ON THE ACTION OF THE MAPPING CLASS GROUP
FOR RIEMANN SURFACES OF INFINITE TYPE

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ABSTRACT. We consider Riemann surfaces of infinite type and their reduced
Teichmüller spaces. The reduced Teichmüller space admits the action of the
reduced mapping class group. Generally, the action is not discrete while it
is faithful. We give sufficient conditions for the discreteness of the action in
terms of the geometry of Riemann surfaces.

1. INTRODUCTION

The mapping class group (or modular group) Mod(R) for a Riemann surface
R is the set of equivalence classes of quasiconformal self-maps of R (cf. [7]). Two
quasiconformal self-maps \( h_1 \) and \( h_2 \) of \( R \) are equivalent if \( h_2^{-1} \circ h_1 \) is homotopic
to the identity by a homotopy that keeps every points of ideal boundary \( \partial R \) fixed
throughout. In the theory of Teichmüller spaces of Riemann surfaces of analytically
finite type, the mapping class group plays an important role in various fields. This
is a group of the biholomorphic automorphisms of the Teichmüller space and it
acts faithfully and discontinuously. On the other hand, it seems that there are
few studies on Mod(R) for a Riemann surface \( R \) of infinite type. Recently, Earle-
Gardiner-Lakic showed in [3] that it acts faithfully on \( T(R) \). In this paper, we
consider the discreteness of the action of the mapping class group. We say that
a subgroup \( G \) of Mod(R) is discrete if the orbit of any point of \( T(R) \) under the
\( G \)-action is discrete.

For a Riemann surface of analytically finite type, Mod(R) is discrete, while in
the case of infinite type, Mod(R) is not necessarily discrete. In particular, if \( R \)
has a boundary curve (border), Mod(R) is not discrete since a slight change of
the boundary value of a quasiconformal map produces a different mapping class in
Mod(R). Thus, it is natural that we consider another group, the reduced mapping
class group. The reduced mapping class group Mod\(^\#\)(R) is the set of homotopy
classes of quasiconformal self-maps of \( R \) whose homotopy maps does not necessarily
keep points of \( \partial R \) fixed. The reduced mapping class group is also important because
it naturally acts on the reduced Teichmüller space.

We explore the problem of discreteness of the reduced mapping class group for
Riemann surfaces of infinite type. Actually, if \( R \) is a Riemann surface of topolog-
ically finite type, then Mod\(^\#\)(R) is discrete. However, Mod\(^\#\)(R) is not discrete in
general. For example, if \( R \) has a sequence of disjoint simple closed geodesics which
are not freely homotopic to a boundary component and whose lengths tend to 0,
then we see that Mod\(^\#\)(R) is not discrete (See §3 and §6). The purpose of this
paper is to give a sufficient condition for discreteness.

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2. THE MAPPING CLASS GROUP FOR THE REDUCED TEICHMÜLLER SPACE

Throughout this paper, we assume that a Riemann surface $R$ is hyperbolic, that is, it is represented by $\mathbb{H}/\Gamma$ for some Fuchsian group $\Gamma$ acting on the upper half plane $\mathbb{H}$. We also assume that the Fuchsian group $\Gamma$ is always non-elementary. In other words, we assume that the group $\Gamma$ is non-abelian. A Riemann surface is called of analytically finite type if the hyperbolic area is finite, and is called of analytically infinite type if the area is not finite.

For an open Riemann surface $R$, a relatively non-compact connected component of the complement of a compact subset of $R$ is called an end. An end $V$ of $R$ is called a hole if it is doubly connected and the hyperbolic area of $V$ is infinite. A doubly connected end of $R$ is called a cusp if the hyperbolic area of $V$ is finite. A cusp $V$ with smooth relative boundary is conformally equivalent to the punctured disk $\{0 < |z| < 1\}$. An ideal boundary of $R$ corresponding to the origin $z = 0$ is called a puncture.

**Notation.** The hyperbolic distance between $A, B \subset \mathbb{H}$ is denoted by $d_{\mathbb{H}}(A, B)$ and the hyperbolic length of a curve $c$ in $\mathbb{H}$ or a Riemann surface $R$ is denoted by $\ell(c)$.

We review the theory of Teichmüller spaces and the mapping class group. See [4], [5] and [7] for the details.

**Definition 1.** Fix a Riemann surface $R$. For a pair $(S, f)$ of a Riemann surface $S$ and a quasiconformal map $f$ of $R$ onto $S$, we say that $(S_1, f_1)$ and $(S_2, f_2)$ are $RT$ (reduced Teichmüller) equivalent if there exists a conformal map $h$ of $S_1$ onto $S_2$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity on $R$.

The reduced Teichmüller space $T^\#(R)$ with the base Riemann surface $R$ is the set of all $RT$ equivalence classes $[S, f]$ of such pairs $(S, f)$ as above.

**Definition 2.** We say that two quasiconformal self-maps $h_1$ and $h_2$ of $R$ are $RT$ equivalent if $h_2^{-1} \circ h_1$ is homotopic to the identity on $R$.

The reduced mapping class group $\text{Mod}^\#(R)$ is the set of all $RT$ equivalence classes $[h]$ of quasiconformal self-maps $h$ of $R$.

Every quasiconformal map of $R = \mathbb{H}/\Gamma$ induces an isomorphism of $\Gamma$ into $\text{PSL}(2, \mathbb{R})$. We see that if two self-maps $h_1$ and $h_2$ are $RT$ equivalent then they induce the same isomorphism modulo $\text{PSL}(2, \mathbb{R})$ conjugacy.

If $R$ is a compact Riemann surface, then the reduced Teichmüller space $T^\#(R)$ is nothing but the ordinary Teichmüller space $T(R)$ of $R$ and the reduced mapping class group $\text{Mod}^\#(R)$ is the ordinary mapping class group $\text{Mod}(R)$.

Similar to the case of $T(R)$, the reduced Teichmüller space $T^\#(R)$ is equipped with the reduced Teichmüller distance $d_T(\cdot, \cdot)$ defined by

$$d_T([S, f], [S', g]) = \frac{1}{2} \inf_{f, g} K(f \circ g^{-1}),$$

where $K(\cdot)$ is the maximal dilatation of a quasiconformal map and the infimum is taken over all quasiconformal maps $f$ and $g$ determining $[S, f]$ and $[S', g]$, respectively. It is known that $T^\#(R)$ is a complete metric space with respect to this $d_T$.

An element $\omega = [h] \in \text{Mod}^\#(R)$ induces an automorphism of $T^\#(R)$ by

$$[S, f] \mapsto [S, f \circ h^{-1}].$$
This is an isometric automorphism with respect to \( d_T \) and denoted by \( \omega \). Namely, we have a homomorphism \( \Mod^\#(R) \rightarrow \Aut T^\#(R) \).

**Remark 1.** In [3], it is proved that for any Riemann surface \( R \) of infinite analytic type (and if \( 2g + n > 4 \) when \( R \) is of finite \((g,n)\)-type), the homomorphism \( \Mod^\#(R) \rightarrow \Aut T^\#(R) \) as above is faithful. Therefore we can identify \( \omega \) with \( \omega \) and omit the asterisk hereafter.

