INNER RADIUS OF UNIVALENCE FOR A STRONGLY STARLIKE DOMAIN

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Abstract. The inner radius of univalence of a domain D with Poincaré density ρ_D is the possible largest number σ such that the condition $\|S_f\|_D = \sup_{w\in D} \rho_D(w)^{-2}|S_f(z)| \le \sigma$ implies the univalence of f for a nonconstant meromorphic function f on D, where S_f is the Schwarzian derivative of f. In this note, we give a lower bound of the inner radius of univalence for strongly starlike domains of order α in terms of the order α.

1. Main result

For a constant $0 \le \alpha \le 1$, a holomorphic function f on the unit disk is called *strongly starlike of order* α if $f'(0) \ne 0$ and if f satisfies the condition

$$\left|\arg \frac{zf'(z)}{f(z)-f(0)}\right| \le \frac{\pi\alpha}{2} \quad (z \in \mathbb{D}^* = \mathbb{D}\setminus\{0\}). \tag{1}$$

Note that a strongly starlike function f is starlike in the usual sense, namely, f is univalent and the image $f(\mathbb{D})$ is starlike with respect to $f(0)$. Every strongly starlike function f of order $\alpha < 1$ is bounded. In fact, Brannan and Kirwan [1] showed that

$$|f(z)-f(0)| \le |zf'(0)|M(\alpha) \quad (z \in \mathbb{D}). \tag{2}$$

Here $M(\alpha)$ is defined by

$$\begin{aligned} M(\alpha) &= \exp\left[\int_0^1 \left\{\left(\frac{1+t}{1-t}\right)^{\alpha} - 1\right\}\frac{dt}{t}\right] \\ &= \exp\left\{2\alpha\sum_{k=0}^{\infty}\frac{1}{(2k+1)(2k+1-\alpha)}\right\} \\ &= \frac{1}{4}\exp\left\{-\frac{\Gamma'((1-\alpha)/2)}{\Gamma((1-\alpha)/2)} - \gamma\right\}, \end{aligned} \tag{3}$$

where Γ is the Euler gamma function and $\gamma = 0.5772\ldots$ is the Euler constant.

A proper subdomain D of the complex plane $\mathbb{C}$ is said to be *strongly starlike of order* α with respect to a point $w_0 \in D$ if D is simply connected and if the conformal map $f : \mathbb{D} \to D$ of D with $f(0) = w_0$ is strongly starlike of order α. A strongly starlike domain of order 1 is nothing but a usual starlike domain. In what follows, without any pain, we always assume that $w_0 = 0$.

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We now introduce a standard domain adapted to the strong starlikeness. For a constant α with $0 < \alpha < 1$, we denote by V_α the bounded domain enclosed by the logarithmic spirals $\gamma_\alpha = \{\exp((-\tan(\pi\alpha/2) + i)\theta); 0 \le \theta \le \pi\}$ and $\bar{\gamma}_\alpha = \{w; \bar{w} \in \gamma_\alpha\}$.

Let D be a proper subdomain of $\mathbb{C}$ containing the origin. It will be convenient to consider the periodic function $R = R_D : \mathbb{R} \to (0, +\infty]$ of period 2π defined by

$$R(\theta) = \sup\{r > 0; [0, re^{i\theta}] \subset D\},$$

where $[a, b]$ denotes the closed line segment joining points a and b in $\mathbb{C}$. Note that R is lower semi-continuous.

In the sequel, we will use the convention $a \cdot D = \{aw; w \in D\}$ for $a \in \mathbb{C}$ and a domain D. Also, set $D^\vee = I(\operatorname{Ext} D)$, where $\operatorname{Ext} D = \widehat{\mathbb{C}} \setminus \overline{D}$ and $I(z) = 1/z$.

The next result will be fundamental for our aim here, whose proof can be found in [7].

