A CONFORMALLY INVARIANT METRIC ON RIEMANN SURFACES ASSOCIATED WITH INTEGRABLE HOLOMORPHIC QUADRATIC DIFFERENTIALS

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Abstract. In this paper, we define a conformally invariant (pseudo-)metric on all Riemann surfaces in terms of integrable holomorphic quadratic differentials and analyze it. This metric is closely related to an extremal problem on the surface. As a result, we have a kind of reproducing formula for integrable quadratic differentials. Furthermore, we establish a new characterization of boundedness of geometry of hyperbolic Riemann surfaces in terms of invariant metrics.

1. Introduction

Throughtout this paper, R denotes a hyperbolic Riemann surface with hyperbolic metric $\rho_R = \rho_R(z)|dz|$ of constant negative Gaussian curvature -4.

For integers m and n, let ω be a differential (m,n)-form on R, namely, ω is a section of the line bundle $\mathcal{K}_R^{\otimes m} \otimes \overline{\mathcal{K}}_R^{\otimes n}$, where $\mathcal{K}_R$ denotes the canonical bundle over R. For a local coordinate $z = \alpha(p)$ from an open set U_α of R onto an open set V_α of $\mathbb{C}$, we will write $\omega = \omega_\alpha(z)dz^m d\bar{z}^n$ on U_α, where ω_α is a function on V_α. Recall that the transition relation $\omega_\beta(w)(w')^m(\overline{w'})^n = \omega_\alpha(z)$, where $w = \beta\circ\alpha^{-1}(z)$ and $w' = dw/dz$. An $(m,0)$-form will be just called an m-form. Traditionally, a 2-form is called a quadratic differential. If no confusion will occur, we sometimes omit the suffix indicating the local coordinate.

We also note that in the case $m = n = 1/2$ the above definition still has meaning because we can well define $(w')^{1/2}(\overline{w'})^{1/2}$ as $|w'|$ and the positivity of coefficients is preserved by the change of local coordinates. We will write $|dz|$ instead of $dz^{1/2}d\bar{z}^{1/2}$. A positive (or non-negative) $(1/2,1/2)$-form on R will be called a (conformal) metric (or pseudo-metric) on R.

We recall that a non-negative $(1,1)$-form $\omega = \omega(z)|dz|^2$ can be regarded as an area element on the surface under the identification $|dz|^2 = dx\wedge dy$, where $z = x+iy$.

Let $f : R \to R'$ be a holomorphic map. Then we can define the pullback of an (m,n)-form ω on R' by f, which will be denoted by $f^*\omega$, by

$$(f^*\omega)_\alpha(z) = \omega_\beta(w)(w')^m(\overline{w}')^n,$$

where α and β are local coordinates of R and R', respectively, such that $f(U_\alpha)\subset U_\beta$ and $w = \beta\circ f\circ\alpha^{-1}(z)$.

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We consider the following vector spaces of differentials on R :

$$\Omega(R) = \{\omega = \omega(z)dz; \text{holomorphic 1-form on } R,\ \|\omega\|_2^2 = \iint_R |\omega(z)|^2 dxdy < \infty\},$$
$$A(R) = \{\varphi = \varphi(z)dz^2; \text{holomorphic 2-form on } R,\ \|\varphi\|_1 = \iint_R |\varphi(z)| dxdy < \infty\},$$
$$B(R) = \{\varphi = \varphi(z)dz^2; \text{holomorphic 2-form on } R,\ \|\varphi\|_\infty = \sup_{p\in R} \rho_R^{-2}|\varphi|(p) < \infty\}.$$

The spaces $A(R)$ and $B(R)$ are important in Teichmüller theory and are sometimes called the Bers spaces. A hyperbolic Riemann surface will be called *non-exceptional* if $\dim A(R) > 0$. Note that an exceptional one is conformally equivalent to a thrice punctured sphere.

We now define two invariant pseudo-metrics b_R and q_R by using the above spaces. We set

$$b_{R,\alpha}(z) = \sup\{|\omega_\alpha(z)|; \omega \in \Omega(R) \text{ with } \|\omega\|_2 \le \sqrt{\pi}\} \text{ and}$$
$$q_{R,\alpha}(z) = \sup\{|\varphi_\alpha(z)|^{1/2}; \varphi \in A(R) \text{ with } \|\varphi\|_1 \le \pi\}$$

for each local coordinate $z = \alpha(p)$ of R.

The first one is known to be expressed by $b_R(z) = \sqrt{\pi K_R(z,z)}$ where $K_R(z,w)dzd\bar{w}$ is the Bergman kernel of R. The second one is new as far as the author knows. These pseudo-metrics satisfy the following fundamental properties.

Lemma 1.1 ([20]). *The metrics b_R and q_R are continuous, normalized, conformally invariant and monotonic. More precisely,*

1. *b_R and q_R are continuous and $\log b_R$ and $\log q_R$ are subharmonic or identically $-\infty$,*
2. *for the unit disk $\mathbb{D} = \{|z| < 1\}$ we have $b_{\mathbb{D}}(z) = q_{\mathbb{D}}(z) = 1/(1-|z|^2)$,*
3. *for a biholomorphic map $f : R \to R'$ we have $f^*b_{R'} = b_R$ and $f^*q_{R'} = q_R$, and*
4. *for a subdomain R_0 of R we have $b_{R_0} \ge b_R$ and $q_{R_0} \ge q_R$.*

The first property follows from a certain compactness property of closed balls of $\Omega(R)$ and $A(R)$. The second property is a consequence of the mean value property of holomorphic functions (see the calculation at the end of Section 2). The other properties can be directly seen.

The next result is one of motivations for the present work.

Proposition 1.2 ([20]). *We have*

$$b_R \le q_R \le h_R,$$

where h_R denotes the Hahn metric on R.

The Hahn metric $h_R = h_{R,\alpha}(z)|dz|$ is conformally invariant and defined by

$$h_{R,\alpha}(z) = \inf_f |(\alpha \circ f)'(0)|^{-1},$$

where the infimum is taken over all injective holomorphic maps f from the unit disk $\mathbb{D}$ into R with $f(0) = \alpha^{-1}(z)$. A systematic treatment of the Hahn metric can be found in [12]. We just recall here a few useful facts.

Lemma 1.3 (Minda [12]). *The Hahn metric is comparable with the quasi-hyperbolic metric for a plane domain. Precisely, the following inequality holds for a proper subdomain* Ω *of* $\mathbb{C}$*:*

$$\frac{1}{4\delta_\Omega(z)} \le h_\Omega(z) \le \frac{1}{\delta_\Omega(z)},$$

where $\delta_\Omega(z) = \min\{|z-a|; a \in \partial\Omega\}$.

Denote by $\iota_R(p)$ the injectivity radius of R at the point p. In other words, $2\iota_R(p) = \inf_\gamma \int_\gamma \rho(z)|dz|$, where the infimum is taken over all smooth closed curves γ passing through p which are not contractible in R. Then we have the following result.

Lemma 1.4 (Gotoh [4]).

$$\frac{\coth \iota_R}{4} \le \frac{h_R}{\rho_R} \le \coth \iota_R.$$

We will need also the following.

Lemma 1.5 (Minda [12]). *Let* $f : \widetilde{R} \to R$ *be a holomorphic covering map. Then* $h_{\widetilde{R}} \le f^* h_R$ *holds.*

Proof of Proposition 1.2. For each $\omega \in \Omega(R)$ with $\|\omega\|_2 \le \sqrt{\pi}$, the square $\omega\otimes\omega$ belongs to $A(R)$ and satisfies $\|\omega\otimes\omega\|_1 = \|\omega\|_2^2 \le \pi$. Therefore, we have $|\omega(z)| = |\omega\otimes\omega(z)|^{1/2} \le q_R(z)$, which implies $b_R(z) \le q_R(z)$. The inequality $q_R \le h_R$ follows from properties 2,3 and 4 in Lemma 1.1 (see [20]). □

For a planar Riemann surface R, it is known that $b_R \equiv 0$ if and only if R does not carry the Green function (i.e., $R \in O_G$). Furthermore, a puncture is removable with respect to square integrable holormophic 1-forms, and hence we have $b_{R-\{p_0\}} \equiv b_R$ for any point $p_0 \in R$. In contrast, we have the following.

