Estimates of hyperbolic metric with applications to Teichmüller spaces

TOSHIYUKI SUGAWA
Department of Mathematics, Kyoto University, 606-8502 Kyoto, Japan
Current Address
Department of Mathematics, University of Helsinki P.O.Box 4 (Yliopistonkatu 5), FIN-00014, Helsinki, Finland
e-mail: sugawa@kusm.kyoto-u.ac.jp

(2000 Mathematics Subject Classification : Primary 30F45; Secondary 30F60, 30A10)

We show a monotonicity property of the hyperbolic metric of a punctured rectangular torus. We will then deduce a lower estimate of the hyperbolic metric of the domain $\mathbb{C} \setminus \{0, 1\}$. We also determine the value of hyperbolic sup-norm of standard quadratic differentials on the once-punctured square torus or the symmetric four-times punctured sphere. This enables us to compute numerically the inner and outer radii of the Bers embedding of the corresponding Teichmüller space under the hypothesis of some conjectural properties of it.

1. MAIN RESULTS

Let $\rho_D(z)|dz|$ denote the hyperbolic metric for a plane domain $D \subset \widehat{\mathbb{C}}$ which has at least three boundary points, or, more generally, for a hyperbolic Riemann surface D. In other words, $\rho_D(f(\zeta))|f'(\zeta)| = 1/(1 - |\zeta|^2)$ for an arbitrary holomorphic universal cover f of D from the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C}; |\zeta| < 1\}$. This quantity has been known to be quite an important conformal invariant in the theory of Complex Analysis. In various places such as Schottky's theorem, Landau's theorem and their refinements, we encounter the need to know about the magnitude of ρ_D. Here is a convenient upper estimate: $\rho_D(z) \le 1/\delta_D(z)$ in the case $D \subset \mathbb{C}$, where $\delta_D(z)$ stands for the Euclidean distance from $z \in D$ to the boundary ∂D. On the other hand, simple lower estimates of ρ_D are less known for general plane domains D. In this direction, the reader can find several results in [3].

When we try to obtain a general lower estimate of ρ_D, the most important thing is probably the estimation of that for the special domain $D_0 = \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ because the principle of hyperbolic metric due to R. Nevanlinna [12, p. 50] yields $\rho_D \ge \rho_{D_0}$ when $D \subset D_0$. We will denote by $\lambda(z)$ the hyperbolic density $\rho_{D_0}(z)$ of this domain throughout this article. Note that the symmetry of the domain D_0 leads to the useful identities

$$\lambda(\bar{z}) = \lambda(z), \quad \lambda(1 - z) = \lambda(z), \quad \text{and} \quad \lambda(1/z)/|z|^2 = \lambda(z). \tag{1.1}$$

This research was carried out during the author's visit to the University of Helsinki under the exchange programme of scientists between the Academy of Finland and the Japan Society for Promotion of Science.

An explicit representation of λ was given by Agard [2] in terms of the complete elliptic integrals of the first kind. An iterative algorithm computing the value of $\lambda(z)$ together with several interesting properties of it is given by Solynin and Vuorinen [16], where the definition of λ is slightly different from ours. Nevertheless it is still difficult to derive a useful estimate for $\lambda(z)$ from those results.

The following inequality was proved by Hempel [4] and Jenkins [5] independently:

$$\lambda(z) \geq \frac{1}{2|z|(|\log|z|| + C_1)}, \tag{1.2}$$

where $C_1 = 1/2\lambda(-1) = \Gamma(1/4)^4/4\pi^2 \approx 4.37688$, $\Gamma(x)$ is the usual Gamma function and equality holds above when $z = -1$. This lower bound has actually the best possible asymptotic behaviour. Indeed, the inclusion $\mathbb{D}^* = \mathbb{D} \setminus \{0\} \subset D_0$ leads to the inequality $\lambda(z) \leq \rho_{\mathbb{D}^*}(z) = 1/2|z|\log(1/|z|)$ for $0 < |z| < 1$ (see [12, p. 246]), which together with (1.2) implies that $\lambda(z) = (1 + o(1))/2|z|\log(1/|z|)$ as $z \to 0$. Our inequality below is, however, still better than the above except for neighbourhoods of the punctures, where the growth order of our bound is apparently smaller than the above.

