Synthetic deformation space of an entire function

Masahiko Taniguchi

Department of Mathematics, Kyoto University,
Kyoto 606, Japan

abstract:

Finitely constructable entire functions are called structurally finite. Structurally finite entire functions have many nice properties. In this note, we define a new kind of the deformation space of a general entire function, and discuss about completeness and stability of such deformation spaces in case of structurally finite entire functions.

1 Deformation spaces

The dynamic Teichmüller space of a holomorphic endomorphism (see [7], [8]) is rather tight to discribe variations of singular affine structures on $\mathbb{C}$. So we relax the connection between the domain and the target, and introduce a different kind of topology to the space of singular affine structures.

Definition 1.1 *Let f be a non-affine entire function. Then the* full deformation set *$FD(f)$ of f is the set of all entire functions g such that there is a quasiconformal self-map ϕ of $\mathbb{C}$ satisfiying the* qc-L^∞ condition*:*

$$\|f - g\circ\phi\|_\infty = \sup_{\mathbb{C}} |f - g\circ\phi| < \infty.$$

Here we may assume that such a ϕ as above is always *normalized*, that is, fixes 0 and 1. Here the following lemma is clear.

Lemma 1.2 *If $g \in FD(f)$, then $f \in FD(g)$.*

Definition 1.3 *For any two function f_1, f_2 in $FD(f)$, we set*

$$d(f_1, f_2) = \inf \left(\log K(\phi_1 \circ \phi_2^{-1}) + \|f_1 \circ \phi_1 - f_2 \circ \phi_2\|_\infty \right),$$

where the infimum is taken over all normalized quasiconformal automorphisms ϕ_1, ϕ_2 of $\mathbb{C}$ satisfying the qc-L^∞ conditions between f and f_1, f_2, respectively.

Lemma 1.4 *The psuedo-distance d is a distance, and $FD(f)$ with this distance is a complete metric space.*

Proof. If $d(f_1, f_2) = 0$, then there are sequences $\{\phi_{j,n}\}$ $(j = 1, 2)$ such that

$$\log K(\phi_{1,n} \circ \phi_{2,n}^{-1}) + \|f_1 \circ \phi_{1,n} - f_2 \circ \phi_{2,n}\|_\infty \to 0$$

as $n, m \to \infty$. Then since $\{\phi_{1,n} \circ \phi_{2,n}^{-1}\}$ is a normal family, they converge to the identity. Moreover, since

$$\|f_1 \circ \phi_{1,n} \circ \phi_{2,n}^{-1} - f_2\|_\infty \to 0,$$

we conclude that f_1 and f_2 coincide with each other. Thus, d is a distance.

Next, to show completeness, let $\{f_n\}$ be an arbitrary Cauchy sequence. Then, there is a sequence of normalized quasiconformal ϕ_n such that

$$\log K(\phi_n \circ \phi_m^{-1}) + \|f_n \circ \phi_n - f_m \circ \phi_m\|_\infty \to 0$$

as $n, m \to \infty$. Fix such a sequence $\{\phi_n\}$. Then a standard argument shows that, taking a subsequence if necessary, we may assume the existence of a normalized quasiconformal automorphism ϕ_∞ such that ϕ_n converge to ϕ_∞ locally uniformly and satisfy

$$\log K(\phi_\infty \circ \phi_n^{-1}) \to 0.$$

In particular, f_n are locally uniformly bounded because of the qc-L^∞ condition between f_1 and f_n, and hence we may assume that f_n converge locally uniformly to some non-constant entire function f_∞. Moreover, the above condition implies that

$$\|f_n \circ \phi_n - f_\infty \circ \phi_\infty\|_\infty \to 0,$$

since $f_n \circ \phi_n(z) - f_m \circ \phi_m(z)$ tends to $f_n \circ \phi_n(z) - f_\infty \circ \phi_\infty(z)$ as $m \to \infty$ for every $z \in \mathbb{C}$. Thus f_n converge to f_∞ in $FD(f)$. ■

Definition 1.5 *We call this distance the* synthetic Teichmüller distance *on* $FD(f)$*. The space* $FD(f)$ *equipped with this synthetic Teichmüller distance is called the* full synthetic deformation space *of* f *and written as* $FSD(f)$*.*

In this paper, we discuss about the full synthetic deformation spaces. But, to discuss some kind of stability, we also consider several other deformation spaces.

In the sequel, we consider only such an f *that the singular values of* f *have a finite number of accumulating points.*

Also, we call a point not virtually evenly covered a *singular value of the covering*. Here for such a function f as above, we say that a point α is *virtually evenly covered* if α is not a critical value and if there is a neighborhood U and a simple path L from the boundary of U to α such that every component D of $f^{-1}(U-L)$ is relatively compact and f is a biholomorphic map of D onto $U-L$.

Recall that, if f belongs to the Speiser class, then singular values of the covering are nothing but *singular values*, i.e. values not evenly covered by f.