Theorem A. *Let D be a proper subdomain of $\mathbb{C}$ with $0 \in D$ and let α be a constant with $0 < \alpha < 1$. Then the following conditions are equivalent.*

(a) *D is strongly starlike of order α with respect to the origin.*
(b) *$D^\vee$ is strongly starlike of order α with respect to the origin.*
(c) *$w \cdot V_\alpha \subset D$ holds for each point $w \in \overline{D}$.*
(d) *The radius function $R = R_D$ is bounded, absolutely continuous, and satisfies $|R'/R| \le \tan(\pi\alpha/2)$ a.e. in $\mathbb{R}$.*

Remark. The implication (a)⇒(d) is essentially due to Fait, Krzyż and Zygmunt [2]. Actually, we will employ their idea which was used to show the quasiconformal extendability of strongly starlike functions.

Let D be a subdomain of $\mathbb{C}$ with the Poincaré (or hyperbolic) metric $\rho_D(w)|dw|$ of constant curvature -4. The *inner radius of univalence* of D, which will be denoted by $\sigma(D)$, is the possible maximal number σ for which the condition $\|S_f\|_D \le \sigma$ implies the univalence of the nonconstant meromorphic function f on D, where S_f denotes the Schwarzian derivative $(f''/f')' - (f''/f')^2/2$ of f and $\|\varphi\|_D = \sup_{w \in D} \rho_D(w)^{-2}|\varphi(w)|$. Note that $\sigma(D)$ is Möbius invariant in the sense that $\sigma(L(D)) = \sigma(D)$ for a Möbius transformation L. In particular, $\sigma(D^\vee) = \sigma(\operatorname{Ext} D)$. The reader may consult the textbook [4] by Lehto as a general reference for the inner radius of univalence and related notions. When D is simply connected, theorems of Ahlfors and Gehring imply that $\sigma(D) > 0$ if and only if D is a quasidisk and, furthermore, $\sigma(D)$ is estimated from below by a positive constant $c(K)$ depending only on K for a K-quasidisk D. However, it is hard to give an explicit lower bound of $\sigma(D)$ for a concrete quasidisk D in general. Our main result is concerned with the inner radius of univalence of strongly starlike domains.

Theorem 1. *A strongly starlike domain D of order α satisfies*

$$\sigma(D) \ge \frac{2}{M(\alpha)^2} \cdot \frac{\cos(\pi\alpha/2)}{1 + \sin(\pi\alpha/2)}, \tag{4}$$

where $M(\alpha)$ is defined by (3).

Remarks. 1. When α tends to 0, the right-hand side above tends to 2. On the other hand, it is known that $\sigma(\mathbb{D}) = 2$. See also the final section.

2. By a result of Fait, Krzyż and Zygmunt [2], we know that a strongly starlike domain D of order α is a $K(\alpha)$-quasidisk, where $(K(\alpha)-1)/(K(\alpha)+1) = \sin(\pi\alpha/2)$. Hence, as a corollary, we have $\sigma(D) \geq c(K(\alpha))$. So, the novelty of this theorem lies in the explicitness of the estimate.

3. From Theorem 1 we observe that $D^\vee$ is strongly starlike of order α under the assumption of Theorem 1. Hence, we obtain $\sigma(\operatorname{Ext} D) \geq 2\cos(\pi\alpha/2)/(1+\sin(\pi\alpha/2))M(\alpha)^2$ simultaneously. We also note that the standard domain V_α has the property $\sigma(V_\alpha) = \sigma(\operatorname{Ext} V_\alpha)$.

2. Mapping function of V_α

Let $\mathcal{S}$ denote the set of holomorphic univalent functions on the unit disk $\mathbb{D}$ normalized by $f(0) = 0$ and $f'(0) = 1$. For $0 \leq \alpha \leq 1$, we define the function k_α in the class $\mathcal{S}$ by the relation

$$\frac{zk'_\alpha(z)}{k_\alpha(z)} = \left(\frac{1+z}{1-z}\right)^\alpha$$

on $\mathbb{D}$. More explicitly, k_α can be expressed by

$$k_\alpha(z) = z\exp\left[\int_0^z \left\{\left(\frac{1+\zeta}{1-\zeta}\right)^\alpha - 1\right\}\frac{d\zeta}{\zeta}\right].$$

This function is known to play a role of the usual Koebe function in the class of normalized strongly starlike functions of order α in many cases. Actually k_1 is nothing but the Koebe function.