Lemma 1.6. *For a non-exceptional hyperbolic Riemann surface* R, *we have* $q_R(p) > 0$ *for each point* p *of* R. *For a proper subdomain* R_0 *of* R, *the metric* b_{R_0} *is not identically equal to* b_R *on* R_0.

Proof. When R is planar, R can be embedded in the complex plane $\mathbb{C}$. Then R omits at least three points, say a, b and c, in the plane because R is non-exceptional. Thus the holomorphic quadratic differential $dz^2/(z-a)(z-b)(z-c)$ on R is integrable and has no zeros there. When R is not planar, the space $\Omega(R)$ has positive dimension. In this case, Virtanen's theorem (see [18, p. 43]) tells us that $b_R(p) > 0$ at each point $p \in R$. By Proposition 1.2 we have the conclusion in this case, too. □

Another motivation comes from the relation between the Bers spaces $A(R)$ and $B(R)$. Once upon a time, it had been conjectured that $A(R) \subset B(R)$ for all Riemann surfaces. This was easily proved in the case that R is of finite type. Lehner is the first person who gave a sufficient condition for this inclusion relation to be valid for Riemann surfaces of inifinite topological type (see [7]). Later, Pommerenke [16] disproved the conjecture by showing that there exists a holomorphic 1-form ω on R which is hyperbolically bounded, i.e., $\rho_R^{-1}|\omega|$ is bounded, but is not square integrable on R for some Riemann surface R.

(Recently, more geometric constructions for counterexamples were given by Ohsawa [15] and Matsuzaki [10].)

Finally, Niebur and Sheingorn [13] gave a complete answer to this problem: $A(R) \subset B(R)$ if and only if $\lambda^*(R) > 0$, where $\lambda^*(R)$ denotes the infimum of hyperbolic lengths of those simple closed curves in R which are homotopic to neither a point nor a puncture. We will say that a Riemann surface R with the property $\lambda^*(R) > 0$ is of *Lehner type*. If the infimum $\lambda(R)$ of hyperbolic lengths of simple closed curves in R which are not homotopic to any point is positive, then R is said to be *of bounded geometry*. Since $2\inf_{p\in R}\iota_R(p) = \lambda(R)$, this property is equivalent to the condition that the (hyperbolic) injectivity radius has a positive lower bound on R.

Note that $\lambda(R) = 0$ if R has a puncture and $\lambda(R) = \lambda^*(R)$ otherwise. We have the following characterization of boundedness of geometry in terms of invariant metrics.

Proposition 1.7 (cf. [19]). *A hyperbolic Riemann surface R is of bounded geometry, i.e., $\lambda(R) > 0$ if and only if the Hahn metric is comparable with the Poincaré metric:*

$$\sup_{p\in R}\frac{h_R}{\rho_R}(p) < +\infty.$$

In this paper, we provide a new characterization of boundedness of geometry as in the following.

Theorem 1.8. *A hyperbolic Riemann surface R is of bounded geometry, i.e., $\lambda(R) > 0$ if and only if*

$$H(R) := \sup_{p\in R}\frac{h_R}{q_R}(p) < +\infty.$$

Note that there is no order relation between ρ_R and q_R in general (see the remark made at the end of Section 3). We will give a proof of this in Section 5 with a more concrete estimate (Theorem 5.1).

If $A(R) \subset B(R)$, we set

$$\kappa(R) = \sup\{\|\varphi\|_\infty; \varphi \in A(R) \text{ with } \|\varphi\|_1 \le \pi\}.$$

Note that $\kappa(R) < +\infty$ by the closed graph theorem. If $A(R)$ is not contained in $B(R)$, we set $\kappa(R) = +\infty$. Recently, Matsuzaki [9] obtained a concrete estimate of this quantity:

$$\frac{1}{2\lambda^*(R)} \le \kappa(R) \le \min\left\{\frac{c_1}{\lambda^*(R)}, c_2\right\}, \tag{1.1}$$

where c_1 and c_2 are positive absolute constants. The following result will enable us to deduce a refinement of Matsuzaki's result in the sence that the ratio q_R/ρ_R may be regarded as a localization of the global quantity $\kappa(R)$. In fact, as an application, we reproduce Matsuzaki's inequality with concrete constants (Theorem 4.6) in Section 4.

Proposition 1.9. *We have the relation*

$$\kappa(R) = \sup_{p\in R}\left(\frac{q_R}{\rho_R}(p)\right)^2.$$

Proof. Setting $A_0(R) = \{\varphi \in A(R); \|\varphi\|_1 \le \pi\}$, we calculate

$$\kappa(R) = \sup_{\varphi \in A_0(R)} \|\varphi\|_\infty = \sup_{\varphi \in A_0(R)} \sup_{p \in R} \rho_R^{-2} |\varphi|(p) = \sup_{p \in R} \sup_{\varphi \in A_0(R)} \rho_R^{-2} |\varphi|(p) = \sup_{p \in R} \rho_R^{-2} q_R^2(p).$$

□

In particular, we have the following characterization of Riemann surfaces of Lehner type, which should be compared with Proposition 1.7 and Theorem 1.8.

Corollary 1.10. *A hyperbolic Riemann surface R is of Lehner type, i.e., $\lambda^*(R) > 0$ if and only if the metric q_R is dominated by the Poincaré metric ρ_R, that is,*

$$\sup_{p \in R} \frac{q_R}{\rho_R}(p) < +\infty.$$

We now briefly explain the structure of the present paper. In Section 2, we consider an extremal problem in connection with our metric q_R to obtain a reproducing formula for $A(R)$. Section 3 will be devoted to a preliminary investigation of q_R for doubly connected domains R. Using the results in Section 3, we deduce upper and lower estimates of q_R around a short geodesic and a puncture in Section 4. In Section 5, we will prove the characterization theorem (Theorem 1.8) for boundedness of geometry (equivalently, uniform perfectness of the "boundary") as was promised. We will provide a lower estimate of q_R in terms of the (logarithmic) capacity metric for $R \notin O_G$ in Section 6. Section 7 will treat the connection with the usual kernel function for $A(R)$.

2. Extremal problem for quadratic differentials

Which quadratic differential does attain the supremum in the definition of the metric q_R? By definition, we see

$$q_{R,\alpha}(z_0)^2 = \sup_{\varphi \in A_0(R)} \operatorname{Re} \varphi_\alpha(z_0), \tag{2.1}$$

where $A_0(R) = \{\varphi \in A(R); \|\varphi\|_1 \le \pi\}$. The extremal problem to find a differential $\varphi \in A_0(R)$ attaining the supremum in (2.1) has a unique solution and leads to a kind of reproducing formula for integrable holomorphic quadratic differentials on R.

Theorem 2.1. *Let $z = \alpha(p)$ be a local coordinate of a non-exceptional Riemann surface R. For each point $z_0 \in V_\alpha$ there exists a unique $\varphi \in A(R)$ such that*

$$q_{R,\alpha}(z_0)^2 = \varphi_\alpha(z_0).$$

Furthermore, $\|\varphi\|_1 = \pi$ and the following formula holds:

$$\psi_\alpha(z_0) = \frac{q_{R,\alpha}(z)^2}{\pi} \iint_R \frac{|\varphi|}{\varphi} \psi \quad (\psi \in A(R)). \tag{2.2}$$

Conversely, if a non-zero element φ in $A(R)$ satisfies relation (2.2)*, then $\pi\varphi/\|\varphi\|_1$ is a solution of the extremal problem* (2.1).

Proof. We omit the suffix α below. Take a sequence φ_n in $A_0(R)$ such that $\operatorname{Re} \varphi_n(z_0) \to q_R(z_0)^2$ as $n \to \infty$. Since $A_0(R)$ is a normal family, we may assume that the sequence φ_n

converges to a holomorphic quadratic differential φ uniformly on each compact subset of R. Recalling Lemma 1.6, we can see

$$q_R(z_0)^2 = \mathrm{Re}\,\varphi(z_0) = \varphi(z_0) > 0. \tag{2.3}$$

Fatou's lemma yields

$$\|\varphi\|_1 \le \varliminf_{n\to\infty} \|\varphi_n\|_1 \le \pi.$$

In fact, $\|\varphi\|_1 = \pi$ because of the extremality of φ. Thus we have shown the existence of an extremal differential φ.