Definition 1.6 *The set* $S(f)$ *consisting of all entire functions* g *such that there are bijections between critical points counted with their multiplicity, and between singular values of the covering counted with their coincidence, of* g *and* f*.*

Note that, when singular values of the covering are countable in number, then $S(f)$ has a set of the natural local parameters, which gives a local injection to $\mathbb{C}^\infty$.

Definition 1.7 *For an entire function* f*, let* $\mathrm{Top}(f)$ *be the set of all entire functions topologically equivalent to* f*, and* $\mathrm{QC}(f)$ *the set of all entire functions quasiconformally equivalent to* f*.*

Here we say that g *is* topologically (resp., quasiconformally) equivalent to f *if there are self-homeomorphisms (resp., quasiconformal maps)* ϕ *and* ψ *of* $\mathbb{C}$ *such that* $g = \psi \circ f \circ \phi$*.*

Now clearly,

$$\mathrm{QC}(f) \subset \mathrm{Top}(f) \subset S(f).$$

2 The case of structurally finite entire functions

To construct entire functions or singular affine structures, we need two kinds of *building blocks*; ones are *quadratic blocks*

$$az^2 + bz + c : \mathbb{C} \to \mathbb{C} \qquad (a \neq 0)$$

and the others are *exponential blocks* (exp-blocks)

$$a \exp bz + c : \mathbb{C} \to \mathbb{C} \qquad (ab \neq 0).$$

Remark 2.1 *The trivial covering structure is given by similarities:*

$$az + b : \mathbb{C} \to \mathbb{C} \qquad (a \neq 0).$$

We call such one a $\mathbb{C}$-block.

Definition 2.2 (Maskit surgery by connecting functions)
Let $f_j : \mathbb{C} \to \mathbb{C}$ $(j = 1, 2)$ be two entire functions, and A_j be the set of singular values of f_j. Assume that there is a cross-cut L in $\mathbb{C}$ (i.e. the image L of a continuous proper injection of $\mathbb{R}$ into $\mathbb{C}$) such that

1. *$L \cap A_1$ is coincident with $L \cap A_2$, and is either empty or consists of a single point z_0, which is an isolated point of each A_j,*
2. *L separates $A_1 - \{z_0\}$ from $A_2 - \{z_0\}$, and*
3. *if $L \cap A_1 = L \cap A_2 = \{z_0\}$, then $\{z_0\}$ is a critical value of each f_j: for a small disk U with center z_0 such that $U \cap A_j = \{z_0\}$, $f_j^{-1}(U)$ has a relatively compact component W_j which contains a critical point for each f_j.*

Then we say that an entire function $f : \mathbb{C} \to \mathbb{C}$ is constructed from f_1 and f_2 by a Maskit surgery *with respect to L, and to $\{W_j\}$ when they exist, if the following assumptions are satisfied: Let D_j be the component of $\mathbb{C} - L$ containing $A_j - \{z_0\}$. Then there exist*

1. *components $\tilde{D}_1$ and $\tilde{D}_2$ of $f_1^{-1}(D_2)$ and $f_2^{-1}(D_1)$, respectively, such that $f_j : \tilde{D}_j \to D_{3-j}$ is biholomorphic and $\tilde{D}_j \cap W_j \neq \emptyset$ if $L \cap A_j$ are non-empty,*

2. *a cross-cut* $\tilde{L}$ *in* $\mathbb{C}$ *such that* f *gives a homeomorphism of* $\tilde{L}$ *onto* L*, and*

3. *a conformal map* ϕ_j *of* $\mathbb{C} - \tilde{D}_j$ *onto* U_j *such that* $f_j = f \circ \phi_j$ *on* $\mathbb{C} - \tilde{D}_j$ *for each* j*, where* U_1 *and* U_2 *are components of* $\mathbb{C} - \tilde{L}$*.*

Definition 2.3 *We say that an entire function is* structurally finite *if it is constructed from a finite number of building blocks by Maskit surgeries.*

We say that a structurally finite function is of type (p, q) *if it is constructed from* p *quadratic blocks and* q *exp-blocks.*

Remark 2.4

1. *Suppose that* f *is structurally finite, then every* $g \in S(f)$ *is of the same type as* f*.*

2. *Suppose that* f *belongs to the Speiser class, then*

$$\mathrm{QC}(f) = \mathrm{Top}(f).$$

Compare with the Maskit combinations in [6]. Also, note that structural finiteness is characterized by the following topological condition.

Definition 2.5 *We call that a holomorphic endomorphism of* $\mathbb{C}$ *covers* $\mathbb{C}$ almost evenly *if there are only a finite number of points which are not evenly covered, and at every such point* α*, there are only a finite number of components of* $f^{-1}(B)$ *where* f *is not a homeomorphism onto* B *for every small disk* B *with center* α*.*

Proposition 2.6 (Topological Characterization) *Every structurally finite entire function covers* $\mathbb{C}$ *almost evenly. Conversely, every entire function which covers* $\mathbb{C}$ *almost evenly is structually finite.*

Proof. The first assertion is clear.