Noting $k_\alpha(1) = M(\alpha)$, we consider the function

$$g_\alpha(z) = k_\alpha(z)/M(\alpha) = \exp\left[\int_1^z \left(\frac{1+\zeta}{1-\zeta}\right)^\alpha \frac{d\zeta}{\zeta}\right]. \tag{5}$$

The following fact is useful to note. Although this result was stated in [7], we give a direct proof here for completeness.

Lemma 1. $g_\alpha(\mathbb{D}) = V_\alpha$ *for* $0 < \alpha < 1$.

Proof. If we set $g_\alpha(e^{it}) = r(t)e^{i\Theta(t)} = R(\theta)e^{i\theta}$, then we have $e^{it}g'_\alpha(e^{it})/g_\alpha(e^{it}) = \Theta'(t) - ir'(t)/r(t)$. Since $\arg(zg'_\alpha(z)/g_\alpha(z)) = \pi\alpha/2$ for $z = e^{it}$ with $t \in (0,\pi)$, we obtain

$$\frac{R'(\theta)}{R(\theta)} = \frac{r'(t)}{r(t)\Theta'(t)} = -\tan\frac{\pi\alpha}{2},$$

which yields $\log R(\theta) = -\theta\tan(\pi\alpha/2)$ for $\theta = \Theta(t) \in (0,\pi)$. In the same way, we have $\log R(-\theta) = -\theta\tan(\pi\alpha/2)$ for $\theta \in (0,\pi)$. These imply that the radius function R of $g_\alpha(\mathbb{D})$ agrees with that of V_α, and hence $g_\alpha(\mathbb{D}) = V_\alpha$. ∎

3. Proof of Main Theorem

First, we recall the construction of a quasiconformal reflection in the boundary of a strongly starlike domain given by [2]. Let D be a strongly starlike domain of order $\alpha \in (0,1)$ with respect to the origin and let R be its radius function. Then we can take the quasiconformal reflection λ in ∂D defined by

$$\lambda(re^{i\theta}) = \frac{R(\theta)^2}{r} e^{i\theta}$$

for all $r > 0$ and $\theta \in \mathbb{R}$. We then calculate

$$\partial\lambda = i\frac{RR'}{r^2} \quad \text{and} \quad \bar{\partial}\lambda = \frac{e^{2i\theta}}{r^2}(iRR' - R^2) \tag{6}$$

at $w = re^{i\theta}$.

Now we use a general estimate on the radius of univalence in terms of quasiconformal reflections to prove our main result. The following estimate is first shown by Lehto [3] in a restricted situation and is generalized to the present form by [6].

Theorem B. *Let D be a quasidisk with quasiconformal reflection λ in ∂D. Then the following inequality holds:*

$$\sigma(D) \geq \varepsilon(\lambda, D) := 2 \operatorname*{ess.inf}_{w \in D} \frac{|\bar{\partial}\lambda(w)| - |\partial\lambda(w)|}{|\lambda(w) - w|^2 \rho_D(w)^2}. \tag{7}$$

Let us return to our case. By (6) and Theorem A (d), we obtain the estimates

$$\begin{aligned} |\bar{\partial}\lambda(w)| - |\partial\lambda(w)| &= \frac{R^2}{r^2}\left(\sqrt{1 + |R'/R|^2} - |R'/R|\right) \\ &\geq \frac{R^2}{r^2}\left(\sqrt{1 + \tan^2(\pi\alpha/2)} - \tan(\pi\alpha/2)\right) \\ &= \frac{R^2}{r^2} \cdot \frac{\cos(\pi\alpha/2)}{1 + \sin(\pi\alpha/2)} \end{aligned}$$

and

$$|\lambda(w) - w| = \frac{R^2}{r} - r = \frac{R^2 - r^2}{r}$$

for almost all $w = re^{i\theta} \in D$.