Next we prove equality (2.2). Let $\psi \in A(R)$ be linearly independent of φ over the real field and consider the function $f(t) = \|\varphi + t\psi\|_1$ of $t \in \mathbb{R}$. Then the extremality of φ implies the inequality

$$\frac{\mathrm{Re}\,(\varphi(z_0) + t\psi(z_0))}{f(t)} \le \frac{\mathrm{Re}\,\varphi(z_0)}{f(0)} = \frac{q_R(z_0)^2}{\pi}.$$

Since $||\varphi + t\psi| - |\varphi|| \le |t||\psi|$ and since $|a + tb| = |a| + t|a|\mathrm{Re}\,(b/a) + o(t)$ as $t \to 0$ in $\mathbb{R}$ for complex numbers a and b, Lebesgue's dominated convergence theorem produces

$$f(t) = f(0) + t\,\mathrm{Re} \iint_R \frac{|\varphi|}{\varphi}\psi + o(t)$$

as $t \to 0$. Hence we have

$$q_R(z_0)^2 + t\,\mathrm{Re}\,\psi(z_0) \le \frac{q_R(z_0)^2}{\pi}\left(\pi + t\,\mathrm{Re} \iint_R \frac{|\varphi|}{\varphi}\psi + o(t)\right)$$

as $t \to 0$. Since t may take both signs, this inequality forces

$$\mathrm{Re}\,\psi(z_0) = \pi^{-1} q_R(z_0)^2 \mathrm{Re} \iint_R |\varphi|\psi/\varphi.$$

This equality also holds for $e^{i\theta}\psi$ instead of ψ, so we can eliminate the symbol "Re" from this equality. Thus (2.2) is now established.

Finally, we show the uniqueness of the extremal differential φ. Suppose that $\varphi_1 \in A_0(R)$ is also an extremal differential at z_0, i.e., $q_R(z_0)^2 = \varphi_1(z_0)$. Then, applying $\psi = \varphi_1$ to (2.2), we obtain

$$q_R(z_0)^2 = \varphi_1(z_0) = \frac{q_R(z_0)^2}{\pi}\mathrm{Re} \iint_R \frac{|\varphi|}{\varphi}\varphi_1.$$

We now estimate

$$\pi = \mathrm{Re} \iint_R \frac{|\varphi|}{\varphi}\varphi_1 \le \left|\iint_R \frac{|\varphi|}{\varphi}\varphi_1\right| \le \iint_R |\varphi_1| \le \pi.$$

Therefore, equalities hold above and the argument of $|\varphi|\varphi_1/\varphi$ must be identically 0, which implies that φ_1/φ is a positive constant. Thus $\varphi_1 = \varphi$ and the uniqueness is now proved. The latter part of the theorem is trivial. □

Remark. The extremal differential φ above depends on the choice of the local coordinate α. To be more precise, we write $\varphi = \varphi_R^{\alpha,z_0}$. If we take another local coordinate β around

$p_0 = \alpha^{-1}(z_0)$, then we have

$$\varphi_R^{\beta, w(z_0)} \frac{\overline{w'(z_0)}}{w'(z_0)} = \varphi_R^{\alpha, z_0},$$

where $w(z) = \beta \circ \alpha^{-1}(z)$. In other words, the extremal differential at p can be written as

$$\varphi_R(r, p) = \varphi_{R,\alpha,\beta}(\zeta, z) d\zeta^2 \frac{d\bar{z}}{dz},$$

where $\zeta = \alpha(r)$, $z = \beta(p)$ and $\varphi_{R,\alpha,\beta}(\zeta, z) = \varphi_{R,\alpha}^{\beta,z}(\zeta)$. This means that φ_R is a section of the line bundle $\mathcal{K}_R^{\otimes 2} \boxtimes (\mathcal{K}_R^* \otimes \overline{\mathcal{K}}_R)$ over $R \times R$, where $\boxtimes$ stands for the external tensor product.

It is convenient to record the following translation formula under the notation in the above remark.

Lemma 2.2. *Let D and D' be plane domains and $F : D \to D'$ be a conformal map. Then we have*

$$\varphi_{D'}(F(\zeta), F(z)) F'(\zeta)^2 \frac{\overline{F'(z)}}{F'(z)} = \varphi_D(\zeta, z).$$

In the case that R is the unit disk $\mathbb{D}$ with the canonical coordinate $\alpha(z) = z$, from the mean value property of integrable holomorphic functions on $\mathbb{D}$:

$$f(0) = \frac{1}{\pi} \iint_{\mathbb{D}} f(z) dx dy,$$

it follows that $\varphi \equiv 1$ is the extremal quadratic differential at the origin. By Lemma 2.2, we see that

$$\varphi_{\mathbb{D}}(\zeta, z) = \frac{(1 - |z|^2)^2}{(1 - \bar{z}\zeta)^4}.$$

Therefore, we have a reproducing formula for $A(\mathbb{D})$ by (2.2):

$$\psi(z) = \frac{1}{\pi(1 - |z|^2)^2} \iint_{\mathbb{D}} \psi(\zeta) \left(\frac{1 - \bar{z}\zeta}{1 - z\bar{\zeta}} \right)^2 d\xi d\eta.$$

3. Estimate on annulus

In this section, we deduce an estimate of $q_{\mathbb{A}}(w)$ for a round annulus $\mathbb{A} = \mathbb{A}_r = \{w \in \mathbb{C}; r < |w| < 1/r\}$ for $0 < r < 1$.

Theorem 3.1. *For the annulus $\mathbb{A} = \{w; r < |w| < 1/r\}$, we have the estimate*

$$Q_1(w) \leq q_{\mathbb{A}}(w)^2 \leq Q_2(w), \quad r < |w| < 1/r,$$

where

$$Q_1(w) = \max \left\{ \left(\frac{r}{1 - (r|w|)^2} \right)^2, \left(\frac{r}{|w|^2 - r^2} \right)^2, \frac{1}{4|w|^2 \log 1/r}, \frac{r}{2|w|(1 - r^2)}, \frac{r}{2|w|^3(1 - r^2)} \right\}$$

and Q_2 is determined by the relations $Q_2(1/w)|w|^{-4} = Q_2(w)$ and

$$Q_2(w) = \max \left\{ \frac{1}{2|w|^2(\log \frac{1}{r|w|} + r|w| - 1)}, \frac{1}{2|w|^2(\log \frac{r}{|w|} + \frac{|w|}{r} - 1)} \right\}$$

for $|w| \le 1$. *We also have* $Q_2(w) \le 9Q_1(w)$ *for* $w \in \mathbb{A}$.

Before proceeding to the proof, we make several observations. Recall that the pullback j^* by the automorphism $j(z) = 1/z$ of $\mathbb{A}$ is defined by $j^*P(w) = P(1/w)|w|^{-4}$ for a $(1,1)$-form $P = P(w)|dw|^2$ on $\mathbb{A}$. We denote by P_1, P_2, P_3, P_4 and P_5 the five functions in the above definition of Q_1 in the order of appearance, which are all regarded as $(1,1)$-forms on $\mathbb{A}$. Note that $j^*P_1 = P_2$, $j^*P_3 = P_3$ and $j^*P_4 = P_5$. In particular, we know that Q_1, Q_2 and $q_{\mathbb{A}}^2$ are all invariant under the pullback by the analytic involution j. Setting $m(t) = -\log t + t - 1$ for $t > 0$, we have the expression

$$Q_2(w) = \begin{cases} 1/2|w|^2 m(|w|/r) & \text{if } r < |w| \le r_0 \\ 1/2|w|^2 m(r|w|) & \text{if } r_0 \le |w| \le 1 \\ 1/2|w|^2 m(r/|w|) & \text{if } 1 \le |w| \le 1/r_0 \\ 1/2|w|^2 m(1/r|w|) & \text{if } 1/r_0 \le |w| < 1/r, \end{cases}$$