Next, let f be an almost evenly covered entire function. Then we take a cross-cut L which togather with ∞ is freely homotopic to a simple closed curve C surrounding a single singular value, with a homotopy keeping the set of singular values fixing pointwise, in the Riemann sphere $\hat{\mathbb{C}}$. Then there are a finite number of components of $f^{-1}(L)$ which divides the singular affine structures non-trivially.

Applying such decompositions (of Klein types) a finitely many times, we can decompose the given almost evenly covered f into a finite number of exp-blocks and *simple critical blocks*

$$z \mapsto a(z - z_0)^k + c, \quad (a \neq 0)$$

with degree $k\,(\geq 2)$.

Now, let f be such a simple critical block, and L a cross-cut passing through the critical value c of f. Take a suitable pair of components of $f^{-1}(L - \{c\})$ whose images by f cover $L - \{c\}$. Then we can decompose f into two simple critical blocks with degrees k_1 and k_2 satisfying $k_1 + k_2 = k + 1$. Applying such decompositions (of Maskit types) a finite number of times, we can decompose f into $k - 1$ quadratic blocks. ■

Theorem 2.7 (Inclusion Theorem) *For a structurally finite entire function f, the full deformation set $FD(f)$ contains all structurally finite entire functions of the same type as that of f.*

Corollary 2.8 *For a structurally finite f,*

$$S(f) \subset FD(f).$$

The proof of Inclusion Theorem will be given in the final section. Actually, the proof also implies the following

Corollary 2.9 *If a structurally finite f of type (p, q) has $p + q$ distinct singular values, then*

$$\mathrm{QC}(f) = \mathrm{Top}(f) = S(f).$$

Definition 2.10 *We define the set $SF_{p,q}$ (with $p + q \geq 1$) by setting*

$$SF_{p,q} = \left\{ \int_0^z (c_p t^p + \cdots + c_0) e^{a_q t^q + \cdots + a_1 t} dt + b \right\}$$

with $c_p a_q \neq 0$ if $q > 0$, and we regard that $SF_{p,0} = Poly_{p+1}$; the set of all polynomials of degree exactly $p + 1$.

Such primitive functions have already appeared as typical examples in various contexts. See for instance, [1], [2], [3], and [9].

Now the topological characterization shows the following

Corollary 2.11 *Every element of $SF_{p,q}$ is structurally finite and of type (p,q).*

Thus, Inclusion Theorem implies that $FD(f)$ contains $SF_{p,q}$ for every $f \in SF_{p,q}$. In particular, the synthetic Teichmüller distance is finite on $SF_{p,q} \times SF_{p,q}$, and we may equip $SF_{p,q}$ with the synthetic Teichmüller topology.

Definition 2.12 *For $f \in SF_{p,q}$, we set $SD(f) = SF_{p,q}$, and equip it with the sythetic Teichmüller topology, which we call the* synthetic deformation space *of f.*

Actually, we can show the following

Theorem 2.13 (Representation Theorem) *An entire function is structurally finite and of type (p,q) if and only if it belongs to $SF_{p,q}$.*

For a proof, see [14], where the proof relies on the proof of Inclusion Theorem given here.

Proposition 2.14 *For every $P \in SF_{p,0} = Poly_{p+1}$, $SD(P) = FSD(P)$.*

Proof. Fix $Q \in FSD(P)$. Then since P is proper, so is Q. Indeed, if there were a compact ball B such that $Q^{-1}(B)$ is non-compact, then $U = (Q \circ \phi)^{-1}(B)$ would be non-compact, where ϕ is a quasiconformal map satisfying the qc-L^∞ condition between P and Q. Hence $\sup_U |P|$ should be $+\infty$. But since $\sup_U |P| \le \|P - Q \circ \phi\| + \sup_B |z| < +\infty$, this is impossible.

Hence Q is polynomial, and then the degree should be equal to $p+1$. ∎

On the other hand, we have

Example 2.15 (Melting of $\mathbb{C}$-decorations)

$$f_j(z) = \left(1 + \frac{z}{j}\right) e^z \in SF_{1,1}$$

converge to $g(z) = e^z \in SF_{0,1}$ with respect to the synthetic Teichmüller topology as j tend to $+\infty$.

Similarly,

$$g_j(z) = e^{2z} + \frac{2}{j}\, e^z$$

are structurally infinite, but converge to $g(2z)$ with respect to the synthetic Teichmüller topology as j tend to $\pm\infty$.

Thus we can see that, even for an $f \in SF_{p,q}$ with $q > 0$, $FD(f)$ always so large that it contains structurally infinite functions, and $SD(f)$ is neither closed nor open in $FSD(f)$, except for the case that $p = 0$. Hence we need to consider slightly large complete subspaces. The following kind of compactness gives such ones, and also shows that $SD(f)$ can be considered as a stratum of $FSD(f)$.

