin^{-1}\left(\frac{C_1}{|z|}\right).$$

Hence we have the following

Lemma 5 *T contains*

$$T' = \{z \in \Pi \mid |z| \geq R_1,\ |(p-q+1)\theta + \operatorname{Im} Q(z) + A - 2m\pi| \leq \frac{\pi}{8q},\ m \in \mathbb{Z}\}.$$

Proof. If $z \in T'$, the condition 1 for R_1 gives that

$$|\arg\, f(z) - 2m\pi| < \sin^{-1}\left(\frac{C_1}{|z|}\right) + \frac{\pi}{8q} < \frac{\pi}{4q}$$

with some $m \in \mathbb{Z}$, which implies that $f(z) \in \Pi$. Thus T contains T'. ∎

If $p - q + 1 = 0$, set

$$T'' = \{z \in \Pi \mid |z| \geq R_1,\ |\operatorname{Im} Q(z) + A - 2m\pi| \leq \frac{\pi}{16q},\ m \in \mathbb{Z}\},$$

which is clearly contained in T'.

If $p - q + 1 \neq 0$, for every $j = -2|p-q+1|, \cdots, 2|p-q+1| - 1$, set

$$B_j = \frac{(2j+1)\pi}{16q|p-q+1|},$$

and

$$S_j = \left\{re^{i\theta} \in \Pi \mid r \geq R_1,\ B_j - \frac{\pi}{16q|p-q+1|} \leq \theta \leq B_j + \frac{\pi}{16q|p-q+1|}\right\}.$$

Then we have the following

Lemma 6 *If $p - q + 1 \neq 0$, then*

$$\bigcup_{j=-2|p-q+1|}^{2|p-q+1|-1} S_j$$

is coincident with $S' = S \cap \{|z| \geq R_1\}$, and $T' \cap S_j$ contains

$$T''_j = \{z \in S_j \mid |(p-q+1)B_j + \operatorname{Im} Q(z) + A - 2m\pi| \leq \frac{\pi}{16q},\ m \in \mathbb{Z}\}.$$

Proof. The first assertion is clear. Next if $z \in T''_j$, then

$$\begin{aligned} & |(p-q+1)\theta + \operatorname{Im} Q(z) + A - 2m\pi| \\ < \ & \frac{\pi}{16q} + |(p-q+1)B_j + \operatorname{Im} Q(z) + A - 2m\pi| \le \frac{\pi}{8q}, \end{aligned}$$

which implies that $z \in T'$. ∎

Thus set

$$T'' = \bigcup_{j=-2|p-q+1|}^{2|p-q+1|-1} T''_j,$$

if $p - q + 1 \neq 0$, and we conclude that $T'' \subset T'$.

Next set

$$\eta_1 = \frac{\pi}{32q}$$

and T_* be the subset of the image $Q(T'')$ of T'' under $Q(z)$ consisiting of all such z that the disk with center z and radius η_1 is contained in $Q(T'')$. Further set

$$T^\circ = Q^{-1}(T_*) \cap S'.$$

Then we have the following

Lemma 7 *Every disk B with center $z_B \in T^\circ$ and radius $|z_B|^{-q}$ is contained in T''.*

Proof. By the condition 4-(a) for R, we have (as in the proof of Lemma 3) that

$$\left|\log \frac{Q'(z)}{Q'(z_B)}\right| \le |z_B|^{-q} \max_{z \in B} |N(Q)(z)| < \max_{z \in B} |N(Q)(z)| < 2q|z_B|^{-1}$$

for every $z \in B$, which implies that the distortion of Q on B satisfies

$$V(Q, B) < 4q|z_B|^{-1} < \epsilon.$$

Thus $Q(B)$ is nearly disk-shaped (with respect to ϵ), and in particular, $Q(B)$ is contained in a disk with center $Q(z_B) \in T_*$ and radius

$$\frac{5}{4}\,|Q'(z_B)||z_B|^{-q} < \frac{5}{2}\,q|z_B|^{-1} < \eta_1$$

by the conditions 4-(a) and 1 for R. Hence we conclude that $Q(B) \subset Q(T'')$, which means that $B \subset T''$. ∎

Here, we recall the following well-known covering lemma. See for instance, [7] Lemma in 1.6 of Chapter I.