Theorem 1. *For the hyperbolic metric $\lambda(z)|dz|$ of D_0, the lower estimate*

$$\lambda(z) \geq C_2|z|^{-3/4}|z-1|^{-1/2}$$

can be obtained and equality holds precisely when $z = -1$. Here, $C_2 = \sqrt{2}\lambda(-1) = 2\sqrt{2}\pi^2/\Gamma(1/4)^4 \approx 0.161555$.

Remark 1. Setting $m_1(z) = 1/2|z|(|\log|z|| + C_1)$ and $m_2(z) = C_2|z|^{-3/4}|z-1|^{-1/2}$, we also have the estimate $\lambda(z) \geq m_j(1-z)$ for $j = 1, 2$. It is quite elementary to show that $m_j(z) \geq m_j(1-z)$ if and only if $\operatorname{Re} z \leq 1/2$. By the property $m_j(1/z) = m_j(z)/|z|^2$, we note the relation $m_2(1/z)/m_1(1/z) = m_2(z)/m_1(z)$. A numerical experiment suggests that $m_2(z) \geq m_1(z)$ if $1/r_0 \leq |z| \leq r_0$, where $r_0 = 48$. We have checked also by numerical experiments that $\max\{\lambda(z)/m_2(z); 1/r_0 \leq |z| \leq r_0, \operatorname{Re} z \leq 1/2\} = \lambda(1/r_0)/m_2(1/r_0) \approx 1.2281$, while $\max\{\lambda(z)/m_1(z); 1/r_0 \leq |z| \leq r_0, \operatorname{Re} z \leq 1/2\} = \lambda(1/2)/m_1(1/2) = 2 + 8\pi^2\Gamma(1/4)^{-4}\log 2 \approx 2.3167$.

As an immediate corollary of the above theorem, we have the next theorem of Landau type.

Corollary 2. *Let $f(z) = a_0 + a_1 z + z_2 z^2 + \cdots$ be a holomorphic function on the unit disk which omits the values 0 and 1. Then the inequality*

$$|a_1| \leq C_3|a_0|^{3/4}|a_0 - 1|^{1/2}$$

follows, where equality holds precisely when f is a universal covering of D_0 and $a_0 = -1$. Here, $C_3 = \Gamma(1/4)^4/2\sqrt{2}\pi^2 \approx 6.1894$.

Our main inequality comes from a monotonicity property of the hyperbolic density for a symmetric subdomain of a rectangular torus with respect to a horizontal or a vertical lines. This property is, as we shall see in the next section, a direct consequence of Weitsman's theorem on circularly symmetric domains.

A (marked) *rectangular torus* T is the complex torus given by $T = \mathbb{C}/\mathbb{Z}[ai]$ for a positive number $a > 0$, where $\mathbb{Z}[ai]$ is the lattice group generated by 1 and ai over $\mathbb{Z}$, which will be regarded also as the subset $\{m + nai; m, n \in \mathbb{Z}\}$ of $\mathbb{C}$. We will denote by $[z]$ the image of $z \in \mathbb{C}$ under the canonical projection $p : \mathbb{C} \to T = \mathbb{C}/\mathbb{Z}[ai]$.

Note that a proper subdomain of T is always hyperbolic whereas T itself is not hyperbolic. For a proper subdomain D of T, we denote by $\widetilde{D}$ the inverse image of D under the projection $p : \mathbb{C} \to T$. Since the hyperbolic density $\rho_{\widetilde{D}}(z)$ is invariant under translations of the lattice, we can regard it as a function on D and we will denote it by $\rho_D([z])$, namely, $\rho_D([z]) = \rho_{\widetilde{D}}(z)$.