Secondly, we estimate ρ_D from above. Fix $w = re^{i\theta}$ and set $R = R(\theta)$. If we think of the domain $W = w_0 \cdot V_\alpha$, where $w_0 = Re^{i\theta} \in \partial D$, from Theorem A (c), we have $W \subset D$. The monotoneity property of the Poincaré metric then implies $\rho_D(w) \leq \rho_W(w)$. Now we write $\tau_\alpha = \rho_{V_\alpha}$. Then $\rho_W(w) = \tau_\alpha(w/w_0)/|w_0| = \tau_\alpha(r/R)/R$. Consequently, we have $\rho_D(w) \leq \tau_\alpha(r/R)/R$.

Summarizing the above, we have the estimate

$$\begin{aligned} \frac{|\bar{\partial}\lambda(w)| - |\partial\lambda(w)|}{|\lambda(w) - w|^2 \rho_D(w)^2} &\geq \frac{R^2}{r^2} \cdot \left(\frac{r}{R^2 - r^2}\right)^2 \cdot \frac{R^2}{\tau_\alpha(r/R)^2} \cdot \frac{\cos(\pi\alpha/2)}{1 + \sin(\pi\alpha/2)} \\ &= \frac{1}{(1 - (r/R)^2)^2 \tau_\alpha(r/R)^2} \cdot \frac{\cos(\pi\alpha/2)}{1 + \sin(\pi\alpha/2)}. \end{aligned}$$

Hence,

$$\varepsilon(\lambda, D) \geq \frac{2}{\sup_{0<u<1}(1-u^2)^2\tau_\alpha(u)^2} \cdot \frac{\cos(\pi\alpha/2)}{1+\sin(\pi\alpha/2)}.$$

Therefore, if we can show the following lemma, the proof of our main theorem will be finished.

Lemma 2. *The Poincaré density* τ_α *of* V_α *satisfies*

$$\sup_{0<u<1} (1-u^2)\tau_\alpha(u) = M(\alpha).$$

Proof. Since $g_\alpha : \mathbb{D} \to V_\alpha$ is biholomorphic by Lemma 1, we have $(1-|z|^2)^{-1} = \tau_\alpha(g_\alpha(z))|g'_\alpha(z)|$ for $z \in \mathbb{D}$. Note here that $u = u(x) = g_\alpha(x) > 0$ and $g'_\alpha(x) > 0$ for positive x. If we set

$$Q(x) = (1-u(x)^2)\tau_\alpha(u(x)) = \frac{1-u(x)^2}{(1-x^2)u'(x)}$$

for $x \in (0,1)$, we have only to show that Q is non-increasing in the interval $(0,1)$ because $\lim_{x\to 0} Q(x) = \tau_\alpha(0) = 1/|g'_\alpha(0)| = M(\alpha)$.

Since $xu'/u = \{(1+x)/(1-x)\}^\alpha$, we have the expression

$$Q = \frac{1-u^2}{1-x^2} \cdot \frac{x}{u} \cdot \left(\frac{1-x}{1+x}\right)^\alpha = \frac{x}{u} \cdot \frac{1-u^2}{(1+x)^{1+\alpha}(1-x)^{1-\alpha}}.$$

Taking the logarithmic derivative, we obtain

$$x\frac{Q'}{Q} = 1 - \frac{xu'}{u} \cdot \frac{1+u^2}{1-u^2} + \frac{2x(x-\alpha)}{1-x^2} = \frac{1-2\alpha x+x^2}{1-x^2} - \frac{1+u^2}{1-u^2}\left(\frac{1+x}{1-x}\right)^\alpha.$$

Therefore, $Q' \leq 0$ if and only if

$$\frac{1-u^2}{1+u^2} \leq \left(\frac{1+x}{1-x}\right)^\alpha \cdot \frac{1-x^2}{1-2\alpha x+x^2} = \frac{(1+x)^{1+\alpha}(1-x)^{1-\alpha}}{1-2\alpha x+x^2} =: P(x).$$

By representation (5), we see

$$\frac{1-u^2}{1+u^2} = \tanh\left[-\int_1^x \left(\frac{1+t}{1-t}\right)^\alpha \frac{dt}{t}\right].$$

Hence, the assertion $Q' \geq 0$ on the interval $(0,1)$ is further equivalent to the validity of the statement that

$$\int_x^1 \left(\frac{1+t}{1-t}\right)^\alpha \frac{dt}{t} \leq \operatorname{arctanh} P(x)$$

holds whenever $P(x) < 1$.