**Definition 3.** We say that a subgroup \( G \) of \( \Mod^\#(R) \) is discrete if every sequence \( \{\omega_n\} \subset G \) satisfying \( \lim_{n \to \infty} \omega_n(p) = q \) for some pair of points \( p, q \) in \( T^\#(R) \) is eventually a constant sequence, that is, there exists an \( N \in \mathbb{N} \) such that \( \omega_n = \omega_N \) for every \( n \geq N \).

3. Examples

As we noted in the introduction, if \( R \) is a compact Riemann surface, then the action of \( \Mod(R) \) on \( T(R) \) is discrete. Contrary to this case, there are various kinds of examples which show non-discreteness of \( \Mod^\#(R) \) for a Riemann surface \( R \) of infinite type.

**Example 1.** Suppose that \( R \) has a sequence \( \{c_n\} \) of disjoint simple closed geodesics that are not peripheral (i.e. that are not freely homotopic to a boundary component) and that these hyperbolic lengths tend to 0. Then the Dehn twist along each \( c_n \) gives an element \( \omega_n \) of \( \Mod^\#(R) \) such that the sequence \( \{\omega_n(p_0)\} \) converges to \( p_0 \) as \( n \to \infty \), where \( p_0 = [R, id] \) is the base point of \( T^\#(R) \). Hence \( \Mod^\#(R) \) is not discrete.

There exists a Riemann surface \( R \) without short geodesics but containing a point with arbitrarily large injectivity radius with respect to the hyperbolic metric such that \( \Mod^\#(R) \) is not discrete.

**Example 2.** Set

\[
R = \mathbb{C} - \bigcup_{n=1}^{\infty} \bigcup_{m \in \mathbb{Z}} \{ \frac{m}{n} + (2n + 1)\sqrt{-1} \}.
\]

Then we can see that \( R \) has no short geodesics by considering the classical extremal domains, whereas the injectivity radii at \( 2ni \) tend to \( \infty \).

Further, set

\[
f_n(z) = \begin{cases} 
x - (y - 2n - 2)/n + y\sqrt{-1} & (2n + 1 \leq y < 2n + 2) \\
x + (y - 2n)/n + y\sqrt{-1} & (2n \leq y < 2n + 1) \\
x + y\sqrt{-1} & \text{elsewhere.}
\end{cases}
\]

Then \( f_n \) are quasiconformal self-maps of \( R \) and the maximal dilatations of \( \{f_n\} \) tend to \( 1 \). Thus \( \Mod^\#(R) \) is not discrete.

There is another example of a planar Riemann surface \( R \) without cusps but containing a point with arbitrarily large injectivity radius with respect to the hyperbolic metric such that \( \Mod^\#(R) \) is not discrete.

**Example 3.** For each \( n \geq 2 \), we set

\[
I_n = [-1, 1] \cup \bigcup_{k=1}^{\infty} I_{n,k},
\]

This is an isometric automorphism with respect to \( d_T \) and denoted by \( \omega \). Namely, we have a homomorphism \( \Mod^\#(R) \rightarrow \Aut T^\#(R) \).
where
\[ I_{n,k} = \{ x + (1 - 1/n)k\sqrt{-1} \mid -1 \leq x \leq 1 \} \]
\[ \cup \{ x + (1 + (k - 1)/n)\sqrt{-1} \mid -1 \leq x \leq 1 \}. \]

We take infinitely many copies \( \{ R_n \} \) of \( \mathbb{C} - \{ y\sqrt{-1} \mid y \leq -1 \} \) and set \( R_n' = R_n - I_n \) for each \( n \geq 2 \). We make a Riemann surface \( R \) by gluing the right hand side of \( \{ y\sqrt{-1} \mid y < -1 \} \) on \( R_n' \) with the left hand side of \( \{ y\sqrt{-1} \mid y < -1 \} \) on \( R_{n-1}' \) \( (n = 3, 4, \ldots) \) along the imaginary axis.

Consider a quasiconformal map \( f_n \) of \( R_n' \) defined by
\[
\begin{align*}
f_n(z) &= \begin{cases} 
  x + (1 - 1/n)y\sqrt{-1} & (0 < y \leq 1) \\
  x + (y - 1/n)\sqrt{-1} & (y > 1) \\
  x + y\sqrt{-1} & \text{elsewhere.}
\end{cases}
\end{align*}
\]

It is easily seen that the maximal dilatations of \( f_n \) converge to 1 as \( n \to \infty \). Obviously, \( f_n \) is extended to a quasiconformal self-map of \( R \) by setting it the identity on \( R - R_n' \) and we denote it by the same letter \( f_n \). Thus the quasiconformal map \( f_n \) gives an element \([f_n]\) of \( \text{Mod}^\#(R) \) such that \([f_n](p_0)\) converges to \( p_0 \) as \( n \to \infty \), where \( p_0 = [R, \text{id}] \) is the base point of \( T^\#(R) \). Hence we conclude that \( \text{Mod}^\#(R) \) is not discrete.

Even if a Riemann surface \( R \) has no short geodesics and no points with arbitrarily large injectivity radius, \( \text{Mod}^\#(R) \) may not be discrete as the following example shows:

**Example 4.** Consider a torus \( S \) with two geodesic borders with the same length to each other. We take infinity many copies \( \{ S_n \}_{n=-\infty}^\infty \) of \( S \). We denote the two geodesic borders of \( S_n \) by \( \ell_{n,1} \) and \( \ell_{n,2} \). Construct a Riemann surface \( R \) by gluing the \( \ell_{n-1,2} \) with \( \ell_{n,1} \) and gluing \( \ell_{n,2} \) with \( \ell_{n+1,1} \) for each \( n \). Let \( f \) be a conformal self-map of \( R \) which sends a point \( a \in S_n \) to a point of \( S_{n+1} \) corresponding to the same point of \( S \) as \( a \), and we set \( f_n := f^n \). Then we see that \([f_n] \neq \text{id}\) as an element of \( \text{Mod}^\#(R) \). However, \([f_n](p_0) = p_0 \) for all \( n \), where \( p_0 = [R, \text{id}] \in T^\#(R) \) because \( f_n : R \to R \) is a conformal mapping. Hence, \( \text{Mod}^\#(R) \) is not discrete.

4. Main Results

As Example 1 shows, for the discreteness of the mapping class group, it is necessary that there exists no sequence of geodesics on the Riemann surface whose lengths converge to zero. Example 2, 3 show that some conditions for the injectivity radius are required for the discreteness.

**Definition 4.** For a given \( M > 0 \), we say that a point \( p \) of \( R \) belongs to a subset \( R_M \) of \( R \) if there exists a non-trivial simple closed curve \( c_p \) containing \( p \) such that the hyperbolic length of \( c_p \) is less than \( M \). The set \( R_\epsilon \) is called the \( \epsilon \)-thin part of \( R \) if \( \epsilon > 0 \) is smaller than the Margulis constant.

Now, we exhibit our main results.

**Theorem 1.** Let \( R \) be a Riemann surface with non-abelian fundamental group. Suppose that \( R \) satisfies the following two conditions

1. There exists an \( \epsilon > 0 \) such that the \( \epsilon \)-thin part of \( R \) consists only of cusp neighborhoods. (i.e. the \( \epsilon \)-thin part equals to the \( \epsilon \)-cuspidal part.)
2. There exist a constant $M$ and a connected component $R_M^*$ of $R_M^*$ such that the natural map of $\pi_1(R_M^*)$ to $\pi_1(R)$ which is induced by the inclusion map of $R_M^*$ to $R$ is surjective.