We offer a result on the property of symmetric subdomains of a rectangular torus. Here we will say that a subdomain D of T is (horizontally) *symmetric* if, for any $x \in \mathbb{R}$, the set $\{t \in [-a/2, a/2]; [x+ti] \in D\}$ is an open interval of the form $(-\delta, \delta)$ for some $0 \le \delta \le a/2$ or the whole interval $[-a/2, a/2]$.

Theorem 3. *Let D be a horizontally symmetric proper subdomain of the rectangular torus $T = \mathbb{C}/\mathbb{Z}[ai]$ with $a > 0$. Then, for a fixed $x \in \mathbb{R}$, the hyperbolic density $\rho_D([z]) = \rho_D([x + yi])$ is strictly increasing in $0 < y < a/2$ and strictly decreasing in $-a/2 < y < 0$ as long as $[x+yi] \in D$. In particular, $\rho_D([x+yi]) \ge \rho_D([x])$ for $x \in \mathbb{R}$ and $y \in [-a/2, a/2]$ with $[x + yi] \in D$.*

As a special case, we can extract the following corollary from the above.

Corollary 4. *The hyperbolic density $\rho_X([z])$ of the once-punctured rectangular torus $X = (\mathbb{C} \setminus L)/L$, where $L = \mathbb{Z}[ai]$ for some $a > 0$, attains its minimum at the unique point $[(1 + ai)/2]$.*

This knowledge leads to a result on the hyperbolic sup-norm of quadratic differentials on X. In general, for a holomorphic quadratic differential $\varphi = \varphi(z)dz^2$ on a hyperbolic Riemann surface R, we define the hyperbolic sup-norm of it by

$$\|\varphi\|_R = \sup_{z \in R} \rho_R(z)^{-2} |\varphi(z)|.$$

Note that the quantity $\rho_R^{-2}|\varphi|$ is a function on R, namely, it does not depend on the choice of a local coordinate. Holomorphic quadratic differentials on R of finite norm form a complex Banach space, which will be denoted by $B_2(R)$. This is known to be important as the ambient space of the Teichmüller space of R (see Section 3).

We consider the standard quadratic differential $\varphi_0 = dz^2$ on the once-punctured rectangular torus $X = (\mathbb{C} \setminus L)/L,\ L = \mathbb{Z}[ai]$, where z denotes standard local coordinates of X provided by local inverses of the projection $p : \mathbb{C} \setminus L \to X$. By the definition and Corollary 4, we have

$$\|\varphi_0\|_X = \sup_{[z] \in X} \rho_X([z])^{-2} = \sup_{z \in \widetilde{X}} \rho_{\widetilde{X}}(z)^{-2} = \rho_X([(1 + ai)/2])^{-2}.$$

In the case $a = 1$, the explicit expression of $\rho_{X_0}([(1 + i)/2])$ can be given, where $X_0 = (\mathbb{C} \setminus \mathbb{Z}[i])/\mathbb{Z}[i]$:

$$\rho_{X_0}([(1 + i)/2]) = \frac{2\pi^{3/2}}{\Gamma(1/4)^2} \approx 0.84721. \tag{1.3}$$

The proof of this will be given in Section 2. We remark here that other than this special case it seems to be quite difficult to calculate the value of $\rho_X([(1 + ai)/2])$ even in a numerical way.

As an immediate consequence, we have

Theorem 5. *The standard quadratic differential* $\varphi_0 = dz^2$ *on the once-punctured square torus* $X_0 = (\mathbb{C} \setminus L)/L,\ L = \mathbb{Z}[i]$, *has the norm*

$$\|\varphi_0\|_{X_0} = \rho_{X_0}([(1+i)/2])^{-2} = \frac{\Gamma(1/4)^4}{4\pi^3} \approx 1.3932.$$

It is easy to see $B_2(D_0) = 0$. Thus, a four-times punctured sphere is one extreme among the domains Y with nontrivial $B_2(Y)$. Actually, when $Y = \widehat{\mathbb{C}} \setminus \{0, 1, \infty, b\}$ with $b \neq 0, 1, \infty$, $B_2(Y)$ is the vector space spanned by the standard quadratic differential $\varphi(z) = 1/z(z-1)(z-b)$ over $\mathbb{C}$. In the case $Y_0 = \widehat{\mathbb{C}} \setminus \{0, 1, \infty, 1/2\}$, using the classical fact that Y_0 and $X_0 = (\mathbb{C} \setminus \mathbb{Z}[i])/\mathbb{Z}[i]$ are commensurable with respect to covering, we can deduce the following result from Theorem 5.

