ON THE INNER RADIUS OF UNIVALENCE FOR AN ANNULUS

SHINRYO NONIN AND TOSHIYUKI SUGAWA

ABSTRACT. For the annulus $\{z \in \mathbb{C}, 1 < |z| < e^m\}$, $0 < m \leq \infty$, we give estimates of the inner radius of univalence with respect to Schwarzian derivatives. The estimates have sharp order when m tends to 0 or $+\infty$.

1. MAIN RESULT

Let D be a domain in the Riemann sphere $\widehat{\mathbb{C}}$ with at least three boundary points. Then D inherits the hyperbolic metric $\rho_D(z)|dz|$ of constant negative curvature -4 from its universal covering space $\mathbb{D} = \{|z| < 1\}$ with the metric $\rho_{\mathbb{D}}(\zeta) = 1/(1 - |\zeta|^2)$. The Bers space $B_2(D)$ is defined as the complex Banach space of holomorphic functions on D with the hyperbolic norm

$$\|\varphi\|_D = \sup_{z \in D} |\varphi(z)|\rho_D(z)^{-2}.$$

For a non-constant meromorphic function f on D, if $\|S_f\|_D < +\infty$, then f must be locally univalent. The remarkable result due to Beardon and Gehring [1] states that $\|S_f\|_D \leq 12$ holds for an arbitrary univalent meromorphic function f on D. Here, the number 12 is sharp even when D is simply connected. Note that this result extends the Kraus-Nehari theorem stating that $\|S_f\|_{\mathbb{D}} \leq 6$ for a univalent function on the unit disk $\mathbb{D}$. Consider the quantity

$$\sigma(D) = \inf\{\|S_f\|_D; f \text{ is non-univalent meromorphic in } D\},$$

which is called the *inner radius of univalence* of D. In other words, $\sigma(D)$ is the possible largest number $\sigma \geq 0$ with the property that the condition $\|S_f\|_D \leq \sigma$ implies the univalence of a meromorphic function f on D. For a simply connected domain D, it is well known that $\sigma(D) > 0$ if and only if D is a quasidisk. Note that the quantity $\sigma(D)$ can be regarded as the distance from the point S_f to the boundary in the universal Teichmüller space. Also, explicit values of the inner radii of univalence are known for several domains such as (round) disks, sectors, Euclidean triangles and regular polygons. Note also that the inner radius of univalence is a Möbius invariant, namely, $\sigma(L(D)) = \sigma(D)$ for any Möbius transformation L. For these facts and more information, see [5].

On the other hand, only few things are known for multiply connected domains. Amongst them, the following result is probably most important.

Theorem A (Osgood [6]). *For a finitely connected domain D, the inner radius of univalence of D is positive if and only if each boundary component of D is either a quasicircle or a singleton.*

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As for explicit values or estimates of $\sigma(D)$, however, very little seems to be known even when D is doubly connected, or, more concretely, when D is a circular annulus. As far as the authors know, only one result in this direction is the following due to Kobayashi and Wada [3].

Theorem B. *Let f be a non-constant meromorphic function on a neighbourhood of the unit circle $\mathrm{S}^1 = \partial\mathbb{D}$. If $|S_f(z)| < 3/2$ holds for all $z \in \mathrm{S}^1$, then f is injective in S^1. The bound $3/2$ is sharp.*

In this note, we investigate the inner radius of univalence $I(m) = \sigma(D)$ of a circular annulus D of modulus $m > 0$. By means of the Möbius invariance of $\sigma(D)$, we can define $I(m)$ more specifically by

$$I(m) = \sigma(A_m), \quad \text{where } A_m = \{z \in \mathbb{C}; 1 < |z| < e^m\}.$$

Here we set $A_\infty = \{z \in \mathbb{C}; 1 < |z|\}$, and therefore, $I(\infty) = \sigma(A_\infty) = \sigma(\mathbb{D}^*)$, where $\mathbb{D}^*$ is the once punctured disk $\mathbb{D}\setminus\{0\}$. Our main results are collected in the following theorems.

Theorem 1.1. *The function $I(m)$ is non-decreasing in $0 < m \le \infty$ and satisfies*

$$\frac{1}{2}(1 - e^{-m/2})^2 \le I(m) \le \min\left\{I(\infty), \frac{6m^2}{\pi^2}\right\}$$

for $0 < m < \infty$.

From this result, we have the following asymptotic behaviour of $I(m)$ as $m \to 0$:

$$\frac{1}{8} \le \liminf_{m\to 0^+} \frac{I(m)}{m^2} \le \limsup_{m\to 0^+} \frac{I(m)}{m^2} \le \frac{6}{\pi^2}.$$

Note that $(6/\pi^2) : (1/8) = 48/\pi^2 \approx 4.8634$.

One hopes that the once-punctured disk $\mathbb{D}^*$ is easiest to compute the value of the inner radius of univalence other than simply connected domains. We could, however, get only the following expression for $I(\infty)$ and some estimates for it.

Theorem 1.2. *The value $I(\infty)$ can be computed by the formula*

$$I(\infty) = \inf_{\varphi \in \partial T(1)} \|\varphi\|_{\mathbb{D}^*}. \tag{1.1}$$

It also satisfies the inequality

$$8e^{-2} = 1.0826\cdots \le I(\infty) \le 2. \tag{1.2}$$

Here $\partial T(1)$ is the boundary of the universal Teichmüller space $T(1)$, which is defined as the set of all $\varphi \in B_2(\mathbb{D})$ for which there exists a univalent function on $\mathbb{D}$ with a quasiconformal extension to $\widehat{\mathbb{C}}$ such that $S_f = \varphi$, where the boundary of $T(1)$ is taken in the Banach space $B_2(\mathbb{D})$. See [5] for more details.

In particular, we have the following corollary.