Theorem 2.16 (Compactness Theorem) *Suppose that $f \in SF_{p,q}$ with $q > 0$, and let $\{f_j\}$ be a seqeunce in $SD(f)$ converging to some g in $FSD(f)$. Then g is structurally finite, and of type (p', q) with $p' \leq p$.*

In particular,

$$SF_{\leq p,q} = \cup_{p' \leq p} SF_{p',q}$$

with the synthetic Teichmüller topology is a complete metric space, and hence a completion of $SD(f)$.

Proof. Let $S = \{\alpha_1, \cdots, \alpha_m\}$ be a finite set of distinct singular values of g, and let r be the minimum among all the distances between distinct points in S. Fix an $\epsilon > 0$ smaller than $r/8$ and, by considering sufficiently large j only, we may assume

$$\|g - f_j \circ \phi_j\|_\infty < \epsilon$$

for every j with a suitable normalized quasiconformal map ϕ_j.

Now let D_ℓ be the disk with center α_ℓ and radius $r/4$, i.e. the $r/4$-neighborhood of α_ℓ. Then $g^{-1}(D_\ell)$ has a component where g is not biholomorphic. Let V be such a component. If V is relatively compact, then $\phi_j(V)$ is also relatively compact. Set $a_V = g^{-1}(\alpha_\ell) \cap V$. Then the winding number of $f_j \circ \phi_j(\partial V)$ around $f_j \circ \phi_j(a_V)$ is equal to that of $g(\partial V)$ around α_ℓ. Hence f_j has critical points in $\phi_j(V)$ and the number counted with their multiplicities is the same as that of g at a_V.

Next, suppose that V is not relatively compact. Since $f_j \circ \phi_j(V)$ are bounded and f_j are structurally finite, f_j should have an asymptotic value in $\phi_j(V)$. Thus the number d_ℓ of components of $g^{-1}(D_\ell)$ where g is not biholomorphic is finite, and

$$\sum_{\ell=1}^{m} d_\ell \leq p + q.$$

Since S is arbitrary, this implies that g is structurally finite. In particular, every asymptotic value of g corresponds to a logarithmic singularity. And it remains to show that the type of g is (p', q') with $p' \leq p$, $q' = q$.

Here, we have shown that $p' \leq p$, $q' \leq q$. Also note that, if a point α satisfies that every component of $g^{-1}(D)$ is relatively compact, where D is the $r/4$-neighborhood of α, there are asymptotic values of no f_j in the $r/8$-neighborhood D' of α. Indeed, if some f_j has an asymptotic value in D', then $f_j^{-1}(D')$ has a non-compact component V'. But $\phi_j(V')$ should be contained in $g^{-1}(D)$, which is impossible. Thus, if $q' < q$, then we would have a disk neighborhood D of an asymptotic value of g such that some component W of $g^{-1}(V)$ is simply connected, ∂W is a cross-cut, and $\phi_j(W)$ contains two paths determining different asymptotic values of some f_j. But then f_j can not be bounded on $\phi_j(W)$. This contradiction shows that $q' = q$. ■

Now $SF_{\leq p,q}$ has another natural topology induced from the coefficents space of representatives. For instance, we define the line element ds by

$$ds = \frac{\sum_{m=0}^{p} |dc_m|}{\sum_{m=0}^{p} |c_m|} + \frac{|da_q|}{|a_q|} + \sum_{n=1}^{q-1} |da_n| + |db|$$

at every

$$f(z) = \int_0^z (c_p t^p + \cdots + c_0) e^{a_q t^q + \cdots + a_1 t} dt + b$$

in $SF_{\leq p,q}$. This distance is complete, and we call the induced topology the *coefficient topology* on $SF_{\leq p,q}$. Thus Compactness Theorem shows the following

Corollary 2.17 (Equivalence Theorem) *The synthetic Teichmüller topology is equivalent to the coefficient topology on $SF_{p,q}$ for every p and q.*

Finally, we state the following theorem about the size of the Julia set. Proofs will be given in [15].

Theorem 2.18 *For every transcendental structurally finite entire function f, the Hausdorff dimension of the Julia set of f is two.*

Remark 2.19 *Compare with a theorem of Stallard ([12] II): For every transcendental entire function with bounded singular values, the Hausdorff dimension of $J(f)$ is greater than 1.*

3 The case of structurally infinite entire functions

In the case of structurally infinite entire functions, the situation becomes more complicated. We will show this by several examples.

First, entire functions in the same full synthetic deformation space are not necessarily in the same dynamic Teichmüller space. In particular, we can relax singular orbit relations in a connected component of the full synthetic deformaiton space.

Proposition 3.1 *Set*

$$f_1(z) = \sin z$$

and

$$f_2(z) = \frac{z-c}{z}\sin z$$

with a sufficiently small positive c. Then

$$f_2 \in FD(f_1) - S(f_1)$$

Proof. Critical values of f_2 accumulates to ± 1, which are the critical values of f_1. Also the critical points of f_1 and f_2 are asymptotically the same. Hence we can construct a quasiconformal map ϕ of $\mathbb{C}$ which sends critical points of f_1 to those of f_2 and satisfies that

$$f_1 = f_2 \circ \phi$$

outside the preimage of a compact set under f_1. ■

Here note that

$$g(z) = \frac{\sin z}{z}$$

and f_2 are topologically non-equivalent, for the critical values of g accumulate to a single point 0. On the other hand, set $B = \{|z| < 2\}$, then there is a conformal map ϕ such that $f_2 = g \circ \phi$ outside $f_2^{-1}(B)$, which can be extended to a homeomorphism of $\mathbb{C}$ onto itself. Hence, we can find a homeomorphism ψ of $\mathbb{C}$ such that $\|f_2 - g \circ \psi\|_\infty$ is finite.