Theorem 6. *For the domain* $Y_0 = \widehat{\mathbb{C}} \setminus \{0, 1, \infty, 1/2\}$,

$$\left\| \frac{1}{z(z-1)(z-1/2)} \right\|_{Y_0} = \frac{\Gamma(1/4)^8}{16\pi^4} \approx 19.157071,$$

where the supremum in the definition of the norm is attained by $z = (1 \pm i)/2$.

The last two results have applications to Teichmüller spaces, which will be given in Section 3 and will be based on some interesting conjectures on the Teichmüller spaces of the once-punctured square torus. Section 2 is devoted to the proof of Theorem 3, from which we can deduce the other theorems.

The author would like to thank A. Yu. Solynin and M. Vuorinen for their helpful comments. He is also grateful to H. Miyachi for a useful remark.

2. Proof of main results

We begin with showing Theorem 3. Let D be a symmetric subdomain of T as in the statement and let U be a connected component of $\widetilde{D} = p^{-1}(D)$. Consider the covering map $f : \mathbb{C} \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ defined by $f(z) = \exp(2\pi z/a)$ and set $V = f(U)$. Then the restriction $f : U \to V$ becomes a covering, too. Hence, we have

$$\rho_D([z]) = \rho_U(z) = \rho_V(f(z))|f'(z)| = \frac{2\pi|w|}{a}\rho_V(w),$$

where $w = f(z)$. Note that, by assumption, the domain V is circularly symmetric, precisely speaking, for each positive number r the set $\{\theta \in [-\pi, \pi]; re^{i\theta} \in V\}$ is an open interval of the form $(-\tau, \tau)$ for some $0 \leq \tau \leq \pi$ or the whole interval $[-\pi, \pi]$. We now apply the following theorem, which was first proved by A. Weitsman [17] up to strictness. The strictness was complemented by A. Yu. Solynin [15] (see also [16]).

Theorem A. *For a circularly symmetric hyperbolic domain* V *in* $\widehat{\mathbb{C}}$, *the hyperbolic density* $\rho_V(re^{i\theta})$ *of* V *is strictly increasing in* $0 \leq \theta \leq \pi$ *as long as* $re^{i\theta} \in V$ *for a fixed* $r > 0$ *except for the cases when* $V = \{|z| < c\}, V = \{|z| > d\}$ *and* $V = \{c < |z| < d\}$, *where* $0 \leq c < d \leq +\infty$. *In the exceptional cases, the hyperbolic density is constant in* θ.

Since $|w| = |f(z)|$ is constant along the vertical line $\mathrm{Re}\, z =$const., we immediately obtain the expected conclusion.

Next, we prove the rests of our theorems. The argument will be based on the known commensurability of X_0 and Y_0 (see, for a detailed exposition, [7]). We consider the four-times punctured square torus $Z_0 = T_0 \setminus \{[0],[1/2],[i/2],[(1+i)/2]\}$, where $T_0 = \mathbb{C}/\mathbb{Z}[i]$. Then the mapping $f : [z] \mapsto [2z]$ is an unbranched 4-sheeted cover of X_0 from Z_0. On the other hand, the mapping $g : [z] \mapsto (\wp(z) - \wp(i/2))/(\wp(1/2) - \wp(i/2))$ is an unbranched 2-sheeted cover of Y_0 from Z_0, where $\wp(z)$ is the Weierstrass $\wp$-function with period lattice $L = \mathbb{Z}[i]$. Therefore, we have the relation $f^*\rho_{X_0} = \rho_{Z_0} = g^*\rho_{Y_0}$.