For a simple closed geodesic $c$ on $R$, we set

$$\text{Mod}^\#_c(R) = \{ [f] \in \text{Mod}^\#(R) \mid f(c) \text{ is freely homotopic to } c \}.$$ 

Then $\text{Mod}^\#_c(R)$ is discrete on $T^\#(R)$.

**Remark 2.** Example 1 shows that the first condition of Theorem 1 is necessary for the discreteness, while Example 2 shows that it is not sufficient for the discreteness of $\text{Mod}^\#(R)$. The Riemann surface $R$ in Example 4 satisfies both conditions of Theorem 1 but $\text{Mod}^\#(R)$ is not discrete.

**Remark 3.** The region $R_M$ is not necessarily connected for large $M$ even if the natural map : $\pi_1(R_M^*) \rightarrow \pi_1(R)$ is surjective. Moreover, in Example 7 of §6, we give a Riemann surface $R$ and divergent sequences $\{M_n\}, \{M'_n\}$ such that

- $M_n < M'_n < M_{n+1} < M'_{n+1}$ ($n = 1, 2, \ldots$).
- The natural map : $\pi_1(R_{M_n}) \rightarrow \pi_1(R)$ is surjective and $R_{M_n}$ is connected for all $n$.
- The natural map : $\pi_1(R_{M'_n}) \rightarrow \pi_1(R)$ is surjective but $R_{M'_n}$ is not connected for all $n$.

For a hyperbolic Riemann surface $R = \mathbb{H}/\Gamma$, we consider the convex core $C(\Gamma)$ of the limit set of $\Gamma$, that is, the hyperbolic convex envelope of $\Lambda(\Gamma) \subset \mathbb{R} \cup \{\infty\}$ in $\mathbb{H}$. Since the convex core $C(\Gamma)$ is $\Gamma$-invariant, it determines a region $C(R)$ in $R$ and we call the region the convex core of $R$.

**Definition 5.** We say that a Riemann surface $R$ has $\epsilon$-uniform geometry if the following two conditions are satisfied for some $\epsilon > 0$:

1. The $\epsilon$-thin part of $R$ consists of cusp neighborhoods.
2. The injectivity radius on the convex core $C(R)$ of $R$ is less than $\epsilon^{-1}$.

Since $C(R)$ is connected and it contains any closed geodesic on $R$, from Theorem 1 we have the following immediately.

**Corollary 1.** Let $R$ be a Riemann surface with $\epsilon$-uniform geometry for some $\epsilon > 0$. Then $\text{Mod}^\#_c(R)$ is discrete for any simple closed geodesic $c$ on $R$.

**Remark 4.** The conditions in Theorem 1 do not imply the uniform geometry. For example, set $R = \mathbb{C} - \{n : n \in \mathbb{Z}\}$. Then $R$ has a Fuchsian model of the first kind, and hence the convex core $C(R)$ coincides with $R$. But since $R$ has points with arbitrarily large injectivity radius, $R$ does not have $\epsilon$-uniform geometry for any $\epsilon > 0$. On the other hand, it is easily seen that $R$ satisfies the conditions in Theorem 1.

**Remark 5.** The conditions having uniform geometry were first stated as no short geodesics and no large disk condition. Nakanishi and Yamamoto[8] shows that under these conditions the out radius of the Teichmüller space is strictly less than 6. Ohtake[9] uses these conditions to show that the norm of the Poincaré series is strictly less than one which generalizes a result in McMullen[6].
It is important to give conditions for the mapping class group to be discrete. By using the above results, we have the following.

**Theorem 2.** Let $R$ be a Riemann surface satisfying the conditions in Theorem 1 or Corollary 1. Suppose that either the genus of $R$, the number of cusps or the number of holes is positive finite. Then $\text{Mod}^\#(R)$ is discrete.

For a planar Riemann surface, Theorem 2 does not necessarily hold. See Example 6 in §6.

5. **Proofs of main results**

First of all, we note the geometry of a component of $R_M$.

**Proposition 1.** For $M > 0$, let $R^*_M$ be a connected component of $R_M$ defined in Definition 4 and $R_\varepsilon$ the $\varepsilon$-thin part of $R$ for some small $\varepsilon < M$. We assume that $R^*_M - R_\varepsilon$ is not of type $(0, 3)$. Then there exists a constant $M_1 > 0$ depending only on $M$ and $\varepsilon$ such that for any point $p \in R^*_M - R_\varepsilon$ there exists a simple closed curve $c_p$ passing through $p$ with $\ell(c_p) < M_1$ which does not surround a puncture of $R$.

**Proof.** Let $\Gamma$ be a Fuchsian group representing $R$. Take an arbitrary point $p$ in $R^*_M - R_\varepsilon$. From the definition, we may find a simple closed curve $c_p \ni p$ whose length is less than $M$. If $c_p$ is not homotopic to a simple closed curve which surrounds a puncture of $R$, then there is nothing to prove.

Thus, we suppose that $c_p$ surrounds a puncture of $R$. Then, a parabolic transformation $\gamma \in \Gamma$ represents $c_p$. We may assume that $\gamma(z) = z + 1$. We take $\delta(\varepsilon), \delta(M) > 0$ so that

$$d(\delta(\varepsilon)\sqrt{-1}, \delta(M)\sqrt{-1} + 1) = \varepsilon,$$

$$d(\delta(M)\sqrt{-1}, \delta(M)\sqrt{-1} + 1) = M.$$

We put

$$S(M, \varepsilon) = \{z \in \mathbb{H} \mid \delta(M) \leq \text{Im } z \leq \delta(\varepsilon), 0 \leq \text{Re } z \leq 1\}.$$

Since $\ell(c_p) < M$ and $p \notin R_\varepsilon$, a lift $C_p$ of $c_p$ contains a point in $S(M, \varepsilon)$.

Let $L_z \in \mathbb{H}$ denote the geodesic arc from $z$ to $z + 1$. If the projection $l_z$ in $R$ of $L_z$ via the canonical projection $\pi : \mathbb{H} \to R = \mathbb{H}/\Gamma$ is not simple for some $z \in S(M, \varepsilon)$, then $l_z$ contains a non-trivial simple closed curve $c'_z$ with $\ell(c'_z) < \ell(l_z) < M$. Noting that $l_z$ is the projection of the geodesic arc $L_z$, we verify that $c'_z$ is not homotopic to $c_p$. Connecting $c'_z$ and $c_p$, we have a simple closed curve passing through $p$ with length less than $M_1 = 2(M + d(\delta(\varepsilon)\sqrt{-1}, \delta(M)\sqrt{-1}))$ which does not surround a puncture of $R$.