Corollary 1.3 (Asymptotic behaviour).

$$I(m) \asymp m^2 \quad \text{as } m \to 0, \text{ and}$$

$$\frac{1}{2} \le \lim_{m\to+\infty} I(m) \le 8e^{-2}.$$

Our method developed below is indirect, and hence, does not provide any concrete construction of quasiconformal extensions for those functions f on A_m such that $\|S_f\|_{A_m} < I(m) = \sigma(A_m)$. A constructive proof would be desirable such as the Ahlfors-Weill extension. We, however, should keep in mind with the obstacle that the Schwarzian differential equation $S_f = \varphi$ for a given φ does not necessarily have a single-valued solution f on the multiply connected domain (cf. §2).

2. Fundamental properties of $I(m)$

In this section, we deduce several fundamental properties of the function $I(m)$, from which our main results will follow except for the lower bound for $I(m)$ in Theorem 1.1.

We begin by stating a few preliminary results. The first simple lemma will be used frequently below. The proof is immediately done by the inequality $\rho_D \le \rho_{D_1}$ on D_1 for $D_1 \subset D$ (see [6]).

Lemma 2.1. *For a function $\varphi \in B_2(D)$ and for a subdomain D_1 of D, the inequality $\|\varphi\|_{D_1} \le \|\varphi\|_D$ holds.*

In particular, for a non-constant meromorphic function f on the domain D, we obtain $\|S_f\|_{D_1} \le \|S_f\|_D$ for any subdomain D_1 of D.

The next preliminary result is elementary, but, may have its own right.

Lemma 2.2. *Let A be a ring domain and C be a smooth simple closed curve in A separating two boundary components of A. Suppose that a locally univalent holomorphic function f on A is univalent in each component of $A \setminus C$. Then, f is necessarily univalent in A.*

Remark. The local univalence cannot be dropped in general. For instance, consider the function $f(z) = z + 1/z$ in the annulus $A = \{z; r < |z| < 1/r\}$. Then, f is univalent in each component of $A \setminus \{|z| = 1\}$. On the other hand, f is not univalent in A because f has branch at $z = \pm 1$.

Proof. Let A' and A'' be the two components of $A \setminus C$. Then $f(A')$ and $f(A'')$ are both ring domains and have the curve $f(C)$ as a boundary component in common. Since f is locally univalent and orientation preserving, $f(C)$ must be a Jordan curve and separates $f(A')$ from $f(A'')$. It is now easy to see that f is univalent in the whole A. □

Proof of monotoneity of $I(m)$. Using the above lemmas, we prove that $I(m)$ is non-decreasing in $0 < m \le \infty$. First we consider in $(0, \infty)$. It suffices to show that $I(m) \le I(m')$ provided that $m < m' < 2m$. Suppose that a non-constant meromorphic function f on $A_{m'}$ satisfies $\|S_f\|_{A_{m'}} \le I(m)$. Set $A = A_{m'}$ and $C = \{|z| = e^{m'/2}\}$. Then, by Lemma 2.1, we get $\|S_f\|_{B'} \le I(m)$ and $\|S_f\|_{B''} \le I(m)$, where $B' = A_m$ and $B'' = \{e^{m'-m} < |z| < e^m\}$. By definition of $I(m)$, we observe that f is univalent in B' and in B''. The hypotheses in Lemma 2.2 are now fulfilled. Hence, f is univalent in $A_{m'}$. Since f was arbitrary as far as the condition $\|S_f\|_{A_{m'}} \le I(m)$ is satisfied, we conclude that $I(m') = \sigma(A_{m'}) \ge I(m)$.

Secondly, we show the inequality $I(m) \le I(\infty)$ for any $0 < m < \infty$. Suppose that a non-constant meromorphic function f on A_∞ satisfies $\|S_f\|_{A_\infty} \le I(m)$ for some finite m. Then we have $\|S_f\|_{B_n} \le I(m)$, where $B_n = \{e^{(n-1)m} < |z| < e^{nm}\}$. In particular, f

is univalent in each B_n for $n = 1, 2, \ldots$. By the argument in the proof of Lemma 2.2, we see that the sequence of ring domains $f(B_n)$ is nesting, namely, each $f(B_n)$ separates $f(B_{n-1})$ from $f(B_{n+1})$. Therefore, we conclude that f is univalent in A_∞. This means that $I(m) \le I(\infty)$ for any finite m. □

The authors would like to raise the following apparently simple questions.

Problem 1. Is $I(m)$ continuous in $0 < m \le \infty$?
Problem 2. Is $I(m)$ strictly increasing in $0 < m < \infty$?

To illustrate the sharp contrast between the simply connected case and the multiply connected case, we give the following observation here. If D is simply connected, then the Schwarzian differential equation $S_f = \varphi$ for a given holomorphic φ on D has always a (single-valued) meromorphic solution in D. The same thing, however, no longer holds for a multiply connected domain. In fact, let $\varphi(z) = z^{-2}/2$. It is clear that $\varphi \in B_2(A_m)$ for all $m \in (0, \infty)$. For complex parameter c, we see that the equation $S_f = c\varphi$ has a single-valued meromorphic solution f in A_m if and only if c is written in the form $c = (1-n^2)$ for some non-zero integer n. This is explained by the fact that the function $f_\alpha(z) = (z^\alpha - 1)/\alpha$ defined in a neighbourhood of $z = 1$ satisfies $S_{f_\alpha} = (1-\alpha^2)\varphi$ and that f_α is analytically continued to a single-valued function in A_m if and only if α is a non-zero integer. In particular, the equation $S_f = c\varphi$ has no univalent (single-valued) solution in A_m unless $c = 0$.

Therefore, even if the norm of φ in $B_2(D)$ is very small, there does not necessarily exist a single-valued meromorphic function f satisfying $S_f = \varphi$ in the case when D is multiply connected.