where $r_0 = 2(\log 1/r)/(1/r - r)$. Note that $r_0 = t/\sinh t$ if we set $r = e^{-t}$ and that $r < r_0 < 1$. We also remark that $|w| \le r_0$ if and only if $P_3(w) \le P_5(w)$. Observe that the function $m(t)$ is convex and takes its minimum at $t = 1$. For fixed $t_0 \in (0,1)$ and $t_1 \in (1, +\infty)$, the following estimates will be useful later:

$$m(t) \ge \frac{m(t_0)}{\log 1/t_0} \log \frac{1}{t} \quad \text{for } 0 < t \le t_0, \tag{3.1}$$

$$m(t) \ge \frac{(1-t)^2}{2} \quad \text{for } t_0 \le t \le 1, \tag{3.2}$$

$$m(t) \ge \frac{(t-1)^2}{2t_1} \quad \text{for } 1 \le t \le t_1, \text{ and} \tag{3.3}$$

$$m(t) \ge \frac{m(t_1)}{t_1} t \quad \text{for } t_1 \le t. \tag{3.4}$$

Taking $w = 1$, we have the following

Corollary 3.2. *For* $\mathbb{A} = \{r < |w| < 1/r\}$, *we have*

$$\max\left\{ \left(\frac{r}{1-r^2}\right)^2, \frac{1}{4\log 1/r} \right\} \le q_{\mathbb{A}}(1)^2 \le \frac{1}{2(-\log r + r - 1)}.$$

Proof of Theorem 3.1. First, we show the inequality $Q_1 \le q_{\mathbb{A}}^2$. Since $\mathbb{D}_{1/r} = \{|z| < 1/r\} \supset \mathbb{A}$, the monotonicity of the metric q implies $\rho_{\mathbb{D}_{1/r}} = q_{\mathbb{D}_{1/r}} \le q_{\mathbb{A}}$ in $\mathbb{A}$. Namely, $P_1^{1/2} = r/(1 - r^2|w|^2) \le q_{\mathbb{A}}$. We next consider the differential $\varphi_j(w)dw^2 = w^j dw^2$ on $\mathbb{A}$ for an integer j. By definition, we have $\pi|\varphi_j(w)|/\|\varphi_j\|_1 \le q_{\mathbb{A}}(w)^2$. An easy calculation shows that $\|\varphi_{-2}\|_1 = 4\pi \log 1/r$ and $\|\varphi_{-1}\|_1 = 2\pi(1/r - r)$. Thus we have $P_3 \le q_{\mathbb{A}}^2$ and $P_4 \le q_{\mathbb{A}}^2$. The other inequalities $P_2 \le q_{\mathbb{A}}^2$ and $P_5 \le q_{\mathbb{A}}^2$ follow from the invariance observed above.

Secondly, we show the inequality $q_{\mathbb{A}}^2 \le Q_2$. Let $\varphi(w)dw^2$ be any element of $A(\mathbb{A})$. Take $w \in \mathbb{A}$ and fix it. By the symmetry, we may assume $|w| \le 1$. Let s and t be any numbers such that $r < s < |w| < t < 1/r$. Cauchy's integral formula then implies

$$w^2\varphi(w) = \frac{1}{2\pi i}\int_{|\zeta|=t} \frac{\zeta^2\varphi(\zeta)}{\zeta - w} d\zeta - \frac{1}{2\pi i}\int_{|\zeta|=s} \frac{\zeta^2\varphi(\zeta)}{\zeta - w} d\zeta = \psi_0(w) + \psi_1(w).$$

We calculate

$$|\psi_0(w)| \le \frac{1}{2\pi}\int_0^{2\pi} \frac{t^2|\varphi(te^{i\theta})|}{t-|w|} t d\theta.$$

Integrating both sides of $|\psi_0(w)|(t-|w|)/t^2 \le \int_0^{2\pi} |\varphi(te^{i\theta})|td\theta$ with respect to t from $|w|$ to $1/r$, we obtain

$$|\psi_0(w)|m(r|w|) \le \frac{1}{2\pi}\int_{|w|}^{1/r}\int_0^{2\pi} |\varphi(te^{i\theta})|td\theta dt = \frac{1}{2\pi}\iint_{|w|<|\zeta|<1/r} |\varphi(\zeta)|d\xi d\eta.$$

Similarly, we have

$$|\psi_1(w)|m(|w|/r) \le \frac{1}{2\pi}\iint_{r<|\zeta|<|w|} |\varphi(\zeta)|d\xi d\eta.$$

Combining these two inequalities, we see

$$|w^2\varphi(w)| \le |\psi_0(w)| + |\psi_1(w)| \le \frac{1}{2\pi}\max\left\{\frac{1}{m(r|w|)}, \frac{1}{m(|w|/r)}\right\} \|\varphi\|_1.$$

The last inequality yields $|w|^2 q_{\mathbb{A}}(w)^2 \le \frac{1}{2}\max\{1/m(r|w|), 1/m(|w|/r)\}$, which is nothing but the desired one.

Finally, we show $Q_2(w) \le 9Q_1(w)$. We assume $|w| \le 1$ as well. First we show $1/2|w|^2 m(r|w|) \le 9Q_1(w)$. When $r|w| \ge 1/2$, by (3.2) we have

$$\frac{1}{2|w|^2 m(r|w|)P_1(w)} \le \frac{(1-r^2|w|^2)^2}{r^2|w|^2(1-r|w|)^2} = \left(1+\frac{1}{r|w|}\right)^2 \le 9.$$

When $r|w| \le 1/2$, by (3.1) we have

$$\frac{1}{2|w|^2 m(r|w|)P_3(w)} = \frac{2\log 1/r}{m(r|w|)} \le \frac{2\log 2\cdot\log 1/r}{m(1/2)\log 1/r|w|} \le \frac{2\log 2}{m(1/2)} = 7.17\cdots < 9.$$

So we have shown $1/2|w|^2 m(r|w|) \le 9Q_1(w)$ for $r < |w| \le 1$.

Next we prove $1/2|w|^2 m(|w|/r) \le 9Q_1(w)$. When $|w|/r \ge 2$, by (3.4) we obtain

$$\frac{1}{2|w|^2 m(|w|/r)P_5(w)} \le \frac{2|w|^3(1-r^2)}{r}\cdot\frac{2}{2|w|^2 m(2)|w|/r} \le \frac{2}{m(2)} = 6.51\cdots < 9.$$

On the other hand, when $|w|/r \le 2$, by (3.3), we have

$$\frac{1}{2|w|^2 m(|w|/r)P_2(w)} \le \frac{2(|w|^2-r^2)^2}{r^2|w|^2(|w|/r-1)^2} = 2\left(1+\frac{r}{|w|}\right)^2 < 8 < 9.$$

Hence, we see $1/2|w|^2 m(|w|/r) \le 9Q_1(w)$ for $r < |w| \le 1$. Now the proof is complete. □

We have also an estimate on the punctured disk $\mathbb{D}^* = \mathbb{D} - \{0\}$ in a similar way as above. Note that the function $w\varphi(w)$ is always holomorphic in the unit disk for each $\varphi \in A(\mathbb{D}^*)$.

Theorem 3.3. *We have* $Q_1(w) \le q_{\mathbb{D}^*}(w)^2 \le Q_2(w)$ *for* $w \in \mathbb{D}^*$, *where*

$$Q_1(w) = \max\left\{\frac{1}{(1-|w|^2)^2}, \frac{1}{2|w|}\right\} \quad \text{and} \quad Q_2(w) = \frac{1}{2|w|^2(|w|^{-1}-1-\log|w|^{-1})}.$$

In particular, $q_{\mathbb{D}^*}(w)^2 = \frac{1+o(1)}{2|w|}$ *as* $w \to 0$. *Furthermore,* $Q_2(w) \le 8Q_1(w)$ *holds for* $w \in \mathbb{D}^*$.

Corollary 3.4. *The metric* q_R *is incomplete at a puncture of* R.

Proof. Suppose that $\iota : \mathbb{D}^* \to R$ is an injective holomorphic map such that the induced map $\iota_* : \pi_1(\mathbb{D}^*, *) \to \pi_1(R, *)$ is injective. Therefore, the puncture of $\mathbb{D}^*$ corresponds to a puncture of R. Let $\gamma_0 : (0, 1/2] \to \mathbb{D}^*$ be the curve defined by $\gamma_0(t) = t$. Then $\gamma = \iota \circ \gamma_0$ is a proper map from $(0, 1/2]$ into R.