Also note that both of f_2 and g are bounded on the positive real axis, but that g has an asymptotic value, while f_2 does not.

Proposition 3.2

$$\frac{\sin z}{z} \in S(f_2) - FD(f_2) \bigcup \mathrm{Top}(f_2).$$

Proof. The preimages of a point, say $z = R$ with a sufficiently large R, under f_2 have eventually almost periodic real parts and have bounded imaginary parts. Thus the modulus of any 4 preimages neighboring to each other is bounded and away from 0. On the other hand, though the preimages of

$z = R$ under g again have eventually almost periodic real parts, they have the imaginary parts tending to $\pm\infty$, which implies that the moduli of 4 preimages neighboring to each other tends to 0 or $+\infty$ as they tend to ∞. Thus we conclude that no quasiconformal maps satisfy the qc-L^∞ condition between f_2 and g, and hence that $g \notin FD(f_2)$. ■

Next, topologically equivalent entire functions are not necessarily in the same full synthetic deformation space.

Proposition 3.3 *Set*

$$h_1(z) = z \sin z$$

and

$$h_2(z) = \int_0^z t \prod_{n=1}^{\infty} \left(1 - \frac{t}{r_n}\right) dt,$$

where $\{r_n\}$ tend to $+\infty$ so rapidly that $|h_2(r_n)|$ are strictly increasing and tend to $+\infty$, and h_2 is of order 0.

Then

$$h_2 \in \mathrm{Top}(h_1) - FD(h_1) \bigcup \mathrm{QC}(h_1).$$

Proof. First suppose that there were a normalized quasiconformal map ϕ such that

$$\|h_1 - h_2 \circ \phi\|_\infty < +\infty.$$

Let $\{a_n\}$ and $\{b_n\}$ be the zeros of h_1 and h_2. Then by the Koebe distortion theorem, $\phi^{-1}(b_n)$ should be asymptotically equal to a_n, for $h_1 \circ \phi^{-1}(b_n)$ are bounded. Then, since the order of f_1 is one, that of h_2 should be positive by the Hölder continuity of quasiconformal maps. But this is a contradiction.

Next suppose that there were quasiconformal maps ϕ and ψ such that $h_2 = \psi \circ h_1 \circ \phi$. Here we may assume that ψ is normalized. Then a_n should be equal to $\phi(b_n)$, which implies again that the order of h_2 should be positive. ■

Proposition 3.4 *Set $h_3(z) = 2h_1(z)$. Then*

$$h_3 \in \mathrm{QC}(h_1) - FD(h_1)$$

Proof. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be the critical values of h_1 and h_3. Then both of α_n and β_n tend to ∞. Also note that, for arbitraliry fixed R, α_n for every sufficiently large n has a neighborhood $U_n = \{|z - \alpha_n| < R\}$ such that the component of $h_1^{-1}(U_n)$ containing the critical point over α_n contains no other critical points. The same assertion holds for h_2 and $\{\beta_n\}$.

Suppose that there were a quasiconformal map ϕ of $\mathbb{C}$ such that

$$M = \|h_1 - h_3 \circ \phi\|_\infty < +\infty.$$

Then as in the proof of Compactness Theorem, we can see that, for every sufficiently large n, the $(R-M)$-neighborhoods of α_n contain some β_m. This is impossible. ■

4 Proof of Inclusion Theorem

To describe the structure of an entire function, we use the following kind of a configuration graph. (Compare with dessins d'enfents, for instance, in [11].)

Definition 4.1 (Configuration tree) *A* configuration tree *is a planar tree with the initial vertex (and hence whose edges have an orientation towards the initial vertex) and colored as follows:*

1. *There are two kind of vertices; white ones and black ones.*
2. *There are three kind of edges; white ones, black ones, and red ones.*
3. *Every connected component of the set of all white vertices and white edges is a subtree $\mathbb{R}$ with vertices $\mathbb{Z}$, which we call a $\mathbb{Z}$-*unit*.*
4. *Every edge outside $\mathbb{Z}$-units is colored black or red, according as the edge starts from a black vertex or from a white vertex.*

Also a configuration tree is associated with the configuration data*.*

1. *The singularity data; the center locus attached to every $\mathbb{Z}$-unit and the decoration locus attached to every black edge, and*
2. *a spider at ∞ ([4]), which assigns each distinct singularity datum a mutually disjoint path to the infinity.*

We call a pair of a red edge and the black vertex pointed by it a *reduction pair*. And if the red edge in a reduction pair has the opposite orientation under a change of the initial vertex, we delete the pair, and attach a new pair to every white vertex from which a black edge starts.