Let us proceed to the investigation of the value of $I(\infty)$. We need first the following elementary lemma.

Lemma 2.3. *The function $q(x) = \rho_{\mathbb{D}}(x)/\rho_{\mathbb{D}^*}(x)$ is increasing in $0 < x < 1$ and satisfies $q(0^+) = 0$ and $q(1^-) = 1$.*

Proof. The function q is explicitly given by $q(x) = 2x\log(1/x)/(1-x^2)$. By differentiation, we have

$$\frac{1-x^2}{2}q'(x) = \frac{1+x^2}{1-x^2}\log\frac{1}{x} - 1.$$

Thus, $q'(x) > 0$ if and only if $Q(x) := \log(1/x) - (1-x^2)/(1+x^2) > 0$. We now compute

$$Q'(x) = -\frac{(1-x^2)^2}{x(1+x^2)^2} < 0.$$

Hence, $Q(x) > Q(1) = 0$. The boundary behaviour of q is obvious. □

Next, in order to prove Theorem 1.2, we need the following result concerning the Bers spaces.

Lemma 2.4. *The closed subspace $B_2^0(\mathbb{D}^*) = \{\varphi \in B_2(\mathbb{D}^*); \limsup_{z\to 0}|\varphi(z)| < \infty\}$ of $B_2(\mathbb{D}^*)$ is canonically isomorphic to $B_2(\mathbb{D})$. More precisely, the restriction map $B_2(\mathbb{D}) \to$*

$B_2^0(\mathbb{D}^*)$ *is bijective and the inequalities*

$$\left(\frac{2}{e}\right)^2 \|\varphi\|_{\mathbb{D}} \le \|\varphi\|_{\mathbb{D}^*} \le \|\varphi\|_{\mathbb{D}} \tag{2.1}$$

hold for every $\varphi \in B_2(\mathbb{D})$. *Equality holds in the left-hand side if and only if* φ *is a constant. In particular, the constant* $(2/e)^2 = 0.5413\ldots$ *is best possible.*

Proof. The restriction map $B_2(\mathbb{D}) \to B_2^0(\mathbb{D}^*)$ is a contraction, namely, $\|\varphi\|_{\mathbb{D}^*} \le \|\varphi\|_{\mathbb{D}}$ for $\varphi \in B_2(\mathbb{D})$ by Lemma 2.1. On the other hand, since every $\varphi \in B_2^0(\mathbb{D}^*)$ extends holomorphically to the origin, the surjectivity of the restriction map follows immediately. The Banach open mapping theorem now ensures the existence of a constant $C > 0$ for which $C\|\varphi\|_{\mathbb{D}} \le \|\varphi\|_{\mathbb{D}'}$ holds. We, however, deduce the latter part directly.

Fix t in $0 < t < 1$. Take an arbitrary $\varphi \in B_2(\mathbb{D})$ with $\|\varphi\|_{\mathbb{D}^*} = 1$. For $|z| = t$, we have the estimate $|\varphi(z)| \le \rho_{\mathbb{D}^*}(t)^2$. By the maximum modulus principle, the last inequality is still valid for $|z| \le t$. Therefore, we obtain $(1 - |z|^2)^2|\varphi(z)| \le \rho_{\mathbb{D}^*}(t)^2$ for $|z| \le t$. On the other hand, for $|z| > t$, we have $(1 - |z|^2)^2|\varphi(z)| \le q(|z|)^{-2} \le q(t)^{-2} = (1 - t^2)\rho_{\mathbb{D}^*}(t)^2$ by Lemma 2.3. In summary, we get the inequality $\sup(1 - |z|^2)^2|\varphi(z)| \le \rho_{\mathbb{D}^*}(t)^2$ for a fixed t. As is easily seen, the minimum of the density $\rho_{\mathbb{D}^*}(t)$ is attained at $t = 1/e$ and its value is $e/2$. We now choose $t = 1/e$ to obtain $\|\varphi\|_{\mathbb{D}} \le e^2/4$. In this way, we have shown the desired inequality.

The case when equality holds is easily analyzed by the above proof. □

Note that $B_2^0(\mathbb{D}^*)$ is a subspace of $B_2(\mathbb{D}^*)$ of codimension one. In fact, it is easy to show that $B_2(\mathbb{D}^*) = B_2^0(\mathbb{D}^*) \oplus \langle z^{-1} \rangle$ (see the proof of the following lemma). The next lemma tells us that the factor $1/z$ does nothing with the Schwarzian derivative of single-valued functions.

Lemma 2.5. *Let* f *be a meromorphic function on* $\mathbb{D}^*$ *satisfying* $\|S_f\|_{\mathbb{D}^*} < \infty$. *Then* f *meromorphically extends to* $\mathbb{D}$ *and satisfies* $\|S_f\|_{\mathbb{D}} < \infty$.

Proof. Set $\varphi = S_f$. Since $|\varphi(z)| \le \|\varphi\|_{\mathbb{D}^*}(2|z|\log(1/|z|))^{-2}$ holds for $z \in \mathbb{D}^*$, the function $z\varphi(z)$ has removable singularity at the origin. Hence, φ has an expression in the form $\varphi(z) = c/z + \varphi_0(z)$, where c is a constant and $\varphi_0 \in B_2^0(\mathbb{D}^*)$. It is well known that the function f can be expressed, at least locally, by the ratio of two linearly independent solutions of the differential equation

$$2y'' + \left(\frac{c}{z} + \varphi_0(z)\right) y = 0 \tag{2.2}$$

in the domain $\mathbb{D}^*$. We now use the general theory of the second order linear ODE. See, for instance, [2, Chap. 5.2] as a good source of knowledge. Since $z = 0$ is a regular singular point of the differential equation, a solution to (2.2) around the origin can be given by the power series $y = z^\lambda \sum_{n=0}^\infty a_n z^n$ with positive radius of convergence, where $a_0 \ne 0$. Substituting this form to (2.2), we get the equations