We now investigate the behaviour of the function P on $(0,1)$. Since

$$\frac{P'(x)}{P(x)} = \frac{4(x-\alpha)(\alpha x-1)}{(1-x^2)(1-2\alpha x+x^2)},$$

P is increasing in $(0,\alpha)$ and decreasing in $(\alpha,1)$. Noting $P(0) = 1$ and $P(1) = 0$, we observe that $P(x) > 1$ for $x \in (0,\beta)$ and that $0 < P(x) < 1$ for $x \in (\beta,1)$ for some number β between α and 1. Here, we use the following elementary fact.

Lemma 3. *Let S and T be continuous functions on the interval $(\beta, 1]$ that are positive, have continuous integrable derivatives on $(\beta, 1)$ and satisfy $S(1) = T(1) = 0$ and $S'(x) \leq T'(x)$ for $x \in (\beta, 1)$. Then $S(x) \geq T(x)$ for $x \in (\beta, 1]$.*

Thus it is enough to show the inequality

$$-\frac{1}{x}\left(\frac{1+x}{1-x}\right)^{\alpha} \geq \frac{d}{dx}\mathrm{arctanh}P(x) = \frac{P'(x)}{1-P(x)^2}$$
$$= \frac{-4(x-\alpha)(1-\alpha x)}{(1-2\alpha x+x^2)^2-(1+x)^{2+2\alpha}(1-x)^{2-2\alpha}}\left(\frac{1+x}{1-x}\right)^{\alpha}$$

for $x \in (\beta, 1)$. This inequality is equivalent to

$$(1-2\alpha x+x^2)^2-(1+x)^{2+2\alpha}(1-x)^{2-2\alpha} \leq 4x(x-\alpha)(1-\alpha x)$$
$$\Leftrightarrow (1+x)^{2+2\alpha}(1-x)^{2-2\alpha} \geq (1-2\alpha x+x^2)^2 - 4x(x-\alpha)(1-\alpha x) = (1-x^2)^2$$
$$\Leftrightarrow \left(\frac{1+x}{1-x}\right)^{2\alpha} \geq 1.$$

The last inequality is certainly valid for $x \in (0, 1)$. So, now the proof is complete. □

Remark. We can see from the proof that $\varepsilon(\lambda, V_\alpha) = 2\cos(\pi\alpha/2)/(1+\sin(\pi\alpha/2))M(\alpha)^2$ holds, where λ is the quasiconformal reflection constructed for V_α as above.

4. Upper estimate of the radius of univalence

Set $\xi(\alpha) = 2\cos(\pi\alpha/2)/(1+\sin(\pi\alpha/2))M(\alpha)^2$ and let $\eta(\alpha)$ be the possible largest value for which $\eta(\alpha) \leq \sigma(D)$ holds for any strongly starlike domain D of order α. Theorem 1 says that $\xi(\alpha) \leq \eta(\alpha)$ for $0 < \alpha < 1$. In this section, we give an upper bound for $\eta(\alpha)$ in order to examine how close the bound $\xi(\alpha)$ is to $\eta(\alpha)$. To this end, we give a rough upper estimate of $\sigma(V_\alpha)$.

Theorem 2. *For $0 < \alpha < 1$, the inequality $\sigma(V_\alpha) \leq 2(1-\alpha)^2$ holds.*

Proof. We consider the holomorphic function $f(w) = \log(1-w)$ on the domain $\mathbb{C}\setminus[1, +\infty)$. Although f is univalent, $f(V_\alpha)$ has an outward pointing cusp. So, $f(V_\alpha)$ is not a quasidisk. On the other hand, for a quasidisk D, if $\|S_f\|_D < \sigma(D)$, we know that $f(D)$ is also a quasidisk (see [4, p. 120]). Hence, we conclude $\sigma(V_\alpha) \leq \|S_f\|_{V_\alpha}$ for the above f.