Now we use the standard notation $e_1 = \wp(1/2), e_2 = \wp(i/2)$ and $e_3 = \wp((1+i)/2)$. Then the mapping g can be described by $g \circ p = A \circ \wp$, where $A(z) = (z - e_2)/(e_1 - e_2)$. Note that $e_1 + e_2 = 0, e_3 = 0$ and thus $A(e_3) = 1/2$. As for the value of e_1, we know the following result (see, for example, [1, p. 658]):

$$e_1 = \wp(1/2) = \frac{\Gamma(1/4)^4}{8\pi} \approx 6.87519. \tag{2.1}$$

Writing $\psi_0 = dz^2/z(z-1)(z-1/2)$, we have $A^*\psi_0 = \psi_0(A(z))A'(z)^2 = (e_1-e_2)dz^2/(z-e_1)(z-e_2)(z-e_3)$. Noting the fundamental relation $(\wp')^2 = 4(\wp-e_1)(\wp-e_2)(\wp-e_3)$, we get $(g\circ p)^*\psi_0 = (A\circ\wp)^*\psi_0 = (e_1-e_2)(d\wp)^2/(\wp-e_1)(\wp-e_2)(\wp-e_3) = 4(e_1-e_2)dz^2 = 8e_1dz^2$. On the other hand, we have $(f\circ p)^*\varphi_0 = 4dz^2$, hence $g^*\psi_0 = 2e_1 f^*\varphi_0$. Now we observe

$$(|\psi_0|\rho_{Y_0}^{-2})(g([z])) = 2e_1\left\{|f^*\varphi_0|\rho_{Z_0}^{-2}\right\}([z]) = 2e_1(|\varphi_0|\rho_{X_0}^{-2})(f([z])).$$

Note that the inverse image of $[(1+i)/2]$ under the map f is the set of four points $[(1+i)/2 \pm 1/4 \pm i/4]$, whose image under g consists of two points $(1\pm i)/2$. Then, by Corollary 4, we have

$$\|\psi_0\|_{Y_0} = 4\rho_{Y_0}((1\pm i)/2)^{-2} = 2e_1\rho_{X_0}([(1+i)/2])^{-2} = 2e_1\|\varphi_0\|_{X_0}. \tag{2.2}$$

We now use the fact that the map $h(z) = 4z(1-z)$ is an unbranched cover of D_0 from Y_0 to show the relation $\rho_{Y_0}(z) = h^*\lambda(z) = \lambda(h(z))|h'(z)|$. In particular, letting $z = (1\pm i)/2$, we have $\rho_{Y_0}((1\pm i)/2) = 4\lambda(2) = 4\lambda(-1) = 8\pi^2/\Gamma(1/4)^4$. Now (1.3) and Theorem 6 follow from the last relation, (2.2) and (2.1).

The assertion of Theorem 6 means that the function $|z(z-1)(z-1/2)|\rho_{Y_0}(z)^2$ attains its minimum at the precisely two points $z = (1\pm i)/2$. Note that these two points project to the same point 2 under the map h. If we set $w = h(z)$, we have $(2z-1)^2 = 1-w$. Then, we calculate

$$\begin{aligned}|z(z-1)(z-1/2)|\rho_{Y_0}(z)^2 &= 8|z(z-1)(2z-1)^3|\lambda(h(z))^2 = 2|w||w-1|^{3/2}\lambda(w)^2 \\ &= 2M(1-w)^2,\end{aligned}$$

where we set $M(w) = |w|^{3/4}|w-1|^{1/2}\lambda(w)$. Therefore, we can conclude that the function M has the unique minimum point $z = -1$ in D_0. This is nothing but the statement of Theorem 1.