$$\lambda(\lambda - 1) = 0 \quad \text{and} \tag{2.3}$$

$$2(\lambda + 1)\lambda a_1 + c a_0 = 0. \tag{2.4}$$

Now we assume that $c \neq 0$. From (2.3), λ is either 0 or 1. If $\lambda = 0$, then (2.4) enforces $c = 0$. Thus, λ must be 1. We denote by y_1 the solution y given in the above form with $\lambda = 1$ and $a_0 = 1$. Note that y_1 is a single-valued holomorphic solution to (2.2) around the origin. Then, by the change of unknown functions given by $y = y_1\eta$, it turns out that non-constant η should satisfy $\eta' = by_1^{-2}$ for some constant $b \neq 0$. We may assume that $b = 1$. Since $a_1 = -c/4$ by (2.4), we observe

$$\eta' = \frac{1}{y_1^2} = \frac{1}{z^2}\left(1 - \frac{c}{4}z + \dots\right)^{-2} = \frac{1}{z^2} + \frac{c}{2z} + \dots.$$

In particular, η is never single-valued unless $c = 0$. Since the single-valued function f is locally written as $L \circ \eta$ for some Möbius transformation L, c must be 0. Hence, $\varphi = \varphi_0 \in B_2^0(\mathbb{D}^*)$. The last assertion follows from Lemma 2.4. □

The condition $c = 0$ certainly corresponds to the special case of Theorem 4.1 in [4] when $m = 1$ in their notation though this case was excluded. Actually, the assertion in the above lemma follows also from their proof of that theorem. We, however, decided to include the proof here because the involved calculation is much simper than in the general case considered in [4].

Proof of (1.1). We are now in a position to give a proof for (1.1) in Theorem 1.2. We recall that $I(\infty) = \sigma(\mathbb{D}^*)$ is the infimum of $\|S_f\|_{\mathbb{D}^*}$ over all non-univalent meromorphic functions f on $\mathbb{D}^*$ with $\|S_f\|_{\mathbb{D}^*} < \infty$. By Lemma 2.5, the range of f is same as the set of non-univalent meromorphic functions on $\mathbb{D}$ with finite Schwarzian norm. Fix any non-zero element $\varphi \in B_2(\mathbb{D})$. Let f_t be a meromorphic function in $\mathbb{D}$ determined by $S_{f_t} = t\varphi$ and by the normalization $f_t(0) = f_t'(0) - 1 = f_t''(0) = 0$. It is well known that $f_t(z)$ is holomorphic both in t and z as map with values taken in the Riemann sphere $\widehat{\mathbb{C}}$. Consider the sets $U = \{t \in \mathbb{C}; t\varphi \in \partial T(1)\}$ and $V = \{t \in \mathbb{C}; f_t \text{ non-univalent in } \mathbb{D}\}$. Put $r_0 = \inf\{|t|; t \in U\}$ and $r_1 = \inf\{|t|; t \in V\}$. It is enough to show that $r_0 = r_1$. Since $t\varphi \in T(1)$ whenever $|t| < r_0$, it is clear that $r_0 \leq r_1$. On the other hand, f_t is univalent whenever $|t| < r_1$ and $f_0 = \mathrm{id}$. This implies that f_t is a holomorphic motion of $\mathbb{D}$ over $|t| < r_1$. Therefore, $t\varphi \in T(1)$ for $|t| < r_1$. (This is a special case of a theorem of I. V. Zhuravlev, which was originally proved by using Grunsky's inequality. See [8] for more details.) We now conclude that $r_1 \leq r_0$, and hence, $r_0 = r_1$. □

Proof of (1.2). We next prove (1.2) in Theorem 1.2. The Kraus-Nehari theorem and the Ahlfors-Weill theorem implies that $2 \leq \|\varphi\| \leq 6$ for every $\varphi \in T(1)$ and the constants 2 and 6 are sharp (see [5]). Hence, the inequality $I(\infty) \leq 2$ follows from (1.1). On the other hand, we obtain the reverse inequality $I(\infty) \geq 8e^{-2}$. Indeed, for $\varphi \in \partial T(1)$, we have $\|\varphi\|_{\mathbb{D}^*} \geq (2/e)^2\|\varphi\|_{\mathbb{D}} \geq 2(2/e)^2$ by (2.1). □

The authors have tried to find a non-univalent function on $\mathbb{D}$ such that $\|S_f\|_{\mathbb{D}^*} < 2$, but have not ever succeeded. We would like to add one more comment. If $\varphi \in B_2(\mathbb{D})$ satisfies the condition

$$\|\varphi\|_{\mathbb{D}} = \limsup_{|z|\to 1}(1 - |z|^2)^2|\varphi(z)|,$$

then $\|\varphi\|_{\mathbb{D}^*} = \|\varphi\|_{\mathbb{D}}$ necessarily holds because of the fact $q(1^-) = 1$ in Lemma 2.3. Therefore, if we could have a non-univalent function with $\|S_f\|_{\mathbb{D}^*} < 2$, then the supremum of $(1-|z|^2)^2|S_f(z)|$ over $z \in \mathbb{D}$ must be attained in the interior of $\mathbb{D}$.