Finally, we suppose that $l_z$ is simple for any $z \in S(M, \varepsilon)$. Let us consider a geodesic $L_z$ for $z \in \mathbb{H}$ \cap $\{z \in \mathbb{H} \mid \text{Im } z = \delta(M)\}$, where $\mathbb{H}$ is a lift of $R^*_M$ with $\mathbb{H} \cap S(M, \varepsilon) \neq \emptyset$. From the definition, $\ell(L_z) = M$. Therefore, there exists a simple closed curve $c_z$ in $R^*_M$ passing through $\pi(z)$ with $\ell(c_z) < M = \ell(L_z)$. Obviously, the curve $c_z$ is not homotopic to $l_z = \pi(L_z)$ because $l_z$ is the shortest simple closed curve which passes through $\pi(z)$ and surrounds the puncture. Therefore, connecting $c_z$ and $c_p$ as before, we have a non-trivial simple closed curve passing through $p$ with length less than $M_1$ not surrounding a puncture of $R$. □

To prove the main results, the following proposition on hyperbolic geometry is crucial.
Proposition 2. Let $\Gamma$ be a Fuchsian model on the upper half plane $\mathbb{H}$ of a Riemann surface $R$. Assume that $\Gamma$ is non-elementary.

Let $M$ and $D$ be positive constants, and $g_1, g_2$ and $g_3$ be distinct hyperbolic elements in $\Gamma$ such that

1. translation lengths of $g_j$ ($j = 1, 2, 3$) are less than $M$,
2. the projections of the axes $\ell_j$ of $g_j$ to $R$ are simple closed geodesics, and
3. the distances between a point $z_1$ on $\ell_1$ and $\ell_j$ ($j = 2, 3$) are less than $D$.

Let $f$ be a quasiconformal self-map of $\mathbb{H}$ such that $f \circ \Gamma \circ f^{-1} = \Gamma$, and $\chi$ an isomorphism of $\Gamma$ induced by $f$.

Suppose that

$$\chi(g_1) = g_1, \quad \chi(g_2) = g_2, \quad \chi(g_3) \neq g_3.$$  

Then there is a constant $A > 1$ depending only on $M$ and $D$ such that

$$K(f) \geq A.$$

To prove this proposition, we prepare some well-known results (cf. [1], [5]).

Lemma 1 (Teichmüller). Let $f$ be a quasiconformal selfmap of $\mathbb{C}$ fixing $0$ and $1$, and suppose that there is a point $z_0$ in $\mathbb{C} - \{0, 1\}$ such that

$$\log M = d_1(z_0, f(z_0)) > 0.$$  

Then $K(f) \geq M^2$, where $d_1(\cdot, \cdot)$ is the hyperbolic distance on $\mathbb{C} - \{0, 1\}$.

Lemma 2 (Wolpert). Let $f$ be a quasiconformal mapping of a Riemann surface $S$ onto another Riemann surface $S$, and $c$ be a simple closed geodesic on $S$ with hyperbolic length $L$. Then the hyperbolic length of a closed geodesic on $S$ homotopic to $f(c)$ is not greater than $K(f)L$.

Lemma 3 (Collar lemma 1). For a given $M > 0$, let $g$ and $g'$ be arbitrary two hyperbolic elements of $\Gamma$ with translation lengths less than $M$. Suppose that the projections of the axes of $g$ and $g'$ to $R$ are simple closed geodesics which intersect to each other. Then the axes make an angle greater than $C > 0$ depending only on $M$.

Lemma 4 (Collar lemma 2). For a given $M > 0$, let $g$ and $g'$ be arbitrary two distinct hyperbolic elements of $\Gamma$ with translation lengths less than $M$. Suppose that the projections of the axes of $g$ and $g'$ to $R$ are simple closed geodesics which coincide or disjoint. Then the axes have a distance greater than $C > 0$ depending only on $M$.

Proof of Proposition 2. We may assume that fixed points of $g_1$ are $0$ and $\infty$, and that $z_1 = \sqrt{-1} \in \mathbb{H}$, hence $d_{\mathbb{H}}(\sqrt{-1}, \ell_j) \leq D$ for $j = 2, 3$. We may also assume that the maximal dilatation of $f$ is less than 2. Then at least one of the fixed points of $g_j$ ($j = 2, 3$) is not in $U = \{x \in \mathbb{R} \mid |x| < \delta \text{ or } |x| > 1/\delta\}$ for sufficiently small $\delta > 0$ which depends only on $M$ and $D$. Indeed, if both fixed points of $g_j$ are in $U_1 = \{x \in \mathbb{R} \mid |x| < \delta\}$ for small $\delta > 0$, then it contradicts $d_{\mathbb{H}}(\sqrt{-1}, \ell_j) \leq D$. The same argument works when both fixed points are in $U_2 = \{x \in \mathbb{R} \mid |x| > 1/\delta\}$. Next, suppose that one fixed point of $g_j$ is in $U_1$ and the other is in $U_2$. If $\ell_1 \cap \ell_j \neq \emptyset$, then it contradicts Lemma 3. If $\ell_1 \cap \ell_j = \emptyset$, it contradicts Lemma 4. Therefore, we verify that at least one of the fixed points of $g_j$ ($j = 2, 3$) is not in $U$. By using the same argument, we see that there exists a constant $\delta' > 0$ depending only on $M$ and $D$ such that all fixed points of $g_2$ and $g_3$ are in $\{x \in \mathbb{R} \mid \delta' < |x| < 1/\delta'\}$. 


From the above consideration, we verify that the Euclidean diameter \( \text{diam}(\ell_3) \) of \( \ell_3 \) is greater than some \( r = r(M, D) > 0 \) which depends only \( M \) and \( D \). Set \( g_4 = f \circ g_3 \circ f^{-1} \). By the assumption we have \( g_4 \neq g_3 \). Then, we see that there exists a constant \( C = C(M, D) > 0 \) depending only on \( M \) and \( D \) such that an inequality

\[
|b - f(b)| > C
\]

holds for a fixed point \( b \) of \( g_4 \). Indeed, since \( K(f) < 2 \), the translation length of \( g_4 \)
is less than \( 2M \) by Lemma 2. Noting that \( \text{diam}(\ell_3) > r \), we see that if \( \ell_3 \cap \ell_4 \neq \emptyset \), then we have the assertion from Lemma 3 and that Lemma 4 yields the assertion if \( \ell_3 \cap \ell_4 = \emptyset \).

Take a fixed point \( a \) of \( g_2 \) with

\[
\delta < |a| < \frac{1}{\delta}'.
\]

Let \( \phi \) be a Möbius transformation with \( \phi(0) = 0, \phi(a) = 1 \) and \( \phi(\infty) = \infty \). As we noted,

\[
\delta' < |b| < \frac{1}{\delta'}.
\]

Hence, (1) and (2) implies that

\[
d_a(b, f(b)) > \log L
\]

holds for some constant \( L > 1 \) depending only on \( M \) and \( D \), where \( d_a(.) \) is the hyperbolic distance on \( \mathbb{C} - \{0,a\} \). Considering \( \{0,a,b\} \) instead of \( \{0,1,z_0\} \) for \( z_0 = \phi(b) \) in Lemma 1, we verify that the assertion follows for \( A = L^2 \).

Next, we show a fundamental property of \( \text{Mod}_c^\#(R) \).

**Proposition 3.** Let \( R \) be a Riemann surface. For an arbitrary simple closed geodesic \( c \), let \( \{[f_n]\} \) be a sequence of transformations of \( \text{Mod}_c^\#(R) \) that satisfies \( \lim_{n \to \infty} K(f_n) = 1 \). Then there exists a subsequence \( \{[f_{n_k}]\} \) of \( \{[f_n]\} \) such that \( \{f_{n_k}\} \) locally uniformly converges to a conformal self-map \( f \) of \( R \) which determines an transformation \([f]\) \( \in \text{Mod}_c^\#(R) \).