Remark 2. Noting the symmetricity properties

$$M(\bar z) = M(z) \quad \text{and} \quad M(1/z) = M(z)$$

of M coming from (1.1), we can apply the topological method developed by Jenkins in [5] to show the assertion in Theorem 1, if, for example, we could know that the function M has no local minimum on the upper half plane, for which the author has no proof unfortunately. If we could do so, as a by-product, we would have the claim that $|z-1|^{1/2}\lambda(z)$ decreases on the unit circle $z = e^{i\theta}$ when θ increases in $(0, \pi]$, which is stronger than Lemma 2 in [5]. The last claim seems true for the circle $z = re^{i\theta}$ with arbitrary $r > 0$. Note also that $\lambda(z)$ decreases and $|z-1|\lambda(z)$ increases on the circle $z = re^{i\theta}$ when θ increases in $(0, \pi]$. The first assertion follows from Weitsman's theorem (Theorem A) and the second follows from Corollary 2.14 (1) in the paper [16] by Solynin and Vuorinen. A more subtlety can be seen by recalling the fact that $|z(z-1)|^{1/2}\lambda(z)$ is increases when z moves downward along the left-upper quarter of the ellipse $|z| + |z-1| = c$ for any constant $c > 1$ (see Corollary 3.5 in [16]).

3. Applications to Teichmüller spaces

We conclude this note with applications to Teichmüller spaces. We remind the reader of basic definitions of Teichmüller spaces. As a standard textbook, we refer to [8]. Let R be a hyperbolic Riemann surface with Fuchsian group Γ as its covering transformation group on the unit disk $\mathbb{D}$. Then the space $B_2(R)$ can be canonically identified with the space $B_2(\mathbb{D}, \Gamma)$ consisting of those $\varphi \in B_2(\mathbb{D})$ for which $(\varphi \circ \gamma)(\gamma')^2 = \varphi$ holds for any $\gamma \in \Gamma$, via the pull-back q^* under the natural projection $q : \mathbb{D} \to R$ with covering group Γ. The Bers embedding of the Teichmüller space $T(R)$ of R is defined as the set of those $\varphi \in B_2(R)$ whose pull-backs are the Schwarzian derivatives $S_f = (f''/f')' - (f''/f')^2/2$ of univalent meromorphic maps $f : \mathbb{D} \to \widehat{\mathbb{C}}$ which extend to (Γ-compatible) quasiconformal self-maps of the Riemann sphere $\widehat{\mathbb{C}}$. (For convenience, we adopt here a slightly different definition of the Teichmüller space from the standard one. In the usual terminology, our $T(R)$ should be called (the Bers embedding of) the Teichmüller space of the mirror image R^* of R.)

The inner radius $\mathrm{Rad}_{\mathrm{i}}(T(R))$ and outer radius $\mathrm{Rad}_{\mathrm{o}}(T(R))$ of $T(R)$ are defined as the numbers $\inf\{\|\varphi\|_R; \varphi \in \partial T(R)\}$ and $\sup\{\|\varphi\|_R; \varphi \in \partial T(R)\}$, respectively. It is well known that $2 \le \mathrm{Rad}_{\mathrm{i}}(T(R)) < \mathrm{Rad}_{\mathrm{o}}(T(R)) \le 6$. Furthermore, when R is of finite analytic type, $2 < \mathrm{Rad}_{\mathrm{i}}(T(R))$ and $\mathrm{Rad}_{\mathrm{o}}(T(R)) < 6$ (see [18] and [14], respectively). Other than the extreme cases of $\mathrm{Rad}_{\mathrm{i}}(T(R)) = 2$ or $\mathrm{Rad}_{\mathrm{o}}(T(R)) = 6$, no concrete values of the inner and outer radii have been known.

We offer numerical computations of these radii in the case of X_0 and Y_0, which have been treated above. Note that the spaces $T(X_0)$ and $T(Y_0)$ are isomorphic by an isometry of the ambient spaces because of their commensurability (see [7]). In particular, their inner and outer radii are respectively equal. By numerical computations of the shape of $T(X_0)$ in [7] (see also Figure 1), the following assertion seems to be true, however no rigorous proofs are known.