Proof of $I(m) \leq 6m^2/\pi^2$. We end this section with the proof of the right-side inequality in Theorem 1.1. The inequality $I(m) \leq I(\infty)$ has been done above. The remaining part is the inequality $I(m) \leq 6m^2/\pi^2$. We recall the explicit formula for ρ_{A_m} (see, for instance, [1]):

$$\frac{1}{\rho_{A_m}(z)} = \frac{2m|z|}{\pi} \sin\left(\pi \frac{\log|z|}{m}\right). \tag{2.5}$$

We consider the non-univalent function $f(z) = z^2$ on A_m. Then f has the Schwarzian derivative $S_f(z) = -3/2z^2$. (The sharpness in Theorem B is due to this function.) Using (2.5), we compute

$$\|S_f\|_{A_m} = \sup_{1<x<e^m} \frac{6m^2}{\pi^2} \sin^2\left(\pi \frac{\log x}{m}\right) = \frac{6m^2}{\pi^2}.$$

□

3. Lower estimate

In this section, we give a proof for the left-side inequality in Theorem 1.1. By Lemma 2.1, our problem reduces to the case of simply connected domain. Note that this step is similar to the argument of Osgood developed in [6]. Indeed, let B_m be the upper half of A_m, namely, $B_m = \{z \in A_m; \operatorname{Im} z > 0\}$. Then we have the following.

Lemma 3.1. *The inequality $\sigma(A_m) \geq \sigma(B_m)$ holds.*

Proof. Let f be a non-constant meromorphic function on A_m satisfying $\|S_f\|_{A_m} < \sigma(B_m)$. Suppose that f is not univalent in A_m. Then, there are points z_1 and z_2 in A_m with $z_1 \neq z_2$ such that $f(z_1) = f(z_2)$. By using a suitable rotation, we may assume that both z_1 and z_2 belong to $\overline{B_m}$. Since $\|S_f\|_{B_m} \leq \|S_f\|_{A_m} < \sigma(B_m)$ by Lemma 2.1, the restriction of f to B_m admits a quasiconformal extension to $\widehat{\mathbb{C}}$ (see [5, III.5.3]). This is impossible because $f(z_1) = f(z_2)$, and thus, f must be univalent in B_m. We now conclude that $\sigma(A_m) \geq \sigma(B_m)$. □

By virtue of Ahlfors' characterization of quasicircles, the domain B_m is a K-quasidisk for some $K = K(m)$. Therefore, we would have an estimate of the type $\sigma(B_m) \geq c(K)$, where $c(K)$ is a positive constant depending only on K. However, we have no precise information about $K = K(m)$ and $c(K)$. To get rid of such difficulty, we will construct a quasiconformal reflection in ∂B_m. The following is the key result for our proof.

Theorem C (Ahlfors-Lehto univalence criterion [7]). *Let D be a quasidisk with quasiconformal reflection λ in ∂D. Then, the following estimate is obtained:*

$$\sigma(D) \geq \varepsilon(\lambda, D) := 2 \operatorname*{ess.inf}_{z\in D} \frac{|\bar{\partial}\lambda(z)| - |\partial\lambda(z)|}{|\lambda(z) - z|^2 \rho_D(z)^2}.$$

We remark that the quantity $\varepsilon(\lambda, D)$ is invariant under the Möbius conjugation, in other words, $\varepsilon(L\circ\lambda\circ L^{-1}, L(D)) = \varepsilon(\lambda, D)$ for a Möbius transformation L.

In order to make the symmetry of B_m clear, we transform B_m by the Möbius map $L(z) = (z-ie^{m/2})/(z+ie^{m/2})$. Then the resulting domain can be described by $D = \mathbb{D}\setminus(C_+\cup C_-)$, where $C_+ = \{z\in\mathbb{C}; |z-\coth(m/2)|\le 1/\sinh(m/2)\}$ and $C_- = \{z; -z\in C_+\}$. Note that the circle ∂C_+ intersects the unit circle $\partial\mathbb{D}$ perpendicularly at the points $e^{i\alpha}$ and $e^{-i\alpha}$, where α is the number between 0 and $\pi/2$ determined by $\cos\alpha = \tanh(m/2)$, equivalently, $\tan(\alpha/2) = e^{-m/2}$.

In order to construct a quasiconformal reflection in ∂D, we divide D into three parts D_0, D_+, D_- by setting $D_0 = D\setminus(W_+\cup W_-)$, $D_+ = D\cap W_+ = W_+\setminus C_+$, and $D_- = \{z; -z\in D_+\} = D\cap W_- = W_-\setminus C_-$, where $W_+ = \{z; |2z-\cos\alpha| < \cos\alpha\}$ and $W_- = \{z; -z\in W_+\}$. Note also that the circle ∂W_+ intersects $\partial\mathbb{D}$ at the same points $e^{i\alpha}$ and $e^{-i\alpha}$ as above. We construct the quasiconformal reflection λ in ∂D in such a way that λ is compatible with the reflection $j(z) = -\bar z$ in the imaginary axis, namely, $\lambda\circ j = j\circ\lambda$. Roughly speaking, our reflection will be the usual reflection in the unit circle in D_0 and the usual reflection in the circle $\partial C_\pm$ followed by the angle-stretching map of expansion $K = (3\pi-2\alpha)/(\pi-2\alpha)$ in $D_\pm$.

To give a precise definition, we introduce the auxiliary transformation $M(z) = i(e^{i\alpha}z - 1)/(z-e^{i\alpha})$. A simple calculation shows that $D' = M(D)$, $D'_0 = M(D_0)$, $D'_+ = M(D_+)$ and $D'_- = M(D_-)$ are expressed by

$$\begin{aligned} D' &= \{z\in\mathbb{C}; \operatorname{Im} z > 0, \operatorname{Re} z > 0, |z - i\coth m| > 1/\sinh m\},\\ D'_0 &= \{z\in\mathbb{C}\setminus W'_-; \pi/2-\alpha < \arg z < \pi/2\},\\ D'_+ &= \{z\in\mathbb{C}; 0 < \arg z < \pi/2-\alpha\}, \quad \text{and}\\ D'_- &= \{z\in W'_-; |z - i\tanh m| < 1/\sinh m\}. \end{aligned}$$