The monotonicity of q (Lemma 1.1) implies $\iota^* q_R \le q_{\mathbb{D}^*}$. Thus we have

$$\int_\gamma q_R = \int_{\gamma_0} \iota^* q_R \le \int_{\gamma_0} q_{\mathbb{D}^*}(z)|dz| \le \text{const.} \int_0^{1/2} t^{-1/2} dt < \infty.$$

□

Remark. From the estimates in this section, we know that there is no natural order relation between q_R and ρ_R. Actually, by Corollary 3.2, we see that

$$\frac{q_{\mathbb{A}_r}(1)^2}{\rho_{\mathbb{A}_r}(1)^2} \sim \frac{(\log 1/r)^2}{r - 1 + \log 1/r} \sim \log \frac{1}{r}$$

as $r \to 0+$. On the other hand, since $\rho_{\mathbb{D}^*}(w) = 1/2|w|\log(1/|w|)$, we see that

$$\frac{q_{\mathbb{D}^*}(w)^2}{\rho_{\mathbb{D}^*}(w)^2} \sim |w|(\log 1/|w|)^2 = o(1)$$

as $w \to 0$.

4. Estimate on a collar

As is well recognized, the presence of short geodesics has a significant effect on the geometry of the unit ball of the Banach space $A(R)$ (see, for example, McMullen [11] or Ohsawa [14]). In this section, we estimate the magnitude of q_R on the collar of a short geodesic in R by utilizing the analysis made in the preceeding section.

Let γ be a simple closed geodesic of R with hyperbolic length ℓ. We now take a holomorphic universal covering $u : \mathbb{H} \to R$ of R with the covering transformation $g_0(z) = e^{2\ell} z$ covering the geodesic γ, where $\mathbb{H}$ deontes the upper half plane $\{z \in \mathbb{C}; \operatorname{Im} z > 0\}$. We denote by G the covering transformation group of u and by G_0 the subgroup of G generated by g_0. Note that G is a torsion-free discrete subgroup of $\operatorname{Aut}(\mathbb{H}) = \mathrm{PSL}(2, \mathbb{R})$. The angular region $\mathbb{H}_\theta = \{z \in \mathbb{H}; |\arg(z/i)| < \theta\}$ for $0 < \theta \le \pi/2$ is invariant under the action of G_0. The covering map $E_\ell(z) = \exp(\pi i \ell^{-1} \log(z/i))$ maps $\mathbb{H}_\theta$ onto the annulus $\mathbb{A}(\theta) = \{\exp(-\pi\theta/\ell) < |w| < \exp(\pi\theta/\ell)\}$ and corresponds to the quotient map $\mathbb{H}_\theta \to \mathbb{H}_\theta/G_0$. For abbreviation, we write $r = \exp(-\pi^2/2\ell)$ and $\mathbb{A} = \mathbb{A}(\pi/2) = \{r < |w| < 1/r\}$. The induced covering $f_\gamma : \mathbb{A} \to R$ determined by $u = f_\gamma \circ E_\ell$ is called the *annular covering* of R associated with γ.

If $\mathbb{H}_\theta$ is precisely invariant under G_0 in G, equivalently, if the map f_γ is injective in $\mathbb{A}(\theta)$, the image $C_\gamma(\theta) = f_\gamma(\mathbb{A}(\theta))$ is called the *collar* of γ with angle θ. For the collar $C_\gamma(\theta)$, the map $\alpha_\gamma := f_\gamma^{-1} : C_\gamma(\theta) \to \mathbb{A}(\theta)$ is served as a local coordinate of R, which will be called the canonical coordinate for γ. The collar lemma is an important tool for our aim. The proof can be found, for example, in [1].

Lemma 4.1 (The collar lemma). *A simple closed geodesic of length ℓ has a collar of angle θ, where θ is determined by* $\tan\theta \sinh \ell = 1$.

Since $\arctan(\sinh(x))$ is concave, the function $\arctan(\sinh(x))/x$ is monotonically decreasing in $x > 0$. In particular, we have $1 > \arctan(\sinh(x))/x$, and hence $\pi/2 - \theta = \arctan(\sinh(\ell)) < \ell$, where θ is the angle in the statement of Lemma 4.1. Therefore, $\mathbb{A}(\theta) \supset \mathbb{A}_1 := \mathbb{A}(\theta_1) = \{e^\pi r < |w| < 1/e^\pi r\}$, where $\theta_1 = \pi/2 - \ell$. We now have the following weak version of the collar lemma.

Corollary 4.2. *Let $f_\gamma : \mathbb{A} = \{r < |w| < 1/r\} \to R$ be the annular covering of R associated with a simple closed geodesic γ with length $\ell < \pi/2$, where $r = \exp(-\pi^2/2\ell)$. Then f_γ is injective in $\mathbb{A}_1 = \{e^\pi r < |w| < 1/e^\pi r\}$.*

Set $\mathbb{A}_2 = \mathbb{A}(\theta_2) = \{e^{2\pi} r < |w| < 1/e^{2\pi} r\}$ for a simple closed geodesic γ with length $\ell < \pi/4$, where $\theta_2 = \pi/2 - 2\ell$. The next result will be used to control quadratic differentials on the collar $C_\gamma(\theta_2)$.

Lemma 4.3. *Let γ be a simple closed geodesic with length $\ell \le \pi/8 = 0.3926\ldots$ in R. Then*

$$q_{\mathbb{A}_1}(w)^2 \le \frac{3}{16\pi|w|^2} \quad (w \in \mathbb{A}_2).$$

Proof. By the symmetry, we may assume $e^{2\pi} r \le |w| \le 1$, where $r = \exp(-\pi^2/2\ell)$. Under the notation in Theorem 3.1, we obtain $2|w|^2 q_{\mathbb{A}_1}(w)^2 \le 2|w|^2 Q_2(w) = 1/\min\{m(e^\pi r|w|), m(|w|/e^\pi r)\}$. Noting $e^\pi r|w| \le e^\pi r < 1$ and $1 < e^\pi \le |w|/e^\pi r$, we have

$$(4.1) \qquad m(e^\pi r|w|) \ge m(e^\pi r) > \frac{\pi^2}{2\ell} - \pi - 1 \ge \frac{\pi^2}{2\ell} - \frac{(\pi+1)\pi}{8\ell} = \frac{\pi(3\pi - 1)}{8\ell} > \frac{\pi^2}{3\ell} \ge \frac{8\pi}{3}$$

and $m(|w|/e^\pi r) \ge m(e^\pi) = 18.999\cdots > 8\pi/3$. Hence we see $|w|^2 q_{\mathbb{A}_1}(w)^2 > 3/16\pi|w|^2$. ☐

We also need a "puncture version" of the collar lemma, which is a consequence of the Shimizu-Leutbecher lemma (see, for example, [8, II.C.5]). Let γ be a simple short loop around a puncture in R. We take a holomorphic universal covering $u : \mathbb{D} \to R$ such that the covering transformation group G of u contains the parabolic element $g_0(z) = z + 1$ which covers γ. Note that the domain $\mathbb{H}(k) = \{z \in \mathbb{H}; \operatorname{Im} z > k\}$ is invariant under g_0 for $k \ge 0$. Set $\mathbb{D}^*(k) = E_0(\mathbb{H}(k)) = \{0 < |w| < e^{-2\pi k}\}$, where $E_0(z) = \exp(2\pi i z)$. Moreover we set $\mathbb{D}^*_1 = \mathbb{D}^*(1)$ and $\mathbb{D}^*_2 = \mathbb{D}^*(\sqrt{2})$. We denote by $f_\gamma : \mathbb{D}^* = \mathbb{D}^*(0) \to R$ the annular covering associated with γ, namely, $u = f_\gamma \circ E_0$.

Lemma 4.4. *Let γ be a simple loop corresponding to a puncture of R. Then the annular covering $f_\gamma : \mathbb{D}^* \to R$ is injective in $\mathbb{D}^*_1$.*

The biholomorphic map $\alpha_\gamma = (f_\gamma)^{-1} : f_\gamma(\mathbb{D}^*_1) \to \mathbb{D}^*_1$ will be called the canonical coordinate of R around the puncture corresponding to γ.