**Proof.** Let \( \Gamma \) be a Fuchsian model of \( R \). First we suppose that \( c \) is not homotopic to a boundary component of \( R \). Then there exists a simple closed geodesic \( c' \) on \( R \) with \( c \cap c' \neq \emptyset \). Hence Lemma 5 (with \( K = c \) and \( C = c' \)) below shows the desired result.

Next suppose that \( c \) is homotopic to a boundary component of \( R \). We may assume that the Riemann surface \( R \) is not topologically finite. Hence, there exists a pair of pants \( S \) in \( R \) such that the boundary consists of dividing simple closed geodesics \( c_1 \) and \( c_2 \) other than \( c \), and neither \( c_1 \) nor \( c_2 \) is homotopic to a boundary component of \( R \). If \( f_n(c_1) \) are homotopic to \( c_1 \) or \( c_2 \) for infinitely many \( n \), then we can apply the previous argument. Hence we assume that \( f_n(c_1) \) are homotopic to neither \( c_1 \) nor \( c_2 \) for all \( n \). Since \( f_n(c) \) are homotopic to \( c \), and \( f_n(S) \) are still pairs of pants bounded by \( f_n(c_1), f_n(c_2) \) and \( f_n(c) \) for all \( n \), we conclude that \( f_n(c_1) \cap \hat{S} \neq \emptyset \) or \( f_n(c_2) \cap \hat{S} \neq \emptyset \). We may assume that \( f_n(c_1) \cap \hat{S} \neq \emptyset \). Then the following Lemma 5 again shows the desired result.

**Lemma 5.** Let \( \{[f_n]\} \) be a sequence of quasiconformal self-maps of a hyperbolic Riemann surface \( R \) that satisfies \( \lim_{n \to \infty} K(f_n) = 1 \). Suppose that there exist compact subsets \( C \) and \( K \) of \( R \) such that \( f_n(C) \cap K \neq \emptyset \) for all \( n \). Then there
exist a subsequence \( \{f_{n_j}\} \) of \( \{f_n\} \) and a conformal self-map \( f \) of \( R \) such that \( \{f_{n_j}\} \) converge to \( f \) locally uniformly on \( R \).

**Proof.** From the assumption, there exists a sequence \( \{p_n\} \) on \( C \) such that \( f_n(p_n) \in K \). Since \( C \) and \( K \) are compact, there exist \( p \in C \) and \( q \in K \) such that \( p_n \to p \) and \( f_n(p_n) \to q \) as \( n \to \infty \). Take lifts of \( p_n, p \) and \( q \) in \( \mathbb{H} \), say \( \tilde{p}_n, \tilde{p} \) and \( \tilde{q} \), respectively, so that \( \tilde{p}_n \to \tilde{p} \) as \( n \to \infty \). We can take lifts \( \tilde{f}_n : \mathbb{H} \to \mathbb{H} \) of \( f_n \) satisfying \( \tilde{f}_n(\tilde{p}_n) \to \tilde{q} \).

Since \( \{\tilde{f}_n\} \) is a normal family, a subsequence \( \{\tilde{f}_{n_j}\} \) of \( \tilde{f}_n \) converges locally uniformly, and the limit function \( \tilde{f} \) is either a quasiconformal self-map of \( \mathbb{H} \) or a constant in \( \mathbb{R} \cup \{\infty\} \). Since \( \tilde{f}(\tilde{p}) = \tilde{q} \) is in \( \mathbb{H} \), \( \tilde{f} \) is not a constant. Thus, it follows from \( \lim_{n \to \infty} K(f_n) = 1 \) that \( \tilde{f} \) is a conformal self-map of \( \mathbb{H} \). Hence, \( \{f_{n_j}\} \) converges locally uniformly to a conformal self-map \( f \) of \( R \) which is the projection of \( \tilde{f} \). \( \square \)

Before proving our main theorems, we shall give a sufficient condition for discreteness of a sequence of \( \text{Mod}^\#(R) \) under the conditions in Theorem 1.

**Proposition 4.** Let \( R \) be a Riemann surface satisfying the two conditions in Theorem 1, and \( \{f_n\} \) be a sequence of quasiconformal self-maps of \( R \) satisfying the following conditions:

- \( \{(f_n)_*\} \) converges to the identity, where \( (f_n)_* : \pi_1(R) \to \pi_1(R) \) is an isomorphism induced by \( f_n \).
- \( \lim_{n \to \infty} K(f_n) = 1 \).

Then, as an element of \( \text{Mod}^\#(R) \), \([f_n] = [\text{id}] \) for sufficiently large \( n \).

**Proof.** Let \( \Gamma \) be a Fuchsian model of \( R \), and \( \tilde{f}_n \) a lift of \( f_n \) for each \( n \). We may take \( \tilde{f}_n \) so that the isomorphisms \( \chi_n : \Gamma \to \Gamma \) induced by \( \tilde{f}_n \) converge to the identity.

Since the natural map \( \pi_1(R^M_1) \to \pi_1(R) \) is surjective for sufficiently large \( M > 0 \), we may find two distinct simple closed geodesics \( L_1^0 \) and \( L_2^0 \) on \( R^M_1 \) so that their lengths are less than \( M \). Let \( \gamma_j \ (j = 1, 2) \) be hyperbolic elements of \( \Gamma \) which represent \( L_j^0 \). Since \( \chi_n \to \text{id} \) \((n \to \infty)\), \( \chi_n(\gamma_1) = \gamma_1 \) and \( \chi_n(\gamma_2) = \gamma_2 \) for sufficiently large \( n \).

Suppose that \( \chi_n \) are not eventually the identity. Then we prove the proposition by drawing a contradiction. We may find a \( \gamma_n \in \Gamma \) so that \( \chi_n(\gamma_n) \neq \gamma_n \). The following lemma shows more, that is, we may take better one as \( \gamma_n \).

**Lemma 6.** For sufficiently large \( n \), there exists a hyperbolic element \( \gamma_n \) of \( \Gamma \) that satisfies the following two conditions:

1. \( \chi_n(\gamma_n) \neq \gamma_n \).
2. The projection of the axis of \( \gamma_n \) on \( R \) is a simple closed geodesic with length less than \( M \).

**Proof.** Since \( \chi_n \neq \text{id} \), there exists an element \( \alpha_n \) of \( \Gamma \) such that \( \chi_n(\alpha_n) \neq \alpha_n \). We will show that either \( \alpha_n \circ \gamma_1 \circ \alpha_n^{-1} \) or \( \alpha_n \circ \gamma_2 \circ \alpha_n^{-1} \) is a desired element. It is obvious that both of them satisfy the second condition of the lemma. Hence, it suffices to show that one of them satisfies the first condition.