Here $W'_- = M(W_-)$ is the interior of the circle which passes through three points $M(-e^{i\alpha}) = i\cos\alpha$, $M(-e^{-i\alpha}) = i/\cos\alpha$ and $M(0) = e^{i(\pi/2-\alpha)}$, namely,

$$W'_- = \left\{ z\in\mathbb{C}; \left| z - \frac{1}{2\cosh(m/2)} - i\coth m \right| < \frac{1}{2\sinh(m/2)} \right\}.$$

Define the map $\nu' : D'_+ \to \{z\in\mathbb{C}; -3\pi/2+\alpha < \arg z < 0\}$ by $\nu'(z) = |z|(\bar z/|z|)^K$, in other words, $\nu'(re^{i\theta}) = re^{-iK\theta}$, where $K = (3\pi-2\alpha)/(\pi-2\alpha)$ and set $\nu = M^{-1}\circ\nu'\circ M$. We are now in a position to give the definition of our quasiconformal reflection λ. Set

$$\lambda(z) = \begin{cases} 1/\bar z & \text{for } z\in D_0,\\ \nu(z) & \text{for } z\in D_+,\\ (j\circ\nu\circ j)(z) & \text{for } z\in D_-. \end{cases}$$

We also set $\lambda' = M\circ\lambda\circ M^{-1}$ on D'. By definition, the map λ' is nothing but the reflection $j(z) = -\bar z$ on D'_0. Noting $j(z) = \nu'(z)$ on the ray $\arg z = \pi/2-\alpha$, we see that λ' is continuous in D', and hence, it is a (sense-reversing) homeomorphism from D' onto $\widehat{\mathbb{C}}\setminus\overline{D'}$. Since the angle-stretching map of expansion $K>1$ is K-quasiconformal, λ' is, moreover, an anti-quasiconformal homeomorphism. Thus, λ maps the domain D anti-quasiconformally onto its exterior $\widehat{\mathbb{C}}\setminus\overline{D}$ in such a way that λ keeps the boundary fixed pointwise. Defining as λ^{-1} on $\widehat{\mathbb{C}}\setminus\overline{D}$, we get the desired quasiconformal reflection

$\lambda : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ in ∂D. At this point, we can single out the following as a consequence of the above construction.

Proposition 3.2. *The quasidisk* $B_m = \{z; 1 < |z| < e^m, \operatorname{Im} z > 0\}$ *admits a* K*-quasiconformal reflection in its boundary, where* $K = (3\pi - 2\alpha)/(\pi - 2\alpha)$ *and* $\alpha = \operatorname{arccos} \tanh(m/2)$.

Let us calculate the quantity $\varepsilon(\lambda, D)$. By the symmetric property $\lambda \circ j = j \circ \lambda$, it is enough for us to take the infimum only over $z \in D_0 \cup D_+$. Furthermore, by the Möbius invariance of the quantity $(|\bar\partial\lambda| - |\partial\lambda|)/|\lambda(z) - z|^2 \rho_D(z)^2$ (see [5]), we have

$$\varepsilon(\lambda, D) = 2 \operatorname*{ess.inf}_{z \in D'_0 \cup D'_+} \frac{|\bar\partial\lambda'(z)| - |\partial\lambda'(z)|}{|\lambda'(z) - z|^2 \rho_{D'}(z)^2}.$$

Regarding ν' as the function $\bar z^{(1+K)/2} z^{(1-K)/2}$, we compute

$$\bar\partial\nu'(z) = \frac{(1+K)}{2}\left(\frac{\bar z}{z}\right)^{(K-1)/2} \text{ and } \partial\nu'(z) = \frac{(1-K)}{2}\left(\frac{\bar z}{z}\right)^{(K+1)/2}.$$

In particular, we have $|\bar\partial\lambda'| - |\partial\lambda'| = 1$ in D'_+. The last identity holds also in the interior of D'_0. Hence, we obtain

$$\varepsilon(\lambda, D) = \frac{2}{Q^2}, \text{ where } Q = \sup_{z \in D'_0 \cup D'_+} |\lambda'(z) - z| \rho_{D'}(z). \tag{3.1}$$

Although we can give a conformal map of the domain D' (equivalently, of B_m) in terms of Jacobi's elliptic functions in an explicit way, it seems difficult to compute the quantity $\rho_{D'}$ directly. Therefore, we will estimate $\rho_{D'}$ by using several available techniques and then obtain a consequence on Q. Set $M(z) = |\lambda'(z) - z| \rho_{D'}(z)$. We divide the estimation into several parts.

Case 1. $z \in D'_0$; In this case, as we noted, $\lambda'(z) = -\bar z$. We recall the well-known inequality $\rho_{D'}(z) \le 1/\delta(z)$, where $\delta(z) = \operatorname{dist}(z, \partial D')$ denotes the Euclidean distance from z to the boundary of D'. Therefore, we obtain the estimate

$$Q(z) \le \frac{2x}{\delta(z)}, \quad z = x + iy.$$

We now estimate $\delta(z)$ from below. Let a be a point in $\partial D'$ satisfying $\delta(z) = |z - a|$. We now further divide this case into three subcases according to the place of a.
Case 1a. $\operatorname{Re} a = 0$; Then clearly $\delta(z) = x$, and hence, $Q(z) \le 2$.
Case 1b. $\operatorname{Im} a = 0$; Then $\delta(z) = y$. Since $y \ge x \tan(\pi/2 - \alpha) = x/\tan\alpha$, we obtain $Q(z) \le 2x/y \le 2\tan\alpha = 2/\sinh(m/2)$.
Case 1c. $|a - i\coth m| = 1/\sinh m$; Then $\delta(z) = |z - i\coth m| - 1/\sinh m$. Now consider the set $U = \{w \in \mathbb{C}; |w - i\coth m| - 1/\sinh m < c \operatorname{Re} w\}$ for a constant $0 < c < 1$. We recognize that U is the ellipse expressed by

$$(1 - c^2)\left(u - \frac{c}{(1-c^2)\sinh m}\right)^2 + (v - \coth m)^2 < \frac{1}{(1-c^2)\sinh^2 m}, \quad w = u + iv.$$