The next lemma is a variant of "thick-thin decomposition".

Lemma 4.5. *Let $\varepsilon_0 = \frac{1}{4}\log 2 = 0.17328\ldots$. If a point p in R has injectivity radius $\iota_R(p) < \varepsilon_0$, then p is contained in either $f_\gamma(\mathbb{A}_2)$ for a simple closed geodesic γ of length less than $\pi/8$ or $f_\gamma(\mathbb{D}^*_2)$ for a simple loop γ corresponding to a puncture.*

Proof. Let γ_1 be a non-contractible closed curve passing through p which minimizes the hyperbolic length, i.e., $\int_{\gamma_1} \rho_R = 2\iota_R(p)$. It is easily verified that γ_1 is simple. First we consider the case that γ_1 has a freely homotopic simple closed geodesic γ of length ℓ. By assumption, we note that $\ell \le 2\iota_R(p) < 2\varepsilon_0 < \pi/8$. The hyperbolic distance d between $ite^{i\theta}$ and $ite^{i\theta+2\ell}$, where $t > 0$, satisfies $\tanh d = 1/\sqrt{1+\cos^2\theta/\sinh^2\ell}$. If $\pi/2 - 2\ell \le |\theta| \le \pi/2$ we have $\cos\theta \le \sin(2\ell)$. So, we have $\cos\theta/\sinh\ell \le \sin(2\ell)/\sinh\ell < 2\ell/\ell = 2$. Thus $d > \operatorname{arctanh}(1/\sqrt{5}) > \operatorname{arctanh}(1/3) = 2\varepsilon_0$, which means that if a closed curve passing through a point in the outside of $f_\gamma(\mathbb{A}_2) = f_\gamma(\mathbb{A}(\pi/2 - 2\ell))$ is freely homotopic to γ, then its length is necessarily at least $2\varepsilon_0$. Hence $p \in f_\gamma(\mathbb{A}_2)$ in this case.

Since the hyperbolic distance d between ik and $ik+1$, where $0 < k \le \sqrt{2}$ satisfies $d = \operatorname{arctanh}(1/\sqrt{1+4k^2}) \ge \operatorname{arctanh}(1/3) = 2\varepsilon_0$, the case that γ_1 corresponds to a puncture can be handled in the same manner as above. □

As an application of our metric q_R, now we are ready to prove the following theorem, which is a more concrete form of Matsuzaki's result. Note that we can read more detailed information on the relationship between $A(R)$ and $B(R)$ from our proof presented below. Recall also the fact that $1/2\lambda^*(R) \le \kappa(R)$ from (1.1).

Theorem 4.6.

$$\kappa(R) \le \max\left\{\frac{6}{\lambda^*(R)}, 34\right\}.$$

Proof. By Proposition 1.9, we have only to show that $(q_R/\rho_R)^2 \le \max\{6/\lambda^*(R), 34\}$. Consider the function $F = q_R/\rho_R$ on R. Let $p \in R$ be arbitrary. If $\iota_R(p) \ge \varepsilon_0 = \operatorname{arctanh}(1/3)/2$, by Proposition 1.2 and Lemma 1.4, we have $F(p) \le (h_R/\rho_R)(p) \le \coth\iota_R(p) \le \coth\varepsilon_0 = 3+2\sqrt{2}$. Note here that $(3+2\sqrt{2})^2 = 33.970\cdots < 34$.

Otherwise, by virtue of Lemma 4.5, we have the following two cases:

- *Case 1:* $p \in f_\gamma(\mathbb{A}_2)$ *for a simple closed geodesic* γ *with length* $\ell < \pi/8$.

Let w be a point in $\mathbb{A}_2$ with $f_\gamma(w) = p$. We may assume $|w| \le 1$ and write $\log|w| = -\pi\theta/\ell$, where $0 \le \theta \le \pi/2 - 2\ell$. Note that $(f_\gamma)^*\rho_R(w) = \rho_{\mathbb{A}}(w) = \ell/2\pi|w|\cos\theta$. By Theorem 3.1, we obtain

$$F(p)^2 \le \frac{q_{\mathbb{A}_1}(w)^2}{\rho_{\mathbb{A}}(w)^2} \le \frac{2\pi^2}{\ell^2}\max\left\{\frac{\cos^2\theta}{m(e^\pi r|w|)}, \frac{\cos^2\theta}{m(|w|/e^\pi r)}\right\}.$$

First, we see $\cos^2\theta/m(e^\pi r|w|) \le 1/m(e^\pi r) < 3\ell/\pi^2$ from (4.1). Second, by (3.4), $m(|w|/e^\pi r) \ge m(e^\pi)|w|/e^{2\pi}r$ holds, and hence

$$\frac{\cos^2\theta}{m(|w|/e^\pi r)} \le \frac{e^{2\pi}r}{m(e^\pi)}e^{\pi\theta/\ell}\cos^2\theta.$$

Since the function $e^{\pi x/2\ell}\cos x$ is monotonically increasing in $0 \le x \le \pi/2 - 2\ell$, we have $\cos^2\theta/m(|w|/e^\pi r) \le \sin^2(2\ell)/m(e^\pi) \le 4\ell^2/m(e^\pi)$. Hence, we have

$$F(p)^2 \le \frac{2\pi^2}{\ell^2}\max\left\{\frac{3\ell}{\pi^2}, \frac{4\ell^2}{m(e^\pi)}\right\} = \max\left\{\frac{6}{\ell}, \frac{8\pi^2}{m(e^\pi)}\right\}.$$

We note here $8\pi^2/m(e^\pi) = 4.1558\ldots$.

- *Case 2:* $p \in f_\gamma(\mathbb{D}_2^*)$ *for a simple closed curve* γ *around a puncture.*

Let w be a point in $\mathbb{D}_2^*$ with $f_\gamma(w) = p$. Note that $0 < |w| < e^{-2\sqrt{2}\pi}$ and that $f_\gamma^* \rho_R(w) = \rho_{\mathbb{D}^*}(w) = 1/2|w| \log(1/|w|)$. We now use Theorem 3.3 to show

$$F(p)^2 \leq \frac{q_{\mathbb{D}_1^*}(w)^2}{\rho_{\mathbb{D}^*}(w)^2} = (2|w| \log |w|)^2 e^{4\pi} q_{\mathbb{D}^*}(e^{2\pi} w) \leq \frac{(2|w| \log |w|)^2}{2|w|^2 m(1/e^{2\pi}|w|)} = \frac{2(\log |w|)^2}{m(1/e^{2\pi}|w|)}.$$

The inequality $m(1/e^{2\pi}|w|) \geq e^{-2\sqrt{2}\pi} m(e^{2(\sqrt{2}-1)\pi})/|w|$ follows from (3.4), and then

$$F(p)^2 \leq \frac{2e^{2\sqrt{2}\pi}|w|(\log |w|)^2}{m(e^{2(\sqrt{2}-1)\pi})}.$$

We can easily check that the function $x(\log x)^2$ is monotonically increasing in $0 < x \leq e^{-2\sqrt{2}\pi} (< e^{-2})$, so we have

$$F(p)^2 \leq \frac{16\pi^2}{m(e^{2(\sqrt{2}-1)\pi})} = 15.957\ldots.$$

Summing up these estimates, we obtain the desired inequality. □

We next concentrate on the lower estimate of q_R near γ. To this end, we use the Petersson series for γ. An estimate similar to ours was given by Wolpert [21]. Another approach appealing to Ahlfors' trick can be found in [10]. We provide an estimate independently for the sake of explicitness.

We denote by $A(\mathbb{H}, G)$ the Banach space of those holomorphic automorphic 2-forms $\varphi = \varphi(z)dz^2$ on $\mathbb{H}$ for a torsion-free Fuchsian group G which are integrable on $\mathbb{H}/G$, namely, $\varphi(g(z))g'(z)^2 = \varphi(z)$ for all $g \in G$ and $\|\varphi\|_{A(\mathbb{H},G)} = \iint_\omega |\varphi(z)| dxdy < +\infty$, where ω is a fundamental domain for G with $\mathrm{Area}(\partial\omega) = 0$. Note that $A(\mathbb{H}, G)$ is naturally isomorphic to the Banach space $A(\mathbb{H}/G)$ via the pullback by the canonical projection $\mathbb{H} \to \mathbb{H}/G$.

