ON THE ACTION OF THE MAPPING CLASS GROUP FOR RIEMANN SURFACES OF INFINITE TYPE

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Abstract. We consider Riemann surfaces of infinite type and their reduced Teichmüller spaces. The reduced Teichmüller space admits the action of the reduced mapping class group. Generally, the action is not discrete while it is faithful. We give sufficient conditions for the discreteness of the action in terms of the geometry of Riemann surfaces.

1. Introduction

The mapping class group (or modular group) $\mathrm{Mod}(R)$ for a Riemann surface R is the set of equivalence classes of quasiconformal self-maps of R (cf. [7]). Two quasiconformal self-maps h_1 and h_2 of R are equivalent if $h_2^{-1} \circ h_1$ is homotopic to the identity by a homotopy that keeps every points of ideal boundary ∂R fixed throughout. In the theory of Teichmüller spaces of Riemann surfaces of analytically finite type, the mapping class group plays an important role in various fields. This is a group of the biholomorphic automorphisms of the Teichmüller space and it acts faithfully and discontinuously. On the other hand, it seems that there are few studies on $\mathrm{Mod}(R)$ for a Riemann surface R of infinite type. Recently, Earle-Gardiner-Lakic showed in [3] that it acts faithfully on $T(R)$. In this paper, we consider the discreteness of the action of the mapping class group. We say that a subgroup G of $\mathrm{Mod}(R)$ is discrete if the orbit of any point of $T(R)$ under the G-action is discrete.

For a Riemann surface of analytically finite type, $\mathrm{Mod}(R)$ is discrete, while in the case of infinite type, $\mathrm{Mod}(R)$ is not necessarily discrete. In particular, if R has a boundary curve (border), $\mathrm{Mod}(R)$ is not discrete since a slight change of the boundary value of a quasiconformal map produces a different mapping class in $\mathrm{Mod}(R)$. Thus, it is natural that we consider another group, the reduced mapping class group. The reduced mapping class group $\mathrm{Mod}^{\#}(R)$ is the set of homotopy classes of quasiconformal self-maps of R whose homotopy maps does not necessarily keep points of ∂R fixed. The reduced mapping class group is also important because it naturally acts on the reduced Teichmüller space.

We explore the problem of discreteness of the reduced mapping class group for Riemann surfaces of infinite type. Actually, if R is a Riemann surface of topologically finite type, then $\mathrm{Mod}^{\#}(R)$ is discrete. However, $\mathrm{Mod}^{\#}(R)$ is not discrete in general. For example, if R has a sequence of disjoint simple closed geodesics which are not freely homotopic to a boundary component and whose lengths tend to 0, then we see that $\mathrm{Mod}^{\#}(R)$ is not discrete (See §3 and §6). The purpose of this paper is to give a sufficient condition for discreteness.

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2. THE MAPPING CLASS GROUP FOR THE REDUCED TEICHMÜLLER SPACE

Throughout this paper, we assume that a Riemann surface R is *hyperbolic*, that is, it is represented by $\mathbb{H}/\Gamma$ for some Fuchsian group Γ acting on the upper half plane $\mathbb{H}$. We also assume that the Fuchsian group Γ is always non-elementary. In other words, we assume that the group Γ is non-abelian. A Riemann surface is called *of analytically finite type* if the hyperbolic area is finite, and is called *of analytically infinite type* if the area is not finite.

For an open Riemann surface R, a relatively non-compact connected component of the complement of a compact subset of R is called an *end*. An end V of R is called a *hole* if it is doubly connected and the hyperbolic area of V is infinite. A doubly connected end of R is called a *cusp* if the hyperbolic ares of V is finite. A cusp V with smooth relative boundary is conformally equivalent to the punctured disk $\{0 < |z| < 1\}$. An ideal boundary of R corresponding to the origin $z = 0$ is called a *puncture*.

Notation. The hyperbolic distance between $A, B \subset \mathbb{H}$ is denoted by $d_{\mathbb{H}}(A, B)$ and the hyperbolic length of a curve c in $\mathbb{H}$ or a Riemann surface R is denoted by $\ell(c)$.

We review the theory of Teichmüller spaces and the mapping class group. See [4], [5] and [7] for the details.

Definition 1. Fix a Riemann surface R. For a pair (S, f) of a Riemann surface S and a quasiconformal map f of R onto S, we say that (S_1, f_1) and (S_2, f_2) are *RT (reduced Teichmüller) equivalent* if there exists a conformal map h of S_1 onto S_2 such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity on R.

The *reduced Teichmüller space* $T^{\#}(R)$ with the base Riemann surface R is the set of all RT equivalence classes $[S, f]$ of such pairs (S, f) as above.

Definition 2. We say that two quasiconformal self-maps h_1 and h_2 of R are *RT equivalent* if $h_2^{-1} \circ h_1$ is homotopic to the identity on R.

The *reduced mapping class group* $\mathrm{Mod}^{\#}(R)$ is the set of all RT equivalence classes $[h]$ of quasiconformal self-maps h of R.

Every quasiconformal map of $R = \mathbb{H}/\Gamma$ induces an isomorphism of Γ into $\mathrm{PSL}(2, \mathbb{R})$. We see that if two self-maps h_1 and h_2 are RT equivalent then they induce the same isomorphism modulo $\mathrm{PSL}(2, \mathbb{R})$ conjugacy.

If R is a compact Riemann surface, then the reduced Teichmüller space $T^{\#}(R)$ is nothing but the ordinary Teichmüller space $T(R)$ of R and the reduced mapping class group $\mathrm{Mod}^{\#}(R)$ is the ordinary mapping class group $\mathrm{Mod}(R)$.

Similar to the case of $T(R)$, the reduced Teichmüller space $T^{\#}(R)$ is equipped with the reduced Teichmüller distance $d_T(\cdot, \cdot)$ defined by

$$d_T([S, f], [S', g]) = \frac{1}{2} \inf_{f,\ g} \log K(f \circ g^{-1}),$$

where $K(\cdot)$ is the maximal dilatation of a quasiconformal map and the infimum is taken over all quasiconformal maps f and g determining $[S, f]$ and $[S', g]$, respectively. It is known that $T^{\#}(R)$ is a complete metric space with respect to this d_T. An element $\omega = [h] \in \mathrm{Mod}^{\#}(R)$ induces an automorphism of $T^{\#}(R)$ by

$$[S, f] \mapsto [S, f \circ h^{-1}].$$

This is an isometric automorphism with respect to d_T and denoted by ω_*. Namely, we have a homomorphism $\mathrm{Mod}^\#(R) \to \mathrm{Aut}\, T^\#(R)$.

Remark 1. In [3], it is proved that for any Riemann surface R of infinite analytic type (and if $2g+n>4$ when R is of finite (g,n)-type), the homomorphism $\mathrm{Mod}^\#(R) \to \mathrm{Aut}\, T^\#(R)$ as above is faithful. Therefore we can identify ω_* with ω and omit the asterisk hereafter.