Since the major axis of the ellipse U is the line segment $\{u + i\coth m; -1/(1+c)\sinh m < u < 1/(1-c)\sinh m\}$, it is contained in the disk W'_- when

$$\frac{1}{(1-c)\sinh m} \le \frac{1}{2}\left(\frac{1}{\sinh(m/2)} + \frac{1}{\cosh(m/2)}\right).$$

The last condition is equivalent to $c \leq 1 - e^{-m/2}$. Since z lies in the outside of W'_-, we conclude that $\delta(z) = |z - i\coth m| - 1/\sinh m \geq (1 - e^{-m/2})x$, and therefore, $Q(z) \leq 2/(1 - e^{-m/2})$.

Because $2 < 2/\sinh(m/2) < 2/(1 - e^{-m/2})$ holds for $m > 0$, we finally obtain the estimate $Q(z) \leq 2/(1 - e^{-m/2})$ in Case 1.

Case 2. $z \in D'_+$; In this case, the expression $z = re^{i\theta}$ in polar coordinates is more convenient. Note that $0 < \theta < \pi/2 - \alpha$. For later convenience, we set $\gamma = \pi/2 - \alpha$. Recall that $\cos\gamma = \sin\alpha = 1/\cosh(m/2)$. Since $\lambda'(z) = r^{-iK\theta}$, where $K = (3\pi - 2\alpha)/(\pi - 2\alpha) = (\pi + \gamma)/\gamma$, it follows that $|\lambda'(z) - z| = r|e^{-iK\theta} - e^{i\theta}| = 2r\sin((K + 1)\theta/2)$.

Let $\beta \in (0, \pi/2)$ be the angle satisfying $\cos\beta = 1/\cosh m$. Then, the sector $\Omega = \{w; 0 < \arg w < \beta\}$ is contained in D'. Also, $z \in \Omega$ because $\cos\theta < \cos\gamma < \cos\beta$. Since by the transformation $w \mapsto w^{\pi/\beta}$ the sector Ω is mapped onto the upper half-plane, we compute

$$\rho_\Omega(z) = \frac{\pi}{2\beta}\frac{|z|^{\pi/\beta - 1}}{\mathrm{Im}\,(z^{\pi/\beta})} = \frac{\pi}{2\beta r\sin(\pi\theta/\beta)}.$$

By the inequality $\rho_{D'}(z) < \rho_\Omega(z)$, we obtain the estimate

$$Q(z) < \frac{\pi}{\beta}\frac{\sin(\frac{K+1}{2}\theta)}{\sin(\frac{\pi}{\beta}\theta)}. \tag{3.2}$$

The next lemma ensures the inequality $\pi/\beta < (K + 1)/2 = \pi/2\gamma + 1$.

Lemma 3.3. *The inequality*

$$\frac{\pi}{\arccos(1/\cosh m)} < \frac{\pi}{2\arccos(1/\cosh(m/2))} + 1$$

holds for each number $0 < m < \infty$.

Proof. Let $\beta, \gamma \in (0, \pi/2)$ be the numbers determined by $\cos\beta = 1/\cosh m$ and $\cos\gamma = 1/\cosh(m/2)$ as above. Then $\cos(2\gamma) = 2\cos^2\gamma - 1 = 4/(1 + \cosh m) - 1 = 4/(1 + 1/\cos\beta) - 1 = (3\cos\beta - 1)/(\cos\beta - 1)$. Since the required inequality is equivalent to $2\gamma < \pi\beta/(\pi - \beta)$, it is enough to show that the function

$$F(x) = \frac{3\cos x - 1}{\cos x + 1} - \cos\left(\frac{\pi x}{\pi - x}\right)$$

is positive in $0 < x < \pi/2$. A simple computation yields that $\sin(\pi x/(\pi - x)) \geq \sin x$ for $0 < x \leq 3\pi/8$. Therefore,

$$\begin{aligned} F'(x) &= \frac{\pi^2}{(\pi - x)^2}\sin\left(\frac{\pi x}{\pi - x}\right) - \frac{4\sin x}{(1 + \cos x)^2} \\ &\geq \left(\frac{\pi}{\pi - x} + \frac{2}{1 + \cos x}\right)\left(\frac{\pi}{\pi - x} - \frac{2}{1 + \cos x}\right)\sin x \end{aligned}$$

holds in $0 < x \leq 3\pi/8$. Since the concavity of the function $1 + \cos x$ implies the inequality $1 + \cos x > 2 - 2x/\pi$ in $0 < x < \pi/2$, the positivity of $F'(x)$ follows in $0 < x < 3\pi/8$. In particular, $F(x) > F(0) = 0$ holds for $0 < x \leq 3\pi/8$.

We now concentrate on the case when $3\pi/8 < x < \pi/2$. Since the function $\pi x/(\pi - x)$ is convex, the inequality $\pi > \pi x/(\pi - x) > 4x - \pi$ can be obtained in this interval. Hence,

$$F(x) > \frac{3\cos x - 1}{\cos x + 1} - \cos(4x - \pi) = \frac{3\cos x - 1}{\cos x + 1} + \cos(4x).$$

Setting $G(x) = 2(1 + \cos x)F(x) = 6\cos x - 2 + 2\cos(4x) + \cos(3x) + \cos(5x)$, we see that

$$\begin{aligned} G''(x) &= -6\cos x - 32\cos(4x) - 9\cos(3x) - 25\cos(5x) \\ &= -6\cos x - 32\cos(4x) - 16\cos(5x) - 18\cos(4x)\cos x < 0 \end{aligned}$$

in $3\pi/8 < x < \pi/2$. The last inequality means that G is concave, and hence, $G(x) > \min\{G(3\pi/8), G(\pi/2)\} = 0$ in $3\pi/8 < x < \pi/2$. Therefore, we obtain the positivity of $F(x)$ in this case, too. □

We need one more elementary lemma.