Let γ be a simple closed curve of length ℓ on R. Choose a Fuchsian group G uniformizing R in such a way that the element $g_0(z) = e^{2\ell} z$ in G covers γ. Under the above notation and normalization, the Petersson series $\varphi_\gamma \in A(R)$ for γ is defined by

$$f^* \varphi_\gamma = \Theta_{G_0 \backslash G} \varphi_0,$$

where $\varphi_0 = dz^2/z^2$ and $\Theta_{G_0\backslash G}$ denotes the relative Poincaré series operator for $G_0 \backslash G$ (see [6] for details), namely, for $\varphi \in A(\mathbb{H}, G_0)$

$$\Theta_{G_0\backslash G} \varphi = \sum_{g \in G_0 \backslash G} (\varphi \circ g)(g')^2,$$

where the summation is taken over all representatives of right cosets of G_0 in G. Note that $\|\Theta_{G_0\backslash G}\varphi\|_{A(\mathbb{H},G)} \leq \|\varphi\|_{A(\mathbb{H},G_0)}$. Since $\|\varphi_0\|_{A(\mathbb{H},G_0)} = 2\pi\ell$, we have $\|\varphi_\gamma\|_{A(R)} \leq 2\pi\ell$.

Let μ_0 be the $(1, 1)$-form $|\varphi_0|$ and set $\mu = \Theta_{G_0\backslash G}\mu_0 = \sum_{g \in G_0\backslash G} g^*\mu_0 = \mu_0 + \mu_1$, where $g^*\mu_0 = (\mu_0 \circ g)|g'|^2$. Let ω be a fundamental domain for G such that $\mathbb{H}_{\theta_1} \cap W \subset \omega$, where $W = \{z \in \mathbb{H}; e^{-\ell} < |z| < e^{\ell}\}$. Noting $\iint_\omega g^*\mu_0 = \iint_{g(\omega)} \mu_0$ and $(g_0)^*\mu_0 = \mu_0$, we calculate $\iint_\omega \mu = \iint_W \mu_0 = 2\pi\ell$. On the other hand, we have $\iint_\omega \mu_0 \geq \iint_{\mathbb{H}_{\theta_1} \cap W} \mu_0 = 4\theta_1 \ell$. Hence, $\iint_\omega \mu_1 \leq 2(\pi - 2\theta_1)\ell = 4\ell^2$. Set $\psi_0 = -(\ell/\pi)^2 dw^2/w^2$, then we directly see $E_\ell^*\psi_0 = \varphi_0$. If we write $f_\gamma^*\varphi_\gamma = \psi_0 + \psi_1$, then we have $\|\psi_1\|_{A(\mathbb{A}_1)} \leq \iint_{\mathbb{H}_{\theta_1} \cap W} \mu_1 \leq 4\ell^2$.

Now we assume that ℓ is sufficiently small, for example, say $\ell \leq \pi/8$. By Lemma 4.3, we obtain

$$|\psi_1(w)| \leq \frac{q_{\mathbb{A}_1}(w)^2}{\pi} \|\psi_1\|_{A(\mathbb{A}_1)} \leq \frac{3}{16\pi^2|w|^2} 4\ell^2 = \frac{3\ell^2}{4\pi^2|w|^2}$$

for $w \in \mathbb{A}_2$. Thus,

$$|f_\gamma^* \varphi_\gamma(w)| \geq |\psi_0(w)| - |\psi_1(w)| \geq \frac{\ell^2}{4\pi^2|w|^2}$$

for $w \in \mathbb{A}_2$. Recalling the fact $\|\varphi_\gamma\|_{A(R)} \leq 2\pi\ell$, we obtain $q_{R,\alpha_\gamma}(w)^2 = f_\gamma^* q_R(w)^2 \geq \pi |f_\gamma^*\varphi_\gamma(w)|/\|\varphi_\gamma\|_{A(R)} \geq \ell/8\pi^2|w|^2$. Thus we have shown the following estimate, which may be served for the future use.

Proposition 4.7. *For a simple closed geodesic γ with length $\ell \leq \pi/8$, we have $f_\gamma^* q_R(w)^2 \geq \ell/8\pi^2|w|^2$ for $w \in \mathbb{A}_2$.*

5. A characterization of boundedness of geometry

We can now obtain the following estimate, from which we can deduce Theorem 1.8.

Theorem 5.1. *Let R be a non-exceptional hyperbolic Riemann surface and denote by $H(R)$ the supremum of the function h_R/q_R on R. Then, there exists an absolute constant C such that*

$$\frac{1}{\lambda(R)} \leq CH(R)^2.$$

On the other hand, if $\lambda(R) > 0$ we have

$$H(R) \leq K(\lambda(R)) < +\infty,$$

where $K(\lambda(R))$ is a positive constant depending only on $\lambda(R)$.

Proof. First, we prove the first inequality. Suppose that R has a puncture and let γ be a simple loop around the puncture. By Lemma 4.4 and Theorem 3.3, we then have $f_\gamma^* q_R(w)^2 \leq q_{\mathbb{D}_1^*}(w)^2 \sim 1/|w|$ as $w \to 0$. On the other hand, from Lemmas 1.3 and 1.5, we can deduce $f_\gamma^* h_R(w) \geq h_{\mathbb{D}^*}(w) \sim 1/|w|$ as $w \to 0$ Thus we have $(q_R/h_R)(f_\gamma(w))^2 = O(|w|)$ as $w \to 0$, which implies $H(R) = +\infty$. Hence, we conclude that R has no punctures if $H(R) < +\infty$.

Next, let γ be a simple closed geodesic with length $\ell \leq \pi/8$. Then, by Corollaries 4.2 and 3.2, $f_\gamma^* q_R(1) \leq q_{\mathbb{A}_1}(1) \leq 1/2m(e^\pi r)$. Using (4.1), we have $m(e^\pi r) \geq \pi^2/3\ell$. Moreover, we get $f_\gamma^* h_R(1) \geq h_{\mathbb{A}}(1) \geq 1/4(1-r) \geq 1/4(1-e^{-4\pi})$, and then $f_\gamma^* h_R(1)^2 \geq 1/16(1-e^{-4\pi})^2 > 3/16\pi$. So, now we can see $(h_R/q_R)(f_\gamma(1))^2 \geq \pi/8\ell$. Hence, we have $1/\lambda(R) \leq \max\{8H(R)^2/\pi, 8/\pi\}$. Noting that $h_R/q_R \geq 1$ always holds by Proposition 1.2, we conclude that $1/\lambda(R) \leq CH(R)^2$ with $C = 8/\pi = 2.5464\ldots$.

Now we show the latter part. Suppose that, for a positive number λ, there are no constants $K(\lambda)$ which satisfy the property in the statement above. Then there exist a sequence of Riemann surfaces R_n with $\lambda(R_n) \geq \lambda$ and a sequence of points $p_n \in R_n$ ($n = 1, 2, \ldots$) such that

$$\frac{h_{R_n}}{q_{R_n}}(p_n) \to +\infty$$

as $n \to \infty$. Let G_n be a Fuchsian group acting on $\mathbb{D}$ such that $\mathbb{D}/G_n = R_n$ and $u_n(0) = p_n$, where $u_n : \mathbb{D} \to \mathbb{D}/G = R$ is the natural projection. Now we can use the following result due to Chabauty [2] (see also [5]).

Lemma 5.2. *Let G_n be a sequence of Fuchsian groups acting on $\mathbb{D}$. If there exists a neighborhood V of the identity element* id *in the Lie group* $\operatorname{Aut}\mathbb{D} \cong \mathrm{PSL}(2,\mathbb{R})$ *such that $V \cap G_n = \{\mathrm{id}\}$ for all n. Then there exists a subsequence G_{n_j} which converges geometrically to a discrete subgroup G of* $\operatorname{Aut}\mathbb{D}$.