Further, if a white vertex is the initial one, then we may attach a reduction pair and regard that *the newly attached black vertex is the initial one.* Thus we may always assume that *the initial vertex is black.*

We say that such a new configuration tree is obtained by a *change of the initial vertex*, and that two configuration trees are *equivalent*, if, after suitable changes of the initial vertices of both, they are identical including colors.

Definition 4.2 (Realizability) *We say that a configuration tree T is* realizable *(with suitable configuration data) if there is an entire function f which gives a tree equivalent to T under the following injunctions; a black edge and its starting black vertex represent a Maskit surgery attaching a quadratic block, while a red edge and its starting $\mathbb{Z}$-unit represent a Maskit surgery attaching an exp-block where corresponding cross-cuts cut no legs of the spider.*

We call T a configuration tree of f (with respect to the given configuration data).

We sometimes say that a black edge and the starting black vertex represent a $\mathbb{C}$-*decoration* and the decorated $\mathbb{C}$-block, respectively.

Also note that f may have several non-equivalent configuration tree with the same configuration data, which represent diferent orders of attaching blocks.

Definition 4.3 (Core) *The* core *of a configuration tree is the smallest connected closed subtree containing all black vertices and non-white edges. And we call a tree is* virtually compact *if the core is compact.*

A virtually compact tree is locally finite, and has a finite number of ends. Moreover, we can easily conclude the following theorem.

Theorem 4.4 *Every configuration tree of a structurally finite function is virtually compact. Conversely, every virtually compact configuration tree is realizable by a structurally finite entire function.*

Definition 4.5 *For every vertex v of a configuration tree T with the initial vertex v_0, we write by $Age(v)$ the number of all non-white edges in the simple path from v to v_0, and call it the* age of v.

Corollary 4.6 *If T is a configuration tree of a structurally finite entire function, then Age is a bounded function on the set of all vertices in T.*

Now suppose that f is a structurally finite entire function of type (p, q).

Definition 4.7 *We say that a structurally finite entire function of type* (p, q) *is* simple *if it has* $p + q$ *distinct singurality data.*

Then by standard arguments, we can show the following

Proposition 4.8 *Every structurally finite entire function* f *can be approximated in* $SD(f)$ *by simple functions of the same type.*

Hence we may assume without loss of generality, that the given f is simple. (For existence of simple functions, see [14].) Furthermore, considering a quasiconformal equivalent of f, we may also assume that *all singularity data are real and positive.*

Here and in the sequel, we call an entire function g a *quasiconformal equivalent* of f if g is quasiconformally equivalent to f.

Definition 4.9 *The* standard function $f_{p,q}$ of type (p, q) *is a* $\mathbb{C}$*-block attached* p *quadratic blocks with decoration loci* $\{1, \cdots, p\}$ *and* q *exp-blocks with center loci* $\{p + 1, \cdots, p + q\}$*. Here and in general, if the singularity data are real, we always consider a* canonical spider at ∞ *which is a spider with all legs parallel to the imaginary axis and going down to the real axis.*

Next, we consider to deform the given f in $SD(f)$ without changing the topological type (i.e. to deform f in $SD(f) \cap \mathrm{Top}(f)$).

Definition 4.10 *we say that* g *is* $SD(f)$-admissible *if* g *is a quasiconformal equivalent of* f *and* $d(g, f) < +\infty$*.*

A typical deformation to obtain an $SD(f)$-admissible function is rearrangement of the legs of the spider.

Definition 4.11 *Let* (α, E) *be the pair of a singularity datum* α *and a set* E *of singularity data of* f *such that there are no other singularity data in the minimal interval* $I = I_{\alpha,E}$ *containing* α *and* E*, and* α *is an end point of* I*.*

Then we say that an $SD(f)$*-admissible* g *is obtained from* f *by a* simple move *of* (α, E) *if* g *is a quasiconformal equivalent of* f *with quasiconformal maps* ϕ_1 *and* ϕ_2 *(i.e.* $g = \phi_1 \circ f \circ \phi_2$*) such that*

1. ϕ_1 *induces a cyclic permutation of the set* $\{\alpha\} \cup E$ *such that* α *moves to the other side of* I*, and fixes all the other singularity data pointwise,*

2. *every leg of the canonical spider to a singurarity datum β is mapped by ϕ_1 to the leg to $\phi_1(\beta)$, except for the leg ℓ_α to α, and*
3. *the image $\phi_1(\ell_\alpha)$ and the leg to $\phi_1(\alpha)$ in the canonical spider form a cross-cut, which separates E from all singularity data other than $\{\alpha\} \cup E$.*

Now the following proposition implies Inclusion Theorem.

Proposition 4.12 *Under the same circumstances as above, the standard function $f_{p,q}$ of type (p, q) is $SD(f)$-admissible.*

In particular, $f_{p,q}$ is entire, and

$$d(f, f_{p,q}) < +\infty.$$

This proposition in turn follows from two lemmas below: Rearrangement Lemma and Reduction Lemma. Here we may assume that the initial vertex of the configuration tree of f is black.