Definition 3. We say that a subgroup G of $\mathrm{Mod}^\#(R)$ is *discrete* if every sequence $\{\omega_n\} \subset G$ satisfying $\lim_{n\to\infty} \omega_n(p) = q$ for some pair of points p, q in $T^\#(R)$ is eventually a constant sequence, that is, there exists an $N \in \mathbb{N}$ such that $\omega_n = \omega_N$ for every $n \geq N$.

3. Examples

As we noted in the introduction, if R is a compact Riemann surface, then the action of $\mathrm{Mod}(R)$ on $T(R)$ is discrete. Contrary to this case, there are various kinds of examples which show non-discreteness of $\mathrm{Mod}^\#(R)$ for a Riemann surface R of infinite type.

Example 1. Suppose that R has a sequence $\{c_n\}$ of disjoint simple closed geodesics that are not peripheral (i.e. that are not freely homotopic to a boundary component) and that these hyperbolic lengths tend to 0. Then the Dehn twist along each c_n gives an element ω_n of $\mathrm{Mod}^\#(R)$ such that the sequence $\{\omega_n(p_0)\}$ converges to p_0 as $n \to \infty$, where $p_0 = [R, id]$ is the base point of $T^\#(R)$. Hence $\mathrm{Mod}^\#(R)$ is not discrete.

There exists a Riemann surface R without short geodesics but containing a point with arbitrarily large injectivity radius with respect to the hyperbolic metric such that $\mathrm{Mod}^\#(R)$ is not discrete.

Example 2. Set

$$R = \mathbb{C} - \bigcup_{n=1}^{\infty} \bigcup_{m\in\mathbb{Z}} \left\{ \frac{m}{n} + (2n+1)\sqrt{-1} \right\}.$$

Then we can see that R has no short geodesics by considering the classical extremal domains, whereas the injectivity radii at $2ni$ tend to ∞.

Further, set

$$f_n(z) = \begin{cases} x - (y-2n-2)/n + y\sqrt{-1} & (2n+1 \leq y < 2n+2) \\ x + (y-2n)/n + y\sqrt{-1} & (2n \leq y < 2n+1) \\ x + y\sqrt{-1} & \text{elsewhere.} \end{cases}$$

Then f_n are quasiconformal self-maps of R and the maximal dilatations of $\{f_n\}$ tend to 1. Thus $\mathrm{Mod}^\#(R)$ is not discrete.

There is an another example of a planar Riemann surface R without cusps but containing a point with arbitrarily large injectivity radius with respect to the hyperbolic metric such that $\mathrm{Mod}^\#(R)$ is not discrete.

Example 3. For each $n \geq 2$, we set

$$I_n = [-1,1] \cup \bigcup_{k=1}^{\infty} I_{n,k},$$

where

$$\begin{aligned} I_{n,k} &= \{x+(1-1/n)^k\sqrt{-1} \mid -1 \le x \le 1\} \\ &\bigcup \{x+(1+(k-1)/n)\sqrt{-1} \mid -1 \le x \le 1\}. \end{aligned}$$

We take infinitely many copies $\{R_n\}$ of $\mathbb{C}-\{y\sqrt{-1} \mid y \le -1\}$ and set ${R_n}' = R_n - I_n$ for each $n \ge 2$. We make a Riemann surface R by gluing the right hand side of $\{y\sqrt{-1} \mid y < -1\}$ on ${R_n}'$ with the left hand side of $\{y\sqrt{-1} \mid y < -1\}$ on ${R_{n-1}}'$ $(n = 3, 4, \dots)$ along the imaginary axis.

Consider a quasiconformal map f_n of ${R_n}'$ defined by

$$f_n(z) = \begin{cases} x+(1-1/n)y\sqrt{-1} & (0 < y \le 1) \\ x+(y-1/n)\sqrt{-1} & (y > 1) \\ x+y\sqrt{-1} & \text{elsewhere.} \end{cases}$$

It is easily seen that the maximal dilatations of f_n converge to 1 as $n \to \infty$. Obviously, f_n is extended to a quasiconformal self-map of R by setting it the identity on $R-{R_n}'$ and we denote it by the same letter f_n. Thus the quasiconformal map f_n gives an element $[f_n]$ of $\mathrm{Mod}^{\#}(R)$ such that $\{[f_n](p_0)\}$ converges to p_0 as $n \to \infty$, where $p_0 = [R, id]$ is the base point of $T^{\#}(R)$. Hence we conclude that $\mathrm{Mod}^{\#}(R)$ is not discrete.

Even if a Riemann surface R has no short geodesics and no points with arbitrarily large injectivity radius, $\mathrm{Mod}^{\#}(R)$ may not be discrete as the following example shows:

Example 4. Consider a torus S with two geodesic borders with the same length to each other. We take infinity many copies $\{S_n\}_{n=-\infty}^{\infty}$ of S. We denote the two geodesic borders of S_n by $\ell_{n,1}$ and $\ell_{n,2}$. Construct a Riemann surface R by gluing the $\ell_{n-1,2}$ with $\ell_{n,1}$ and gluing $\ell_{n,2}$ with $\ell_{n+1,1}$ for each n. Let f be a conformal self-map of R which sends a point $a \in S_n$ to a point of S_{n+1} corresponding to the same point of S as a, and we set $f_n := f^n$. Then we see that $[f_n] \ne id$ as an element of $\mathrm{Mod}^{\#}(R)$. However, $[f_n](p_0) = p_0$ for all n, where $p_0 = [R, id] \in T^{\#}(R)$ because $f_n : R \to R$ is a conformal mapping. Hence, $\mathrm{Mod}^{\#}(R)$ is not discrete.

4. MAIN RESULTS

As Example 1 shows, for the discreteness of the mapping class group, it is necessary that there exists no sequence of geodesics on the Riemann surface whose lengths converge to zero. Example 2, 3 show that some conditions for the injectivity radius are required for the discreteness.

Definition 4. For a given $M > 0$, we say that a point p of R belongs to a subset R_M of R if there exists a non-trivial simple closed curve c_p containing p such that the hyperbolic length of c_p is less than M. The set R_ϵ is called the *ϵ-thin part* of R if $\epsilon > 0$ is smaller than the Margulis constant.

Now, we exhibit our main results.

Theorem 1. *Let R be a Riemann surface with non-abelian fundamental group. Suppose that R satisfies the following two conditions*

1. *There exists an $\epsilon > 0$ such that the ϵ-thin part of R consists only of cusp neighborhoods. (i.e. the ϵ-thin part equals to the ϵ-cuspidal part.)*

2. *There exist a constant M and a connected component R^*_M of R_M such that the natural map of $\pi_1(R^*_M)$ to $\pi_1(R)$ which is induced by the inclusion map of R^*_M to R is surjective.*

For a simple closed geodesic c on R, we set

$$\mathrm{Mod}^{\#}_c(R) = \{[f] \in \mathrm{Mod}^{\#}(R) \mid f(c) \textit{ is freely homotopic to } c\}.$$

Then $\mathrm{Mod}^{\#}_c(R)$ is discrete on $T^{\#}(R)$.