Conjecture 1. *For the punctured square torus $X_0 = (\mathbb{C} \setminus L)/L,\ L = \mathbb{Z}[i]$, and $\varphi_0 = dz^2$,*

$$\mathrm{Rad}_{\mathrm{i}}(T(X_0)) = \sup\{t > 0; t\varphi_0 \in T(X_0)\}.$$

In the below, we assume this conjecture to be true. M. Porter [13] showed that the set $\{t \in \mathbb{R}; t\varphi_0 \in T(X_0)\}$ is actually an open interval and its positive endpoint is approximately 1.552. Therefore, combining this with Theorem 5, we observe $\mathrm{Rad}_{\mathrm{i}}(T(X_0)) \approx 1.552 * \|\varphi_0\|_{X_0} \approx 2.162$. Recently, by another method, it is calculated in [7] that $\sup\{t > 0; t\psi_0 \in T(Y_0)\} \approx 0.11289$. So, we see that $\mathrm{Rad}_{\mathrm{i}}(T(Y_0)) \approx 2.1626$ by Theorem 6, which agrees with the above.

On the other hand, we can observe the following interesting statement by computer experiments based on the results in [7]. Let $s_j,\ j = 0, 1, 2, \dots$ be the Fibonacci sequence determined by $s_0 = 0, s_1 = 1$ and $s_n = s_{n-1} + s_{n-2}$ for $n > 1$. Then, the endpoints of rational pleating rays with slope s_n/s_{n+1} seem the best approximation of the farthest boundary point in the first quadrant from the origin (see Figure 1). The pleating ray is sometimes called a bending locus. For the terminology, see [7]. The ray here is quite similar to that of Keen and Series [6] for the Maskit slice; our ray is based on the bending coordinate of the Bers slice of a punctured torus described by McMullen [11, Theorem 7.5].

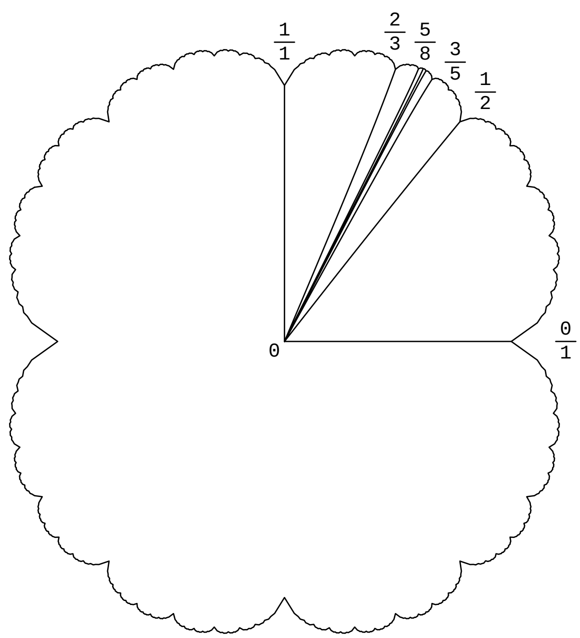

FIGURE 1. Bending loci corresponding to the Fibonacci sequence

As is well known, the limit of the sequence s_n/s_{n+1} is the reciprocal of the golden ratio, that is, $s_n/s_{n+1} \to (\sqrt{5}-1)/2$.

Conjecture 2. *Let X_0 be a once-punctured square torus. The farthest boundary point of $T(X_0)$ from the origin is one of the endpoints of the pleating rays with slope $(\pm\sqrt{5}\pm 1)/2$, where all signatures are possible.*

Let $t_n\varphi_0$ be the endpoint of the rational pleating ray with slope s_n/s_{n+1} for $n = 0, 1, \dots$. By numerical calculations, we have the following table (Table 1).