Lemma 8 *Let X be a measurable set in $\mathbb{C}$, and $\mathcal{B} = \{B_j\}$ be a countable covering of X by disks with bounded radii. Then we can find an absolute constant C (depending neither on X nor on $\mathcal{B}$) and a subset $\{B_{j_k}\}$ of $\mathcal{B}$ such that B_{j_k} are mutually disjoint and*

$$\sum_k m(B_{j_k}) \geq Cm(X),$$

where $m(E)$ is the the area $\int_E dxdy$ for every measurable set E in $\mathbb{C}$.

Lemma 9 *There is a $\delta > 0$ such that, for every disk D with center $z_D \in S'$ and radius $r(D)$ greater than 2π, we have*

$$\rho(D \cap T^\circ, D) \geq \delta.$$

Here and in the sequel, the *density* of a measurable subset X of a mesurable set Y is defined by

$$\rho(X, Y) = \frac{m(X)}{m(Y)}.$$

Proof. First note that T_* contains the intersections

$$E = \bigcup_j S_j^* \cap T_j^{strips},$$

of

$$S_j^* = \{z \in Q(S_j) \mid d(z, \mathbb{C} - Q(S_j)) \geq \eta_1\}$$

(where $d(z, E)$ is the distance from z to the set E) and periodic parallel strips with period 2π;

$$T_j^{strips} = \left\{ |(p - q + 1)B_j + \operatorname{Im} z + A - 2m\pi| \leq \frac{\pi}{32q} \left(= \frac{\pi}{16q} - \eta_1 \right), m \in \mathbb{Z} \right\}$$

for every j. Hence we can find a $\delta' > 0$ such that

$$\rho(D^* \cap E, D^*) \geq \delta'$$

for every disk D^* with center in $Q(S')$ and radius greater than 2π.

Indeed, for every $z \in Q(S')$ and $r \geq 2\pi$, set

$$F(z, r) = \rho(D(z, r) \cap E, D(z, r)),$$

where $D(z, r)$ is the disk with center z and radius r. Then $F(z, r)$ is always positive by the condition 2 for R_1, and continuous on $Q(S') \times [2\pi, +\infty)$. Also it is easy to see that $F(z,r)$ is bounded away from 0 when $|z|$ or r tends to $+\infty$.

In particular, if $q=1$ (and hence $Q(z)=z$), then the assertion of Lemma 9 is clear from the definition of T° with $\delta=\delta'$. So we consider the case that $q>1$. Let D' be the disk with center z_D and radius $r(D)-1/2$. Take a covering of the compact set $D'\cap S'$ by a finite number of open disks with center in $D'\cap S'$ and radius $1/2$. Then by the covering lemma (Lemma 8), we can find a finite set $\{B_m\}_{m=1}^M$ consisting of disjoint disks with center z_m in $D'\cap S'$ and radius $1/2$ satisfying that $\tilde{D}=\cup_{m=1}^M B_m$ is contained in D and

$$m(\tilde{D})\geq C\,m(D'\cap S').$$

On the other hand, it is clear from the shape of S' that there is a positive constant δ_0 depending only on S' such that

$$\rho(D''\cap S',D'')\geq\delta_0$$

for every D'' with center in S' and radius greater than $2\pi-1/2$.