Lemma 4.13 (Rearrangement Lemma) *Suppose that f have a configuration tree T_f with the black initial vertex and with $Age \leq 1$. Then the standard function $f_{p,q}$ of type (p, q) is $SD(f)$-admissible.*

Proof. We may assume, by considering a quasiconformal equivalent of f, that f has the configuration tree with the same singularity data as that of $f_{p,q}$.

If some $k \in \mathbb{Z}$ is a center locus and $k+1$ is a decoration locus of f, then by applying the simple move of the pair $(k+1, \{k\})$ we obtain an $SD(f)$-admissible g_1. But the configuration tree of g_1 (with respect to the canonical spider) has k as a decoration locus and $k+1$ as the center locus, and $Age = 2$ for vertices in the corresponding $\mathbb{Z}$-unit W, while $Age = 1$ for every other non-initial vertex.

Next, by applying the simple move of the pair $(k+1, \{k\})$ to g_1, we obtain an $SD(f)$-admissible g_2 with the same configuration tree as that of g_1 with respect to the canonical spider. Here, if black edges and $\mathbb{Z}$-units of the configuration tree are numbered so that the ordered singularity data of g_1 is the identitical bijection from the numbers to the data, then that of g_2 is the permutation

$$(k, k+1).$$

Again, by applying the simple move of the pair $(k+1, \{k\})$ to g_2, we obtain an $SD(f)$-admissible g with the same configuration tree as that of f

with respect to the canonical spider, and hence $Age \leq 1$. But k is now a decoration locus of g, while $k+1$ is a center locus of g.

Repeating such rearrangements, we can conclude that the standard function $f_{p,q}$ is $SD(f)$- admissible. ■

Lemma 4.14 (Reduction Lemma) *There is an $SD(f)$-admissible simple g having a configuration tree T_g with $Age \leq 1$.*

To prove this lemma, we assume that

$$\max Age = M > 1$$

on the configuration tree of f. Let $V = \{v_1, \cdots, v_n\}$ be the set of all vertices with $Age = M$ that are connected directly (i.e. by single non-white edges) either to the same black vertex v corresponding to a decoration locus α or to the same $\mathbb{Z}$-unit W with center locus α.

Then v or every $w \in W$ have $Age = M - 1 > 0$, and v or some $v_0 \in W$, which we call the *root* of W, is connected directly to some vertex with age $M - 2$ (which may be the initial vertex).

Here the following fact is easily seen.

Lemma 4.15 (Commutability Lemma I) *Suppose that between two singularity data α and β there are no other singularity data, and that α and β, respectively, correspond to a vertex v_α in V and to v_β not in $V \cup \{v\}$ or $V \cup W$. Then we can find an $SD(f)$-admissible simple g with the same configuration tree (with respect to the canonical spider) and the same set of the singularity data as those of f, but data corresponding to v_α and v_β are permuted.*

Proof. By by the simple move of the pair $(\alpha, \{\beta\})$, we obtain a desired g. ■

Thus all the singularity data for V can be gathered near α, and we conclude the following

Lemma 4.16 *There is an $SD(f)$-admissible simple f_1 with the same configuration tree (with respect to the canonical spider) and the same set of the singularity data as those of f such that there is an open interval I in $\mathbb{R}$ in which the singularity data are exactly α and those, say $\{\alpha_j\}$, of V*

Here we may assume that α_j are increasing, and further that $\{\alpha, \alpha_1, \cdots, \alpha_n\}$ equals to $\{1, \cdots, n+1\}$ with $\alpha = n' + 1 > 1$. Then by the simple move of $\{1, \{2, \cdots, n+1\}\}$, we have the following

Lemma 4.17 *There is an $SD(f)$-admissible f_2 with the same configuration tree (with respect to the canonical spider) and the same set of the singularity data as those of f_1 such that the ordered singularity data of f_2 corresponding to $V \cup \{v\}$ or $V \cup W$ is the cyclic permutation*

$$(1, \cdots, n+1)^{-1}$$

of that of f_1.

Repeating such rearrangements, we have an $SD(f)$-admissible f_3 with the same configuration tree (with respect to the canonical spider) and the same set of the singularity data as those of f such that the ordered singularity data of f_3 corresponding to $V \cup \{v\}$ or $V \cup W$ is the cyclic permutation

$$(1, \cdots, n+1)^{-n'}$$

of that of f_1.

Under these circumstances, we divide into two cases.

1) The case that α *corresponds to a black vertex* v: In this case, by applying the similar rearrangements which corresponds to the permutation

$$(1, \cdots, n+1)^{-n'-1},$$

we can obtain an $SD(f)$-admissible f_V with the same set of the singularity data as that of f, but $Age \leq M-1$ on vertices in $\{v\} \cup V$.

2) The case that α *corresponds to the root* w_0: In this case, we need another index for v_j.