Suppose that \( \chi_n \) fixes \( \alpha_n \circ \gamma_j \circ \alpha_n^{-1} \ (j = 1, 2) \). Then

\[
\beta_n \circ \gamma_j \circ \beta_n^{-1} = \gamma_j, \quad (j = 1, 2)
\]

where \( \beta_n = \alpha_n^{-1} \circ \chi_n(\alpha_n) \). Thus, \( \beta_n \) fixes all fixed points of \( \gamma_1 \) and \( \gamma_2 \). Since \( \gamma_1 \) and \( \gamma_2 \) are non-commutative, the M"obius transformation \( \beta_n \) fixes four points and it must be the identity map. This contradicts \( \chi_n(\alpha_n) \neq \alpha_n \). \( \square \)
Now, under the assumption of Theorem 1, we show the existence of elements in
\( \Gamma \) satisfying the assumption of Proposition 2 for \( \chi_n \).

**Lemma 7.** For each \( n \), there exist hyperbolic elements \( g_j, n \) \((j = 1, 2, 3)\) with axes
\( \ell_j, n \) such that they satisfy the following four conditions.

1. the projections \( L_j, n \) of \( \ell_j, n \) to \( R \) are simple closed geodesics.
2. there is a constant \( M \) independent of \( n \) such that the lengths of \( L_j, n \) are less
than \( M \).
3. there is a constant \( D \) independent of \( n \) such that the distances between a point
on \( \ell_1, n \) and \( \ell_j, n \) \((j = 2, 3)\) are less than \( D \), and
4. \( \chi_n (g_j, n) = g_j, n \) for \( j = 1, 2 \), and \( \chi_n (g_3, n) \neq g_4, n \).

**Proof.** Let \( \gamma_n \) be an element in Lemma 6. We denote by \( \ell_1, 0 \), \( \ell_2, 0 \) and \( \ell_n \) the axes
of \( \gamma_1, \gamma_2 \) and \( \gamma_n \), respectively. Let \( R^*_M \) be a lift of \( R^*_M \).

Fix a point \( z_1 \) on \( \ell_1, 0 \). There exists the nearest point \( z_n \) on \( \ell_n \) from \( z_1 \). Since \( z_1 \)
and \( z_n \) belong to \( R^*_M \), and \( R^*_M \) is connected, there exists a curve \( C_n \) on \( R^*_M \) which
connects \( z_1 \) and \( z_n \). Furthermore, we can take the curve \( C_n \) so that the projection
of \( C_n \) is in \( R^*_M - R \).

First, we observe a fundamental property of \( R_M \).

For an arbitrary point \( p_0 \) in \( R^*_M - R \), there exists a non-trivial simple closed
curve \( C_{p_0} \) passing through \( p_0 \) with \( \ell (C_{p_0}) \leq M \). Replacing \( M > 0 \) so large if
necessary, we may assume that \( C_{p_0} \) is not homotopic to a puncture (Proposition 1).
Then there exists a simple closed geodesic \( L_{p_0} \) which is homotopic to \( C_{p_0} \). The
length of \( L_{p_0} \) is greater than \( \epsilon \). Hence there exists a constant \( B \) depending on only
\( \epsilon \) and \( M \) such that \( d_{\ell_0} (p_0, L_{p_0}) \leq B \). This implies that for every \( z_0 \in R^*_M \) which
is not projected to \( R_\epsilon \), there is an axis \( \ell_0 \) of a hyperbolic element of \( \Gamma \) such that
\( d_{\ell_0} (z_0, \ell_0) \leq B \) and that the projection to \( R \) is a simple closed geodesic with length
less than \( M \).

Here, we consider the following two cases for \( d_{\ell_0} (z_1, \ell_n) \).

1: \( d_{\ell_0} (z_1, \ell_n) \leq 4 (B + M + 1) \).

In this case, we set \( g_1, n = \gamma_1, g_2, n = \gamma_2 \) and \( g_3, n = \gamma_n \). Then the third condition
of the lemma holds for \( D = \max (4 (B + M + 1), d_{\ell_0} (z_1, \ell_2, 0)) \). Other three conditions
are trivial from the choice of these transformations.

2: \( d_{\ell_0} (z_1, \ell_n) > 4 (B + M + 1) \).

In this case, there are points \( z'_n \) and \( z''_n \) on \( C_n \) that satisfy \( d_{\ell_0} (z_n, z'_n) = d_{\ell_0} (z'_n, z_n) =
2 (B + M + 1) \). Since \( z'_n \) and \( z''_n \) are points on \( R_M^\ast \) which are not projected to
\( R_\epsilon \), it follows from the above observation that there exists an axis \( \ell'_n \) (resp. \( \ell''_n \))
of \( \gamma'_n \) (resp. \( \gamma''_n \)) in \( \Gamma \) such that \( d_{\ell_0} (z'_n, \ell'_n) \leq B \) (resp. \( d_{\ell_0} (z''_n, \ell''_n) \leq B \)). Since
\( d_{\ell_0} (z'_n, z''_n) = 2 (B + M + 1) \), we see that \( \ell'_n \) and \( \ell''_n \) are distinct. Take a point
\( w_n \in \ell'_n \) so that \( d_{\ell_0} (z'_n, w_n) \leq B \).

If \( \chi_n (\gamma'_n) = \gamma'_n \) and \( \chi_n (\gamma''_n) = \gamma''_n \), set \( g_1, n = \gamma'_n, g_2, n = \gamma''_n \) and \( g_3, n = \gamma_n \). Noting
that
\( d_{\ell_0} (w_n, \ell'_n) \leq 2 (B + M + 1) + 2B \)
and
\( d_{\ell_0} (w_n, \ell_n) \leq 2 (B + M + 1) + B, \)

we see that the third condition of the lemma holds for \( D = 4B + 2M + 1 \). If
\( \chi_n (\gamma'_n) \neq \gamma'_n \) or \( \chi_n (\gamma''_n) \neq \gamma''_n \), replace \( \gamma_n \) by \( \gamma'_n \) or \( \gamma''_n \). Repeating this argument, we get desired elements. \( \square \)
We proceed to prove Proposition 4. Using \(g_{1,n}, g_{2,n}\) and \(g_{3,n}\) obtained in Lemma 7, we have

\[
K(f_n) \geq A = A(M, D) > 1
\]

from Proposition 2. Since constants \(M\) and \(D\) are independent of \(n\), this contradicts \(\lim_{n \to \infty} K(f_n) = 1\). Hence we have proved this proposition.

Proof of Theorem 1. Let \(p_0 = [R, \text{id}]\) be the base point of \(T^\#(R)\). We first suppose that there exists a sequence \(\{g_n\}\) of quasiconformal self-maps of \(R\) which determine distinct elements of \(\text{Mod}^\#(R)\) such that \(\lim_{n \to \infty} g_n(p_0) = p\) for some \(p\) in \(T^\#(R)\).

Consider the sequence \(\{f'_n = g_{n+1}^{-1} \circ g_n\}\). Then we see that \(f'_n(p_0)\) converges to \(p_0\). Thus there exist quasiconformal mappings \(f_n : R \to R\) \((n = 1, 2, \ldots)\) such that \(f_n\) is RT-equivalent to \(f'_n\) and that \(\lim_{n \to \infty} K(f_n) = 1\). From Proposition 3, there exists a conformal self-map \(f\) of \(R\) such that \([f_n \circ f] \in \text{Mod}^\#(R)\) and \(f_n \circ f\) converge to the identity on \(R\) locally uniformly. Since \(\lim_{n \to \infty} K(f_n \circ f) = \lim_{n \to \infty} K(f_n) = 1\), it follows from Proposition 4 that \([f_n \circ f] = [\text{id}]\) for sufficiently large \(n\). Hence \([f_n] = [f^{-1}]\) for sufficiently large \(n\). This contradicts the assumption that all \(f_n\) are distinct.