Hence we conclude that

$$\rho(\tilde{D},D)\geq C\rho(D'\cap S',D)\geq\delta_1=\left(1-\left(\frac{1}{4\pi}\right)\right)^2 C\delta_0.$$

Now for every B_m, we can show as in the proof of Lemma 7 that $Q(B_m)$ is nearly disk-shaped (with respect to ϵ), and in particular, $Q(B_m)$ contains a disk with center $Q(z_m)$ and radius

$$\frac{3}{4}\,|Q'(z_m)|\cdot\frac{1}{2}>\frac{3}{16}q|z_m|^{q-1}>\frac{3q}{16}R\left(>\frac{3q}{16}8\pi q>3q\pi\right)>2\pi$$

by the conditions 1 and 4-(a) for R, and is contained in a disk with the same center and radius $(5/8)|Q'(z_B)|$. (Here recall that $q\geq 2$.) Hence, we conclude that

$$\rho(Q(B_m)\cap E,Q(B_m))\geq\left(\frac{3}{5}\right)^2\delta'.$$

Since

$$\left|\frac{Q'(z)}{Q'(z_m)}-1\right|<\frac{1}{4}$$

on D_m, we conclude that

$$\rho(B_m\cap T^\circ,B_m)\geq\left(\frac{3}{5}\right)^4\delta'.$$

Thus in the case that $q > 1$, we have the assertion with

$$\delta = \left(\frac{3}{5}\right)^4 \delta' \delta_1.$$

■

Now we give a proof of the main theorem.

For this purpose, take a disk B with center $z_B \in T'$ and radius $|z_B|^{-q}$ arbitrarily. Then we have shown in Lemma 3 that $f(B)$ is nearly disk-shaped (with respect to ϵ), and contains a disk B' with center $f(z_B) \in S'$, whose absolute value can be written as

$$\frac{|a|}{q}|z_B|^{p-q+1}|(1 + e_f(z_B))e^{Q(z_B)}|$$

with $|e_f(z_B)| < 1/2$ by Lemma 4 and the condition 1 for R_1, and radius

$$r(B') = \frac{3}{4}|f'(z_B)||z_B|^{-q} - \frac{1}{8} > \frac{5}{8}|f'(z_B)||z_B|^{-q}$$

by the condition 4-(d) for R, and is contained in a disk B'' with the same center and radius

$$\frac{5}{4}|f'(z_B)||z_B|^{-q}.$$

Note that $r(B')$ is greater than 2π by the condition 4-(d) for R and that the distance between B' and $\mathbb{C} - f(B)$ is greater than $1/8$ by the definition.

In particular, Lemma 9 gives that

$$\rho(B' \cap T^\circ, B') \geq \delta,$$

which implies that

$$\rho(B' \cap T^\circ, B'') \geq \frac{\delta}{4}.$$

Thus we conclude the following

Lemma 10 *There is a positive constant δ'' depending only on $f(z)$ and R_1 such that, for every disk B with center $z_B \in T'$ and radius $|z_B|^{-q}$, there exists a packing $\{B_m\}_{m=1}^M$ of $T' \cap f(B)$ with a finite number of disjoint disks B_m in $T''(\subset T')$ with center z_m in T° and radius $|z_m|^{-q}$ such that*

$$\rho\left(\bigcup_{m=1}^{M} B_m, f(B)\right) \geq \delta''.$$

Moreover, for every m,

$$|f(z_m)| \geq \frac{|f(z_B)|}{2}.$$

Proof. First, we cover T° by open disks with center $z \in T^\circ$ and radius $|z|^{-q}$. Then since the above B' is compact, we can select a finite cover of $B' \cap T^\circ$. Then by Lemmas 7 and 8, we can find a packing $\{B_m\}_{m=1}^M$ of T'' by a finite number of disks B_m with center $z_m \in T^\circ$ and radius $|z_m|^{-q}$ such that

$$\rho\left(\bigcup_{m=1}^{M} B_m, f(B)\right) \geq C\rho(B' \cap T^\circ, B'') \geq \frac{1}{4}C\delta.$$

Here note that, since $|z_m|^{-q} < 1/8$ by the condition 1 for R, every such disk B_m is contained in $f(B)$, and hence $\{B_m\}_{m=1}^M$ is actually a packing of $T'' \cap f(B)$. Thus we have the first assertion with $\delta'' = C\delta/4$.