Now we estimate $\|S_f\|_{V_\alpha}$. First note that $V_\alpha \subset W := \{w; |\arg(1-w)| < (1-\alpha)\pi/2\}$. By the monotoneity of the Poincaré metric, we have $\|S_f\|_{V_\alpha} \leq \|S_f\|_W$. Since $(1-w)^{1/(1-\alpha)}$ maps W conformally onto the right half plane, we compute

$$\rho_W(w) = \frac{|1-w|^{\alpha/(1-\alpha)}}{2(1-\alpha)\mathrm{Re}\left[(1-w)^{1/(1-\alpha)}\right]}.$$

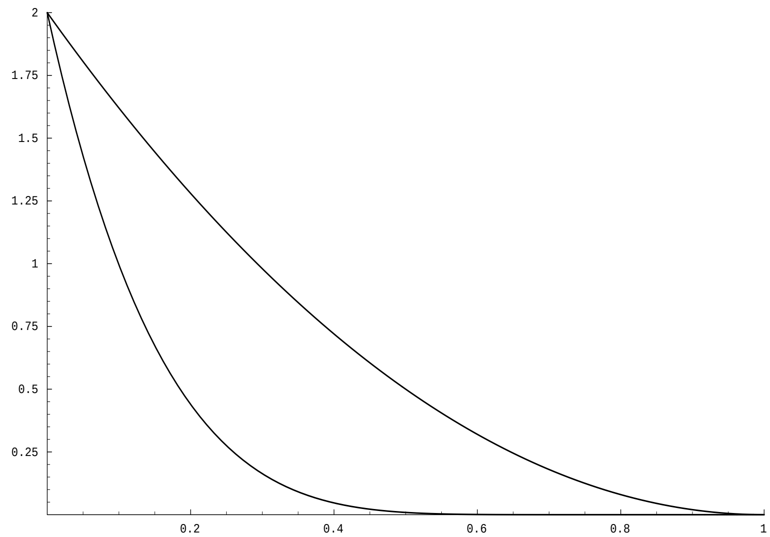
On the other hand, $S_f(w) = 1/2(1-w)^2$. Thus, we calculate

$$\|S_f\|_W = \sup_{w\in W} 2(1-\alpha)^2 \mathrm{Re}\left[\frac{(1-w)^{1/(1-\alpha)}}{|1-w|^{1/(1-\alpha)}}\right] = 2(1-\alpha)^2.$$

Now the proof is completed. □

Remark. Since V_α has a corner of opening $(1-\alpha)\pi$, the above estimate follows also from Lemma 3.3 in the paper [5] by Miller-Van Wieren.

Note that $\xi(\alpha) \le \eta(\alpha) \le \sigma(V_\alpha) \le 2(1-\alpha)^2$. We exhibit the graphs of the functions $\xi(\alpha)$ and $2(1-\alpha)^2$ below.



References

1. D. A. Brannan and W. E. Kirwan, *On some classes of bounded univalent functions*, J. London Math. Soc. (2) **1** (1969), 431–443.
2. M. Fait, J. G. Krzyż, and J. Zygmunt, *Explicit quasiconformal extensions for some classes of univalent functions*, Comment. Math. Helv. **51** (1976), 279–285.
3. O. Lehto, *Remarks on Nehari's theorem about the Schwarzian derivative and schlicht functions*, J. Analyse Math. **36** (1979), 184–190.
4. ———, *Univalent Functions and Teichmüller Spaces*, Springer-Verlag, 1987.
5. L. Miller-Van Wieren, *On Nehari disks and the inner radius*, Comment. Math. Helv. **76** (2001), 183–199.
6. T. Sugawa, *A remark on the Ahlfors-Lehto univalence criterion*, to appear in Ann. Acad. Sci. Fenn. A I Math.
7. ———, *A self-duality of strong starlikeness*, preprint.

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