Remark 2. Example 1 shows that the first condition of Theorem 1 is necessary for the discreteness, while Example 2 shows that it is not sufficient for the discreteness of $\mathrm{Mod}^{\#}_c(R)$. The Riemann surface R in Example 4 satisfies both conditions of Theorem 1 but $\mathrm{Mod}^{\#}(R)$ is not discrete.

Remark 3. The region R_M is not necessarily connected for large M even if the natural map : $\pi_1(R_M) \to \pi_1(R)$ is surjective. Moreover, in Example 7 of §6, we give a Riemann surface R and divergent sequences $\{M_n\}$, $\{M'_n\}$ such that

- $M_n < M'_n < M_{n+1} < M'_{n+1}$ $(n = 1, 2, \dots)$.
- The natural map : $\pi_1(R_{M_n}) \to \pi_1(R)$ is surjective and R_{M_n} is connected for all n.
- The natural map : $\pi_1(R_{M'_n}) \to \pi_1(R)$ is surjective but $R_{M'_n}$ is not connected for all n.

For a hyperbolic Riemann surface $R = \mathbb{H}/\Gamma$, we consider the *convex core* $C(\Gamma)$ of the limit set of Γ, that is, the hyperbolic convex envelope of $\Lambda(\Gamma) \subset \mathbb{R} \cup \{\infty\}$ in $\mathbb{H}$. Since the convex core $C(\Gamma)$ is Γ-invariant, it determines a region $C(R)$ in R and we call the region *the convex core* of R.

Definition 5. We say that a Riemann surface R has *ϵ-uniform geometry* if the following two conditions are satisfied for some $\epsilon > 0$:

1. The ϵ-thin part of R consists of cusp neighborhoods.
2. The injectivity radius on the convex core $C(R)$ of R is less than ϵ^{-1}.

Since $C(R)$ is connected and it contains any closed geodesic on R, from Theorem 1 we have the following immediately .

Corollary 1. *Let R be a Riemann surface with ϵ-uniform geometry for some $\epsilon > 0$. Then $Mod^{\#}_c(R)$ is discrete for any simple closed geodesic c on R.*

Remark 4. The conditions in Theorem 1 do not imply the uniform geometry. For example, set $R = \mathbb{C} - \{n : n \in \mathbb{Z}\}$. Then R has a Fuchsian model of the first kind, and hence the convex core $C(R)$ coincides with R. But since R has points with arbitrarily large injectivity radius, R does not have ϵ-uniform geometry for any $\epsilon > 0$. On the other hand, it is easily seen that R satisfies the conditions in Theorem 1.

Remark 5. The conditions having uniform geometry were first stated as no short geodesics and no large disk condition. Nakanishi and Yamamoto[8] shows that under these conditions the out radius of the Teichmüller space is strictly less than 6. Ohtake[9] uses these conditions to show that the norm of the Poincaré series is strictly less than one which generalizes a result in McMullen[6].

It is important to give conditions for the mapping class group to be discrete. By using the above results, we have the following.

Theorem 2. *Let R be a Riemann surface satisfying the conditions in Theorem 1 or Corollary 1. Suppose that either the genus of R, the number of cusps or the number of holes is positive finite. Then $\mathrm{Mod}^{\#}(R)$ is discrete.*

For a planar Riemann surface, Theorem 2 does not necessarily hold. See Example 6 in §6.

5. PROOFS OF MAIN RESULTS

First of all, we note the geometry of a component of R_M.

Proposition 1. *For $M > 0$, let R_M^* be a connected component of R_M defined in Definition 4 and R_ϵ the ϵ-thin part of R for some small $\epsilon < M$. We assume that $R_M^* - R_\epsilon$ is not of type $(0,3)$. Then there exists a constant $M_1 > 0$ depending only on M and ϵ such that for any point $p \in R_M^* - R_\epsilon$ there exists a simple closed curve c_p passing through p with $\ell(c_p) < M_1$ which does not surround a puncture of R.*

Proof. Let Γ be a Fuchsian group representing R. Take an arbitrary point p in $R_M^* - R_\epsilon$. From the definition, we may find a simple closed curve $c_p \ni p$ whose length is less than M. If c_p is not homotopic to a simple closed curve which surrounds a puncture of R, then there is nothing to prove.

Thus, we suppose that c_p surrounds a puncture of R. Then, a parabolic transformation $\gamma \in \Gamma$ represents c_p. We may assume that $\gamma(z) = z + 1$. We take $\delta(\epsilon), \delta(M) > 0$ so that

$$d_{\mathbb{H}}(\delta(\epsilon)\sqrt{-1}, \delta(\epsilon)\sqrt{-1} + 1) = \epsilon,$$
$$d_{\mathbb{H}}(\delta(M)\sqrt{-1}, \delta(M)\sqrt{-1} + 1) = M.$$

We put

$$S(M, \epsilon) = \{z \in \mathbb{H} \mid \delta(M) \le \mathrm{Im}\, z \le \delta(\epsilon), 0 \le \mathrm{Re}\, z \le 1\}.$$

Since $\ell(c_p) < M$ and $p \notin R_\epsilon$, a lift C_p of c_p contains a point in $S(M, \epsilon)$.

Let L_z ($z \in \mathbb{H}$) denote the geodesic arc from z to $z + 1$. If the projection l_z in R of L_z via the canonical projection $\pi : \mathbb{H} \to R = \mathbb{H}/\Gamma$ is not simple for some $z \in S(M, \epsilon)$, then l_z contains a non-trivial simple closed curve c'_z with $\ell(c'_z) < \ell(l_z) < M$. Noting that l_z is the projection of the geodesic arc L_z, we verify that c'_z is not homotopic to c_p. Connecting c'_z and c_p, we have a simple closed curve passing through p with length less than $M_1 = 2(M + d_{\mathbb{H}}(\delta(\epsilon)\sqrt{-1}, \delta(M)\sqrt{-1}))$ which does not surround a puncture of R.

Finally, we suppose that l_z is simple for any $z \in S(M, \epsilon)$. Let us consider a geodesic L_z for $z \in \widetilde{R_M^*} \cap \{z \in \mathbb{H} \mid \mathrm{Im}\, z = \delta(M)\}$, where $\widetilde{R_M^*}$ is a lift of R_M^* with $\widetilde{R_M^*} \cap S(M, \epsilon) \neq \emptyset$. From the definition, $\ell(L_z) = M$. Therefore, there exists a simple closed curve c_z in R_M^* passing through $\pi(z)$ with $\ell(c_z) < M = \ell(L_z) = \ell(l_z)$. Obviously, the curve c_z is not homotopic to $l_z = \pi(L_z)$ because l_z is the shortest simple closed curve which passes through $\pi(z)$ and surrounds the puncture. Therefore, connecting c_z and c_p as before, we have a non-trivial simple closed curve passing through p with length less than M_1 not surrounding a puncture of R. □

To prove the main results, the following proposition on hyperbolic geometry is crucial.