Next, every point z in $f(B)$, Lemmas 3, 4 and the conditions 4-(d) for R, 1 for R_1 imply that

$$|f(z)| > |f(z_B)| - \frac{5}{4}|f'(z_B)||z_B|^{-q} > \frac{|f(z_B)|}{2},$$

which shows the second assertion. ∎

Now we define a family $\mathcal{E}_k$ consisting of a finite number of disjoint compact sets inductively. For every B as in Lemma 10, we denote by $\mathcal{P}(f(B))$ the packing of $f(B)$ obtained in Lemma 10. Fix such a disk B_* and set $\mathcal{E}_0 = \{B_*\}$. Define

$$\mathcal{E}_k = \{G \mid G \subset F \in \mathcal{E}_{k-1}, \quad f^k(G) \in \mathcal{P}(f^k(F))\}.$$

Then

$$E_\infty = \bigcap_{k=1}^{\infty}\left(\bigcup_{G \in \mathcal{E}_k} G\right)$$

consists of escaping points whose orbit is contained in S, and Lemma 10 implies that

$$\rho\left(\left(\bigcup_{G \in \mathcal{E}_k, G \subset F} f^k(G)\right), f^k(F)\right) \geq \delta''$$

for every k and $F \in \mathcal{E}_{k-1}$.

Here we note the following distortion lemma.

Lemma 11 (Distortion Lemma) *Let $\{D_j\}_{j=0}^n$ be a sequence of disks with radii ℓ_j and $\{f_j\}$ be a sequence of univalent functions on neighborhoods of D_j. Assume that $\ell_j \leq \ell_0 < 1/5$,*

$$f_j(D_j) \supset D_{j+1}, \qquad \ell_j \cdot |f_j'(z)| > 1$$

on D_j, and

$$\ell_j \cdot \max_{z\in D_j} |N(f_j)(z)| < \epsilon$$

for every j. Then letting

$$\phi = \phi_{n,0} = (f_n \circ \cdots \circ f_0)^{-1} : D = D_{n+1} \to D'_0 = \phi(D_{n+1}),$$

we have

$$\left|\frac{\phi'(z)}{\phi'(z_D)} - 1\right| < \epsilon$$

on D, and hence the distortion $V(\phi, D)$ of $\phi(z)$ is bounded by ϵ.

Proof. Let w_0 be the center of D, and on D we set

$$\phi_{n,j} = (f_n \circ \cdots \circ f_j)^{-1} \qquad (j = 0, \cdots, n)$$

and $D'_j = \phi_{n,j}(D)$. Then since

$$|\phi'_{n,j}(z)| < \ell_j \cdots \ell_n$$

on D, the diameter $d(D'_j)$ of D'_j is not greater than

$$\ell_j \cdots \ell_n(2\ell_{n+1}) \le \ell_0^{n-j+1}(2\ell_j).$$

On the other hand,

$$\frac{\phi'(w)}{\phi'(w_0)} = \prod_{j=0}^{n} \frac{(f_j)'(\phi_{n,j}(w_0))}{(f_j)'(\phi_{n,j}(w))},$$

and each factor on D has the following estimate:

$$\begin{aligned}\left|\log \frac{(f_j)'(\phi_{n,j}(w_0))}{(f_j)'(\phi_{n,j}(w))}\right| &= \left|\int_{\phi_{n,j}(w)}^{\phi_{n,j}(w_0)} \frac{f''_j(t)}{f'_j(t)} dt\right| \\ \le\ d(D'_j) \max_{z\in D_j} |N(f_j)(z)| &< 2\ell_0^{n-j+1}\epsilon.\end{aligned}$$

Thus we have

$$\left|\log \frac{\phi'(w)}{\phi'(w_0)}\right| < \sum_{j=0}^{n} 2\ell_0^{n-j+1}\epsilon < \frac{2\ell_0\epsilon}{1-\ell_0},$$

which implies the assertion as in the proof of Lemma 3, since $2\ell_0/(1-\ell_0) < 1/2$ if $\ell_0 < 1/5$. ∎

Lemma 12 *We can find a $\delta_* > 0$ independent of k such that*

$$\rho\left(\left(\bigcup_{G\in\mathcal{E}_k} G\right)\cap F, F\right) \geq \delta_*,$$

for every $F \in \mathcal{E}_{k-1}$.