Letting $t_\infty = \lim_{n\to\infty} t_n$, we observe also that

$$\left| \frac{t_n - t_\infty}{t_{n+1} - t_\infty} \right|^2 \to \lambda_\infty \approx 4.79 \quad (n \to \infty).$$

slope s_n/s_{n+1}	absolute value of t_n
0/1	0.1128908854
1/1	0.1283517305
1/2	0.1405040305
2/3	0.1471612491
3/5	0.1504617577
5/8	0.1520337855
8/13	0.1527685892
13/21	0.1531081637
21/34	0.1532643959
34/55	0.1533359695
55/89	0.1533687471
89/144	0.1533837257
144/233	0.1533905763
233/377	0.1533937047
377/610	0.1533951350
610/987	0.1533957881

Table 1. Distance of the endpoints from the origin

Hence, we find that $\mathrm{Rad}_o(T(X_0))$ is approximately $0.153396 * \|\varphi_0\|_{Y_0} \approx 2.93862$. So, we have obtained the following

Observation . *The Bers embedding of the Teichmüller space of the once-punctured square torus X_0 or the symmetric four-times punctured sphere $Y_0 = \widehat{\mathbb{C}} \setminus \{0, 1/2, 1, \infty\}$ has the inner and outer radii, approximately,* 2.1626 *and* 2.9386, *respectively.*

A related conjecture involving the Fibonacci sequence has been made by D. Wright and C. McMullen in 1989 concerning the self-similarity of the Bers boundary about the limits of pseudo-Anosov mappings. They have observed this picture through the Maskit embedding of the Teichmüller space of a once-punctured torus, which is thought to be close to the Bers embedding and relatively easy to draw by computers. See [9] or [10, Chapter 10] for more details.

References

1. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, 1972.
2. S. Agard, *Distortion theorems for quasiconformal mappings*, Ann. Acad. Sci. Fenn. A. I. Math. **413** (1968), 1–12.
3. A. F. Beardon and Ch. Pommerenke, *The Poincaré metric of plane domains*, J. London Math. Soc. (2) **18** (1978), 475–483.
4. J. A. Hempel, *The Poincaré metric on the twice punctured plane and the theorems of Landau and Schottky*, J. London Math. Soc. (2) **20** (1979), 435–445.
5. J. A. Jenkins, *On explicit bounds in Landau's theorem.* II, Canad. J. Math. **33** (1981), 559–562.
6. L. Keen and C. Series, *Pleating coordinates for the Teichmüller space of a punctured torus*, Bull. Amer. Math. Soc. (N.S.) **26** (1992), 141–146.
7. Y. Komori and T. Sugawa, *Bers embedding of the Teichmüller space of a once-punctured torus*, preprint.
8. O. Lehto, *Univalent Functions and Teichmüller Spaces*, Springer-Verlag, 1987.

9. C. McMullen, *Rational maps and Kleinian groups*, Proceedings of the International Congress of Mathematicians (Kyoto, 1990), Springer, 1991, pp. 889–900.
10. ———, *Renormalization and 3-manifolds which fiber over the circle*, Ann. of Math. Studies, vol. 142, Princeton, 1994.
11. ———, *Complex earthquakes and Teichmüller theory*, J. Amer. Math. Soc. **11** (1998), 283–320.
12. R. Nevanlinna, *Eindeutige Analytischer Funktionen*, Springer-Verlag, 1953.
13. R. M. Porter, *Computation of a boundary point of Teichmüller space*, Bol. Soc. Mat. Mexicana **24** (1979), 15–26.
14. H. Sekigawa, *The outradius of the Teichmüller space*, Tôhoku Math. J. (2) **30** (1978), 607–612.
15. A. Yu. Solynin, *Functional inequalities via polarization*, St. Petersburg Math. J. **8** (1997), 1015–1038.
16. A. Yu. Solynin and M. Vuorinen, *Estimates for the hyperbolic metric of the punctured plane and applications*, to appear in Israel J. Math.
17. A. Weitsman, *A symmetric property of the Poincaré metric*, Bull. London Math. Soc. **11** (1979), 295–299.
18. H. Yamamoto, *Inner radii of Teichmüller spaces of finitely generated Fuchsian groups*, Kodai Math. J. **14** (1991), 350–357.