Proposition 2. *Let Γ be a Fuchsian model on the upper half plane $\mathbb{H}$ of a Riemann surface R. Assume that Γ is non-elementary.*

Let M and D be positive constants, and g_1, g_2 and g_3 be distinct hyperbolic elements in Γ such that

1. *translation lengths of g_j $(j = 1, 2, 3)$ are less than M,*
2. *the projections of the axes ℓ_j of g_j to R are simple closed geodesics, and*
3. *the distances between a point z_1 on ℓ_1 and ℓ_j $(j = 2, 3)$ are less than D.*

Let f be a quasiconformal self-map of $\mathbb{H}$ such that $f \circ \Gamma \circ f^{-1} = \Gamma$, and χ an isomorphism of Γ induced by f.

Suppose that

$$\chi(g_1) = g_1, \quad \chi(g_2) = g_2, \quad \chi(g_3) \neq g_3.$$

Then there is a constant $A > 1$ depending only on M and D such that

$$K(f) \geq A.$$

To prove this proposition, we prepare some well-known results (cf. [1], [5]).

Lemma 1 (Teichmüller)**.** *Let f be a quasiconformal selfmap of $\mathbb{C}$ fixing 0 and 1, and suppose that there is a point z_0 in $\mathbb{C} - \{0, 1\}$ such that*

$$\log M = d_1(z_0, f(z_0)) > 0.$$

Then $K(f) \geq M^2$, where $d_1(\, , \,)$ is the hyperbolic distance on $\mathbb{C} - \{0, 1\}$.

Lemma 2 (Wolpert)**.** *Let f be a quasiconformal mapping of a Riemann surface R onto an another Riemann surface S, and c be a simple closed geodesic on R with hyperbolic length L. Then the hyperbolic length of a closed geodesic on S homotopic to $f(c)$ is not greater than $K(f)L$.*

Lemma 3 (Collar lemma 1)**.** *For a given $M > 0$, let g and g' be arbitrary two hyperbolic elements of Γ with translation lengths less than M. Suppose that the projections of the axes of g and g' to R are simple closed geodesics which intersect to each other. Then the axes make an angle greater than $C > 0$ depending only on M.*

Lemma 4 (Collar lemma 2)**.** *For a given $M > 0$, let g and g' be arbitrary two distinct hyperbolic elements of Γ with translation lengths less than M. Suppose that the projections of the axes of g and g' to R are simple closed geodesics which coincide or disjoint. Then the axes have a distance greater than $C > 0$ depending only on M.*

Proof of Proposition 2. We may assume that fixed points of g_1 are 0 and ∞, and that $z_1 = \sqrt{-1} \in \mathbb{H}$, hence $d_{\mathbb{H}}(\sqrt{-1}, \ell_j) \leq D$ for $j = 2, 3$. We may also assume that the maximal dilatation of f is less than 2. Then at least one of the fixed points of g_j $(j = 2, 3)$ is not in $U = \{x \in \mathbb{R} \mid |x| < \delta \text{ or } |x| > 1/\delta\}$ for sufficiently small $\delta > 0$ which depends only on M and D. Indeed, if both fixed points of g_j are in $U_1 = \{x \in \mathbb{R} \mid |x| < \delta\}$ for small $\delta > 0$, then it contradicts $d_{\mathbb{H}}(\sqrt{-1}, \ell_j) \leq D$. The same argument works when both fixed points are in $U_2 = \{x \in \mathbb{R} \mid |x| > 1/\delta\}$. Next, suppose that one fixed point of g_j is in U_1 and the other is in U_2. If $\ell_1 \cap \ell_j \neq \emptyset$, then it contradicts Lemma 3. If $\ell_1 \cap \ell_j = \emptyset$, it contradicts Lemma 4. Therefore, we verify that at least one of the fixed points of g_j $(j = 2, 3)$ is not in U. By using the same argument, we see that there exists a constant $\delta' > 0$ depending only on M and D such that all fixed points of g_2 and g_3 are in $\{x \in \mathbb{R} \mid \delta' < |x| < 1/\delta'\}$.

From the above consideration, we verify that the Euclidean diameter $\mathrm{diam}(\ell_3)$ of ℓ_3 is greater than some $r = r(M, D) > 0$ which depends only M and D. Set $g_4 = f \circ g_3 \circ f^{-1}$. By the assumption we have $g_4 \neq g_3$. Then, we see that there exists a constant $C = C(M, D) > 0$ depending only on M and D such that an inequality

$$|b - f(b)| > C \tag{1}$$

holds for a fixed point b of g_3. Indeed, since $K(f) < 2$, the translation length of g_4 is less than $2M$ by Lemma 2. Noting that $\mathrm{diam}(\ell_3) > r$, we see that if $\ell_3 \cap \ell_4 \neq \emptyset$, then we have the assertion from Lemma 3 and that Lemma 4 yields the assertion if $\ell_3 \cap \ell_4 = \emptyset$.

Take a fixed point a of g_2 with

$$\delta < |a| < \frac{1}{\delta}. \tag{2}$$

Let ϕ be a Möbius transformation with $\phi(0) = 0, \phi(a) = 1$ and $\phi(\infty) = \infty$. As we noted,

$$\delta' < |b| < \frac{1}{\delta'}.$$

Hence, (1) and (2) implies that

$$d_a(b, f(b)) > \log L$$

holds for some constant $L > 1$ depending only on M and D, where $d_a(,)$ is the hyperbolic distance on $\mathbb{C} - \{0, a\}$. Considering $\{0, a, b\}$ instead of $\{0, 1, z_0\}$ for $z_0 = \phi(b)$ in Lemma 1, we verify that the assertion follows for $A = L^2$. □

Next, we show a fundamental property of $\mathrm{Mod}_c^{\#}(R)$.

Proposition 3. *Let R be a Riemann surface. For an arbitrary simple closed geodesic c, let $\{[f_n]\}$ be a sequence of transformations of $\mathrm{Mod}_c^{\#}(R)$ that satisfies $\lim_{n\to\infty} K(f_n) = 1$. Then there exists a subsequence $\{[f_{n_j}]\}$ of $\{[f_n]\}$ such that $\{f_{n_j}\}$ locally uniformly converges to a conformal self-map f of R which determines an transformation $[f] \in \mathrm{Mod}_c^{\#}(R)$.*

Proof. Let Γ be a Fuchsian model of R. First we suppose that c is not homotopic to a boundary component of R. Then there exists a simple closed geodesic c' on R with $c \cap c' \neq \emptyset$. Hence Lemma 5 (with $K = c$ and $C = c'$) below shows the desired result.