Here we say that G_n geometrically converges to G if the following two conditions are satisfied:

1. For each $\gamma \in G$ there exists a sequence of elements $\gamma_n \in G_n$ such that $\gamma_n \to \gamma$ as $n \to \infty$.
2. If $\gamma_n \in G_n$ converges to an element γ of $\operatorname{Aut}\mathbb{D}$, then $\gamma \in G$.

We return to our situation. By this lemma, we may assume that G_n geometrically converges to some Fuchsian group G. Note also that G is torsion-free by assumption. Now let $R = \mathbb{D}/G$ and $p = u(0)$, where $u : \mathbb{D} \to R$ is the natural projection. As is easily seen, $\lambda(R) \geq \lambda$, in particular, R is non-exceptional. Therefore, by Lemma 1.1, we can find a $\varphi \in A(R)$ which does not vanish at the point p. Then, by the surjectivity of the Poincaré series operator (see, for example, [6]), there exists a $\psi \in A(\mathbb{D})$ such that $u^*\varphi = \Theta_G\psi$. We may assume that $\|\psi\|_{A(\mathbb{D})} = \pi$. Set $\tilde{\varphi}_n = \Theta_{G_n}\psi$ and choose $\varphi_n \in A(R_n)$ so that $\tilde{\varphi}_n = u_n^*\varphi_n$ for $n = 1, 2, \dots$. Then, by [11, Proposition A.2.4], $\tilde{\varphi}_n$ tend to $\tilde{\varphi}$ uniformly on each compact set in $\mathbb{D}$ as $n \to \infty$. Since, $\tilde{\varphi}(0) \neq 0$, we have $|\tilde{\varphi}_n(0)| \geq c_0$ for sufficiently large n, where c_0 is a positive constant. Noting $\|\varphi_n\|_{A(R_n)} \leq \|\psi\|_{A(\mathbb{D})} = \pi$, we obtain $u_n^* q_{R_n}(0)^2 \geq |\tilde{\varphi}_n(0)| \geq c_0$ for sufficiently large n. On the other hand, by Lemma 1.5, $u_n^* h_{R_n}(0) \geq u_n^*\rho_{R_n}(0)\coth(\lambda)/4 = \coth(\lambda)/4$. Thus we have arrived at the contradiction $(h_{R_n}/q_{R_n})(p_n)^2 \leq \coth^2(\lambda)/16c_0$ for n large enough. □

Remark. As Proposition 4.7 suggests, we may state the conjecture: $H(R)^2 \leq C\max\{\lambda(R)^{-1}, 1\}$ for some absolute constant C. The only one obstacle to employ the same method as above is the presence of the case that R_n tends to the exceptional Riemann surface, say, $\mathbb{C} - \{0, 1\}$. This might be overcome by a refinement of estimate of q_R, when Proposition 4.7 would be of use.

6. A lower estimate of q_R in terms of the logarithmic capacity

It seems quite hard to control the metric q_R from below in general. In this section, we propose a lower estimate of q_R by using the Poincaré metric ρ_R and the capacity metric c_R in the case that $R \notin O_G$, i.e., R carries the Green function. Throughout this section, we always assume that $R \notin O_G$.

We denote by c_R the capacity metric (or the logarithmic capacity) of R; precisely, letting α be a local coordinate of R, the metric c_R is defined by the relation

$$G_R(p, p_0) = -\log|\alpha(p) - \alpha(p_0)| - \log c_{R,\alpha}(z_0) + o(1)$$

as $p \to p_0$, where $z_0 = \alpha(p_0)$ and $G_R(p, p_0)$ is the Green function of R with pole at p_0. Classically, $-\log c_R$ is called the Robin constant.

Our result is as follows.

Theorem 6.1. *For $R \notin O_G$, the inequality $q_R^2 \geq \rho_R c_R$ holds.*

Proof. Fix $p_0 \in R$ and take a holomorphic universal covering $u : \mathbb{D} \to R$ from the unit disk $\mathbb{D}$ with covering transformation group G so that $u(0) = p_0$. Note that G is a Fuchsian group of convergence type. In particular, the Blaschke product

$$B(z) = \prod_{g \in G - \{1\}} \frac{|g(0)|}{-g(0)} \frac{z - g(0)}{1 - \overline{g(0)}z}$$

converges in $\mathbb{D}$. Also note that $u^* c_R(0) = B(0)$ (see [17]). Set $\varphi = \Theta_G B$. Then we have $\varphi(0) = B(0) = u^* c_R(0) = u^* \rho_R(0) u^* c_R(0)$. Since $\|\varphi\|_{A(\mathbb{D},G)} \leq \|B\|_{A(\mathbb{D})} \leq \pi$, we obtain $u^* q_R(0)^2 \geq |\varphi(0)| = u^* \rho_R(0) u^* c_R(0)$, which means $q_R^2 \geq \rho_R c_R$ at p_0. □

7. Relationship with the kernel function for $A(R)$

We explain the magnitude of q_R can be described approximately by the usual kernel function for $A(R)$. First, we recall necessary definitions. Let $(\varphi, \psi)_R$ be the natural pairing between $A(R)$ and $B(R)$, namely,

$$(\varphi, \psi)_R = \frac{3}{\pi} \iint_R \rho_R^{-2} \varphi \overline{\psi} = \frac{3}{\pi} \iint_R \rho_R(\zeta)^{-2} \varphi(\zeta) \overline{\psi(\zeta)} d\xi d\eta$$

for $\varphi \in A(R)$ and $\psi \in B(R)$. (The extra factor $3/\pi$ is only for convention.)

Set $K_{\mathbb{D}}(\zeta, z) = (1 - \bar{z}\zeta)^{-4}$ for $\zeta, z \in \mathbb{D}$. Let G be a torsion-free Fuchsian group acting on the unit disk $\mathbb{D}$ with $\mathbb{D}/G = R$. Then we set

$$F(\zeta, z) = \sum_{g \in G} K_{\mathbb{D}}(g(\zeta), z) g'(\zeta)^2$$

for z and $\zeta \in \mathbb{D}$. We denote by u the canonical projection $\mathbb{D} \to \mathbb{D}/G = R$ and define the kernel function $K_R(r, p)$ as a section of the line bundle $\mathcal{K}_R^{\otimes 2} \boxtimes \overline{\mathcal{K}}_R^{\otimes 2}$ over $R \times R$ which is determined by

$$(u \times u)^* K_R(\zeta, z) = F(\zeta, z)$$

for $\zeta, z \in \mathbb{D}$. In other words, $K_R(u(\zeta), u(z)) u'(\zeta)^2 \overline{u'(z)^2} = F(\zeta, z)$ holds locally.

Proposition 7.1 ([6] or [3])**.** *The kernel function $K_R(r, p)$ is holomorphic in r and anti-holomorphic in p and satisfies the following properties.*

1. $K_R(r, p) = \overline{K_R(p, r)}$,
2. $\|K_R(\cdot, p)\|_{A(R)} = \iint_R |K_R(\zeta, p)| d\xi d\eta \leq \pi \rho_R(p)^2$, *where* $\zeta = \xi + i\eta$,
3. $K_R(\cdot, p) = K_R(\zeta, p) d\zeta^2 \in B(R)$ *for each fixed* $p \in R$, *and*
4. $\varphi(p) = (\varphi, K_R(\cdot, p))_R$ *for each* $\varphi \in A(R)$.

We set $\mu_R(p) = \|K_R(\cdot, p)\|_{B(R)} = \sup_{r \in R} \rho_R(r)^{-2} |K_R(r, p)|$. Then, from the above properties, it can be deduced that $\mu_R = \mu_R(z)|dz|^2$ is a $(1,1)$-form on R and the next result follows.

Theorem 7.2. *The inequality $\mu_R \leq q_R^2 \leq 3\mu_R$ is valid on R for an arbitrary non-exceptional hyperbolic Riemann surface R.*

Proof. By property 2 above, we immediately have $\mu_R(p) \le q_R^2(p)$. On the other hand, by property 4, we have $|\varphi(p)| \le \frac{3}{\pi}\mu_R(p)\|\varphi\|_{A(R)}$. This implies $q_R(p)^2 \le 3\mu_R(p)$. □

Setting $M(R) = \sup_{p\in R}(\mu_R/\rho_R^2)(p)$, we have the following

Corollary 7.3.

$$M(R) \le \pi\kappa(R) \le 3M(R).$$

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