Proof. Fix k and $F \in \mathcal{E}_{k-1}$ arbitrarity. Then for every $G_0 \in \mathcal{E}_k$ such that $G_0 \subset F$, we can find a sequence of elements G_{k-j} of $\mathcal{E}_j$ $(j = 0, \cdots, k)$ such that $D_k = B_*$, and for every other j, $D_j = f^j(G_{k-j})$ is a disk in $\mathcal{P}(f^j(G_{k+j-1}))$. Let z_j be the center of D_j, (and hence $|z_j|^{-q}$ is the radius of D_j). Then by the condition 4-(d) for R gives that

$$|z_j|^{-q}|f'(z)|\,(>|f'(z)|(|z|+1)^{-q})\,>1$$

and the conditions 2 and 4-(b) for R give that

$$|z_j|^{-q}|N(f(z))| < 2q|z_j|^{-1} < \epsilon$$

for every $z \in D_j$.

Thus by the above distortion lemma, we conclude that

$$\frac{3}{4}|(f^{-k})'(z_k)| < |(f^{-k})'(z)| < \frac{5}{4}|(f^{-k})'(z_k)|$$

on $D_k = f^k(G_0)$ for every $G_0 \in \mathcal{E}_k$ contained in $F \in \mathcal{E}_{k-1}$. Hence we have the assertion with $\delta_* = (3/5)^2\delta''$. ■

Finally, set

$$d_k = \frac{2}{\psi^{k+1}(6)}$$

with $\psi(x) = \frac{1}{2}\,e^{x^q/2}$. Note that $\psi(x) > x$ and $\psi'(x) > 1$ on $[6, +\infty)$. And for the diameter $d(G)$ of every $G \in \mathcal{E}_k$, we have the following

Lemma 13 *For every $G \in \mathcal{E}_k$,*

$$d(G) < d_k.$$

Proof. First, $|z_{B_*}| > R > \psi(6)$ by the condition 1 for R. Next since, by Lemma 10, every point in $f(B)$ has an absolute value greater than $|f(z_B)|/2$ for every disk B with center z_B in S' and radius $|z_B|^{-q}$, we can see inductively that the inequality

$$|z_{f^{k-1}(F)}| > \psi^k(6)$$

for the center $z_{f^{k-1}(F)}$ of every disk $f^{k-1}(F)$ with $F \in \mathcal{E}_{k-1}$ implies

$$|z_{f^k(G)}| > \frac{|f(z_{f^{k-1}(F)})|}{2} > \psi(|z_{f^{k-1}(F)}|) > \psi^{k+1}(6)$$

by the condition 4-(d) for R, and hence

$$r(f^k(G)) = |f(z_{f^k(G)})|^{-q} < 1/\psi^{k+1}(6),$$

for every $f^k(G) \in \mathcal{P}(f^k(F))$. Since clearly $|(f^{-k})'(z)| < 1$ on $f^k(G)$, we conclude the assertion. ∎

Summing up, we have obtained a sequence $\{\mathcal{E}_k\}$ such that every $G \in \mathcal{E}_k$ is contained in a $F \in \mathcal{E}_{k-1}$ and every $G \in \mathcal{E}_k$ contains at least one element of $\mathcal{E}_{k+1}$. We have also found constants d_k and δ_* such that

$$\rho\left(\left(\bigcup_{G\in\mathcal{E}_k} G\right) \cap F, F\right) \geq \delta_* > 0, \qquad d(G) \leq d_k,$$

$$d_k \to 0, \quad \frac{k}{\log(1/d_k)} = \to 0,$$

for every k and G in $\mathcal{E}_k$. Hence the Fundamental lemma in [2] (also cf. [3] Lemma 3.2.7) shows that the Hausdorff dimension of E_∞ is two.

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