Next suppose that c is homotopic to a boundary component of R. We may assume that the Riemann surface R is not topologically finite. Hence, there exists a pair of pants S in R such that the boundary consists of dividing simple closed geodesics c_1 and c_2 other than c, and neither c_1 nor c_2 is homotopic to a boundary component of R. If $f_n(c_1)$ are homotopic to c_1 or c_2 for infinitely many n, then we can apply the previous argument. Hence we assume that $f_n(c_1)$ are homotopic to neither c_1 nor c_2 for all n. Since $f_n(c)$ are homotopic to c, and $f_n(S)$ are still pairs of pants bounded by $f_n(c), f_n(c_1)$ and $f_n(c_2)$ for all n, we conclude that $f_n(c_1) \cap \bar{S} \neq \emptyset$ or $f_n(c_2) \cap \bar{S} \neq \emptyset$. We may assume that $f_n(c_1) \cap \bar{S} \neq \emptyset$. Then the following Lemma 5 again shows the desired result. □

Lemma 5. *Let $\{f_n\}$ be a sequence of quasiconformal self-maps of a hyperbolic Riemann surface R that satisfies $\lim_{n\to\infty} K(f_n) = 1$. Suppose that there exist compact subsets C and K of R such that $f_n(C) \cap K \neq \emptyset$ for all n. Then there* *exist a subsequence* $\{f_{n_j}\}$ *of* $\{f_n\}$ *and a conformal self-map* f *of* R *such that* $\{f_{n_j}\}$ *converge to* f *locally uniformly on* R.

Proof. From the assumption, there exists a sequence $\{p_n\}$ on C such that $f_n(p_n) \in K$. Since C and K are compact, there exist $p \in C$ and $q \in K$ such that $p_n \to p$ and $f_n(p_n) \to q$ as $n \to \infty$ Take lifts of p_n, p and q in $\mathbb{H}$, say $\tilde{p}_n$ $\tilde{p}$ and $\tilde{q}$, respectively, so that $\tilde{p}_n \to \tilde{p}$ as $n \to \infty$. We can take lifts $\tilde{f}_n : \mathbb{H} \to \mathbb{H}$ of f_n satisfying $\tilde{f}_n(\tilde{p}_n) \to \tilde{q}$. Since $\{\tilde{f}_n\}$ is a normal family, a subsequence $\{\tilde{f}_{n_j}\}$ of $\tilde{f}_n$ converges locally uniformly, and the limit function $\tilde{f}$ is either a quasiconformal self-map of $\mathbb{H}$ or a constant in $\mathbb{R} \cup \{\infty\}$. Since $\tilde{f}(\tilde{p}) = \tilde{q}$ is in $\mathbb{H}$, $\tilde{f}$ is not a constant. Thus, it follows from $\lim_{n\to\infty} K(f_n) = 1$ that $\tilde{f}$ is a conformal self-map of $\mathbb{H}$. Hence, $\{f_{n_j}\}$ converges locally uniformly to a conformal self-map f of R which is the projection of $\tilde{f}$. □

Before proving our main theorems, we shall give a sufficient condition for discreteness of a sequence of $\mathrm{Mod}^{\#}(R)$ under the conditions in Theorem 1.

Proposition 4. *Let* R *be a Riemann surface satisfying the two conditions in Theorem 1, and* $\{f_n\}$ *be a sequence of quasiconformal self-maps of* R *satisfying the following conditions:*

- $\{(f_n)_*\}$ *converges to the identity, where* $(f_n)_* : \pi_1(R) \to \pi_1(R)$ *is an isomorphism induced by* f_n.
- $\lim_{n\to\infty} K(f_n) = 1$.

Then, as an element of $\mathrm{Mod}^{\#}(R)$, $[f_n] = [id]$ *for sufficiently large* n.

Proof. Let Γ be a Fuchsian model of R, and $\tilde{f}_n$ a lift of f_n for each n. We may take $\tilde{f}_n$ so that the isomorphisms $\chi_n : \Gamma \to \Gamma$ induced by $\tilde{f}_n$ converge to the identity.

Since the natural map $\pi_1(R^*_M) \to \pi_1(R)$ is surjective for sufficiently large $M > 0$, we may find two distinct simple closed geodesics L^0_1 and L^0_2 on R^*_M so that their lengths are less than M. Let γ_j $(j = 1, 2)$ be hyperbolic elements of Γ which represent L^0_j. Since $\chi_n \to id$ $(n \to \infty)$, $\chi_n(\gamma_1) = \gamma_1$ and $\chi_n(\gamma_2) = \gamma_2$ for sufficiently large n.

Suppose that χ_n are not eventually the identity. Then we prove the proposition by drawing a contradiction. We may find a $\gamma_n \in \Gamma$ so that $\chi_n(\gamma_n) \neq \gamma_n$. The following lemma shows more, that is, we may take better one as γ_n.

Lemma 6. *For sufficiently large* n, *there exists a hyperbolic element* γ_n *of* Γ *that satisfies the following two conditions:*

1. $\chi_n(\gamma_n) \neq \gamma_n$.
2. *the projection of the axis of* γ_n *on* R *is a simple closed geodesic with length less than* M.

Proof. Since $\chi_n \neq id$, there exists an element α_n of Γ such that $\chi_n(\alpha_n) \neq \alpha_n$. We will show that either $\alpha_n \circ \gamma_1 \circ \alpha_n^{-1}$ or $\alpha_n \circ \gamma_2 \circ \alpha_n^{-1}$ is a desired element. It is obvious that both of them satisfy the second condition of the lemma. Hence, it suffices to show that one of them satisfies the first condition.

Suppose that χ_n fixes $\alpha_n \circ \gamma_j \circ \alpha_n^{-1}$ $(j = 1, 2)$. Then

$$\beta_n \circ \gamma_j \circ \beta_n^{-1} = \gamma_j, \qquad (j = 1, 2)$$

where $\beta_n = \alpha_n^{-1} \circ \chi_n(\alpha_n)$. Thus, β_n fixes all fixed points of γ_1 and γ_2. Since γ_1 and γ_2 are non-commutative, the Möbius transformation β_n fixes four points and it must be the identity map. This contradicts $\chi_n(\alpha_n) \neq \alpha_n$. □

Now, under the assumption of Theorem 1, we show the existence of elements in Γ satisfying the assumption of Proposition 2 for χ_n.