First taking another configuration tree of f if necessary, we may assume that no non-white edges end at w_0. Indeed, if there is such a vertex v^0, we can replace the corresponding edge to one towards the vertex connected directly with w_0. In this new tree, $Age(v^0) = M - 1$.

Then, since the root w_0 divides the belonging $\mathbb{Z}$-unit into two connected components, which in turn divide the set $\{v_j\}$ into two classes $V^{\pm} = \{v_j^{\pm}\}_{j=1}^{m^{\pm}}$ with $m^+ + m^- = n$. Here we consider to reduce $V^{\pm}$ to the empty sets. We discuss V^+ only, for V^- can be treated similarly.

Let $H(v_j^+)$ be the number of white edges in the simple path from v_j^+ to w_0, and call it the *height* of v_j^+. Also note the following

Lemma 4.18 (Commutability Lemma II) *Suppose that between two singularity data α and β there are no other singularity data, and that α and β, respectively, correspond to vertices u^+ in V^+ and to w which is either in*

V^+ with $H(u^+) \neq H(w^+)$ or not in $V^+ \cup W$. Then we can find an $SD(f)$-admissible simple g with the same configuration tree, the same heights, and the same set of the singularity data (with respect to the canonical spider) as those of f, but data corresponding to u^+ and w are permuted.

Thus as before we can gather all the singularity data for V^+ near α. Here we assume that, by applying the simple move of the pair $(\alpha_m^+, E \cup \{\alpha\})$ and then another simple move of the pair $(\alpha, \{\alpha_m^+\})$, we can decrease the height of $v_{m^+}^+$ by 1, where E is the set of the singularity data between α and α_m^+. Also applying suitable simple moves and renumbering V^+ if necessary, we may assume that the singularity data $\{\alpha_j\}$ for V^+ are greater than α and increasing with respect to j.

Then similarly as in the proof of Rearrangement Lemma, we can decrease $\{H(v_j^+)\}$ one by one. And finally, we can obtain an $SD(f)$-admissible g with the same set of the singularity data $\{\alpha_j\}$, such that $[\alpha, \alpha_{m^+}^+]$ contains no other singularity data and that $H(v_j^+) = 1$ for every j. Thus we may assume that $\alpha = 1$, and $\{\alpha_j^+\}$ equals to $\{2, \cdots, m^+ + 1\}$.

Now, by applying the simple move of the pair $(1, \{2, \cdots, m^+ + 1\})$, we can obtain an $SD(f)$-admissible f_V with the same singularity data as that of f, but $Age \leq M - 1$ on vertices in V^+.

Proof of Reduction Lemma. Applying such reductions as above to all vertices with age M, we can obtain an $SD(f)$-admissible simple function with the same set of the singularity data as that of f, but with $Age \leq M - 1$.

Repeating this process a finite number of times, we have the desired g. ■

References

[1] I. N. Baker, *Wandering domains in the iteration of entire functions,* Proc. London Math. Soc. (3) **49** (1984), 563–576.

[2] W. Bergweiler, *Newton's method and a class of meromorphic functions without wandering domains,* Ergod. Th. & Dynam. Sys. **13**, (1993), 231–247.

[3] R.L. Devaney and L. Keen, *Dynamics of meromorphic maps with polynomial Schwarzian derivative,* Ann. Sci. École Norm Sup **22**, (1989), 55–81.

[4] J. H. Hubbard and D. Schleicher, *The Spider algorithm,* AMS Proc. Symp. Appl. Math. **49**, 1994, 155–180.

[5] M. Kisaka, *Some dynamical properties of structurelly finite entire functions,* preprint.

[6] K. Matsuzaki and M. Taniguchi, *Hyperbolic Manifolds and Kleinian Groups,* OUP, 1998.

[7] C.T. McMullen and D. Sullivan, *Quasiconformal Homeomorphisms and Dynamics III: The Teichmüller space of a holomorphic dynamical system,* Adv. Math., **135** (1998) 351-395.

[8] S. Morosawa, Y. Nishimura, M. Taniguchi, and T. Ueda, *Holomorphic Dynamics,* Cambridge Univ. Press, 1999.

[9] R. Nevanlinna, *Analytic Functions,* Springer, 1970.

[10] Y. Okuyama, *On weakly hyperbolic entire functions,* preprint.

[11] L. Schneps, *Dessins d'enfents on the Riemann surface,* L. N. London Math. Soc. **200** 1994, 47–77.

[12] G.M. Stallard, *The Hausdorff dimension of Julia sets of entire functions I,II,III,* I: Ergod. Th. and Dynam. Sys. **11** (1991), 769–777; II: Math. Proc. Camb. Phil. Soc. **119** (1996), 513–536; III: ibid. **122**, 223–244.

[13] M Taniguchi, *Maskit surgery of entire functions,* to appear in RIMS kokyuroku

[14] M Taniguchi, *Explicit representations of structurally finite entire functions,* to appear in Proc. Japan Acad..

[15] M Taniguchi, *The size of the Julia sets of structurally finite entire functions,* preprint.