Finally, we see that the same argument as above is valid for an arbitrary point \(q = [S, f]\) in \(T^\#(R)\). To see this, it suffices to show that the conditions of Theorem 1 is invariant under the quasiconformal deformation. Namely, the following lemma concludes the theorem.

Lemma 8. Let \(f : R \to S\) be a \(K\)-quasiconformal homeomorphism. Suppose that a Riemann surface \(R\) satisfies the both conditions in Theorem 1. Then \(S\) also satisfies them.

Proof. Let \(\tilde{f} : \mathbb{H} \to \mathbb{H}\) be a lift of \(K\)-quasiconformal map \(f\). The quasiconformal map \(\tilde{f}\) can be extended to \(\mathbb{H} \cup \mathbb{R}\) with \(\tilde{f}(\infty) = \infty\) and the restriction \(\tilde{f}|_{\mathbb{R}}\) of \(\tilde{f}\) to \(\mathbb{R}\) is a quasisymmetric function. The Douady-Earle extension \(\Phi(f)\) of \(f|_{\mathbb{R}}\) to \(\mathbb{H}\) is a quasiconformal and bilipschitz map, and the bilipschitz constant \(K'\) depends only on \(K\) (cf. [2]). The projection \(\phi_f : R \to S\) of \(\Phi(f)\) satisfies

\[
(1/K')\ell(c) \leq \ell(\phi_f(c)) \leq K'\ell(c)
\]

for an arbitrary curve \(c\) on \(R\), and \([S, f] = [S, \phi_f]\) in \(T^\#(R)\). Then for an arbitrary point \(a\) in \(\phi_f(R_M^*)\), there exists a non-trivial simple closed curve \(c_0\) containing \(a\) such that \(\ell(c_0) \leq K'M\). Thus, \(\phi_f(R_M^*) \subset S_{K'M}^*\). Therefore, we see that the Riemann surface \(S\) satisfies the second condition in Theorem 1 for a connected component of \(S_{K'M}^*\) containing \(\phi_f(R_M^*)\).

The same argument also shows that the first condition is satisfied by \(S\).

Proof of Theorem 2. Suppose that \(R\) is a Riemann surface of positive finite genus \(g\) and satisfies the conditions in Theorem 1. Further suppose that \(\text{Mod}^\#(R)\) is not discrete. Then there exists a sequence \(\{f_n\}\) of quasiconformal self-maps of \(R\) which determine distinct elements of \(\text{Mod}^\#(R)\) such that \(\lim_{n \to \infty} K(f_n) = 1\). Let \(l\) be a dividing simple closed curve such that one of components of \(R \setminus l\) is a Riemann surface \(S\) of genus \(g\) with only one boundary component. Take a non-dividing simple closed geodesic \(c\) on \(S\). Then \(f_n(c) \cap \bar{S} \neq \emptyset\) for all \(n\). Indeed, if \(f_n(c) \subset \bar{S}^*\), then \(f_n(c)\) should be a dividing curve. Since \(c\) is a non-dividing curve and \(f_n\) is a homeomorphism, it can not occur. Then from Lemma 5, there exists a subsequence
of \( \{ f_n \} \) which converges to a conformal self-map \( f \) of \( R \) locally uniformly on \( R \). Hence we can apply Proposition 4, and we conclude a contradiction.

Next suppose that \( R \) has finite positive number of cusps and satisfies the conditions in Theorem 1. If \( \text{Mod}^\#(R) \) is not discrete, then there exists a sequence \( \{ f_n \} \) as above. Let \( V \) be a cusp neighborhood of a puncture of \( R \). Since \( R \) has only finitely many cusps, we may assume that \( f_n(V) \cap V \neq \emptyset \) for all \( n \) by taking a subsequence of \( \{ f_n \} \). Let \( S \) be a pair of pants in \( R \) such that it contains \( V \) and that the boundaries of \( S \) consist of the puncture and two dividing simple closed geodesics, say \( c_1 \) and \( c_2 \). We may assume that two geodesics \( c_1 \) and \( c_2 \) are not homotopic to a boundary component of \( R \). If \( f_n(c_1) \) is homotopic to \( c_1 \) for infinity many \( n \), then they determine elements of \( \text{Mod}^\#_1(R) \). Hence, they must be discrete from Theorem 1. Assume that \( f_n(c_1) \) is not homotopic to \( c_1 \) for all \( n \). Since \( f_n(V) \cap V \neq \emptyset \) and \( f_n(S) \) is still a pair of pants for each \( n \), we see that \( f_n(c_1) \cap (S \setminus V) \neq \emptyset \) or \( f_n(c_2) \cap (S \setminus V) \neq \emptyset \). We may assume that \( f_n(c_1) \cap (S \setminus V) \neq \emptyset \). Then from Lemma 5 and Proposition 4, we conclude a contradiction.

Finally, suppose that \( R \) has finite positive number of borders and satisfies the conditions in Theorem 1. If \( \text{Mod}^\#(R) \) is not discrete, then there exists a sequence \( \{ f_n \} \) as before. Let \( B \) be a one of borders of \( R \). Since \( R \) has only finite number of borders, we may assume that \( f_n(B) = B \) for all \( n \). Let \( c \) be a simple closed geodesics which is homotopic to \( B \). Then \( f_n(c) \) is homotopic to \( c \). Thus \( f_n \in \text{Mod}^\#(R) \), and \( \{ f_n \} \) is discrete by Theorem 1. This contradicts \( \lim_{n \to \infty} K(f_n) = 1 \). Hence \( \text{Mod}^\#(R) \) is discrete.

\[ \square \]

6. Further examples

In Example 4, we showed that there exists a Riemann surface \( R \) that satisfies the two conditions in Theorem 1, but that \( \text{Mod}^\#(R) \) is not discrete. In this case, there exists a sequence \( \{ \omega_n \} \) of distinct elements of \( \text{Mod}^\#(R) \) such that \( \omega_n(p_0) = p_0 \) for any \( n \), where \( p_0 = [R, id] \in T^\#(R) \). By modifying this example, we see that the following example gives a Riemann surfaces \( R \) that there exists a sequence \( \{ \omega_n \} \) of distinct elements of \( \text{Mod}^\#(R) \) such that \( \lim_{n \to \infty} d_T(\omega_n(p), p) = 0 \) for some \( p \in T^\#(R) \) and \( \omega_n(p) \neq p \) for any \( n \).

**Example 5.** There is a Riemann surface \( R \) without cusps that satisfies two conditions in Theorem 1, but that \( \text{Mod}^\#(R) \) is not discrete.

We will give a sketch of a construction of such an \( R \). Consider a torus \( A_0 \) with two geodesic borders of the same length. Let \( B_0 \) be another torus obtained via the \((1 + \epsilon_0)\) quasiconformal deformation of \( A_0 \) for some \( \epsilon_0 > 0 \). Attach two copies of \( B_0 \) to \( A_0 \) along the borders suitably, and we obtain a Riemann surface \( A_1 \). Hence, it is a Riemann surface of genus 3 with two geodesic borders.