Lemma 7. *For each* n*, there exist hyperbolic elements* $g_{j,n}$ $(j = 1, 2, 3)$ *with axes* $\ell_{j,n}$ *such that they satisfy the following four conditions.*

1. *the projections* $L_{j,n}$ *of* $\ell_{j,n}$ *to* R *are simple closed geodesics.*
2. *there is a constant* M *independent of* n *such that the lengths of* $L_{j,n}$ *are less than* M*.*
3. *there is a constant* D *independent of* n *such that the distances between a point on* $\ell_{1,n}$ *and* $\ell_{j,n}$ $(j = 2, 3)$ *are less than* D*, and*
4. $\chi_n(g_{j,n}) = g_{j,n}$ *for* $j = 1, 2$*, and* $\chi_n(g_{3,n}) \neq g_{3,n}$*.*

Proof. Let γ_n be an element in Lemma 6. We denote by ℓ_1^0, ℓ_2^0 and ℓ_n the axes of γ_1, γ_2 and γ_n, respectively. Let $\widetilde{R_M^*}$ be a lift of R_M^*.

Fix a point z_1 on ℓ_1^0. There exists the nearest point z_n on ℓ_n from z_1. Since z_1 and z_n belong to $\widetilde{R_M^*}$, and $\widetilde{R_M^*}$ is connected, there exists a curve C_n on $\widetilde{R_M^*}$ which connects z_1 and z_n. Furthermore, we can take the curve C_n so that the projection of C_n is in $R_M^* - R_\epsilon$.

First, we observe a fundamental property of R_M.

For an arbitrary point p_0 in $R_M^* - R_\epsilon$, there exists a non-trivial simple closed curve C_{p_0} passing through p_0 with $\ell(C_{p_0}) < M$. Replacing $M > 0$ so large if necessary, we may assume that C_{p_0} is not homotopic to a puncture(Proposition 1). Then there exists a simple closed geodesic L_{p_0} which is homotopic to C_{p_0}. The length of L_{p_0} is greater than ϵ. Hence there exists a constant B depending only on ϵ and M such that $d_{\mathbb{H}}(p_0, L_{p_0}) \leq B$. This implies that for every $z_0 \in \widetilde{R_M^*}$ which is not projected to R_ϵ, there is an axis ℓ_0 of a hyperbolic element of Γ such that $d_{\mathbb{H}}(z_0, \ell_0) \leq B$ and that the projection to R is a simple closed geodesic with length less than M.

Here, we consider the following two cases for $d_{\mathbb{H}}(z_1, \ell_n)$.

1: $d_{\mathbb{H}}(z_1, \ell_n) \leq 4(B + M + 1)$.

In this case, we set $g_{1,n} = \gamma_1$, $g_{2,n} = \gamma_2$ and $g_{3,n} = \gamma_n$. Then the third condition of the lemma holds for $D = \max(4(B + M + 1), d_{\mathbb{H}}(z_1, \ell_2^0))$. Other three conditions are trivial from the choice of these transformations.

2: $d_{\mathbb{H}}(z_1, \ell_n) > 4(B + M + 1)$.

In this case, there are points z_n' and z_n'' on C_n that satisfy $d_{\mathbb{H}}(z_n, z_n') = d_{\mathbb{H}}(z_n', z_n'') = 2(B + M + 1)$. Since z_n' and z_n'' are points on $\widetilde{R_M^*}$ which are not projected to R_ϵ, it follows from the above observation,that there exists an axis ℓ_n' (resp. ℓ_n'') of γ_n' (resp. γ_n'') in Γ such that $d_{\mathbb{H}}(z_n', \ell_n') \leq B$ (resp. $d_{\mathbb{H}}(z_n'', \ell_n'') \leq B$). Since $d_{\mathbb{H}}(z_n', z_n'') = 2(B + M + 1)$, we see that ℓ_n' and ℓ_n'' are distinct. Take a point $w_n \in \ell_n'$ so that $d_{\mathbb{H}}(z_n', w_n) \leq B$.

If $\chi_n(\gamma_n') = \gamma_n'$ and $\chi_n(\gamma_n'') = \gamma_n''$, set $g_{1,n} = \gamma_n'$, $g_{2,n} = \gamma_n''$ and $g_{3,n} = \gamma_n$. Noting that

$$d_{\mathbb{H}}(w_n, \ell_n'') \leq 2(B + M + 1) + 2B$$

and

$$d_{\mathbb{H}}(w_n, \ell_n) \leq 2(B + M + 1) + B,$$

we see that the third condition of the lemma holds for $D = 4B + 2(M + 1)$. If $\chi_n(\gamma_n') \neq \gamma_n'$ or $\chi_n(\gamma_n'') \neq \gamma_n''$, replace γ_n by γ_n' or γ_n''. Repeating this argument, we get desired elements. □

We proceed to prove Proposition 4. Using $g_{1,n}, g_{2,n}$ and $g_{3,n}$ obtained in Lemma 7, we have

$$K(f_n) \geq A = A(M, D) > 1$$

from Proposition 2. Since constants M and D are independent of n, this contradicts $\lim_{n\to\infty} K(f_n) = 1$. Hence we have proved this proposition. $\square$

Proof of Theorem 1. Let $p_0 = [R, id]$ be the base point of $T^{\#}(R)$. We first suppose that there exists a sequence $\{g_n\}$ of quasiconformal self-maps of R which determine distinct elements of $\mathrm{Mod}_c^{\#}(R)$ such that $\lim_{n\to\infty} g_n(p_0) = p$ for some p in $T^{\#}(R)$. Consider the sequence $\{f'_n = g_{n+1}^{-1} \circ g_n\}$. Then we see that $f'_n(p_0)$ converges to p_0. Thus there exist quasiconformal mappings $f_n : R \to R$ $(n = 1, 2, \dots)$ such that f_n is RT-equivalent to f'_n and that $\lim_{n\to\infty} K(f_n) = 1$. From Proposition 3, there exists a conformal self-map f of R such that $[f_n \circ f] \in \mathrm{Mod}_c^{\#}(R)$ and $f_n \circ f$ converge to the identity on R locally uniformly. Since $\lim_{n\to\infty} K(f_n \circ f) = \lim_{n\to\infty} K(f_n) = 1$, it follows from Proposition 4 that $[f_n \circ f] = [id]$ for sufficiently large n. Hence $[f_n] = [f^{-1}]$ for sufficiently large n. This contradicts the assumption that all f_n are distinct.