Lemma 3.4. *Let* $0 < L < M$*. Then the function* $\sin(M\theta)/\sin(L\theta)$ *is decreasing in* $0 < \theta < \pi/2M$*. In particular,* $\sin(M\theta)/\sin(L\theta) < M/L$ *holds there.*

Proof. Set $f(\theta) = \sin(M\theta)/\sin(L\theta)$. Then,

$$\begin{aligned} \sin^2(L\theta)f'(\theta) &= M\cos(M\theta)\sin(L\theta) - L\sin(M\theta)\cos(L\theta) \\ &= -LM(M^2 - L^2)\frac{\theta^2}{2} + O(|\theta|^3), \quad \theta \to 0. \end{aligned}$$

In particular, $f'(\theta) < 0$ when $\theta > 0$ is sufficiently small. On the other hand, $f'(\theta) = 0$ if and only if $g(M\theta) = g(L\theta)$, where $g(t) = (\tan t)/t$. Since $g(t)$ is (strictly) increasing in $0 < t < \pi/2$, it follows that $f'(\theta) \neq 0$ for $0 < \theta < \pi/2M$. Thus, $f'(\theta) < 0$ throughout $0 < \theta < \pi/2M$, which means that $f(\theta)$ is decreasing. □

The above lemma can now be applied to (3.2) to get the upper estimate $Q(z) \le (K + 1)/2 = 2(\pi - \alpha)/(\pi - 2\alpha)$ in Case 2.

Summarizing the above observations, we come to the estimate

$$Q \le \max\left\{\frac{2}{1 - e^{-m/2}}, \frac{2(\pi - \alpha)}{\pi - 2\alpha}\right\}.$$

Finally, we show the inequality

$$\frac{2(\pi - \alpha)}{\pi - 2\alpha} < \frac{2}{1 - e^{-m/2}}. \tag{3.3}$$

Once we obtain this, we have the inequality $\varepsilon(\lambda, D) = 2/Q^2 \ge (1 - e^{-m/2})^2/2$ by recalling (3.1). Theorem C then produces the desired estimate $\sigma(A_m) \ge (1 - e^{-m/2})^2/2$.

Inequality (3.3) is equivalent to $\alpha < \pi/(1 + e^{m/2})$. Since $\sin\alpha = 1/\cosh(m/2)$, the last inequality is further equivalent to $\sin(\pi/(1 + e^{m/2})) > 1/\cosh(m/2) = 2/(e^{m/2} + e^{-m/2})$. If we set $x = 1/(1 + e^{m/2}) \in (0, 1/2)$, we can write the last inequality as

$$\sin(\pi x) > \frac{2x(1 - x)}{1 - 2x + 2x^2},$$

or equivalently,

$$F(x) := \frac{4x(1-x)}{\sin(\pi x)} < 1 + 4\left(x - \frac{1}{2}\right)^2. \tag{3.4}$$

It is known (and actually easily verified) that F is convex and $1 \le F(x) < 4/\pi$ holds in $0 < x < 1$. This information, however, is not sufficient to show (3.4). We show now that $F''(x) < 8$. Then $F'(x) = F'(x) - F'(1/2) < 8(x - 1/2)$ and $F(x) - F(1/2) < 4(x - 1/2)^2$ will follow for $x \in (0, 1/2)$. The final inequality is nothing but (3.4).

A direct calculation gives us

$$F''(x) = -\frac{8\sin^2(\pi x) + 8\pi(1-2x)\sin(\pi x)\cos(\pi x) + 4\pi^2 x(1-x)(\sin^2(\pi x) - 2)}{\sin^3(\pi x)}.$$

Therefore, the inequality $F''(x) < 8$ is equivalent to the positivity of the function

$$G(x) = 2\sin^3(\pi x) + 2\sin^2(\pi x) + 2\pi(1-2x)\sin(\pi x)\cos(\pi x) + \pi^2 x(1-x)(\sin^2(\pi x) - 2).$$

Using the inequality $\cos(\pi x) \ge 1 - 2x$ for $0 < x < 1/2$, we estimate

$$\begin{aligned} G'(x) &= \pi\sin(\pi x)\Big[\big\{2\pi^2 x(1-x) + 6\sin(\pi x)\big\}\cos(\pi x) - 3(1-2x)\pi\sin(\pi x)\Big] \\ &\ge \pi\sin(\pi x)\Big[\big\{2\pi^2 x(1-x) + 6\sin(\pi x)\big\}(1-2x) - 3(1-2x)\pi\sin(\pi x)\Big] \\ &= 3\pi(\pi-2)(1-2x)\sin(\pi x)\left[\frac{2\pi^2}{3(\pi-2)}x(1-x) - \sin(\pi x).\right] \end{aligned}$$

Since $2\pi^2/3(\pi - 2) = 5.76\cdots > 4$, we conclude that $G'(x) > 0$ for $0 < x < 1/2$ from the inequality $F(x) > 1$. Hence, G is increasing and, in particular, $G(x) > G(0) = 0$ in $0 < x < 1/2$. Now we have shown all we needed.

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Department of Mathematics, Kyoto University, 606-8502 Kyoto, Japan

Current address: Department of Mathematics, University of Helsinki, P.O.Box 4 (Yliopistonkatu 5), FIN-00014, Helsinki, Finland

E-mail address: sugawa@kusm.kyoto-u.ac.jp