Next we take a Riemann surface \( B_1 \) which is the \((1 + \epsilon_1)\) quasiconformal deformation of \( A_1 \) for some \( \epsilon_1 > 0 \). Attach two copies of \( B_1 \) to \( A_1 \) along the borders suitably, and we obtain a Riemann surface \( A_2 \) which is a Riemann surface of genus 9 with two geodesic borders. Repeating this process, we have a sequence of Riemann surfaces \( \{ A_n \} \) for some sequence \( \{ \epsilon_n \} \) of positive numbers. We denote the inductive limit of these \( A_n \) by \( R \). If the sequence \( \{ \sum_{n=1}^p \epsilon_n \}_{p=1}^{\infty} \) is bounded, then we see that \( R \) satisfies the above conditions on the injectivity radius.

Let \( f_n \) be a quasiconformal selfmap of \( R \) which sends a part corresponding to \( A_n \) to a part corresponding to \( B_n \). If we take a sequence \( \{ \epsilon_n \} \) converges to zero so rapidly, then we verify that the maximal dilatations of \( f_n \) converge to 1. Thus,
$f_n$ induces an element of $\text{Mod}^\#(R)$ whose orbits of $p_0 = [R, id]$ converge to $p_0$ in $T^\#(R)$.

Next, we modify the above construction to obtain another kind of examples as follows.

**Example 6.** We give a planar surface $R$ with no short geodesics such that $\text{Mod}^\#(R)$ is not discrete.

For instance, set

\[ z_n = \begin{cases} n + \frac{c - 1}{j(n)+1}, & (n \neq 0) \\ 0, & (n = 0) \end{cases} \]

where $j(n)$ is the power of the factor 2 when we decompose $|n|$ to the product of primes. And set $R = \mathbb{C} - \sum_{n=-\infty}^{\infty} \{z_n\}$.

Now for every positive $m$, we take a locally affine quasiconformal selfmap $f_m$ of $R$ such that $\text{Re} f_m(z) = \text{Re} z + 2^m$ (and hence $f_m(z_n) = z(n+2^m)$). Then, since $j(n + 2^m) = j(n)$ if $j(n) < m$, we may take $f_m$ so that the maximal dilatations of $f_m$ tend to 1. Hence $\text{Mod}^\#(R)$ is not discrete.

We shall construct a Riemann surface $R$ and sequences $\{M_n\}$, $\{M'_n\}$ having the properties referred in Remark 3 in §3.

**Example 7.** We consider right-angled hexagons $H_n$ $(n = 1, 2, \ldots)$ in the hyperbolic plane $\mathbb{H}$. The sides of the hexagon $H_n$ are labelled $a_{j,n}$ $(j = 1, 2, \ldots, 6)$ counterclockwise. We construct the hexagon so that $\ell(a_{2,n}) = \ell(a_{6,n})$, $\ell(a_{3,n}) = \ell(a_{5,n}) = 1$ and $\ell(a_{1,n}) = (2n)^{-1}$. Then $\{H_n\}$ converges to a pentagon with a cusp as $n \to \infty$. Thus, we see that

\begin{equation}
\frac{1}{3} d_\mathbb{H}(P_n, a_{2,n}) = d_\mathbb{H}(P_n, a_{6,n}) \leq M < \infty
\end{equation}

holds for some $M$ independent of $n$, where $P_n$ is the midpoint of $a_{4,n}$. Take the perpendicular line $L_{j,n}$ $(j = 2, 6)$ from $P_n$ to $a_{j,n}$. As $n \to \infty$, we see that $d_\mathbb{H}(a_{1,n}, L_{2,n}) = d_\mathbb{H}(a_{1,n}, L_{6,n}) \to \infty$.

Now, we take $k(n)$ copies of $H_n$, say $H_n^1, \ldots, H_n^{k(n)}$, so that

\begin{equation}
\frac{1}{3} d_\mathbb{H}(a_{1,n}, L_{2,n}) \leq 2k(n) \ell(a_{1,n}) = \frac{1}{n} k(n) \leq \frac{1}{2} d_\mathbb{H}(a_{1,n}, L_{2,n}).
\end{equation}

Obviously, $k(n)/n \to \infty$ as $n \to \infty$. We denote the sides of $H_n^i$ corresponding to $a_{j,n}$ by $a_{j,n}^i$ $(i = 1, 2, \ldots, k(n); j = 1, 2, \ldots, 6)$ and glue $H_n^i$ and $H_n^{i+1}$ along $a_{6,n}^i$ and $a_{2,n}^{i+1}$. Then, we have a right-angled $(2k(n)+4)$-gon $D_n$ in $\mathbb{H}$. Label the side of $D_n$ formed by $a_{1,n}^1 \cup \ldots \cup a_{1,n}^{k(n)}$ $b_{1,n}$ and denote the rest of sides by $b_{2,n}, \ldots, b_{2k(n)+4}$ counterclockwise.

We take a copy of $D_n'$ of $D_n$ with sides $b_{j,n}'$ $(j = 1, 2, \ldots, 2k(n)+4)$ corresponding to $b_{j,n}$ of $D_n$. We glue $D_n$ and $D_n'$ along $b_{j,n}$ and $b_{2k(n)+j-1,n}$ for $j = 2, 4, \ldots, 2k(n) + 2$ and $2k(n) + 4$. Then we have a hyperbolic bordered surface $S_n$ of type $(0, k(n) + 1)$. The boundary $\partial S_n$ consists of one long curve $c_{1,n}$ and $k(n)$ short curves $c_{2,n}, \ldots, c_{k(n),n}$. It follows from the construction that

\[ \ell(c_{1,n}) = \frac{k(n)}{n}, \]

\[ \ell(c_{2,n}) = \ell(c_{k(n),n}) = 2, \]

and

\[ \ell(c_{3,n}) = \ldots = \ell(c_{k(n)-1,n}) = 4. \]
From (3), we verify that \((S)_{4M}\) is connected and the natural map of \(\pi_1((S)_{4M})\) to \(\pi_1(S)\) is surjective. On the other hand, it follows from (4) that \((S)_{k(n)/n}\) is not connected while both \((S)_k\) and \((S)_{2k(n)/n+4M}\) are connected.

We take a sequence \(\{j_n\}\) so that

\[ 4M < \frac{k(j_n)}{j_n} < \frac{k(j_{n+1})}{10j_{n+1}}. \quad (n = 1, 2, \ldots) \]

We glue \(S_{j_n}\) and \(S_{j_{n+1}}\) along \(c_{k(j_n)}\) of \(\partial S_{j_n}\) and \(c_{2j_{n+1}}\) of \(\partial S_{j_{n+1}}\). Then we have a bordered Riemann surface \(S\) and a Riemann surface \(R\) whose convex core is \(S\). From the construction we verify that \(R_{M_n}\) is connected for \(M_n = k(j_n)/2j_n\) but \(R_{M'_n}\) is not connected for \(M'_n = k(j_n)/j_n\). Since \(M_n, M'_n > 4M\), the natural maps of \(\pi_1(R_{M_n})\) and \(\pi_1(R_{M'_n})\) to \(\pi_1(R)\) is surjective. Thus, \(R, \{M_n\}\) and \(\{M'_n\}\) are our desired ones.

**References**