Finally, we see that the same argument as above is valid for an arbitrary point $q = [S, f]$ in $T^{\#}(R)$. To see this, it suffices to show that the conditions of Theorem 1 is invariant under the quasiconformal deformation. Namely, the following lemma concludes the theorem. $\square$

Lemma 8. *Let $f : R \to S$ be a K-quasiconformal homeomorphism. Suppose that a Riemann surface R satisfies the both conditions in Theorem 1. Then S also satisfies them.*

Proof. Let $\tilde{f} : \mathbb{H} \to \mathbb{H}$ be a lift of K-quasiconformal map f. The quasiconformal map $\tilde{f}$ can be extended to $\mathbb{H} \cup \hat{\mathbb{R}}$ with $\tilde{f}(\infty) = \infty$ and the restriction $\tilde{f}|\mathbb{R}$ of $\tilde{f}$ to $\mathbb{R}$ is a quasisymmetric function. The Douady-Earle extension $\Phi(\tilde{f})$ of $\tilde{f}|\mathbb{R}$ to $\mathbb{H}$ is a quasiconformal and bilipschitz map, and the bilipschitz constant K' depends only on K (cf. [2]). The projection $\phi_f : R \to S$ of $\Phi(\tilde{f})$ satisfies

$$(1/K')\ell(c) \leq \ell(\phi_f(c)) \leq K'\ell(c)$$

for an arbitrary curve c on R, and $[S, f] = [S, \phi_f]$ in $T^{\#}(R)$. Then for an arbitrary point a in $\phi_f(R_M^*)$, there exists a non-trivial simple closed curve c_0 containing a such that $\ell(c_0) \leq K'M$. Thus, $\phi_f(R_M^*) \subset S_{K'M}$. Therefore, we see that the Riemann surface S satisfies the second condition in Theorem 1 for a connected component of $S_{K'M}$ containing $\phi_f(R_M^*)$.

The same argument also shows that the first condition is satisfied by S.

$\square$

Proof of Theorem 2. Suppose that R is a Riemann surface of positive finite genus g and satisfies the conditions in Theorem 1. Further suppose that $\mathrm{Mod}^{\#}(R)$ is not discrete. Then there exists a sequence $\{f_n\}$ of quasiconformal self-maps of R which determine distinct elements of $\mathrm{Mod}^{\#}(R)$ such that $\lim_{n\to\infty} K(f_n) = 1$. Let l be a dividing simple closed curve such that one of components of $R \setminus l$ is a Riemann surface S of genus g with only one boundary component. Take a non-dividing simple closed geodesic c on S. Then $f_n(c) \cap \bar{S} \neq \emptyset$ for all n. Indeed, if $f_n(c) \subset \bar{S}^c$, then $f_n(c)$ should be a dividing curve. Since c is a non-dividing curve and f_n is a homeomorphism, it can not occur. Then from Lemma 5, there exists a subsequence of $\{f_n\}$ which converges to a conformal self-map f of R locally uniformly on R. Hence we can apply Proposition 4, and we conclude a contradiction.

Next suppose that R has finite positive number of cusps and satisfies the conditions in Theorem 1. If $\mathrm{Mod}^{\#}(R)$ is not discrete, then there exists a sequence $\{f_n\}$ as above. Let V be a cusp neighborhoods of a puncture of R. Since R has only finitely many cusps, we may assume that $f_n(V)\cap V \neq \emptyset$ for all n by taking a subsequence of $\{f_n\}$. Let S be a pair of pants in R such that it contains V and that the boundaries of S consist of the puncture and two dividing simple closed geodesics, say c_1 and c_2. We may assume that two geodesics c_1 and c_2 are not homotopic to a boundary component of R. If $f_n(c_1)$ is homotopic to c_1 for infinity many n, then they determine elements of $\mathrm{Mod}^{\#}_{c_1}(R)$. Hence, they must be discrete from Theorem 1. Assume that $f_n(c_1)$ is not homotopic to c_1 for all n. Since $f_n(V)\cap V \neq \emptyset$ and $f_n(S)$ is still a pair of pants for each n, we see that $f_n(c_1)\cap(\bar{S}\setminus V)\neq\emptyset$ or $f_n(c_2)\cap(\bar{S}\setminus V)\neq\emptyset$. We may assume that $f_n(c_1)\cap(\bar{S}\setminus V)\neq\emptyset$. Then from Lemma 5 and Proposition 4, we conclude a contradiction.

Finally, suppose that R has finite positive number of borders and satisfies the conditions in Theorem 1. If $\mathrm{Mod}^{\#}(R)$ is not discrete, then there exists a sequence $\{f_n\}$ as before. Let B be a one of borders of R. Since R has only finite number of borders, we may assume that $f_n(B)=B$ for all n. Let c be a simple closed geodesics which is homotopic to B. Then $f_n(c)$ is homotopic to c. Thus $f_n \in \mathrm{Mod}^{\#}_{c}(R)$, and $\{f_n\}$ is discrete by Theorem 1. This contradicts $\lim_{n\to\infty} K(f_n)=1$. Hence $\mathrm{Mod}^{\#}(R)$ is discrete. □

6. FURTHER EXAMPLES

In Example 4, we showed that there exists a Riemann surface R that satisfies the two conditions in Theorem 1, but that $\mathrm{Mod}^{\#}(R)$ is not discrete. In this case, there exists a sequence $\{\omega_n\}$ of distinct elements of $\mathrm{Mod}^{\#}(R)$ such that $\omega_n(p_0)=p_0$ for any n, where $p_0=[R,id]\in T^{\#}(R)$. By modifying this example, we see that the following example gives a Riemann surfaces R that there exists a sequence $\{\omega_n\}$ of distinct elements of $\mathrm{Mod}^{\#}(R)$ such that $\lim_{n\to\infty} d_T(\omega_n(p),p)=0$ for some $p\in T^{\#}(R)$ and $\omega_n(p)\neq p$ for any n.

Example 5. There is a Riemann surface R without cusps that satisfies two conditions in Theorem 1, but that $\mathrm{Mod}^{\#}(R)$ is not discrete.

We will give a sketch of a construction of such an R. Consider a torus A_0 with two geodesic borders of the same length. Let B_0 be another torus obtained via the $(1+\epsilon_0)$ quasiconformal deformation of A_0 for some $\epsilon_0>0$. Attach two copies of B_0 to A_0 along the borders suitably, and we obtain a Riemann surface A_1. Hence, it is a Riemann surface of genus 3 with two geodesic borders.

Next we take a Riemann surface B_1 which is the $(1+\epsilon_1)$ quasiconformal deformation of A_1 for some $\epsilon_1>0$. Attach two copies of B_1 to A_1 along the borders suitably, and we obtain a Riemann surface A_2 which is a Riemann surface of genus 9 with two geodesic borders. Repeating this process, we have a sequence of Riemann surfaces $\{A_n\}$ for some sequence $\{\epsilon_n\}$ of positive numbers. We denote the inductive limit of these A_n by R. If the sequence $\{\sum_{n=1}^{p}\epsilon_n\}_{p=1}^{\infty}$ is bounded, then we see that R satisfies the above conditions on the injectivity radius.

Let f_n be a quasiconformal selfmap of R which sends a part corresponding to A_n to a part corresponding to B_n. If we take a sequence $\{\epsilon_n\}$ converges to zero so rapidly, then we verify that the maximal dilatations of f_n converge to 1. Thus,

f_n induces an element of $\mathrm{Mod}^{\#}(R)$ whose orbits of $p_0 = [R, id]$ converge to p_0 in $T^{\#}(R)$.

Next, we modify the above construction to obtain another kind of examples as follows.

Example 6. We give a planar surface R with no short geodesics such that $\mathrm{Mod}^{\#}(R)$ is not discrete.

For instance, set

$$z_n = \begin{cases} n + \frac{\sqrt{-1}}{j(n)+1}, & (n \neq 0) \\ 0, & (n = 0) \end{cases}$$

where $j(n)$ is the power of the factor 2 when we decompose $|n|$ to the product of primes. And set $R = \mathbb{C} - \sum_{n=-\infty}^{\infty} \{z_n\}$.

Now for every positive m, we take a locally affine quasiconformal selfmap f_m of R such that $\mathrm{Re}\, f_m(z) = \mathrm{Re}\, z + 2^m$ (and hence $f_m(z_n) = z_{(n+2^m)}$). Then, since $j(n + 2^m) = j(n)$ if $j(n) < m$, we may take f_m so that the maximal dilatations of f_m tend to 1. Hence $\mathrm{Mod}^{\#}(R)$ is not discrete.

We shall construct a Riemann surface R and sequences $\{M_n\}$, $\{M'_n\}$ having the properties referred in Remark 3 in §4.

Example 7. We consider right-angled hexagons H_n $(n = 1, 2, \dots)$ in the hyperbolic plain $\mathbb{H}$. The sides of the hexagon H_n are labelled $a_{j,n}$ $(j = 1, 2, \dots, 6)$ counterclockwise. We construct the hexagon so that $\ell(a_{2,n}) = \ell(a_{6,n})$, $\ell(a_{3,n}) = \ell(a_{5,n}) = 1$ and $\ell(a_{1,n}) = (2n)^{-1}$. Then $\{H_n\}$ converges to a pentagon with a cusp as $n \to \infty$. Thus, we see that

$$d_{\mathbb{H}}(P_n, a_{2,n}) = d_{\mathbb{H}}(P_n, a_{6,n}) \leq M < \infty \tag{3}$$

holds for some M independent of n, where P_n is the midpoint of $a_{4,n}$. Take the perpendicular line $L_{j,n}$ $(j = 2, 6)$ from P_n to $a_{j,n}$. As $n \to \infty$, we see that $d_{\mathbb{H}}(a_{1,n}, L_{2,n}) = d_{\mathbb{H}}(a_{1,n}, L_{6,n}) \to \infty$.

Now, we take $k(n)$ copies of H_n, say $H_n^1, \dots, H_n^{k(n)}$, so that

$$\frac{1}{3} d_{\mathbb{H}}(a_{1,n}, L_{2,n}) \leq 2k(n)\ell(a_{1,n}) = \frac{1}{n} k(n) \leq \frac{1}{2} d_{\mathbb{H}}(a_{1,n}, L_{2,n}). \tag{4}$$

Obviously, $k(n)/n \to \infty$ as $n \to \infty$. We denote the sides of H_n^i corresponding to $a_{j,n}$ by $a_{j,n}^i$ $(i = 1, 2, \dots k(n); j = 1, 2, \dots, 6)$ and glue H_n^i and H_n^{i+1} along $a_{6,n}^i$ and $a_{2,n}^{i+1}$. Then, we have a right-angled $(2k(n) + 4)$-gon D_n in $\mathbb{H}$. Label the side of D_n formed by $a_{1,n}^1 \cup \dots \cup a_{1,n}^{k(n)}$ $b_{1,n}$ and denote the rest of sides by $b_{2,n}, \dots, b_{2k(n)+4,n}$ counterclockwise.

We take a copy of D'_n of D_n with sides $b'_{j,n}$ $(j = 1, 2, \dots, 2k(n) + 4)$ corresponding to $b_{j,n}$ of D_n. We glue D_n and D'_n along $b_{j,n}$ and $b'_{2k(n)+6-j,n}$ for $j = 2, 4, \dots, 2k(n) + 2$ and $2k(n) + 4$. Then we have a hyperbolic bordered surface S_n of type $(0, k(n)+1)$. The boundary ∂S_n consists of one long curve $c_{1,n}$ and $k(n)$ short curves $c_{2,n}, \dots, c_{k(n),n}$. It follows from the construction that

$$\ell(c_{1,n}) = \frac{k(n)}{n},$$

$$\ell(c_{2,n}) = \ell(c_{k(n),n}) = 2,$$

and

$$\ell(c_{3,n}) = \dots = \ell(c_{k(n)-1,n}) = 4.$$

From (3), we verify that $(S_n)_{4M}$ is connected and the naturel map of $\pi_1((S_n)_{4M})$ to $\pi_1(S_n)$ is surjective. On the other hand, it follows from (4) that $(S_n)_{k(n)/n}$ is not connected while both $(S_n)_{k(n)/2n}$ and $(S_n)_{2k(n)/n+4M}$ are connected.

We take a sequence $\{j_n\}$ so that

$$4M < \frac{k(j_n)}{j_n} < \frac{k(j_{n+1})}{10j_{n+1}}. \qquad (n = 1, 2, \dots)$$

We glue S_{j_n} and $S_{j_{n+1}}$ along $c_{k(j_n),j_n}$ of ∂S_{j_n} and $c_{2,j_{n+1}}$ of $\partial S_{j_{n+1}}$. Then we have a bordered Riemann surface S, and a Riemann surface R whose convex core is S. From the construction we verify that R_{M_n} is connected for $M_n = k(j_n)/2j_n$ but $R_{M'_n}$ is not connected for $M'_n = k(j_n)/j_n$. Since $M_n, M'_n > 4M$, the natural maps of $\pi_1(R_{M_n})$ and $\pi_1(R_{M'_n})$ to $\pi_1(R)$ is surjective. Thus, R, $\{M_n\}$ and $\{M'_n\}$ are our desired ones.

References

[1] P. Buser, *Geometry and Spectra of Compact Riemann Surfaces*, Progress in Mathematics **106**, Birkhäuser Boston-Basel-Berlin, 1992.

[2] A.Douady and C.J. Earle, Conformally natural extension of homeomorphisms of the circle, Acta Math, **157** (1986) 23–48.

[3] C. J. Earle, F. P. Gardiner and N. Lakic, *Teichmüller spaces with asymptotic conformal equivalence,* preprint.

[4] F. P. Gardiner, Teichmüller theory and quadratic differentials, Wiley-Interscience,New York 1987.

[5] Y. Imayoshi and M. Taniguchi, *Introduction to Teichmüller Spaces*, Springer-Tokyo 1992.

[6] C. McMullen, Amenability, Poincaré series and quasiconformal maps, Invent. Math. **97** (1989), 95–127.

[7] S. Nag, *The Complex Analytic Theory of Teichmüller Spaces*, John Wiley & Sons 1988.

[8] T. Nakanishi and H. Yamamoto, On the outradius of the Teichmüller space, Comment. Math. Helvetici **64** (1989), 288–299.

[9] H. Ohtake, On the norm of the Poincaré series operator for a universal covering group, J. Math. Kyoto Univ. **32** (1992), 57–72.

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